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Synchronization in Herbivorous Population Models with Diffusion and Delays

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This paper is dedicated to Prof. Waldyr Oliva

Abstract. We consider two herbivorous species whose dynamics are modeled by Hutchinson's equation, and are migrating to a new environment, where they will need to compete. We show that, if the migration barriers are weak, then both species will persist and furthermore they will asymptotically behave as one specie, that is, they will synchronize.

1 Introduction

A variety of mathematical models for biological processes are most appropriately framed as partial functional differential equations. For example, the reaction-diffusion logistic equation with finite delay, commonly known as Hutchinson's ([13])

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equation,

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = d\dfrac{\partial^2 u(t,x)}{\partial x^2} + bu\left(1 - \dfrac{u(t-\tau)}{K}\right), \ t > 0, x \in (0,1) \\ \left.\begin{array}{c} \dfrac{\partial u(t,x)}{\partial n} = 0 \\ \text{or} \\ u(t,x) = 0 \end{array}\right\}, x = 0, 1, \end{cases} \tag{1.1}$$

where d, b, τ and K are positive constants, has been used to model a one-dimensional herbivorous population and has been studied by many authors, for example Busenberg and Huang [1], Friesecke [6, 7], Golpalsamy, He and Sun [8], Green and Stech [9], Lin and Khan [17], Luckhaus [18], Wu [23], Oliveira [20], Carvalho and Oliveira [3], Murray [19], Hale [11] to name a few.

Among the interesting questions that can be addressed, the subject of the effect of population dispersal is a topic of considerable ecological interest, references can be found in the work of Levin [15, 16] and Murray [19]. In recent years, much attention has been given to the asymptotic behavior of mathematical models which describe population dispersal between patches [5, 14, 4, 22], in particular, the models for single species dispersal between patches of heterogeneous environment with barriers between patches [22], where questions of permanence, global stability, attractors and bifurcations are of interest.

In this work, we wish to consider two herbivorous species that are migrating to an island (where they must compete for resources) and there are barriers to these migrations. We would like to study the behavior of the populations on this new environment. We will assume that the behavior of each specie is governed by Hutchinson's equation (with appropriate parameters) and we will model the interaction in the island as a coupling between the two equations. More precisely we will consider the following system of equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + a_1 u\left(1 - \dfrac{u(t-\tau_1)}{K_{11}} - \dfrac{v(t-\rho_1)}{K_{12}}\right) - M_1(u-v) \\ \dfrac{\partial v}{\partial t} = \Delta v + a_2 v\left(1 - \dfrac{v(t-\tau_2)}{K_{21}} - \dfrac{u(t-\rho_2)}{K_{22}}\right) - M_2(v-u) \end{cases} \tag{1.2}$$

for all $t > 0$ and $x \in (0,1)$, plus either one of the following boundary conditions

$$\text{(N)} \begin{cases} \dfrac{\partial u}{\partial n} = 0 \\ \dfrac{\partial v}{\partial n} = 0 \end{cases} \quad \text{or} \quad \text{(D)} \begin{cases} u = 0 \\ v = 0 \end{cases} \tag{1.3}$$

where $(t,x) \in (0,\infty) \times (0,1)$, K_{11}, K_{12}, K_{21}, K_{22}, a_1, a_2, τ_1, τ_2, ρ_1, ρ_2, M_1, M_2 are positive, and $r = \max\{\tau_1, \tau_2, \rho_1, \rho_2\}$. In order to be a plausible problem from the biological point of view, we will only consider the following initial conditions

$$\left.\begin{array}{l} u(s,x) > 0 \\ v(s,x) > 0 \end{array}\right\} \text{ for all } (s,x) \in [-r,0] \times [0,1]. \tag{1.4}$$

