SPECTRAL ANALYSIS FOR SOME THIRD-ORDER DIFFERENTIAL EQUATIONS: A SEMIGROUP APPROACH

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Abstract. In this paper we consider the ordinary differential equation of third order

$$\frac{d^3u}{dt^3} + A^\theta \frac{d^2u}{dt^2} + A^\varrho \frac{du}{dt} + Au = 0$$

where $0 \leqslant \theta < \varrho \leqslant 1$, subject to initial conditions

$$u(0) = u_0, \quad \frac{du}{dt}\Big|_{t=0}(0) = u_1, \quad \frac{d^2u}{dt^2}\Big|_{t=0}(0) = u_2$$

where X is a separable Hilbert space, $A : D(A) \subset X \to X$ is an unbounded, linear, closed, densely defined, self-adjoint and positive definite operator, and $A^\alpha : D(A^\alpha) \subset X \to X$ denotes the fractional powers for $\alpha \in (0,1)$. We discuss the possibility of these problems be well posed (here well posedness means that there is a strongly continuous semigroup associated to the equation) in a suitable phase space and for different choices of $0 \leqslant \theta < \varrho \leqslant 1$; namely, $\theta + \varrho < 1$ (range for which the problem is not well posed), $\theta + \varrho = 1$ (range for which the problem may be well posed) and $\theta + \varrho > 1$ (range where the problem may be well posed and the associated semigroup may be analytic). Moreover, we present cases, as well as, applications of our results for evolutionary equations.

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Contents

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1. Introduction

In this paper we consider the ordinary differential equation of third order

$$\frac{d^3u}{dt^3}+A^\theta\frac{d^2u}{dt^2}+A^\varrho\frac{du}{dt}+Au=0 \tag{1.1}$$

where $0\leqslant\theta<\varrho\leqslant 1$, subject to initial conditions

$$u(0)=u_0,\quad \frac{du}{dt}\Big|_{t=0}(0)=u_1,\quad \frac{d^2u}{dt^2}\Big|_{t=0}(0)=u_2 \tag{1.2}$$

where X is a separable Hilbert space and $A: D(A)\subset X\to X$ is an unbounded, linear, self-adjoint and positive definite operator, and therefore, A is a positive sectorial operator in the sense of Henry [20, Definition 1.3.1]. This allows us to define the fractional power $A^{-\alpha}$ of order $\alpha\in(0,1)$ according to Amann [3, Formula 4.6.9] and Henry [20, Theorem 1.4.2], as a closed linear operator on its domain $D(A^{-\alpha})$ with inverse A^{α}, see e.g. Hasse [19, Proposition 3.2.1], Henry [20, Theorem 1.4.2] and Krein [23, Section I.5.2, Formula (5.8)].

Denote by $X^\alpha=D(A^\alpha)$ for $\alpha\in[0,1)$, taking $A^0:=I$ on $X^0:=X$ when $\alpha=0$. Recall that X^α is dense in X for all $\alpha\in(0,1]$, for details see Amann [3, Theorem 4.6.5]. The fractional power space X^α endowed with the norm

$$\|\cdot\|_{X^\alpha}:=\|A^\alpha\cdot\|_X$$

is a Banach space. It is not difficult to show that A^α is an unbounded, linear, self-adjoint and positive definite operator and consequently it is the generator of a strongly continuous analytic semigroup on X, see Kreĭn [23] and Tanabe [29] for any $\alpha \in [0,1]$. With this notation, we have $X^{-\alpha} = (X^\alpha)'$ for all $\alpha > 0$, see Amann [3] and Triebel [31] for the characterization of the negative scale.

In recent years, differential equations of third order has been attracting the attention of many researchers, see e.g. [1], [2], [5], [6], [7], [12], [13], [21], [22], [24], [26], [28] and references therein. The applicability of the results associated with these equations is very connected to the Moore-Gibson-Thompson (MGT) equations; namely, equations of the type

$$\tau\partial_t^3 u - c\Delta_D u + \alpha\partial_t^2 u - b\Delta_D\partial_t u = 0$$

where τ, c, α, b are parameters from physical-mathematical modeling of acoustic waves and Δ_D is the Dirichlet Laplacian operator. For more details see e.g. [1], [2], [7], [12], [13], [21], [22], [24], [26], [28] and references therein. We also observe regularized MGT equations like

$$\tau\partial_t^3 u - c\Delta_D u + \alpha\partial_t^2 u - \delta\Delta_D\partial_t^2 u - b\Delta_D\partial_t u = 0$$

treated in the literature with an analytic semigroup approach, where $\tau, c, \alpha, b, \delta$ are parameters from physical-mathematical modeling of acoustic waves, see e.g. [14].

Moreover, we also have works on the analysis of abstract differential equations of third order in time of the type

$$\frac{d^3u}{dt^3} + Au + \eta\frac{d^2}{dt^2}A^{\frac{1}{3}}u + \eta\frac{d}{dt}A^{\frac{2}{3}}u = f(u)$$

where $\eta \geqslant 0$ under the point of view of the theory of fractional powers of operators, see e.g [5, 6] and [16].

Motivated by strictly mathematical questions no physical appeal necessarily, in this paper we deal with a wide class of abstract differential equations of third order in time as in (1.1)-(1.2) on the phase space $Y_{(\theta,\varrho)}$ to be defined below, in the sense of the classical theory of strongly continuous semigroup, see e.g. [27], and establish the range of the parameters were the well posedness may be pursued. We treat (1.1)-(1.2) in a suitable phase space $Y_{(\theta,\varrho)}$ in different cases for $0 \leqslant \theta < \varrho \leqslant 1$ separately; namely, $\theta + \varrho < 1$, $\theta + \varrho = 1$ and $\theta + \varrho > 1$.

Since well-posed problems (in the sense of semigroups of bounded linear operators) and regular solutions (analytic semigroups) are characterized by the spectral properties of a linear operator, the spectral analysis plays a big role to understand problem (1.1). The aim of this paper is to study the spectrum of the operator related to problem (1.1) (see (2.2) and (2.4)) and from the localization of the spectrum, we determine the situations for which ill-posedness surely happens and the situations where the well posedness and/or regularity may happen. Using a tool from Galois Theory (discriminant of a third order polynomial) we cover the study of the spectrum in all cases where $0 \leqslant \theta < \varrho \leqslant 1$. Thus, for any $0 \leqslant \theta < \varrho \leqslant 1$ we know if our problem can be well-posed (that is, the localization of the spectrum allows it) and, in this case, if the solution can be regular.

The article is organized in the following way. In Section 2 we present general facts about the spectral behavior of our problem. In Section 3 we consider the problem of the generation of strongly continuous and analytic semigroup. Here, we treat (1.1)-(1.2) in a suitable phase space in different cases for $0 \leqslant \theta < \varrho \leqslant 1$ separately; namely, $\theta + \varrho < 1$, $\theta + \varrho = 1$ and $\theta + \varrho > 1$. Moreover, some examples are given. Finally, in Section 4 we explore our results to present an application associated with the Dirichlet Laplacian operator Δ_D.

The figure below better illustrates what is happening.

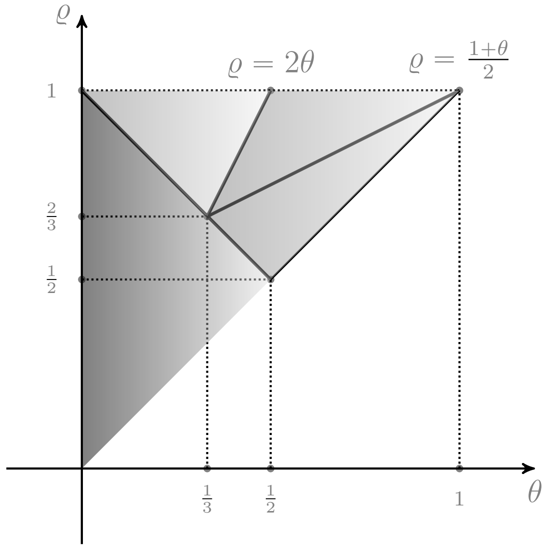

FIGURE 1. $\theta\varrho-$plane

1.) We have the region $\theta + \varrho < 1$ (and $0 \leqslant \theta < \varrho < 1$) in the $\theta\varrho-$plane, where we show that the problem is not well posed. This means that the operator $-\Lambda_{(\theta,\varrho)}$ (see (2.4)) associated to (1.1) does not generate a continuous semigroup;

2.) In the line segment with endpoints $(0, 1)$ and $(\frac{1}{2}, \frac{1}{2})$ in the $\theta\varrho-$plane we show that it is possible that the operator $-\Lambda_{(\theta,\varrho)}$ generates strongly continuous semigroup of bounded linear operators, but it does not generate an analytic semigroup of bounded linear operators, for instance:

2.1.) In $(0, 1)$ we show that it does not generate a strongly continuous semigroup of bounded linear operators;

2.2.) In $(\frac{1}{3}, \frac{2}{3})$ we show that it generates a strongly continuous semigroup of bounded linear operators;

3.) For $1 < \theta + \varrho$, $\varrho < 2\theta$ and $2\varrho < \theta + 1$ and $1 < \theta + \varrho$ and $2\theta < \varrho$ we show that it is possible that the operator $-\Lambda_{(\theta,\varrho)}$ generates strongly continuous semigroup of bounded linear operators however, it does not generate an analytic semigroup;

4.) We have the region $1 < \theta+\varrho$, $\theta+1 \leqslant 2\varrho$ and $\varrho < 2\theta$ (and $0 \leqslant \theta < \varrho < 1$) in the $\theta\varrho-$plane, where we show that it is possible that $-\Lambda_{(\theta,\varrho)}$ can generate an analytic semigroup;

5.) In the line $\varrho = 2\theta$ we show that $-\mathit{\Lambda}_{(\theta,\varrho)}$ can generate an analytic semigroup.

An analysis on the well-posedness and stability of abstract systems with the presence of fractional powers can be found e.g. in [15] and [17]. Motived by these works we summarize our results using the $\theta\varrho-$plane in the Figure 1.

