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The Poincaré Generalized Lemma

Jorge Aragona

IME-USP, Universidade de São Paulo,
São Paulo, Brazil

Abstract. In this note we prove the result: let Ω be a non void open subset of $\mathbb{R}^N$ and f a k-generalized differential form on Ω. Moreover, assume that Ω is star-shaped and that f is a closed form on Ω. Then, there is a $(k-1)$-generalized differential form u on Ω whose differential is f. This result is a part of what I call internal development of the Colombeau theory.

Notation In what follows we will adhere to the following conventions: $\mathbf{I} :=]0,1[\subset \mathbb{R}$; $\bar{\mathbf{I}} := [0,1] \subset \mathbb{R}$, $N \in \mathbb{N}^*$ is fixed and Ω is a non void open subset of $\mathbb{R}^N$.

For every $k \in \mathbb{N}$ such that $1 \le k \le N$ we denote by

$$\mathbf{A}_N^k := Alt\left({}^k\left(\mathbb{R}^N\right);\mathbb{R}\right)$$

the $\mathbb{R}$-vector space of all k-linear alternate forms on $\mathbb{R}^n$ (i.e. its domain is $\mathbb{R}^N \times \overset{k}{\cdots} \times \mathbb{R}^{\mathbb{N}}$). We extend this definition to the case $k = 0$ by setting

$$\mathbf{A}_N^0 = Alt\left({}^0\left(\mathbb{R}^N\right);\mathbb{R}\right) := \mathbb{R}.$$

For a fixed $k \in \mathbb{N}$ such that $1 \le k \le N$, if $I = (i_1, \ldots, i_k)$, with $1 \le i_1, i_2, \ldots, i_k \le N$, we define $I^* := \{i_1, i_2, \ldots, i_k\}$ and the order of I as being the number $|I| := k$. We define

$$dx^I := dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

and for each α such that $1 \le \alpha \le k$ we set

$$dx^{I,\widehat{i_\alpha}} := dx_{i_1} \wedge \ldots \widehat{dx_{i_k}} \wedge \ldots \wedge dx_{i_k} =: \bigwedge_{\nu \ne \alpha} dx_{i_\nu}.$$

Here the symbol $\widehat{}$ over dx_{i_α} indicates that it is omitted.

The symbol $\sum_I'$ means that sommation is restricted to multi-indices

$I = (i_1, i_2, \ldots, i_k)$ verifying $1 \leq i_1 < i_2 < \ldots < i_k \leq N$.
It is well known that the sequence

$$\mathcal{B}_N^k := (dx_i \wedge \ldots \wedge dx_{i_k})_{1 \leq i_1 < \cdots < i_k \leq N} \tag{1}$$

is an $\mathbb{R}$-basis of $\mathbf{A}_N^k$ and using some canonical isomorphisms $\mathcal{B}_N^k$ can be considered a $\mathbb{R}$-basis of $\mathcal{G}_r(\Omega) := \mathcal{G}\left(\Omega, \mathbf{A}_N^k\right)$ which is defined in the sequel $[[2], Ex.7.2.1, (b)$ Real differential forms]. Also, we will need the following spaces (see [2], section 7.2) where $k \geq 1$:

$$\mathcal{E}_M\left[\Omega; \mathbf{A}_N^k\right], \mathcal{N}\left[\Omega; \mathbf{A}_N^k\right] \text{ and } \mathcal{G}_k(\Omega) := \frac{\mathcal{E}_M\left[\Omega, \mathbf{A}_N^k\right]}{\mathcal{N}\left[\Omega; \mathbf{A}_N^k\right]}.$$

If $k = 0$, we have $\mathbf{A}_N^0 = \mathbb{R}$ and hence

$$\mathcal{G}_0(\Omega) = \frac{\mathcal{E}_M[\Omega, \mathbb{R}]}{\mathcal{N}[\Omega, \mathbb{R}]} = \mathcal{G}(\Omega).$$