This problem can be rewritten in abstract form as

$$\dot{\varphi}(t) = (\mathbb{I}\ A)\,\varphi(t) + f(\varphi_t), \quad t > 0 \tag{1.5}$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^2$, A is the Laplacian with the corresponding boundary condition, $\varphi_t : [-r, 0] \to L^2(0,1) \times L^2(0,1)$ denotes the function $\varphi_t(\theta) = \varphi(t+\theta)$ and $f : C([-r,0], H^1(0,1) \times H^1(0,1)) \to L^2(0,1) \times L^2(0,1)$ is given by

$$f(\varphi_t^1, \varphi_t^2) = h(\varphi_t^1, \varphi_t^2) + \begin{pmatrix} -M_1 & M_1 \\ M_2 & -M_2 \end{pmatrix} \begin{pmatrix} \varphi_t^1(0) \\ \varphi_t^2(0) \end{pmatrix}$$

and

$$h(\varphi_t^1, \varphi_t^2) = \begin{pmatrix} a_1 \varphi_t^1(0) \left(1 - \dfrac{\varphi_t^1(-\tau_1)}{K_{11}} - \dfrac{\varphi_t^2(-\rho_1)}{K_{12}}\right) \\ a_2 \varphi_t^2(0) \left(1 - \dfrac{\varphi_t^2(-\tau_2)}{K_{21}} - \dfrac{\varphi_t^1(-\rho_2)}{K_{22}}\right) \end{pmatrix}. \tag{1.6}$$

The global existence and uniqueness of solutions of (1.5) (or (1.2)) has already been established for any initial condition in $C([-r,0], H^1(0,1) \times H^1(0,1))$, and moreover, the solutions are classical after time $t > r$ and all solutions are positive (see Oliveira [20] and Wu [23]).

We will be able to show that, if $U(t)$ is the nonlinear semigroup associated with the solution of (1.5), then $U(t)$ is bounded dissipative, that is, there exists a set $\mathcal{V}$ that attracts any bounded set $\mathcal{U}$ under $U(t)$ ($dist(U(t)\mathcal{U}, \mathcal{V}) \to 0$, as $t \to \infty$) and furthermore, there exists an invariant set $\mathcal{A}$ ($U(t)\mathcal{A} \subset \mathcal{A}$, for $t \geq 0$) which is a *compact global attractor*, that is, a set $\mathcal{A}$ which is a maximal invariant compact set which attracts each bounded set $\mathcal{U}$ (see Hale [10]).

The objective is to study the asymptotic behavior of the system as the parameters change. We wish to show a synchronization property, that is, if the coupling parameters M_1 and M_2 are sufficiently large (which means, in the model, that the barriers to the migrations are sufficiently weak), then asymptotically the solutions will remain close to the diagonal (namely $u = v$), and the distance from the diagonal will be proportional to the difference of the corresponding parameters.

Furthermore, if $M_1 = M$ and $M_2 = M + m$, we can show that, as $M \to \infty$, the family of attractors of (1.2) $\{\mathcal{A}_M,\ M \geq 0\}$ is upper semicontinuous at infinity, where $\mathcal{A}_\infty = \{(u,u) : u \in \mathcal{A}_l\}$ and $\mathcal{A}_l$ is the global attractor of the limit problem

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{2} \left[a_1 u \left(1 - \frac{u(t-\tau_1)}{K_{11}} - \frac{u(t-\rho_1)}{K_{12}}\right) + a_2 u \left(1 - \frac{u(t-\tau_2)}{K_{21}} - \frac{u(t-\rho_2)}{K_{22}}\right)\right]. \tag{1.7}$$

These results have some very nice implications from the biological point of view. One of them deals with the persistence issue. Suppose, as an example, that $\rho_1 = \rho_2 = \tau_1 = \tau_2 = r$ and that u is a species modeled by (1.1) and Dirichlet boundary conditions hold. We know (see [1]) that: if b is small, then (1.1) has the zero solution as a global attractor (in other words, the species becomes extinct). But as we couple u with the other species, also modeled by (1.1) but with a larger parameter b, we get that both species will synchronize and, if the parameter b of this new species is large enough, the species u will not be extinct.

Another application is to prove the existence of periodic orbits for (1.2), which we can get when (1.7) has a stable periodic orbit, and since the existence of periodic orbits for a simplified version of (1.7) has been widely studied (see for instance [23] and [1]), we will get a concrete example of (1.2) where we have such orbits.

We also want to mention that the same analysis could be done if we modeled the population with Lotka-Volterra integral type of equations, that is,

$$\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u(t,x)}{\partial x^2} + bu(1 - \int_{-\infty}^{0} u(t+s)d\sigma(s)),\ t>0, x\in(0,1)\,.$$

The only difference is that the boundedness of the solutions is easier to get.