2. Functional framework

The problems of the type (1.1)-(1.2) can be studies in several functional spaces. Motivated by works [5], [8], [9] and [25] we consider the initial value problem (1.1)-(1.2) on the phase space

$$Y_{(\theta,\varrho)} = X^{\varrho} \times X^{\theta} \times X$$

endowed with the norm

$$\forall\ \mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in X, \quad \|\mathbf{u}\|^2_{Y_{(\theta,\varrho)}} = \|u\|^2_{X^{\varrho}} + \|v\|^2_{X^{\theta}} + \|w\|^2_{X}. \tag{2.1}$$

We can write the problem (1.1)-(1.2) as a first order abstract system on Y of the form

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} + \mathit{\Lambda}_{(\theta,\varrho)}\mathbf{u} = 0, \ t > 0, \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{2.2}$$

where $v = u_t$, $w = u_{tt}$, $\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$, $\mathbf{u}_0 = \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}$, and $\mathit{\Lambda}_{(\theta,\varrho)} : D(\mathit{\Lambda}_{(\theta,\varrho)}) \subset Y_{(\theta,\varrho)} \to Y_{(\theta,\varrho)}$ denotes the unbounded linear operator defined by

$$D(\mathit{\Lambda}_{(\theta,\varrho)}) := X^1 \times X^{\varrho} \times X^{\theta} \tag{2.3}$$

and

$$\mathit{\Lambda}_{(\theta,\varrho)} = \begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ A & A^{\varrho} & A^{\theta} \end{bmatrix} \tag{2.4}$$

where

$$\mathit{\Lambda}_{(\theta,\varrho)} \begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix} := \begin{bmatrix} -\varphi \\ -\psi \\ A\phi + A^{\varrho}\varphi + A^{\theta}\psi \end{bmatrix},$$

for any $\begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix} \in D(\mathit{\Lambda}_{(\theta,\varrho)})$.

The choice of space was motived by the previous papers [5] and [6] jointly with the Example 3.5 which treated the case $\theta = \frac{1}{3}$ and $\varrho = \frac{2}{3}$, where $Y_{(\frac{1}{3},\frac{2}{3})} = X^{\frac{2}{3}} \times X^{\frac{1}{3}} \times X$.

Proposition 2.1. *Let $\mathit{\Lambda}_{(\theta,\varrho)} : D(\mathit{\Lambda}_{(\theta,\varrho)}) \subset Y_{(\theta,\varrho)} \to Y_{(\theta,\varrho)}$ be the unbounded linear operator defined in (2.3)-(2.4), then we have all following.*

(*i*) *$\mathit{\Lambda}_{(\theta,\varrho)}$ is densely defined and closed;*

(*ii*) *Zero belongs to resolvent set $\rho(\mathit{\Lambda}_{(\theta,\varrho)})$ of $\mathit{\Lambda}_{(\theta,\varrho)}$; that is, it has an inverse $\mathit{\Lambda}^{-1}_{(\theta,\varrho)}$ where*

$$\mathit{\Lambda}^{-1}_{(\theta,\varrho)} \begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix} := \begin{bmatrix} A^{\varrho-1} & A^{\theta-1} & A^{-1} \\ -I & 0 & 0 \\ 0 & -I & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} A^{\varrho-1}\phi + A^{\theta-1}\varphi + A^{-1}\psi \\ -\phi \\ -\varphi \end{bmatrix}$$

for any $\begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix} \in Y_{(\theta,\varrho)}$. If, in addition, we assume that A has compact resolvent in X and $0 < \theta < \varrho < 1$, then $\mathit{\Lambda}_{(\theta,\varrho)}$ has compact resolvent in Y.

Proof: (i) Firstly, it is not hard to see that $\Lambda_{(\theta,\varrho)}$ is densely defined. Secondly, if $\begin{bmatrix} \phi_n \\ \varphi_n \\ \psi_n \end{bmatrix} \in D(\Lambda_{(\theta,\varrho)})$ converges to $\begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix} \in D(\Lambda_{(\theta,\varrho)})$ as $n \to \infty$ in $Y_{(\theta,\varrho)}$ and $\begin{bmatrix} -\varphi_n \\ -\psi_n \\ A\phi_n + A^{\varrho}\varphi_n + A^{\theta}\psi_n \end{bmatrix}$ converges to $\begin{bmatrix} f \\ g \\ h \end{bmatrix}$ as $n \to \infty$ in $Y_{(\theta,\varrho)}$, then $A^{1-\theta}\phi_n + A^{\varrho-\theta}\varphi_n + \psi_n \to A^{1-\theta}\phi + A^{\varrho-\theta}\varphi + \psi$ as $n \to \infty$ in X, and using the fact the A^{θ} is a closed operator we guarantee that

$$A\phi_n + A^{\varrho}\varphi_n + A^{\theta}\psi_n \to A\phi + A^{\varrho}\varphi + A^{\theta}\psi$$

and $\Lambda_{(\theta,\varrho)}$ is closed;

(ii) It is not hard to see that zero belongs to resolvent set $\rho(\Lambda_{(\theta,\varrho)})$ of $\Lambda_{(\theta,\varrho)}$ and

$$\Lambda^{-1}_{(\theta,\varrho)} \begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix} := \begin{bmatrix} A^{\varrho-1}\phi + A^{\theta-1}\varphi + A^{-1}\psi \\ -\phi \\ -\varphi \end{bmatrix}$$

for any $\begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix} \in Y_{(\theta,\varrho)}$. Moreover, if we assume that A has compact resolvent in X, and $\begin{bmatrix} \phi_n \\ \varphi_n \\ \psi_n \end{bmatrix} \in Y_{(\theta,\varrho)}$ is a bounded sequence, then $\Lambda^{-1}_{(\theta,\varrho)} \begin{bmatrix} \phi_n \\ \varphi_n \\ \psi_n \end{bmatrix}$ is a convergent subsequence in $Y_{(\theta,\varrho)}$, where $0 < \theta < \varrho < 1$.

The figure below better illustrates what is happening.

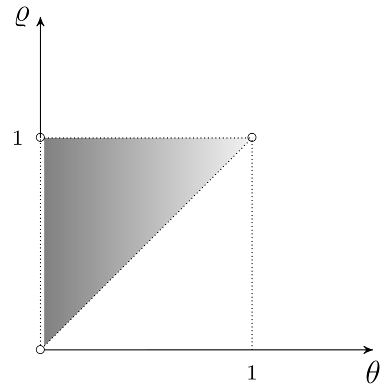

FIGURE 2. Resolvent compactness region

□

Proposition 2.2. *Let $\Lambda_{(\theta,\varrho)} : D(\Lambda_{(\theta,\varrho)}) \subset Y_{(\theta,\varrho)} \to Y_{(\theta,\varrho)}$ be the unbounded linear operator defined in (2.3)-(2.4). If we assume that A has compact resolvent in X and $0 < \theta < \varrho < 1$, then the following assertions are true.*

- (i) *If $\sigma(\Lambda_{(\theta,\varrho)})$ and $\sigma_p(\Lambda_{(\theta,\varrho)})$ denotes the spectrum and point spectrum of the operator $\Lambda_{(\theta,\varrho)}$, respectively; then $\sigma(\Lambda_{(\theta,\varrho)}) = \sigma_p(\Lambda_{(\theta,\varrho)})$ contains at most countably many eigenvalues λ_j;*
- (ii) $\lim_{j\to\infty} |\lambda_j| = \infty$;
- (iii) *For all $\lambda_j \in \sigma(\Lambda_{(\theta,\varrho)})$, then range of $\lambda_j I - \Lambda_{(\theta,\varrho)}$ is closed and $\dim Ker(\lambda_j I - \Lambda_{(\theta,\varrho)}) = codim R(\lambda_j I - \Lambda_{(\theta,\varrho)}) < \infty$.*

Proof: It is an immediate consequence of the Proposition 2.1. □

To better present our results on spectrum of the operator $\Lambda_{(\theta,\varrho)}$ we remember the definition of *approximate point spectrum*, see [30, Chapter V]. If $\lambda \in \mathbb{C}$ is an eigenvalue of $\Lambda_{(\theta,\varrho)}$), then there exists an $x \in D(\Lambda_{(\theta,\varrho)})$ such that $\|x\|_{Y_{(\theta,\varrho)}} = 1$ and $\Lambda_{(\theta,\varrho)}x = \lambda x$.

Definition 2.3. *Let $\Lambda_{(\theta,\varrho)} : D(\Lambda_{(\theta,\varrho)}) \subset Y_{(\theta,\varrho)} \to Y_{(\theta,\varrho)}$ be the unbounded linear operator defined in* (2.3)-(2.4). *We call $\lambda \in \mathbb{C}$ an approximate eigenvalue of $\Lambda_{(\theta,\varrho)}$ if to each $\epsilon > 0$ there corresponds some $x \in D(\Lambda_{(\theta,\varrho)})$ such that $\|x\|_{Y_{(\theta,\varrho)}} = 1$ and $\|\lambda x - \Lambda_{(\theta,\varrho)}x\|_{Y_{(\theta,\varrho)}} < \epsilon$. The set of all such that λ is called the approximate point spectrum of $\Lambda_{(\theta,\varrho)}$.*

Remark 2.4. *We note that $-\Lambda_{(\theta,\varrho)}$ is not a dissipative operator on $Y_{(\theta,\varrho)}$, according to Pazy* [27, Definition 4.1, Chapter 1]. *Indeed, if u be a non-trivial element in X^1 and $\mathbf{u} = \begin{bmatrix} u \\ u \\ 0 \end{bmatrix} \in D(\Lambda_{(\theta,\varrho)})$, then*

$$\begin{aligned}\left\langle -\Lambda_{(\theta,\varrho)}\mathbf{u}, \mathbf{u} \right\rangle_Y &= \left\langle \begin{bmatrix} u \\ 0 \\ -(Au + A^{\varrho}u) \end{bmatrix}, \begin{bmatrix} u \\ u \\ 0 \end{bmatrix} \right\rangle_{Y_{(\theta,\varrho)}} \\ &= \|u\|^2_{X^{\varrho}} > 0.\end{aligned}$$

Explicitly, this means that $-\Lambda_{(\theta,\varrho)}$ is not an infinitesimal generator of a strongly continuous semigroup of contractions on $Y_{(\theta,\varrho)}$.