If $f = \sum_{|I|=k}{}' f_I dx^I \in \mathcal{G}_k(\Omega)$ we have $f_I \in \mathcal{G}(\Omega) \, (= \mathcal{G}(\Omega, \mathbb{R}))$ for each I. If $\widehat{f_I}$ is any representative of f_I then $\widehat{f_I} \in \mathcal{E}_M[\Omega] \, (= \mathcal{E}_M[\Omega; \mathbb{R}])$ for every I. It follows that

$$\widehat{f} := \sum_{|I|=k}{}' \widehat{f}_I dx^I \in \mathcal{E}_M\left[\Omega; \mathbf{A}_N^k\right] (k \geq 1)$$

is a representative of f.

Note that the unitary and commutative ring $\mathcal{E}_M[\overline{\mathbf{I}} \times \Omega; \mathbb{R}]$ is well defined since $\overline{\mathbf{I}} \times \Omega$ is a quasi-regular set (see [1]). Therefore, we can consider the $\mathcal{E}_M\left[\overline{\mathbf{I}} \times \Omega; \mathbb{R}\right]$-free module over $\mathcal{B}_N^k$ (see (1)) that we will denote by

$$\mathcal{E}_M\left[\overline{\mathbf{I}} \times \Omega; \mathbf{A}_N^k\right]. \tag{2}$$

An arbitrary element of $\mathcal{E}_M\left[\mathbf{I} \times \boldsymbol{\Omega}; \mathbf{A}_N^k\right]$ is of the kind

$$F = \sum_{|I|=k}{}' F_I(\varphi, t, x)\, dx^I \tag{3}$$

where $F_I \in \mathcal{E}_M\left[\overline{\mathbf{I}} \times \Omega; \mathbb{R}\right] \;\; \forall\, I, \forall \varphi \in A_0$. Given F as in (3) we define

$$\int_0^1 F := \sum_{|I|=k}{}' \left\{ \int_0^1 F_I(\varphi, t, x)\, dt \right\} dx^I \tag{4}$$

and it is easily seen that

$$\int_0^1 d_x F = d\left(\int_0^1 F \right). \tag{5}$$

Before proving (5) let recall that $\mathcal{C}_k^\infty(\Omega) = \mathcal{C}_k^\infty(\Omega; \mathbb{R})$ is the set of all k-differential forms of class $\mathcal{C}^\infty$ over Ω. An arbitrary element of $\mathcal{C}_k^\infty(\Omega)$ is of the form

$$g = g(x) = \sum_{|I|=k}{}' g_I(x)\, dx^I = \sum_{|I|=k}{}' g_I dx^I$$

with $g_I \in \mathcal{C}^\infty(\Omega) \quad \forall\ I$. If U is an open subset of $\mathbb{R}^p$ (where $p \in \mathbb{N}^*$ is arbitrarily fixed), $\mu \in \mathcal{C}^\infty(U; \Omega)$ and

$$\widehat{f} = \sum_{|I|=p}{}' \widehat{f}_I(\varphi, t, x)\, dx^I \in \mathcal{E}_M\left[\mathbf{I} \times \Omega, A_N^k\right],$$

the pull-back by μ of $\widehat{f}$, which we will denote by $\mu^* \widehat{f}$, is defined by

$$\mu^* f := \sum_{|I|=k}{}' \left(f_I' \circ \mu\right) d\mu^I \in \mathcal{C}_k^\infty(U; \mathbb{R}).$$

Lemma 1. (4) $\Longrightarrow$ (5). *More precisely, given* F *as in* (3) *(that is* $F = \sum_{|I|=k}{}' F_I(\varphi, t, x)\, dx^I$*) we have* (5), *that is*

$$\int_0^1 d_x F = d\left(\int_0^1 F \right) \tag{1.1}$$

(hence (1.1) = (5)*) where we define*

$$\int_0^1 F := \sum_{|I|=k}{}' \left\{ \int_0^1 F_I(\varphi, t, x)\, dt \right\} dx^I \tag{1.2}$$