Since the computations are very similar for Newmann or Dirichlet boundary conditions, from now on, we will only consider Newmann boundary conditions.

The paper will proceed as follows: in section 2, we will use the ideas of Luckhaus [18] to prove that (1.2) is bounded dissipative, where the bounds will be independent of the parameter M. With this, we will establish the existence of the global attractors. The uniform boundedness will allow us to prove, in section 3, two results dealing with synchronization: the first one, following the work due to Rodrigues [21], will give us an estimate of the distance of the solutions to the diagonal, and following the results due to Carvalho, Rodrigues and Dlotko [2] and Carvalho and Oliveira [3], we will study the behavior of the attractors as M goes to infinity. Finally in section 4, we will show that, under certain conditions, if (1.7) has a stable periodic orbit so does (1.2).

2 A priori estimates, uniform bounds and global attractors

We will follow closely the ideas found in Luckhaus [18]. The first thing to be done is a reduction argument to a free boundary. Let $M = \min\{M_1, M_2\}$, $m = |M_1 - M_2|$ and $a = max\{a_1, a_2\}$.

Lemma 2.1 *Let $p \geq 2$ and suppose that there exist $T > r > 0$ and $\alpha \in (0,1)$ such that any solution of*

$$\begin{cases} \left.\begin{array}{l} \dot{u} \leq \Delta u + a_1 u - M_1(u-v) \\ \dot{v} \leq \Delta v + a_2 v - M_2(v-u) \end{array}\right\} \text{ in } (0,T)\times(0,1) \\ \\ u(t,x)u(t-\tau_1,x) = 0,\ (t,x)\in(\tau_1,T)\times(0,1) \\ v(t,x)v(t-\tau_2,x) = 0,\ (t,x)\in(\tau_2,T)\times(0,1) \\ \\ \dfrac{\partial u}{\partial n} = 0 \\ \dfrac{\partial v}{\partial n} = 0 \\ \\ u,v \geq 0 \end{cases} \tag{2.1}$$

satisfies the decay estimate

$$\int_0^1 (u^p(T,x) + v^p(T,x))\,dx \leq \alpha \int_0^1 (u^p(0,x)+v^p(0,x))\,dx. \tag{2.2}$$

Then there exists a constant K such that the solution of (1.2) with $\int_{-r}^{0}\int_0^1 (u^p + v^p)(t,x)dxdt < \infty$ must satisfy $\limsup_{t\to\infty} \int_0^1 (u^p+v^p)(t,x)dx \leq K$.

Proof First of all, by comparison, we can assume that $K_{12} = K_{22} = \infty$, that is, we do not have the terms $u(t)v(t-\rho_1)$ and $v(t)u(t-\rho_2)$. Let (u,v) be a solution of (1.2) with $\int_{-r}^{0}\int_{0}^{1}(u^p+v^p)(t,x)dxdt < \infty$.

Multiplying the first equation of (1.2) by u^{p-1}, the second by v^{p-1}, we get, for all $t>0$ and $x\in(0,1)$,

$$\left\{\begin{array}{rcl} u^{p-1}\dfrac{\partial u}{\partial t} &=& u^{p-1}\Delta u + a_1 u^p\left(1-\dfrac{u(t-\tau_1)}{K_{11}}\right) - M_1 u^{p-1}(u-v) \\ v^{p-1}\dfrac{\partial v}{\partial t} &=& v^{p-1}\Delta v + a_2 v^p\left(1-\dfrac{v(t-\tau_2)}{K_{21}}\right) - M_2 v^{p-1}(v-u) \end{array}\right. \tag{2.3}$$

Now if we add the equations above and integrate by parts, we get

$$\begin{aligned} &\frac{1}{p}\frac{\partial}{\partial t}\int_0^1 (u^p+v^p)(t,x)dx \\ &\le -(p-1)\int_0^1 (u^{p-2}|\nabla u|^2 + v^{p-2}|\nabla v|^2)(t,x)dx + a\int_0^1 (u^p+v^p)(t,x)dx \\ &\quad -M\int_0^1 (u^p+v^p)(t,x)dx + (M+m)\int_0^1 (u^{p-1}v+v^{p-1}u)(t,x)dx \end{aligned} \tag{2.4}$$