However, our analysis does not end here. We may know a little more about spectral behavior of the unbounded linear operator $\Lambda_{(\theta,\varrho)}$ and better understand what can happen in the sense of theory of strongly continuous semigroup of bounded linear operators.

3. Generation of strongly continuous and analytic semigroup

Since our operator $-\Lambda_{(\theta,\varrho)}$ never generates a strongly continuous semigroup of contractions on $Y_{(\theta,\varrho)}$, we can analyze the behavior of the spectrum for each $0 \leqslant \theta < \varrho \leqslant 1$ to understand on which cases the operator can possibly generate a strongly continuous or even an analytic semigroup provided we change the norm in the vector space $Y_{(\theta,\varrho)}$. To cover all cases, we divide them in three sub-cases, where we assume that A has compact resolvent in all of them.

3.1. **The case $\theta + \varrho < 1$.** We will show that this is the only case where $-\Lambda_{(\theta,\varrho)}$ can not generate a strongly continuous semigroup of bounded linear operator on $Y_{(\theta,\varrho)}$.

Theorem 3.1. *Let $\Lambda_{(\theta,\varrho)} : D(\Lambda_{(\theta,\varrho)}) \subset Y_{(\theta,\varrho)} \to Y_{(\theta,\varrho)}$ be the unbounded linear operator defined in* (2.3)-(2.4). *If $\theta + \varrho < 1$ and $0 \leqslant \theta < \varrho < 1$, then $-\Lambda_{(\theta,\varrho)}$ does not generate a strongly continuous semigroup of boundary linear operator on $Y_{(\theta,\varrho)}$.*

Proof: We know that the eigenvalues of $-\Lambda_{(\theta,\varrho)}$ are the roots of the polynomial below

$$p_n(\lambda) = \lambda^3 + \mu_n^{\theta}\lambda^2 + \mu_n^{\varrho}\lambda + \mu_n, \tag{3.1}$$

where μ_n are the eigenvalues of A with $\mu_n \to +\infty$.

Since the discriminant of (3.1) is

$$\Delta = 18\mu_n^{\theta+\varrho+1} + \mu_n^{2\theta+2\varrho} - 4\mu_n^{3\varrho} - 4\mu_n^{3\theta+1} - 27\mu_n^2 \tag{3.2}$$

and $\theta+\varrho<1$, we guarantee that the discriminant is negative for large n; that is, (3.1) has two distinct complex roots z_n and $\overline{z_n}$ and one real root x_n.

We will prove that $Re(z_n)\to+\infty$ as $n\to+\infty$; that is, $-\Lambda_{(\theta,\varrho)}$ does not generate a strongly continuous semigroup of bounded linear operators.

From Girard's formula we guarantee that

$$Re(z_n)=\frac{-\mu_n^\theta-x_n}{2}. \tag{3.3}$$

Take $\epsilon>0$ such that

$$\theta+\varrho+3\epsilon<1 \ \text{ and } \ 2\theta+\epsilon<1. \tag{3.4}$$

If we prove that $p_n(-\mu_n^\theta-\mu_n^\epsilon)>0$, it will follow from

$$\lim_{\lambda\to-\infty}p_n(\lambda)=-\infty$$

and by the Intermediate Value Theorem that

$$x_n<-(\mu_n^\theta+\mu_n^\epsilon), \text{ for large } n$$

and from (3.3) we will guarantee that $Re(z_n)\to+\infty$.

Indeed, it is easy to see that

$$p_n(-\mu_n^\theta-\mu_n^\epsilon)=-\mu_n^{2\theta+\epsilon}-2\mu_n^{\theta+2\epsilon}-\mu_n^{3\epsilon}-\mu_n^{\theta+\varrho}-\mu_n^{\varrho+\epsilon}+\mu_n, \tag{3.5}$$

which implies (from (3.4)) that

$$p_n(-\mu_n^\theta-\mu_n^\epsilon)>0, \text{ for large } n$$

and we conclude the proof. □

Example 3.2 (Case $\theta=\varrho=0$). *In this example we consider $\theta=\varrho=0$, and consequently the third-order differential equation*

$$\frac{d^3u}{dt^3}+\frac{d^2u}{dt^2}+\frac{du}{dt}+Au=0. \tag{3.6}$$

In this case we have

$$Y_{(0,0)}=X\times X\times X$$

endowed with the norm given in (2.1), *and $\Lambda_{(0,0)}:D(\Lambda_{(0,0)})\subset Y_{(0,0)}\to Y_{(0,0)}$ denotes the unbounded linear operator defined by*

$$D(\Lambda_{(0,0)}):=X^1\times X\times X \tag{3.7}$$

and

$$\Lambda_{(0,0)}=\begin{bmatrix}0 & -I & 0\\ 0 & 0 & -I\\ A & I & I\end{bmatrix} \tag{3.8}$$

where

$$\Lambda_{(0,0)}\begin{bmatrix}\phi\\ \varphi\\ \psi\end{bmatrix}:=\begin{bmatrix}-\varphi\\ -\psi\\ A\phi+\varphi+\psi\end{bmatrix}$$

for any $\begin{bmatrix}\phi\\ \varphi\\ \psi\end{bmatrix}\in D(\Lambda_{(0,0)})$.

We already know from Theorem 3.1 that $-\Lambda_{(0,0)}$ *does not generate a strongly continuous semigroup of bounded linear operators. The computation below emphasize this result and show how the spectrum explodes on the right side of the complex plane.*

Let

$$c(\mu) = \sqrt[3]{3\sqrt{3}\sqrt{27\mu^2 - 14\mu + 3} + 27\mu - 7}$$

with $\mu \in \sigma(A)$*. Note that* $\sigma_p(-\Lambda_{(0,0)})$ *is given by*

$$\sigma_1 \cup \sigma_2 \cup \sigma_3$$

where

$$\sigma_1 = \Big\{\lambda \in \mathbb{R}; \lambda = \frac{c(\mu)}{3\sqrt[3]{2}} - \frac{2\sqrt[3]{2}}{3c(\mu)} + \frac{1}{3}\Big\},$$

$$\sigma_2 = \Big\{\lambda \in \mathbb{C}; \lambda = -(1 - i\sqrt{3})\frac{c(\mu)}{6\sqrt[3]{2}} + \frac{\sqrt[3]{2}(1 + i\sqrt{3})}{3c(\mu)} + \frac{1}{3}\Big\},$$

and

$$\sigma_3 = \overline{\sigma_2} = \{\overline{\lambda} \in \mathbb{C}; \lambda \in \sigma_2\}.$$

Indeed, let

$$-\Lambda_{(0,0)}\begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix} = \lambda \begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix}$$

be the eigenvalue problem for $-\Lambda_{(0,0)}$*. Then*

$$\begin{cases} -\varphi = \lambda\phi, \\ -\psi = \lambda\varphi, \\ A\phi + \varphi + \psi = \lambda\psi, \end{cases}$$

that is

$$\lambda^3 - \lambda^2 + \lambda - \mu_n = 0, \tag{3.9}$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ *denotes the ordered sequence of eigenvalues of* A *including their multiplicity. We already know that the real root of the equation* (3.9) *is equal to*

$$\lambda_1(\mu_n) = \frac{c(\mu_n)}{3\sqrt[3]{2}} - \frac{2\sqrt[3]{2}}{3c(\mu_n)} + \frac{1}{3}$$

and the two complex roots of the equation (3.9) *are given by*

$$\lambda_2(\mu_n) = -(1 - i\sqrt{3})\frac{c(\mu_n)}{6\sqrt[3]{2}} + \frac{\sqrt[3]{2}(1 + i\sqrt{3})}{3c(\mu_n)} + \frac{1}{3}$$

and

$$\lambda_3(\mu_n) = -(1 + i\sqrt{3})\frac{c(\mu_n)}{6\sqrt[3]{2}} + \frac{\sqrt[3]{2}(1 - i\sqrt{3})}{3 \cdot c(\mu_n)} + \frac{1}{3}.$$

Note that

$$\lim_{n\to\infty} \lambda_1(\mu_n) = -\infty,$$
$$\lim_{n\to\infty} \lambda_2(\mu_n) = \Big(\frac{1}{2} + i\frac{\sqrt{3}}{2}\Big)\infty,$$
$$\lim_{n\to\infty} \lambda_3(\mu_n) = \Big(\frac{1}{2} - i\frac{\sqrt{3}}{2}\Big)\infty.$$

The figure below better illustrates what is happening.

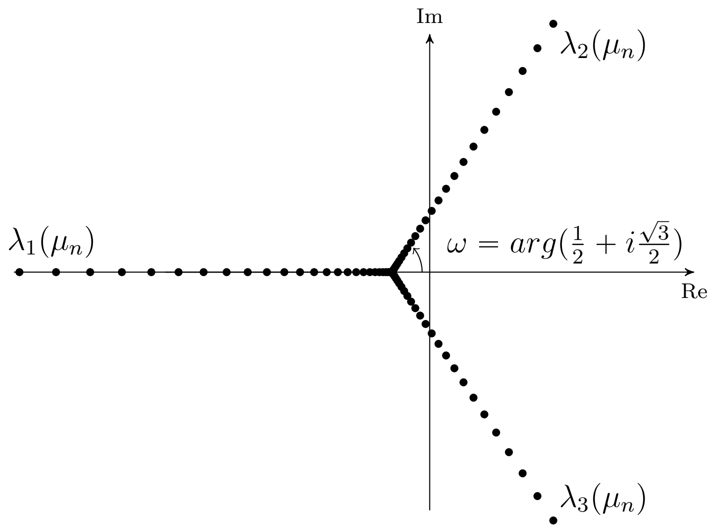

FIGURE 3. Eigenvalues of $-\Lambda_{(0,0)}$

Example 3.3 (Case $\theta = 0$ and $\varrho = \frac{1}{2}$). *In this example we consider $\theta = 0$ and $\varrho = \frac{1}{2}$, and consequently the third-order differential equation*

$$\frac{d^3u}{dt^3} + \frac{d^2u}{dt^2} + A^{\frac{1}{2}}\frac{du}{dt} + Au = 0. \tag{3.10}$$

In this case we have

$$Y_{(0,\frac{1}{2})} = X^{\frac{1}{2}} \times X \times X$$

endowed with the norm given in (2.1), *and $\Lambda_{(0,\frac{1}{2})} : D(\Lambda_{(0,\frac{1}{2})}) \subset Y_{(0,\frac{1}{2})} \to Y_{(0,\frac{1}{2})}$ denotes the unbounded linear operator defined by*

$$D(\Lambda_{(0,\frac{1}{2})}) := X^1 \times X^{\frac{1}{2}} \times X \tag{3.11}$$

and

$$\Lambda_{(0,\frac{1}{2})} = \begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ A & A^{\frac{1}{2}} & I \end{bmatrix}. \tag{3.12}$$

Once more, it follows from Theorem 3.1 that $-\Lambda_{(0,\frac{1}{2})}$ does not generate a continuous semigroup.