(hence (1.2) = (4)*).*

Proof. Fix any $F \in \mathcal{E}_M\left[\overline{\mathbf{I}} \times \Omega; \mathbf{A}_N^k\right]$ as in (3), we will need the canonical form of $d_x F$ which is (for the sake of simplicity we omit the variables φ, t, x):

$$d_x F = \sum_{|I|=k}{}' \sum_{\nu=1}^{N} \frac{\partial F_I}{\partial x_\nu} dx_\nu \wedge dx^I = \sum_{|K|=k+1}{}' \left\{ \sum_{\nu,I} \varepsilon_{\nu I}^K \frac{\partial F_I}{\partial x_\nu} \right\} dx^K \tag{1.3}$$

where $\varepsilon_{\nu I}^K = 0$ if $\nu \in \mathbf{I}^* := \{i_1, \ldots, i_k\}$, where $I = (i_1, \ldots, i_k)$ with $1 \leq i_1 < \ldots < i_k \leq N$ or, conversely $\varepsilon_{\nu I}^K$ is the signature of the permutation which transforms $\nu I := (\nu, i_1, \ldots i_k)$ into the permutation $K = (l_1, l_2, \ldots, l_k, l_{k+1})$ of νI verifying

$$1 \leq l_1 < l_2 < \ldots < l_k < l_{k+1} \leq N.$$

Clearly, the second identity of (1.3) follows from writing $d_x F$ in the canonical form

$$d_x F = \sum_{|K|=k+1}{}' \left\{ \sum_{\nu \cdot I} \varepsilon_{\nu I}^K \frac{\partial F_I}{\partial x_\nu} \right\} dx^K. \tag{1.3$'$}$$

Computation of $\int_0^1 d_x F$:

Here we will use (4) and ($1.3'$) obtaining

$$\int_0^1 d_x F = \sum_{|K|=k+1}{}' \left\{ \int_0^1 \left(\sum_{\nu \cdot I} \varepsilon_{\nu I}^K \frac{\partial F_I}{\partial x_\nu} (\varphi, t, x) \right) dt \right\} dx^I =$$

$$= \sum_{|K|=k+1}{}' \left\{ \sum_{\nu,I} \varepsilon_{\nu I}^K \frac{\partial}{\partial x_\nu} \left(\int_0^1 F_I (\varphi, t, x)\, dt \right) \right\} dx^K =$$

$$= \sum_{|I|=k}{}' \sum_{\nu=1}^{N} \frac{\partial}{\partial x_\nu} \left(\int_0^1 F_I (\varphi, t, x)\, dt \right) \wedge dx_\nu \wedge dx^I.$$

Therefore

$$\int_0^1 d_x F = \sum_{|I|=k}' \sum_{\nu=1}^{N} \frac{\partial}{\partial x_\nu} \left(\int_0^1 F_I(\varphi, t, x)\, dt \right) dx_\nu \wedge dx^I. \tag{1.4}$$

Computation of $d\left(\int_0^1 F\right)$:

Here we will differenciate (4) getting

$$d\left(\int_0^1 F\right) = \sum_{|I|=k}' d\left(\int_0^1 F_I(\varphi, t, x)\, dt\right) \wedge dx^I \tag{1.5}$$

and since clearly we have

$$d\left(\int_0^1 F_I(\varphi, t, x)\, dt\right) = \sum_{\nu=1}^{N} \left[\frac{\partial}{\partial x_\nu}\left(\int_0^1 F_I(\varphi, t, x)\, dt\right)\right] dx_\nu,$$

by inserting the second member of the above identity into the second member of (1.5), we can conclude that

$$d\left(\int_0^1 F\right) = \sum_{|I|=k}' \left[\frac{\partial}{\partial x_\nu}\left(\int_0^1 F_I(\varphi, t, x)\, dt\right)\right] dx_\nu \wedge dx^I.$$

The above identity together (1.5) proves the result. □

The generalized Poincaré lemma is as follows

Proposition 2. *Let Ω be a star-shaped open subset of $\mathbb{R}^N$ and*

$$f = \sum_{|I|=K}' f_I(x)\, dx^I \in \mathcal{G}_k(\Omega)$$

such that $df = 0$ in Ω. Then there is $u \in \mathcal{G}_{k-1}(\Omega)$ such that $du = f$ in Ω.