Using Young's inequality, one gets,

$$\begin{aligned} \frac{1}{p}\frac{\partial}{\partial t}\int_0^1 & (u^p+v^p)(t,x)dx + (p-1)\int_0^1 (u^{p-2}|\nabla u|^2 + v^{p-2}|\nabla v|^2)(t,x)dx \\ & \le (a+m)\int_0^1 (u^p+v^p)(t,x)dx \end{aligned} \tag{2.5}$$

and hence

$$\frac{\partial}{\partial t}\int_0^1 (u^p+v^p)(t,x)dx \le p(a+m)\int_0^1 (u^p+v^p)(t,x)dx. \tag{2.6}$$

At this moment, one can follow the same steps as in Luckhaus [18] and we can see the result in the same way. □

Now is time to verify when the decay estimate (2.2) for the solutions of (2.1) holds, and, once again, the proof follows closely to the proof found in Luckhaus [18].

Lemma 2.2 *Let (u,v) be a solution of (2.1), then*

$$\int_0^1 (u^p+v^p)(T,x)dx \le \left[e^{\left(pT\left(a+m-\frac{(p-1)\pi^2}{p^2}\frac{T^2}{r^2}\right)\right)}\right]\int_0^1 (u^p+v^p)(0,x)dx.$$

Proof The first thing to notice is that in one space dimension we can approximate u and v by Lipschitz functions, satisfying the same inequalities as u and v, and such that the boundary of its supports are smooth manifolds (just follow an analogous procedure contained in Luckhaus [18]). Hence we can check the lemma for the approximations instead for u and v. We will still denote the approximations by u and v.

Let us denote by C^u the interior of $supp(u)$ in $(0,T)\times(0,1)$ and by $\{C_i^u\}$ the set composed by its connected components. We do the same for v. We have that $C^u \cap (C^u - (\tau_1, 0)) = \emptyset$ and $C^v \cap (C^v - (\tau_2, 0)) = \emptyset$. Thus, as in Luckhaus, one gets that $C_i^u \cap (C_i^u - (n\tau_1, 0)) = \emptyset$ and $C_i^v \cap (C_i^v - (n\tau_2, 0)) = \emptyset$, for all connected components (that is, for all i) and all integers n.

Defining $C_t^\bullet = \{x \in (0,1)\colon (t,x) \in C^\bullet\}$, we have that $C_{i,t}^u \cap C_{i,t-n\tau_1}^u = \emptyset$, for all n, and similarly for v. Therefore one gets,

$$\int_0^T |C_{t,i}^\bullet| dt \le r, \text{ if } r < T.$$

Now, we have (as in (2.5)),

$$\begin{aligned}
\frac{1}{p}\frac{\partial}{\partial t}\int_0^1 (u^p + v^p) \le & \int_0^1 u^{p-1}\Delta u + a_1 u^p - M_1 u^{p-1}(u - v) \\
& +v^{p-1}\Delta v + a_2 v^p - M_2 v^{p-1}(v-u) \\
\le & \sum_i \int_{C_{i,t}^u} u^{p-1}\Delta u + \sum_i \int_{C_{i,t}^v} v^{p-1}\Delta v + (a+m)\int_0^1 (u^p + v^p) \\
= & -\sum_i (p-1)\int_{C_{i,t}^u} u^{p-2}|\nabla u|^2 - \sum_i (p-1)\int_{C_{i,t}^v} v^{p-2}|\nabla v|^2 \\
& +(a+m)\int_0^1 (u^p + v^p).
\end{aligned}$$

Note that $\dfrac{p^2}{4} u^{p-2}|\nabla u|^2 = |\nabla u^{\frac{p}{2}}|^2$, and similarly for v. Therefore, using Sobolev estimates, we get