Let

$$c(\mu) = \sqrt[3]{3\sqrt{3}\sqrt{-14\mu^{3/2} + 27\mu^2 + 3\mu} + 27\mu - 9\sqrt{\mu} + 2}$$

with $\mu \in \sigma(A)$. *Note that* $\sigma_p(-\varLambda_{(0,\frac{1}{2})})$ *is given by*

$$\sigma_1 \cup \sigma_2 \cup \sigma_3$$

where

$$\sigma_1 = \Big\{\lambda \in \mathbb{R}; \lambda = -\frac{\sqrt[3]{2}(3\sqrt{\mu}-1)}{3c(\mu)} + \frac{3c(\mu)}{3\sqrt[3]{2}} + \frac{1}{3}\Big\},$$

$$\sigma_2 = \Big\{\lambda \in \mathbb{C}; \lambda = \frac{(1+i\sqrt{3})(3\sqrt{\mu}-1)}{3\cdot 2^{2/3}c(\mu)} - \frac{(1-i\sqrt{3})c(\mu)}{6\sqrt[3]{2}} + \frac{1}{3}\Big\},$$

and

$$\sigma_3 = \overline{\sigma_2} = \{\overline{\lambda} \in \mathbb{C}; \lambda \in \sigma_2\}.$$

Indeed, let

$$-\varLambda_{(0,\frac{1}{2})}\begin{bmatrix}\phi\\ \varphi\\ \psi\end{bmatrix} = \lambda\begin{bmatrix}\phi\\ \varphi\\ \psi\end{bmatrix}$$

be the eigenvalue problem for $-\varLambda_{(0,\frac{1}{2})}$. *Then*

$$\begin{cases} -\varphi = \lambda\phi,\\ -\psi = \lambda\varphi,\\ A\phi + A^{\frac{1}{2}}\varphi + \psi = \lambda\psi, \end{cases}$$

that is

$$\lambda^3 - \lambda^2 + \mu_n^{\frac{1}{2}}\lambda - \mu_n = 0, \tag{3.13}$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ *denotes the ordered sequence of eigenvalues of* A *including their multiplicity. We already know that the real root of the equation* (3.13) *is equal to*

$$\lambda_1(\mu_n) = \frac{c(\mu_n)}{3\sqrt[3]{2}} - \frac{2\sqrt[3]{2}}{3\cdot c(\mu_n)} + \frac{1}{3}$$

and the two complex roots of the equation (3.13) *are given by*

$$\lambda_2(\mu_n) = \frac{(1+i\sqrt{3})(3\sqrt{\mu_n}-1)}{3\cdot 2^{2/3}c(\mu_n)} - \frac{(1-i\sqrt{3})c(\mu_n)}{6\sqrt[3]{2}} + \frac{1}{3}$$

and

$$\lambda_3(\mu_n) = \frac{(1-i\sqrt{3})(3\sqrt{\mu_n}-1)}{3\cdot 2^{2/3}c(\mu_n)} - \frac{(1+i\sqrt{3})c(\mu_n)}{6\sqrt[3]{2}} + \frac{1}{3}.$$

Note that

$$\lim_{n\to\infty}\lambda_1(\mu_n) = -\infty,$$

$$\lim_{n\to\infty}\lambda_2(\mu_n) = (\frac{1}{2} + i\frac{\sqrt{3}}{2})\infty,$$

$$\lim_{n\to\infty}\lambda_3(\mu_n) = (\frac{1}{2} - i\frac{\sqrt{3}}{2})\infty,$$

The figure below better illustrates what is happening.

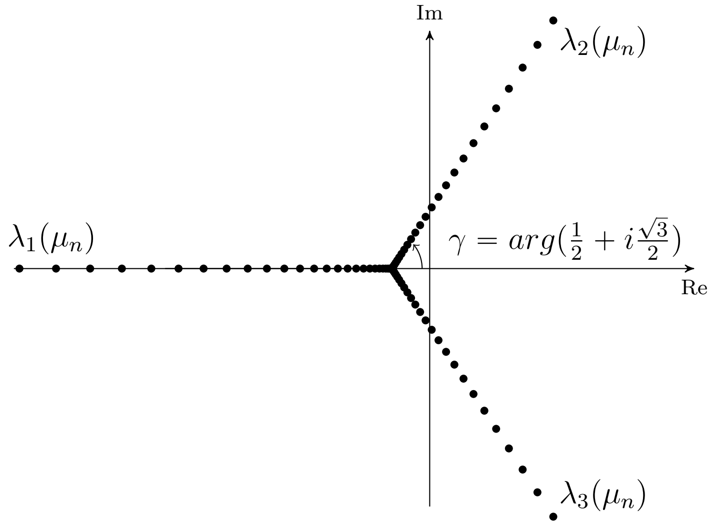

FIGURE 4. Eigenvalues of $-\Lambda_{(0,\frac{1}{2})}$

3.2. **The case** $\theta + \varrho = 1$**.** In this case our polynomial

$$p_n(\lambda) = \lambda^3 + \mu_n^\theta \lambda^2 + \mu_n^{1-\theta}\lambda + \mu_n$$

has $-\mu_n^\theta$ as a real root and therefore (from (3.3)) we guarantee that z_n and $\overline{z_n}$ are imaginary with

$$O(|Im(z_n)|) = \mu_n^{\frac{1-\theta}{2}},^{1}$$

that is, $-\Lambda_{(\theta,\varrho)}$ does not generate an analytic semigroup but it can possibly generate a strongly continuous semigroup of bounded linear operators on Y.

Example 3.4 (Case $\theta = 0$ and $\varrho = 1$)**.** *In this example we consider* $\theta = 0$ *and* $\varrho = 1$, *and consequently the third-order differential equation*

$$\frac{d^3u}{dt^3} + \frac{d^2u}{dt^2} + A\frac{du}{dt} + Au = 0. \tag{3.14}$$

This is perhaps the most familiar case of ordinary differential equation of third order of the type (1.1) *because it sends us to Moore-Gibson-Thompson equations, see e.g.* [1], [2], [7], [12], [13], [21], [22], [24], [26], [28]. *In this case we have*

$$Y_{(0,1)} = X^1 \times X \times X$$

endowed with the norm given in (2.1), *and* $\Lambda_{(0,1)} : D(\Lambda_{(0,1)}) \subset Y_{(0,1)} \to Y_{(0,1)}$ *denotes the unbounded linear operator defined by*

$$D(\Lambda_{(0,1)}) := X^1 \times X^1 \times X \tag{3.15}$$

[1]That is, the term $|Im(z_n)|$ has same order of $\mu_n^{\frac{1-\theta}{2}}$.

and

$$\Lambda_{(0,1)} = \begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ A & A & I. \end{bmatrix} \tag{3.16}$$

Remembering that A has compact resolvent in X, the spectrum of the unbounded linear operator $-\Lambda_{(0,1)}$ is given by

$$\sigma_1 \cup \sigma_2 \cup \sigma_3$$

where

$$\sigma_1 = \{-1\},$$

$$\sigma_2 = \Big\{\lambda \in \mathbb{C}; \lambda = i\sqrt{\mu},\ \mu \in \sigma(A)\},$$

and

$$\sigma_3 = \overline{\sigma_2} = \{\overline{\lambda} \in \mathbb{C}; \lambda \in \sigma_2\}.$$

see e.g. [26]. *For the sake of completeness, we will reproduce the arguments. Let*

$$-\Lambda_{(0,1)} \begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix} = \lambda \begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix}$$

be the eigenvalue problem for $-\Lambda_{(0,1)}$. Then

$$\begin{cases} -\varphi = \lambda\phi, \\ -\psi = \lambda\varphi, \\ A\phi + A\varphi + \psi = \lambda\psi, \end{cases}$$

that is

$$\lambda^3 + \lambda^2 + \mu_n\lambda + \mu_n = (\lambda + 1)(\lambda^2 + \mu_n) = 0, \tag{3.17}$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ denotes the ordered sequence of eigenvalues of A including their multiplicity.

The real root of the equation (3.17) *is equal to*

$$\lambda_1(\mu_n) = -1$$

and the two complex roots of the equation (3.17) *are given by*

$$\lambda_2(\mu_n) = i\sqrt{\mu_n}$$

and

$$\lambda_3(\mu_n) = -i\sqrt{\mu_n}.$$

The figure below better illustrates what is happening.