Proof. Clearly we can assume that $x_0 = 0$, that is Ω is 0-star-shaped. Hence the map

$$\mu := (t, x) \in \mathbf{I} \times \Omega \longmapsto \mu(t, x) := tx \in \Omega$$

is well defined.
Next, fix an arbitrary representative $\widehat{f} = \sum'_{|I|=k} f_I(\varphi, x)\, dx^I \in \mathcal{E}_M\left[\Omega; \mathbf{A}_N^k\right]$ of $f \in \mathcal{G}_k(\Omega)$ which is fixed such that $df = 0$. The pull-back of $\widehat{f}$ for μ is

$$\mu^* \widehat{f} = \sum_{|I|=k}{}' \left(\widehat{f}_I \circ \mu\right) d\mu^I \tag{2.1}$$

and since $d\mu_i = d\mu_i(t,x) = x_i dt + t dx_i \quad (1 \le i \le N)$ we can compute the factor $d\mu^I$ which appears in (2.1) in terms of the dt, dx_i :

$$\begin{gathered} If\ I = (i_1, i_2, \ldots, i_k) \text{ and } 1 \le i_1 < i_2 < \ldots < i_k \le N \\ \text{then } d\mu^I = \sum_{\nu=1}^{k} (-1)^{\nu-1} x_{i_\nu} t^{k-1} dt \wedge dx^{I, \widehat{i_\nu}} + t^k dx^I = \\ dt \wedge \sum_{\nu=1}^{k} (-1)^{\nu-1} x_{i_\nu} t^{k-1} dx^{I, \widehat{i_\nu}} + t^k dx^I. \end{gathered} \tag{2.2}$$

Next, by replacing the terms $d\mu^I$ which appear in (2.1) by the second member of (2.2) we get (note that $\widehat{f}_I(\varphi, \mu(t,x)) = \widehat{f}_I(\varphi, tx)$ since $\mu(t,x) = tx$):

$$\begin{gathered} \mu^* \widehat{f}(\varphi, t, x) = \\ = \sum_{|I|=k}{}' \left(\widehat{f}_I(\varphi, \mu(t,x))\right) \left\{ dt \wedge \sum_{\nu=1}^{k} (-1)^{\nu-1} x_{i_\nu} t^{k-1} dx^{I, \widehat{i_\nu}} + t^k dx^I \right\} \\ dt \wedge \widehat{f}^0(\varphi, t, x) + \widehat{g}(\varphi, t, x) \quad \text{where} \\ \widehat{g} = \widehat{g}(\varphi, t, x) := \sum_{|I|=k}{}' \widehat{f}_I(\varphi, tx)\, t^k dx^I \in \mathcal{E}_M\left[\mathbf{I} \times \Omega; \mathbf{A}_N^k\right] \end{gathered}$$

and (in fact, a more correct notation would be $\widehat{f}^0_{(k)}$ instead of $\widehat{f}^0$).

$$\begin{gathered} \widehat{f}^0 = \widehat{f}^0(\varphi, t, x) := \\ := \sum_{|I|=k}{}' \sum_{\nu=1}^{k} (-1)^{\nu-1} \cdot \widehat{f}_I(\varphi, tx)\, x_{i_\nu} t^{k-1} dx^{I, \widehat{i_\nu}} \in \mathcal{E}_M\left[\mathbf{I} \times \Omega; \mathbf{A}_N^{k-1}\right] \end{gathered} \tag{2.3}$$