$$\begin{aligned}
& \frac{1}{p}\frac{\partial}{\partial t}\int_0^1 (u^p + v^p) \\
\le & -\sum_i \frac{(p-1)\pi^2}{p^2|C_{i,t}^u|^2}\int_{C_{i,t}^u} u^p - \sum_i \frac{(p-1)\pi^2}{p^2|C_{i,t}^v|^2}\int_{C_{i,t}^v} v^p + (a+m)\int_0^1 (u^p + v^p) \\
\le & -\sum_i \frac{(p-1)\pi^2}{p^2(\max_i |C_{i,t}^u|)^2}\int_{C_{i,t}^u} u^p - \sum_i \frac{(p-1)\pi^2}{p^2(\max_i |C_{i,t}^v|)^2}\int_{C_{i,t}^v} v^p \\
& +(a+m)\int_0^1 (u^p + v^p) \\
\le & \left(a + m - \frac{(p-1)\pi^2}{p^2(\max\{\max_i |C_{i,t}^u|, \max_i |C_{i,t}^v|\})^2} \right) \int_0^1 (u^p + v^p) .
\end{aligned}$$

Let $C_t = \max\{\max_i |C_{i,t}^u|, \max_i |C_{i,t}^v|\}$ and $\displaystyle ⨍_0^T = \frac{1}{T}\int_0^T$. Therefore, using Jensen's inequality, we get the following inequalities.

$$\begin{aligned}
& \int_0^1 (u^p(T) + v^p(T)) \\
& \le \left[\exp\left(pT ⨍_0^T \left(a + m - \frac{(p-1)\pi^2}{p^2|C_t|^2} \right) \right) \right] \int_0^1 (u^p(0) + v^p(0)) \\
& \le \left[\exp\left(pT \left(a + m - \frac{(p-1)\pi^2}{p^2(⨍_0^T |C_t|)^2} \right) \right) \right] \int_0^1 (u^p(0) + v^p(0)) \\
& \le \left[\exp\left(pT \left(a + m - \frac{(p-1)\pi^2 T^2}{p^2 r^2} \right) \right) \right] \int_0^1 (u^p(0) + v^p(0)) ,
\end{aligned}$$

which proves the lemma. □

Remark 2.3 There are similar versions of Lemmas 2.1 and 2.2 when we have Dirichlet boundary conditions instead of Neumann boundary conditions, (See [18]).

Remark 2.4 At this point we have established that (1.2) is bounded dissipative in $L^p \times L^p$, for any $p \geq 2$. One important thing to point out is that the bounds do not depend on $M > 0$.

In order to prove the existence of global attractors we will need to establish that (1.2) is bounded dissipative in $H^1 \times H^1$, and furthermore, since we want to establish the synchronization property, the bounds must be independent of M, and that is exactly what we prove in the following lemma.

Lemma 2.5 *There exists a bounded set* $\mathcal{M} \subset C([-r, 0], H^1(0,1) \times H^1(0,1))$ *which attracts each bounded set of* $C([-r, 0], H^1(0,1) \times H^1(0,1))$, *under the flow defined by (1.5). Furthermore* $\mathcal{M}$ *can be taken independent of* M.

Proof To prove the result, we will first make the following change of variables

$$w(t,x) = \frac{v(t,x) - u(t,x)}{2} \quad \text{and } z(t,x) = \frac{v(t,x) + u(t,x)}{2} \tag{2.7}$$

We will assume, without loss of generality, that $M_1 = M + m$ and $M_2 = M$, and thus (w, z) will satisfy the following system, keeping in mind the definition of $h = (h_1, h_2)$ in (1.6),

$$\begin{cases} \dfrac{\partial w}{\partial t} = \Delta w & -2Mw - mw \\ & +\dfrac{1}{2}\left(h_2\left((z-w)_t, (z+w)_t\right) - h_1\left((z-w)_t, (z+w)_t\right)\right) \\ \dfrac{\partial z}{\partial t} = \Delta z & \qquad + mw \\ & +\dfrac{1}{2}\left(h_2\left((z-w)_t, (z+w)_t\right) + h_1\left((z-w)_t, (z+w)_t\right)\right) \end{cases} \tag{2.8}$$

Observe that if we prove the result for the solutions of (2.8) we are also proving the results for the solutions of (1.5).

Let us consider a bounded set $\mathcal{B} \subset C([-r, 0], H^1(0,1) \times H^1(0,1))$. From, Lemmas 2.1 and 2.2, there exist K, independent of M and $\mathcal{B}$, and $t_0 > r$, such that, for any $(w_0, z_0) \in \mathcal{B}$,

$$\|(w_t, z_t)\|_{C([-r,0],L^2(0,1)\times L^2(0,1))}, \ \|(w_t, z_t)\|_{C([-r,0],L^4(0,1)\times L^4(0,1))} \leq K, \tag{2.9}$$

for any $t \geq t_0$.