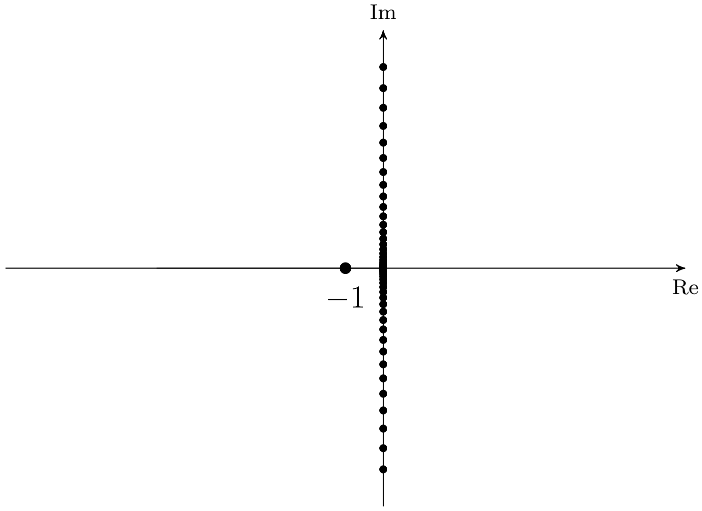

FIGURE 5. Eigenvalues of $-\Lambda_{(0,1)}$

Although the behavior of the spectrum lead us to believe that $-\Lambda_{(0,1)}$ generate a continuous semigroup of linear operators, this is not the case. In order to prove that $-\Lambda_{(0,1)}$ does not generate a strongly continuous semigroup of bounded linear operators on $Y_{(0,1)}$, consider the bounded linear operator $P_{(0,1)} : D(\Lambda_{(0,1)}) \subset Y_{(0,1)} \to Y_{(0,1)}$ given by

$$P_{(0,1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}$$

and note that

$$S_{(0,1)} = -(P_{(0,1)} + \Lambda_{(0,1)})$$

is given by

$$S_{(0,1)} = \begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & 0 \\ A & A & I \end{bmatrix}.$$

From

$$(\lambda I + S_{(0,1)})^{-1} = \begin{bmatrix} \lambda^{-1} I & \lambda^{-2} I & 0 \\ 0 & \lambda^{-1} I & 0 \\ -(\lambda^2+\lambda)^{-1} A & -\lambda^{-2} A & (\lambda+1)^{-1} I \end{bmatrix}$$

and the fact that $\lambda^{-2} I : X \to X^1$ is not bounded we proved that $S_{(0,1)}$ does not generate a strongly continuous semigroup on X. Consequently, $-\Lambda_{(0,1)}$ does not generate a strongly continuous semigroup of bounded linear operators on $Y_{(0,1)}$.

Example 3.5 (Case $\theta = \frac{1}{3}$ and $\varrho = \frac{2}{3}$). *In this example we consider $\theta = \frac{1}{3}$ and $\varrho = \frac{2}{3}$, and consequently the third-order differential equation*

$$\frac{d^3 u}{dt^3} + A^{\frac{1}{3}} \frac{d^2 u}{dt^2} + A^{\frac{2}{3}} \frac{du}{dt} + Au = 0. \tag{3.18}$$

In this case we have

$$Y_{(\frac{1}{3},\frac{2}{3})} = X^{\frac{2}{3}} \times X^{\frac{1}{3}} \times X$$

endowed with the norm given in (2.1), *and* $\Lambda_{(\frac{1}{3},\frac{2}{3})} : D(\Lambda_{(\frac{1}{3},\frac{2}{3})}) \subset Y_{(\frac{1}{3},\frac{2}{3})} \to Y_{(\frac{1}{3},\frac{2}{3})}$ *denotes the unbounded linear operator defined by*

$$D(\Lambda_{(\frac{1}{3},\frac{2}{3})}) := X^1 \times X^{\frac{2}{3}} \times X^{\frac{1}{3}} \tag{3.19}$$

and

$$\Lambda_{(\frac{1}{3},\frac{2}{3})} = \begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ A & A^{\frac{2}{3}} & A^{\frac{1}{3}} \end{bmatrix}. \tag{3.20}$$

Since A has compact resolvent in X, the spectrum of the unbounded linear operator $-\Lambda_{(0,1)}$ *is given by*

$$\sigma_1 \cup \sigma_2 \cup \sigma_3$$

where

$$\sigma_1 = \{-\sqrt[3]{\mu}\},$$

$$\sigma_2 = \Big\{\lambda \in \mathbb{C}; \lambda = i\sqrt[3]{\mu},\ \mu \in \sigma(A)\},$$

and

$$\sigma_3 = \overline{\sigma_2} = \{\overline{\lambda} \in \mathbb{C}; \lambda \in \sigma_2\}.$$

Indeed, let

$$-\Lambda_{(\frac{1}{3},\frac{2}{3})}\begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix} = \lambda\begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix}$$

be the eigenvalue problem for $-\Lambda_{(\frac{1}{3},\frac{2}{3})}$. *Then*

$$\begin{cases} -\varphi = \lambda\phi, \\ -\psi = \lambda\varphi, \\ A\phi + A^{\frac{2}{3}}\varphi + A^{\frac{1}{3}}\psi = \lambda\psi, \end{cases}$$

that is

$$\lambda^3 + \mu_n^{\frac{2}{3}}\lambda^2 + \mu_n^{\frac{1}{3}}\lambda + \mu_n = 0, \tag{3.21}$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ denotes the ordered sequence of eigenvalues of A including their multiplicity.

The real root of the equation (3.21) *is equal to*

$$\lambda_1(\mu_n) = -\sqrt[3]{\mu_n}$$

and the two complex roots of the equation (3.9) *are given by*

$$\lambda_2(\mu_n) = i\sqrt[3]{\mu_n}$$

and

$$\lambda_3(\mu_n) = -i\sqrt[3]{\mu_n}.$$

The figure below better illustrates what is happening.

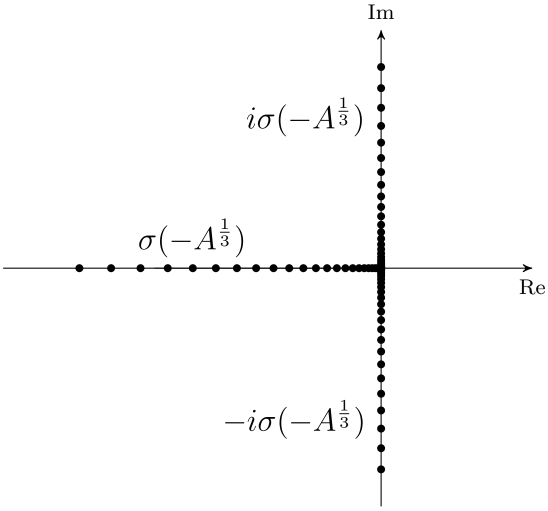

FIGURE 6. Eigenvalues of $-\Lambda_{(\frac{1}{3},\frac{2}{3})}$

In this case we can show that there exists a strongly continuous semigroup of bounded linear operator on $Y_{(\frac{1}{3},\frac{2}{3})}$. Remember that $Y_{(\frac{1}{3},\frac{2}{3})} = X^{\frac{2}{3}} \times X^{\frac{1}{3}} \times X$ and define

$$z = u_{tt} + A^{\frac{2}{3}} u.$$

Note that

$$z_t + A^{\frac{1}{3}} z = 0$$

and denoting $V = \begin{bmatrix} A^{\frac{2}{3}} u \\ A^{\frac{1}{3}} u_t \\ A^{\frac{1}{3}} z \end{bmatrix}$ we have $\dfrac{dV}{dt} = MV$, where

$$M = \begin{bmatrix} 0 & A^{\frac{1}{3}} & 0 \\ -A^{\frac{1}{3}} & 0 & 1 \\ 0 & 0 & -A^{\frac{1}{3}} \end{bmatrix}.$$

Since $M = B + D$, where B is the bounded linear operator given by

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and D is the linear operator given by

$$D = \begin{bmatrix} 0 & A^{\frac{1}{3}} & 0 \\ -A^{\frac{1}{3}} & 0 & 0 \\ 0 & 0 & -A^{\frac{1}{3}} \end{bmatrix}$$

which is dissipative using an equivalent norm on $X \times X \times X$, we guarantee that our original problem (2.2) has a solution $U = \begin{bmatrix} u \\ u_t \\ u_{tt} \end{bmatrix} \in X^{\frac{2}{3}} \times X^{\frac{1}{3}} \times X$ in the sense of the theory of strongly continuous semigroup of bounded linear operators.

Example 3.6 (Case $\theta = \frac{1}{2}$ and $\varrho = \frac{1}{2}$)**.** *In this example we consider $\theta = \varrho = \frac{1}{2}$, and consequently the third-order differential equation*

$$\frac{d^3u}{dt^3} + A^{\frac{1}{2}}\frac{d^2u}{dt^2} + A^{\frac{1}{2}}\frac{du}{dt} + Au = 0. \tag{3.22}$$

In this case we have

$$Y_{(\frac{1}{2},\frac{1}{2})} = X^{\frac{1}{2}} \times X^{\frac{1}{2}} \times X$$

endowed with the norm given in (2.1)*, and $\Lambda_{(0,\frac{1}{2})} : D(\Lambda_{(0,\frac{1}{2})}) \subset Y_{(\frac{1}{2},\frac{1}{2})} \to Y_{(\frac{1}{2},\frac{1}{2})}$ denotes the unbounded linear operator defined by*

$$D(\Lambda_{(\frac{1}{2},\frac{1}{2})}) := X^1 \times X^{\frac{1}{2}} \times X^{\frac{1}{2}} \tag{3.23}$$

and

$$\Lambda_{(\frac{1}{2},\frac{1}{2})} = \begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ A & A^{\frac{1}{2}} & A^{\frac{1}{2}} \end{bmatrix} \tag{3.24}$$

where.

If we assume that A has compact resolvent in X, then the spectrum of the unbounded linear operator $-\Lambda_{(\frac{1}{2},\frac{1}{2})}$ is given by

$$\sigma_1 \cup \sigma_2 \cup \sigma_3$$

where

$$\sigma_1 = \{-\sqrt{\mu_n};\ \mu_n \in \sigma(A)\}\},$$
$$\sigma_2 = \Big\{i\sqrt[4]{\mu_n};\ \mu_n \in \sigma(A)\},$$

and

$$\sigma_3 = \overline{\sigma_2} = \{\overline{\lambda} \in \mathbb{C}; \lambda \in \sigma_2\}.$$

Indeed, let

$$-\Lambda_{(\frac{1}{2},\frac{1}{2})}\begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix} = \lambda\begin{bmatrix} \phi \\ \varphi \\ \psi \end{bmatrix}$$

be the eigenvalue problem for $-\Lambda_{(\frac{1}{2},\frac{1}{2})}$. Then

$$\begin{cases} -\varphi = \lambda\phi, \\ -\psi = \lambda\varphi, \\ A^{\frac{1}{2}}(A^{\frac{1}{2}}\phi + \varphi + \psi) = \lambda\psi, \end{cases}$$

that is

$$\lambda^3 + \mu_n^{\frac{1}{2}}\lambda^2 + \mu_n^{\frac{1}{2}}\lambda + \mu_n = (\lambda + \mu_n^{\frac{1}{2}})(\lambda^2 + \mu_n^{\frac{1}{2}}) = 0, \tag{3.25}$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ denotes the ordered sequence of eigenvalues of A including their multiplicity.