Now, note that the differential forms we are working with are completely general (except the assumption $df = 0$ in Ω, which don't appear in the proof of $(2.3')$ below). Hence from (2.3) it follows that

$$\begin{aligned}&\forall\ \widehat{f}\in\mathcal{E}_M\left[\Omega;\mathbf{A}_N^k\right]\ \exists\ \widehat{g}\in\mathcal{E}_M\left[\mathbf{I}\times\Omega;\mathbf{A}_N^k\right]\quad\text{and}\\&\exists\ \widehat{f}^0\in\mathcal{E}_M\left[\mathbf{I}\times\Omega;\boldsymbol{A}_N^{k-1}\right]\ \text{such that}\ \mu^*\widehat{f}=\widehat{g}+dt\wedge\widehat{f}^0.\end{aligned}\tag{2.3$'$}$$

It is worthwhile to note that in the definitions of $\widehat{g}$ and $\widehat{f}^0$ there appear the claims:

$$\widehat{g}\in\mathcal{E}_M\left[\mathbf{I}\times\Omega,\mathbf{A}_N^k\right]\ \text{ and }\ \widehat{f}^0\in\mathcal{E}_M\left[\mathbf{I}\times\Omega;\mathbf{A}_N^{k-1}\right]\tag{2.3$''$}$$

whose proofs, which follow obviously from the moderation of the functions $\widehat{f}_I$ $(|I|=k)$, are trivial. Moreover, it is easily seen that the forms $\widehat{g}$ and $\widehat{f}^0$ of $(2.3^{'})$ there are unique (that is, determined from $\widehat{f}$). Indeed, these claims follow from the computations in (2.3) and (2.2) which lead to the expression of $\mu^*\widehat{f}$ which appears in $(2.3^{'})$.

Next, from the uniqueness of $\widehat{g}$ and $\widehat{f}^0$ in $(2.3^{'})$, it follows that we can define the linear operator below:

$$T=T_k:\widehat{f}\in\mathcal{E}_M\left[\Omega;\mathbf{A}_N^k\right]\longmapsto\int_0^1\widehat{f}^0_{(k)}\in\mathcal{E}_M\left[\Omega;\mathbf{A}_N^{k-1}\right]\tag{2.4}$$

for every $k=1,2,\ldots,N$ (and $T=T_{n+1}=0$, in this section of this proof, the index k in T_k or in $\widehat{f}^0_{(k)}$ can be of some help).

We are going to prove that

$$T_{k+1}\left(d\widehat{f}\right)+d\left(T_k\widehat{f}\right)=\widehat{f}\ \forall k=1,2,\ldots,N,\ \forall\widehat{f}\in\mathcal{E}_M\left[\Omega;\mathbf{A}_N^{k-1}\right].\tag{2.5}$$

Indeed, from differentiation in $(2.3^{'})$ and the identity $d\left(\mu^*\widehat{f}\right)=\mu^*d\widehat{f}$ we get

$$\mu^*d\widehat{f}=d_x\widehat{g}+\frac{\partial\widehat{g}}{\partial t}dt+d\left(dt\wedge\widehat{f}^0\right)$$

and since (see [4, Prop.19.7,p.147]) $d\left(dt\wedge\widehat{f}^0\right)=-dt\wedge d_x\widehat{f}^0$ we have

$$\mu^*d\widehat{f}=d_x\widehat{g}+dt\wedge\left(-d_xf^{\ 0}+\frac{\partial\widehat{g}}{\partial t}\right).\tag{2.6}$$

Now it is clear that (2.6) is a representation of μ^*df of the kind

$$A + dt \wedge B$$

which is formally equal to the identity $(2.3^{'})$ and since this representation is unique, from the definition of $T = T_k$ we get

$$T_k\left(d\widehat{f}\right) = \int_0^1 \left(-d_x\widehat{f}^0 + \frac{\partial \widehat{g}}{\partial t}\right). \tag{2.7}$$

From (1.1) in Lemma 1 we have

$$\int_0^1 - d_x\widehat{f}^0 = -d\left(\int_0^1 \widehat{f}^0\right) \overset{*}{=} -d\left(T\widehat{f}\right)$$

[$(*)$: Indeed, $\int_0^1 \widehat{f}^0 = T\widehat{f}$ from the definition of $T = T_k$] which implies-from (2.7) :

$$T\left(d\widehat{f}\right) = -d\left(T\widehat{f}\right) + \int_0^1 \frac{\partial \widehat{g}}{\partial t}dt. \tag{2.8}$$

From the definition of $\widehat{g}$ we have:

$$\widehat{g}\,(\varphi, 1, x) = \widehat{f}\,(\varphi, x) \text{ and } \widehat{g}\,(\varphi, 0, x) = 0$$

and therefore, from (2.8) we get

$$T\left(d\widehat{f}\right) + d\left(T\widehat{f}\right) = \widehat{f} \quad \left(\iff T_{k+1}\left(d\widehat{f}\right) + d\left(T_k\widehat{f}\right) = \widehat{f}\right) \tag{2.9}$$

which is an identity in $\mathcal{E}_M\left[\Omega; \boldsymbol{A}_N^k\right]$. Next, we will prove that (2.9) can be extended to $\mathcal{G}_k\left(\Omega\right)$. More precisely we will prove that the operator (see (2.4)):

$$T = T_k : \mathcal{E}_M[\Omega; \mathbf{A}_N^k] \longrightarrow \mathcal{E}_M\left[\Omega; \mathbf{A}_N^{k-1}\right]$$

induces in the quotient another operator

$$T^* = T_k^* : \mathcal{G}_k\left(\Omega\right) \longrightarrow \mathcal{G}_{k-1}\left(\Omega\right)$$

such that the diagram below

$$\begin{array}{ccc} \mathcal{E}_M\left[\Omega;\boldsymbol{A}_N^k\right] & \stackrel{T}{\longrightarrow} & \mathcal{E}_M\left[\Omega;\mathbf{A}_N^{k-1}\right] \\ \pi_1\downarrow & & \downarrow\pi_2 \\ \mathcal{G}_k\left(\Omega\right) & \stackrel{T^*}{- - - \rightarrow} & \mathcal{G}_{k-1}\left(\Omega\right) \end{array} \tag{2.10}$$

commutes (π_1 and π_2 denote the canonical maps).

It is well known that the existence of a such T^* is equivalent to the inclusion $\mathcal{N}\left[\Omega,\mathbf{A}_N^k\right]\subset Ker\left(\pi_2\circ T\right)$ or

$$\mathcal{N}\left[\Omega;\mathbf{A}_N^k\right]\subset\left\{\widehat{h}\in\mathcal{E}_M\left[\Omega;\mathbf{A}_N^k\right]\ \mid T\left(\widehat{h}\right)\in\mathcal{N}\left[\Omega;\mathbf{A}_N^{k-1}\right]\right\}. \tag{2.11}$$

We will prove (2.11). Fix $\widehat{f}\in\mathcal{N}\left[\Omega;\mathbf{A}_N^k\right]$ arbitrary, then $\widehat{f}=\sum\limits_{|I|=k}\widehat{f}_I dx^I$, where $\widehat{f}_I\in\mathcal{N}\left[\Omega;\mathbb{R}\right]$ for all I, which implies ($\mathcal{N}$ is an ideal of $\mathcal{E}_M$) that the function

$$(\varphi,t,x)\in A_0\times(\mathbf{I}{\times}\Omega)\longmapsto(-1)^{\nu-1}\widehat{f}_I\left(\varphi,t,x\right)x_{i_\nu}t^{k-1}\in\mathbb{R}$$

belongs to $\mathcal{N}\left[\mathbf{I}{\times}\Omega;\mathbb{R}\right]$ for all I and ν. Hence

$$\widehat{f}^0\in\mathcal{N}\left[\mathbf{I}{\times}\Omega;\mathbf{A}_N^{k-1}\right]$$

and, as a consequence

$$T\widehat{f}=\int_0^1\widehat{f}^0\in\mathcal{N}\left[\Omega;\mathbf{A}_N^{k-1}\right]$$

which proves (2.11) and hence the existence and uniqueness of the operator $T^*=T_k^*$ such that (2.10) commutes. Finally, we will prove that (2.9) hold with T^* and f instead of T and $\widehat{f}$ respectively. From the commutativity of (2.10) we get