The real root of the equation (3.25) *is equal to*

$$\lambda_1(\mu_n) = -\sqrt{\mu_n}$$

and the two complex roots of the equation (3.25) *are given by*

$$\lambda_2(\mu_n) = i\sqrt[4]{\mu_n}$$

and

$$\lambda_3(\mu_n) = -i\sqrt[4]{\mu_n}.$$

The figure below better illustrates what is happening.

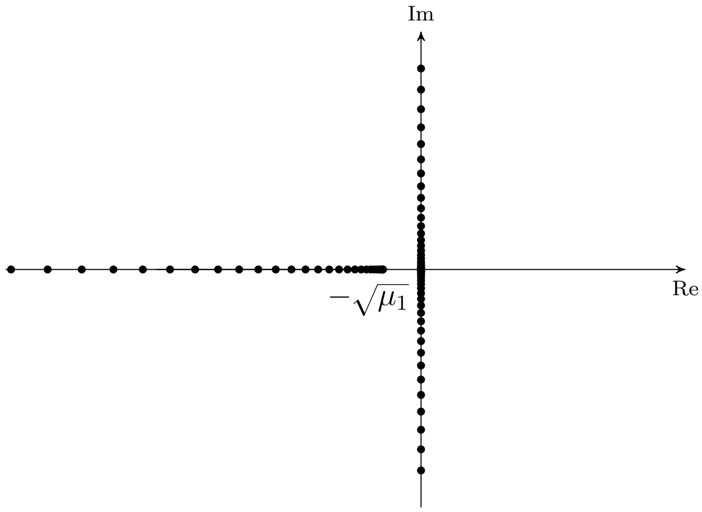

FIGURE 7. Eigenvalues of $-\Lambda_{(\frac{1}{2},\frac{1}{2})}$

3.3. **The case $\theta + \varrho > 1$.** We will show that in this sub-case there exists the possibility of the operator $-\Lambda_{(\theta,\varrho)}$ generate an analytic semigroup on $Y_{(\theta,\varrho)}$. First of all, remember that when $\Delta < 0$ the real solution of (3.1) is given by

$$x_n = -\frac{\mu_n^\theta}{3} + \frac{f(\mu_n)}{3\sqrt[3]{2}} - \frac{\sqrt[3]{2}(3\mu_n^\varrho - \mu_n^{2\theta})}{3f(\mu_n)} \tag{3.26}$$

and one of the complex solution z_n is

$$z_n = -\frac{\mu_n^\theta}{3} - (1 - i\sqrt{3})\frac{f(\mu_n)}{6\sqrt[3]{2}} + (1 + i\sqrt{3})\frac{(3\mu_n^\varrho - \mu_n^{2\theta})}{3\sqrt[3]{4}f(\mu_n)}, \tag{3.27}$$

where

$$f(\mu_n) = \sqrt[3]{9\mu_n^{\theta+\varrho} - 2\mu_n^{3\theta} - 27\mu_n + \sqrt{-27\Delta}}\ . \tag{3.28}$$

We will divide the sub-case $\theta + \varrho > 1$ into two sub-cases:
Sub-case $\varrho < 2\theta$: Remember that in this case we have

$$O(9\mu_n^{\theta+\varrho} - 2\mu_n^{3\theta} - 27\mu_n) = -2\mu_n^{3\theta}.$$

Now we divide this sub-case in the following possibilities:

a) $\theta + 1 = 2\varrho$: In this sub-case we have

$$\Delta = 14\mu_n^{\frac{3\theta+3}{2}} - 3\mu_n^{3\theta+1} - 27\mu_n^2 \tag{3.29}$$

and since $3\theta+1 > \frac{3\theta+3}{2}$ we guarantee that $\Delta < 0$ for large n. Moreover, note that $3\theta+1$ is the larger exponent in (3.29); that is,

$$O(\Delta) = -3\mu_n^{3\theta+1}$$

and from $\dfrac{3\theta+1}{2} = \dfrac{2\theta+2\varrho}{2} < 3\theta$ we have

$$O(f(\mu_n)) = -\sqrt[3]{2}\cdot\mu_n^{\theta}.$$

Hence, it follows from (3.26) and (3.27) that

$$O(z_n) = \mu_n^{-\frac{\theta}{2}}\left(\frac{-1-i\sqrt{3}}{2}\right) \text{ and } O(x_n) = -\mu_n^{\frac{1-\theta}{2}}.$$

b) $\theta+1 > 2\varrho$: Note that in this sub-case we have

$$O(\Delta) = -4\mu_n^{3\theta+1},$$

which ensures that $\Delta < 0$ for large n.

Note that from $\theta+\varrho > 1$ and $\varrho < 2\theta$ we must have $\theta > \dfrac{1}{3}$; that is,

$$\frac{3\theta+1}{2} < 3\theta.$$

Thus

$$O(f(\mu_n)) = -\sqrt[3]{2}\cdot\mu_n^{\theta}$$

and following as in case (3.3) we guarantee that

$$O(Re(z_n)) = -\frac{\mu_n^{\varrho-\theta}}{2},\ O(Im(z_n)) = \left(\mu_n^{1-\theta} - \frac{\mu_n^{2\varrho-2\theta}}{4}\right)^{\frac{1}{2}}$$

and

$$O(x_n) = -\mu_n^{\theta} + \frac{\mu_n^{\varrho}}{\mu_n^{\theta}}.$$

c) $\theta+1 < 2\varrho$: In this case the discriminant Δ in (3.2) is positive for large n (the larger exponent is $2\theta+2\varrho$); that is, p_n has three distinct real roots. From (3.1) it is easy to see that all roots are negative for large n.

Sub-case $\varrho = 2\theta$: In this case we have

$$\Delta = -3\mu_n^{6\theta} + 14\mu_n^{3\theta+1} - 27\mu_n^2$$

and the larger exponent of Δ is 6θ, which guarantees that $\Delta < 0$ for large n; that is, p_n has two complex roots.

Moreover, from

$$9\mu_n^{\theta+\varrho} - 2\mu_n^{3\theta} - 27\mu_n = 7\mu_n^{3\theta} - 27\mu_n$$

we have

$$O(f(\mu_n)) = \sqrt[3]{16}\cdot\mu_n^{\theta}.$$

With some algebraic manipulation we obtain that

$$O(z_n) = \mu_n^{\theta}\left(\frac{-1-i\sqrt{3}}{2}\right) \text{ and } O(x_n) = -\mu_n^{1-2\theta}.$$

Sub-case $\varrho > 2\theta$: In this case the larger exponent of Δ is 3ϱ again, which guarantees that $\Delta < 0$ and p_n has two complex roots. Now will not proceed as the previous cases, since the highest order of $f(\mu_n)$ vanishes and make the computations harder. Our approach will consist in use the Intermediate Value Theorem together with (3.3) to guarantee that $-\Lambda_{(\theta,\varrho)}$ can not be sectorial but can generate a continuous semigroup of bounded linear operators.

Remember that $x_n < 0$ for large n and from (3.3) we have

$$Re(z_n)^2 = \frac{\mu_n^{2\theta} + 2\mu_n^{\theta}x_n + x_n^2}{4}$$

and

$$\left(\frac{Im(z_n)}{Re(z_n)}\right)^2 = \frac{4\mu_n}{-x_n\mu_n^{2\theta} - 2x_n^2\mu_n^{\theta} - x_n^3} - 1. \tag{3.30}$$

Hence, if we show that

$$\frac{4\mu_n}{-x_n\mu_n^{2\theta} - 2x_n^2\mu_n^{\theta} - x_n^3} \to +\infty$$

we guarantee that $-\Lambda_{(\theta,\varrho)}$ can not be sectorial. Note that from $1 < \theta + \varrho$ and $2\theta < \varrho$ we must have $\frac{2}{3} < \varrho$. Moreover, it holds

$$1 + \theta < 2\varrho. \tag{3.31}$$

Indeed, if $\theta \leqslant \frac{1}{3}$ we have

$$1 + \theta \leqslant \frac{4}{3} < 2\varrho$$

and if $\theta > \frac{1}{3}$

$$1 + \theta < 4\theta < 2\varrho.$$

Fix $\epsilon > 0$ such that

$$1 + \theta + \epsilon < 2\varrho, \quad 1 + \epsilon < \theta + \varrho, \quad 2 + 3\epsilon < 3\varrho \;\text{ and }\; 2\theta + \epsilon < \varrho. \tag{3.32}$$

It is easy to see that

$$p_n(-\mu_n^{1-\varrho+\epsilon}) < 0$$

which implies, together with $p_n(0) > 0$ that $x_n > -\mu_n^{1-\varrho+\epsilon}$. Therefore

$$-x_n\mu_n^{2\theta} - 2x_n^2\mu_n^{\theta} - x_n^3 < \mu_n^{1+2\theta-\varrho+\epsilon} + 2\mu_n^{\theta+2-2\varrho+2\epsilon} + \mu_n^{3-3\varrho+3\epsilon}$$

that is,

$$-x_n\mu_n^{2\theta} - 2x_n^2\mu_n^{\theta} - x_n^3 \leqslant 4\mu_n^{1+2\theta-\varrho+\epsilon}$$

and

$$\frac{4\mu_n}{-x_n\mu_n^{2\theta} - 2x_n^2\mu_n^{\theta} - x_n^3} \geqslant \frac{\mu_n}{\mu_n^{1+2\theta-\varrho+\epsilon}}. \tag{3.33}$$

It follows from (3.32) that

$$1 + 2\theta - \varrho + \epsilon < 1$$

and from (3.33) we conclude that

$$\frac{4\mu_n}{-x_n\mu_n^{2\theta} - 2x_n^2\mu_n^\theta - x_n^3} \to +\infty.$$

From (3.30) we guarantee that $-\Lambda_{(\theta,\varrho)}$ can not be sectorial. On the other hand $1 < \theta + \varrho$ implies that $p_n(-\mu_n\theta) < 0$ and consequently

$$-\mu_n^\theta < x_n < 0.$$

Therefore, from (3.3) we obtain that $Re(z_n) < 0$ and, since $x_n < 0$, $-\Lambda_{(\theta,\varrho)}$ can generate a strongly continuous semigroup of bounded linear operators.