$$T^*\left(f\right)=cl\left(T\widehat{f}\right) \tag{2.12}$$

and therefore

$$T^*\left(df\right)=cl\left(T\left(d\widehat{f}\right)\right). \tag{2.13}$$

Note that $d\widehat{f}$ is a representative of df therefore we can write it as $\widehat{df}$, that is

$$d\widehat{f} = \widehat{df} \text{ and } d\left(cl\left(\widehat{\nu}\right)\right) = cl\left(d\widehat{\nu}\right) \text{ (any } \nu) \tag{2.14}$$

which implies (by (2.12) and (2.14)):

$$d\left(T^{*}f\right) = d\left(cl\left(T\widehat{f}\right)\right) = cl\left(d\left(T\widehat{f}\right)\right).$$

Hence

$$d\left(T^{*}f\right) = cl\left(d\left(T\widehat{f}\right)\right). \tag{2.15}$$

Now, from (2.13) and (2.15) we get

$$T^{*}\left(df\right) + d\left(T^{*}f\right) = cl\left(T\left(d\widehat{f}\right)\right) + cl\left(d\left(T\widehat{f}\right)\right) \tag{2.16}$$

and from (2.9), the second member of (2.16) is equal to $cl\left(\widehat{f}\right) = f$, hence from (2.16) it follows that (remember that $T = T_k$)

$$T_{k+1}^{*}\left(df\right) + d(T_{k}^{*}f) = f. \tag{2.17}$$

Now, it is enough to define $u := T_{k}^{*}f \in \mathcal{G}_{k}\left(\Omega\right)$ and remark that since $df = 0$ in Ω and T_{k+1}^{*} is linear, from (2.17) we can conclude that $du = f$ in Ω. □

Next, we will present, as an application of PROP.2, a local existence result for the $\partial\overline{\partial}$ operator. In what follows, Ω denotes an open subset of $\mathbb{C}^{n}$ $(n \in \mathbb{N}^{*}$ fixed$), p, q \in \mathbb{N}$, $0 \leq p, q \leq n$ and $p + q > 0$. We also set $\mathcal{G}_{(0,0)}\left(\Omega\right) = \mathcal{G}\left(\Omega\right)$. We don't recall the definition of the spaces $G_{(p,q)}\left(\Omega\right)$ of the (p, q)-complex differential forms on Ω. If $f \in \mathcal{G}\left(\Omega\right)$ and $\widehat{f} \in \mathcal{E}_{M}\left[\Omega\right]$ is a representative of f, then the conjugate $\overline{\widehat{f}}$ of $\widehat{f}$ belongs to $\mathcal{E}_{M}\left[\Omega\right]$ and obviously, if $\widehat{f}$ and $\widehat{f}_{1}$are two representatives of f, then $\overline{\widehat{f}} - \overline{\widehat{f}_{1}} \in \mathcal{N}\left[\Omega\right]$. We denote by $\overline{f}$ the element of $\mathcal{G}\left(\Omega\right)$ represented by $\overline{\widehat{f}},$ which is called the conjugate of f.

For a given

$$f = \sum_{|I|=p,|J|=q}{}^{'} f_{IJ} dz^{I} \wedge d\overline{z}^{J} \in \mathcal{G}_{(p,q)}\left(\Omega\right)$$

we have

$$\overline{f} = \sum_{|I|=p,|J|=q}' \overline{f_{IJ}} d\overline{z}^I \wedge dz^J \in \mathcal{G}_{(q,p)}(\Omega) .$$

Clearly, by conjugation as in the classical case, all the existence results for the $\overline{\partial}$ operator remain valid for the ∂ operator. From this remark, from Prop.2 and [3,Th.5] we get the following result:

Proposition 3. *Let U be an open subset of $\mathbb{C}^n$ and $g \in \mathcal{G}_{(p,q)}(U)$ such that $dg = 0$ in U, where $1 \leq p, q \leq n$. Then, for each $a \in U$ there are an open neighborhood W of a and $v \in \mathcal{G}_{(p-1,q-1)}(W)$ verifying*

$$\partial\overline{\partial} v = g \text{ in } W.$$

Proof. The proof is easy and consists in the application of the Poincaré Lemma for the operators $d, \overline{\partial}$ and ∂. Fix $a \in U$ arbitrary and consider an open 0-start-shaped neighborhood N_1 of a contained in U. Then, since $dg = 0$ in U, from Prop.2 there exists $h \in \mathcal{G}_{r-1}(N_1)$ such that

$$dh = g \text{ in } N_1 \tag{3.1}$$

where $r := p + q$ is the total degree of g. Now, we have

$$\mathcal{G}_{r-1}(N_1) = \underset{1\leq l\leq r}{\oplus} \mathcal{G}_{(l-1,r-l)}(N_1) \tag{3.2}$$

hence

$$h = \sum_{l=1}^{r} h_{(l-1,r-l)} \text{ which } h_{(l-1,r-l)} \mathcal{G}_{(l-1,r-l)}(N_1) \ \forall\, l = 1, 2, \ldots, r. \tag{3.3}$$

Then we can conclude that in N_1 we have (see (3.1)):

$$g = dh = \left(\partial + \overline{\partial}\right) h = \partial h + \overline{\partial} h = \sum_{l=1}^{r} \partial h_{(l-1,r-l)} + \sum_{l=1}^{r} \overline{\partial} h_{(l-1,r-l)},$$

which implies:

$$\begin{gathered} \mathcal{G}_{(p,q)}(N_1) \ \ni \ g = dh = A + b, \quad \text{where} \\ A := \sum_{l=1}^{r} \partial h_{(l-1,r-l)} \quad \text{and} \quad B := \sum_{l=1}^{r} \overline{\partial} h_{(l-1,r-l)}. \end{gathered} \tag{3.4}$$

Therefore, we can write the identity $dh = A + B$ in (3.4) in the form

$$dh = \partial h_{(p-1,q)} + \partial \overline{h}_{(p,q-1)} \text{ in } N_1 \tag{3.4'}$$

and in N_1 :

$$\partial h_{(l-1,r-l)} = 0 \; \forall \; l \neq p$$

$$\partial \overline{h}_{(l-1,r-l)} = 0 \; \forall \; l \neq p+1.$$

Therefore

$$\partial h_{(p,q-1)} = \overline{\partial} h_{(p-1,q)} = 0 \text{ in } N_1. \tag{3.5}$$

Finally, from [3,Th.5] there exists a bounded open set N_2 verifying $a \in N_2 \subset U$ and there are $u_1, u_2 \in \mathcal{G}_{(p-1,q-1)}(N_2)$ such that

$$\partial u_1 = h_{(p,q-1)} \text{ and } \overline{\partial} u_2 = h_{(p-1,q)} \; textin \; N_2$$

which, by $(3.4^{'})$, implies in $W := N_1 \cap N_2$:

$$g = dh = \partial h_{(p-1,q)} + \overline{\partial} h_{(p,q-1)} = \partial \left(\overline{\partial} u_2 \right) + \overline{\partial} \left(\partial u_1 \right) = \partial \overline{\partial} \left(u_2 - u_1 \right).$$

This proves our result by setting

$$v := u_2 - u_1 \in \mathcal{G}_{(p-1,q-1)}(W).$$

□

References

[1] J.Aragona, Colombeau generalized functions on quasi-regular sets, *Publ. Math. Debrecen,* 3-4, 68 (2006).

[2] J.Aragona and H.A.Biagioni, Intrinsic definition of the Colombeau algebra of genealized functions, *Anal. Math.* 17,2 (1991), 75-132.

[3] J.Aragona and J.F.Colombeau, On the $\overline{\partial}$- Neumann problem for generalized functions, *J. Math. Anal. Appl., 110 (1985), 179-199.*

[4] J.Mujica, *Complex analysis in Banach spaces*, North-Holland Mathematics Studies 120, North-Holland Publishing Co. Amsterdam (1986).