Example 3.7 (Case $\theta = \frac{1}{2}$ and $\varrho = 1$)**.** *In this example we consider $\theta = \frac{1}{2}$ and $\varrho = 1$, and consequently the third-order differential equation*

$$\frac{d^3u}{dt^3} + A^{\frac{1}{2}}\frac{d^2u}{dt^2} + A\frac{du}{dt} + Au = 0. \tag{3.34}$$

In this case we have

$$Y_{(\frac{1}{2},1)} = X^1 \times X^{\frac{1}{2}} \times X$$

endowed with the norm given in (2.1), *and $\Lambda_{(\frac{1}{2},1)} : D(\Lambda_{(\frac{1}{2},1)}) \subset Y_{(\frac{1}{2},1)} \to Y_{(\frac{1}{2},1)}$ denotes the unbounded linear operator defined by*

$$D(\Lambda_{(\frac{1}{2},1)}) := X^1 \times X^1 \times X^{\frac{1}{2}} \tag{3.35}$$

and

$$\Lambda_{(\frac{1}{2},1)} = \begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ A & A & A^{\frac{1}{2}} \end{bmatrix} \tag{3.36}$$

Since $\theta + \varrho > 1$ and $\varrho = 2\theta$, we already know that $-\Lambda_{(\frac{1}{2},1)}$ can be a sectorial operator. The following Theorem shows how the spectrum behaves, more specifically, how it explodes on the left side of the complex plane.

Let

$$c(\mu) = \sqrt[3]{7\mu^{3/2} + \sqrt{-378\mu^{5/2} + 81\mu^3 + 729\mu^2} - 27\mu}$$

with $\mu \in \sigma(A)$. If we assume that A has compact resolvent in X, then the spectrum of the unbounded linear operator $-\Lambda_{(\frac{1}{2},1)}$ is given by

$$\sigma_1 \cup \sigma_2 \cup \sigma_3$$

where

$$\sigma_1 = \Big\{\lambda \in \mathbb{R}; \lambda = -\frac{2\mu\sqrt[3]{2}}{3\cdot c(\mu)} + \frac{c(\mu)}{3\sqrt[3]{2}} - \frac{\sqrt{\mu}}{3}\Big\},$$

$$\sigma_2 = \Big\{\lambda \in \mathbb{C}; \lambda = \frac{\mu\sqrt[3]{2}(1+i\sqrt{3})}{3\cdot c(\mu)} - \frac{c(\mu)(1-i\sqrt{3})}{6\sqrt[3]{2}} - \frac{\sqrt{\mu}}{3}\Big\},$$

and

$$\sigma_3 = \overline{\sigma_2} = \{\overline{\lambda} \in \mathbb{C}; \lambda \in \sigma_2\}.$$

Indeed, let

$$-\Lambda_{(\frac{1}{2},1)}\begin{bmatrix}\phi\\ \varphi\\ \psi\end{bmatrix}=\lambda\begin{bmatrix}\phi\\ \varphi\\ \psi\end{bmatrix}$$

be the eigenvalue problem for $-\Lambda_{(\frac{1}{2},1)}$*. Then*

$$\begin{cases}-\varphi=\lambda\phi,\\ -\psi=\lambda\varphi,\\ A^{\frac{1}{2}}(A^{\frac{1}{2}}\phi+A^{\frac{1}{2}}\varphi+\psi)=\lambda\psi,\end{cases}$$

that is

$$\lambda^3+\mu_n^{\frac{1}{2}}\lambda^2+\mu_n\lambda+\mu_n=0, \tag{3.37}$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ *denotes the ordered sequence of eigenvalues of* A *including their multiplicity. From (3.26) and (3.27) we know that the real root of the equation* (3.37) *is equal to*

$$\lambda_1(\mu_n)=-\frac{2\mu_n\sqrt[3]{2}}{3\cdot c(\mu_n)}+\frac{c(\mu_n)}{3\sqrt[3]{2}}-\frac{\sqrt{\mu_n}}{3}$$

and the two complex roots of the equation (3.37) *are given by*

$$\lambda_2(\mu_n)=\frac{\mu_n\sqrt[3]{2}(1+i\sqrt{3})}{3\cdot c(\mu_n)}-\frac{c(\mu_n)(1-i\sqrt{3})}{6\sqrt[3]{2}}-\frac{\sqrt{\mu_n}}{3}$$

and

$$\lambda_3(\mu_n)=\frac{\mu_n\sqrt[3]{2}(1-i\sqrt{3})}{3\cdot c(\mu_n)}-\frac{c(\mu_n)(1+i\sqrt{3})}{6\sqrt[3]{2}}-\frac{\sqrt{\mu_n}}{3}.$$

Moreover, it holds

$$\begin{aligned}&\lim_{n\to\infty}\lambda_1(\mu_n)=-1,\\ &\lim_{n\to\infty}\lambda_2(\mu_n)=(-\frac{1}{2}+i\frac{\sqrt{3}}{2})\infty,\\ &\lim_{n\to\infty}\lambda_3(\mu_n)=(-\frac{1}{2}-i\frac{\sqrt{3}}{2})\infty.\end{aligned}$$

and the figure below better illustrates what is happening.

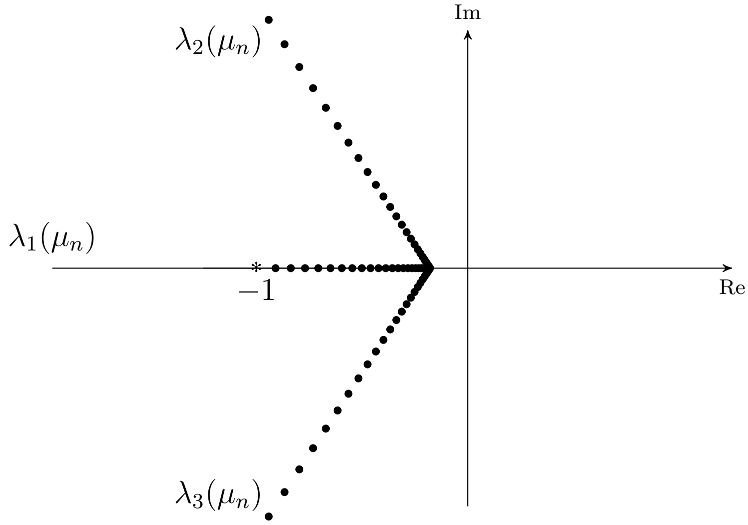

FIGURE 8. Eigenvalues of $-\Lambda_{(\frac{1}{2},1)}$

Therefore, we can conclude that -1 belongs to the approximate point spectrum of $\Lambda_{(\frac{1}{2},1)}$.

Remark 3.8. *Note that if $\Delta > 0$ for large n then p_n has three real roots and it follows easily that they are all negative at the infinity; that is, $-\Lambda_{(\theta,\varrho)}$ can be sectorial in this cases.*

When $\Delta < 0$ we know that $f(\mu_n) \neq 0$ since $f(\mu_n) = 0$ if, and only if all roots of p_n are real and equal to $-\frac{\mu_n^\theta}{3}$. Therefore we can use expressions (3.26) and (3.27) appropriately.

3.4. **Conclusion.** The tables below summarize our results, where ✓ means that **it is possible to generate the respective strongly continuous semigroups** of bounded linear operators on Y, and ✗ means that **it it not possible to generate the respective strongly continuous semigroups** of bounded linear operators on Y.

For $\theta + \varrho < 1$ we have the following results:

Analytic Semigroup	Strongly Continuous Semigroup
✗	✗

TABLE 1. Case $\theta + \varrho < 1$

For $\theta + \varrho = 1$ we have the following results:

Analytic Semigroup	Strongly Continuous Semigroup
✗	✓

TABLE 2. Case $\theta + \varrho = 1$

For $\theta + \varrho > 1$ we have:

Cases	Analytic Semigroup	Strongly Continuous Semigroup
$\varrho < 2\theta$ and $\theta + 1 = 2\varrho$	✓	✓
$\varrho < 2\theta$ and $\theta + 1 > 2\varrho$	✗	✓
$\varrho < 2\theta$ and $\theta + 1 < 2\varrho$	✓	✓
$\varrho = 2\theta$	✓	✓
$\varrho > 2\theta$	✗	✓

Table 3. Case $\theta + \varrho > 1$

Note that we did not characterize all cases where $-\Lambda_{(\theta,\varrho)}$ is the generator of a strongly continuous semigroup or an analytic semigroup. However, despite the difficulties encountered in the non-convered cases, we have the following conjecture:

Conjecture: Assume that $0 < \theta < \varrho < 1$ and $1 < \theta + \varrho$. Then:

(1) If $1 < \theta + \varrho$, $\varrho < 2\theta$ and $\theta + 1 \leqslant 2\varrho$, then $-\Lambda_{(\theta,\varrho)}$ generates an analytic semigroup;
(2) If $\theta = 2\varrho$, then $-\Lambda_{(\theta,\varrho)}$ generates an analytic semigroup;
(3) If $2\theta < \varrho$, then $-\Lambda_{(\theta,\varrho)}$ generates a Gevrey semigroup;
(4) If $\varrho < 2\theta$ and $2\varrho < \theta + 1$, then $-\Lambda_{(\theta,\varrho)}$ generates a Gevrey semigroup,

for more details on Gevrey class semigroups, see e.g. [11].

Moreover, it is possible to get asymptotic eigenvalue expression for the $\theta - \rho$ unit square. The partition of the parameter region rely on this, see e.g. [4] and [18].

4. Applications

We can consider a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth (at least $C^{2,\alpha}$) boundary and $N \in \mathbb{N}$. Let $A = -\Delta_D$ be the unbounded linear operator, where Δ_D denotes the Laplacian operator with homogeneous Dirichlet boundary condition. Its $L^2(\Omega)$-normalized eigenfunctions are denoted w_j, and its eigenvalues counted with their multiplicities are denoted λ_j; that is,

$$-\Delta_D w_j = \mu_j w_j. \tag{4.1}$$

It is well know that $0 < \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_j \leqslant \cdots$, $\mu_j \to \infty$ as $j \to \infty$, and that $-\Delta_D$ is a positive self-adjoint operator in $L^2(\Omega)$ with domain $D(-\Delta_D) = H^2(\Omega) \cap H_0^1(\Omega)$, and that Δ_D generates a compact analytic C^0-semigroup in $L^2(\Omega)$, see Henry [20].

This allows us to define the fractional power $A^{-\alpha}$ of order $\alpha \in (0,1)$ according to Amann [3, Formula 4.6.9] and Henry [20, Theorem 1.4.2], as a closed linear operator on its domain $D(A^{-\alpha})$ with inverse A^{α}. Denote by $X^\alpha = D(A^\alpha)$ for $\alpha \in [0,1]$. The fractional power space X^α endowed with graphic norm

$$\|\cdot\|_{X^\alpha} := \|A^\alpha \cdot\|_X$$

is a Banach space; namely, e.g., if $m\alpha$ is an integer, then

$$X^\alpha = D((-\Delta_D)^\alpha) = H^{2\alpha}(\Omega) \cap H_0^\alpha(\Omega)$$

with equivalent norms, see Cholewa and Dłotko [10, Page 29] and Henry [20, Pages 29 and 30].

With this notation, we have $X^{-\alpha} = (X^{\alpha})'$ for all $\alpha > 0$, see Amann [3] for the characterization of the negative scale.

The scale of fractional power spaces $\{X^{\alpha}\}_{\alpha\in\mathbb{R}}$ associated with A_D safisty

$$X^{\alpha} \subset H^{2\alpha}(\Omega), \quad \alpha \in [0,1],$$

where $H^{2\alpha}(\Omega)$ are the potential Bessel spaces, see Cholewa and Dłotko [10, Page 48].

From Sobolev embedding theorem, we obtain

$$X^{\alpha} \subset L^{r}(\Omega), \text{ for any } r \leqslant \frac{2N}{N-4\alpha}, \ 0 \leqslant \alpha < \frac{N}{4},$$

$$X = L^2(\Omega),$$

$$L^{s}(\Omega) \subset X^{\alpha}, \text{ for any } s \geqslant \frac{2N}{N-4\alpha}, \ -\frac{N}{4} < \alpha \leqslant 0,$$

with continuous embeddings.

Let $\alpha \in (0,1]$. We recall that the fractional powers of the negative Laplacian operator can be calculated through the spectral decomposition: since $X = L^2(\Omega)$ is a Hilbert space and $A = -\Delta_D$ with zero Dirichlet boundary condition in Ω is a self-adjoint operator and is the infinitesimal generator of a C_0-semigroup of contractions on X, it follows that there exists an orthonormal basis composed by eigenfunctions $\{\varphi_j\}_{j\in\mathbb{N}}$ of A. Let ν_j be the eigenvalues of $A = -\Delta_D$, then $(\nu_j^{\alpha}, \varphi_j)$ are the eigenvalues and eigenfunctions of $A^{\alpha} = (-\Delta_D)^{\alpha}$, also with zero Dirichlet boundary condition, respectively.

It is well know that the fractional Laplacian $A^{\alpha} : D(A^{\alpha}) \subset X \to X$ is well defined in the space

$$D(A^{\alpha}) = X^{\alpha} = \Big\{ u = \sum_{j=1}^{\infty} a_j \varphi_j \in L^2(\Omega) : \sum_{j=1}^{\infty} a_j^2 \nu_j^{2\alpha} < \infty \Big\},$$

where

$$A^{\alpha} u = \sum_{j=1}^{\infty} \nu_j^{\alpha} a_j \varphi_j, \qquad u \in D(A^{\alpha}) = X^{\alpha}.$$

Finally, we apply all our results from previous sections to boundary-initial value problem of the type

$$\begin{cases} \partial_t^3 u + (-\Delta_D)u + (-\Delta_D)^{\theta}\partial_t^2 u + (-\Delta_D)^{\varrho}\partial_t u = 0, & x \in \Omega, \ t > 0, \\ u(x,0) = u_0(x), \ \partial_t u(x,0) = u_1(x), \ \partial_t^2 u(x,0) = u_2(x), & x \in \Omega, \ t > 0. \end{cases}$$

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References

[1] L. Abadias, C. Lizama and M. Murillo, *Hölder regularity for the Moore-Gibson-Thompson equation with infinite delay*, Commun. Pure Appl. Anal., **17** (1) (2018) 243–265.

[2] M. S. Alves, C. B., M. V. Ferreira, J. E. Muñoz Rivera, M. Sepúlveda and O. Vera, *Asymptotic behaviour for the vibrations modeled by the standard linear solid model with a thermal effect*, J. Math. Anal. Appl., **399** (2013) 472–479.

[3] H. Amann, "Linear and quasilinear parabolic problems". Volume I: Abstract Linear Theory, Birkhäuser Verlag, Basel, 1995.

[4] K. Ammari, Z. Liu and F. Shel, *Note on stability of an abstract coupled hyperbolic-parabolic system: singular case*. Appl. Math. Lett. 141 (2023).

[5] F. D. M. Bezerra and L. A. Santos, *Fractional powers approach of operators for abstract evolution equations of third order in time*, J. Differential Equations, **269** (2020) 5661–5679.

[6] F. D. M. Bezerra and L. A. Santos and A. N. Carvalho, *Well-posedness for some third-order evolution differential equations: A semigroup approach*, J. Evol. Equ., **22** (2022) 1–18.

[7] A. H. Caixeta, I. Lasiecka, and V. N. D. Cavalcanti, *On long time behavior of Moore-Gibson-Thompson equation with molecular relaxation*, Evol. Equ. Control Theory., **05** (2016) 661.

[8] A. N. Carvalho and J. W. Cholewa, *Local well posedness for strongly damped wave equations with critical nonlinearities*, Bull. Austral. Math. Soc. 66 (2002), 443–463.

[9] A. N. Carvalho, J. W. Cholewa, and T. Dłotko, *Strongly damped wave problems: Bootstrapping and regularity of solutions*, J. Differential Equations 244 (2008) 2310–2333.

[10] J. W. Cholewa, and T. Dłotko, "Global Attractors in abstract parabolic problems", London Mathematical Society, Lecture Note Series, Cambridge University Press, 278, 2000.

[11] S. Chen and R. Triggiani, *Gevrey Class semigroups arising from elastic systems with gentle dissipative: the case* $0 < \alpha < 1$, Proc. Am. Math. Soc. **110** 2 (1990) 401–415.

[12] J. A. Conejero, C. Lizama and F. Rodenas, *Chaotic behaviour of the solutions of the Moore-Gibson-Thompson equation*, Appl. Math. Inf. Sci., **09** (5) (2015) 2233–2238.

[13] F. Dell'Oro, I. Lasiecka and V. Pata, *The Moore-Gibson-Thompson equation with memory in the critical case*, J. Differential Equations, **261** 7 (2016) 4188–4222.

[14] F. Dell'Oro, L. Liverani and V. Pata, *On the regularized Moore-Gibson-Thompson equation*, Discrete Contin. Dyn. Syst. - S, doi: 10.3934/dcdss.2023025.

[15] R. Denk and R. Racke, *L^p-resolvent estimates and time decay for generalised thermoelastic plate equations*, Electron. J. Differ. Equ., Vol. 2006(2006), No. 48, pp. 1–16.

[16] H. O. Fattorini, The Cauchy Problem. "In Encyclopedia of Mathematics and its Applications", Vol. 18. Addison-Wesley Publishing Company: Reading, Massachusetts, 1983.

[17] J. Hao and Z. Liu, *Stability of an abstract system of coupled hyperbolic and parabolic equations*, Z. Angew. Math. Phys. **64**, (2013) 1145–1159. https://doi.org/10.1007/s00033-012-0274-0.

[18] J. Hao, Z. Liu and J. Yong, *Regularity analysis for an abstract system of coupled hyperbolic and parabolic equations*, J. Differential Equations **259** (2015), no. 9, 47634798.

[19] M. Hasse, "The Functional Calculus for Sectorial Operators", Birkhäuser Verlag, 2006.

[20] D. Henry, "Geometric theory of semilinear parabolic equations", Lecture Notes in Mathematics 840, Springer-Verlag, Berlin, 1981.

[21] B. Kaltenbacher, I. Lasiecka and R. Marchand, *Well-posedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound*, Control Cybernet., **40** (4) (2011), 971–988.

[22] B. Kaltenbacher, I. Lasiecka and M. Pospieszalska, *Well-posedness and exponential decay of the energy in the nonlinear Jordan-Moore-Gibson-Thompson equation arising in high intensity ultrasound*, Math. Models Methods Appl. Sci., 22 (1) 1250035 (2012).

[23] S. G. Kreĭn, "Linear Differential Equations in a Banach Space". Transl. Mathem. Monogr., 29, Amer. Math. Soc., 1971.

[24] I. Lasiecka and X. Wang, *Moore-Gibson-Thompson equation with memory, part I: Exponential decay of energy*, Z. Angew. Math. Phys., **67** (05) (2015).

[25] P. Massatt, "Nonlinear Phenomena in Mathematical Sciences", Proceedings of an International Conference on Nonlinear Phenomena in Mathematical Sciences, Held at the University of Texas at Arlington, Arlington, Texas, June 16–20, 1980; 1982, Pages 663–670.

[26] R. Marchand, T. McDevitt and R. Triggiani, *An abstract semigroup approach to the third-order Moore-Gibson-Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability*, Math. Methods Appl. Sci. **35** (15) (2012) 1896–1929.

[27] A. Pazy, "Semigroup of linear operators and applications to partial differential equations", Springer-Verlag, New York, 1983.

[28] M. Pellicera and J. Solà-Morales, *Optimal scalar products in the Moore-Gibson-Thompson equation*, Equ. Control Theory, (2019) **8**(1): 203–220.

[29] H. Tanabe, "Equations of Evolution", Pitman, London, 1979.

[30] A. E. Taylor and D. C. Lay, "Introduction to Functional Analysis", John Willey & Sons, New York, Chichester, Brisbane, Toronto, 1980.

[31] H. Triebel, "Interpolation theory, function spaces, differential operators", Veb Deutscher, 1978.

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