Given A as in (1.5), and any $\beta > 0$, let $\lambda > 0$ be the first eigenvalue of $A - \beta I$.

First of all, it is easy to see that orbits of bounded set of (2.8) are bounded in finite time, and therefore there exists $N > 0$, such that, for all $-r \leq s \leq t_0$, $\|(w(s), z(s))\|_{H^1 \times H^1} \leq N$.

Let $t_1 > t_0$, be such that $e^{-(\lambda+\beta)(t_1 - t_0)} N < 1$.

Let us concentrate now on the first equation of (2.8). Using the variation of constants formula for this equation, one has

$$w(t) = e^{(A-(\beta+2M+m)I)(t-t_0)}w(t_0) + \int_{t_0}^{t} e^{(A-(\beta+2M+m)I)(t-s)}.$$

$$\left(\frac{1}{2}\left(h_2\left((z-w)_s,(z+w)_s\right) - h_1\left((z-w)_s,(z+w)_s\right)\right) + \beta w(s)\right) ds.$$

Taking the H^1 norm on both sides, one gets

$$\|w(t)\|_{H^1} = e^{-(\lambda+\beta+2M+m)(t-t_0)}\|w(t_0)\|_{H^1} + \int_{t_0}^{t} (t-s)^{-\frac{1}{2}} e^{-(\lambda+\beta+2M+m)(t-s)}.$$

$$\left(\frac{1}{2}\|h_2\left((z-w)_s,(z+w)_s\right) - h_1\left((z-w)_s,(z+w)_s\right)\|_{L^2} + \beta\|w(s)\|_{L^2}\right) ds. \tag{2.10}$$

From the definition of h_1 and h_2, we get that, if $t \geq t_1 + r$

$$\begin{aligned}
&\|h_2\left((z-w)_s,(z+w)_s\right) - h_1\left((z-w)_s,(z+w)_s\right)\|_{L^2} \\
= &\left\| a_2 v(s)\left(1 - \frac{v(s-\tau_2)}{K_{21}} - \frac{u(s-\rho_2)}{K_{22}}\right)\right. \\
&\left. - a_1 u(s)\left(1 - \frac{u(s-\tau_1)}{K_{11}} - \frac{v(s-\rho_1)}{K_{12}}\right)\right\|_{L^2} \\
\leq & \; a_2\left[\|v(s)\|_{L^2} + \frac{\|v(s)\|_{L^4}}{4K_{21}} + \frac{\|v(s-\tau_2)\|_{L^4}}{4K_{21}} + \frac{\|v(s)\|_{L^4}}{4K_{22}} + \frac{\|u(s-\rho_2)\|_{L^4}}{4K_{22}}\right] \\
& + a_1\left[\|u(s)\|_{L^2} + \frac{\|u(s)\|_{L^4}}{4K_{11}} + \frac{\|u(s-\tau_1)\|_{L^4}}{4K_{11}} + \frac{\|u(s)\|_{L^4}}{4K_{12}} + \frac{\|v(s-\rho_1)\|_{L^4}}{4K_{12}}\right] \\
\leq & \; a_2 K\left[1 + \frac{1}{2K_{21}} + \frac{1}{2K_{22}}\right] + a_1 K\left[1 + \frac{1}{2K_{11}} + \frac{1}{2K_{12}}\right] \equiv \tilde{K}\,.
\end{aligned}$$

Substituting this in (2.10) one has, if $t \geq t_1 + r$,

$$\begin{aligned}
\|w(t)\|_{H^1} &= e^{-(\lambda+\beta+2M+m)(t-t_0)} N \\
&\quad + \int_{t_0}^{t} (t-s)^{-\frac{1}{2}} e^{-(\lambda+\beta+2M+m)(t-s)} \left(\frac{1}{2}\tilde{K} + \beta K\right) ds \\
&\leq 1 + C\int_0^{\infty} s^{-\frac{1}{2}} e^{-(\lambda+\beta)s} ds,
\end{aligned}$$

where C is independent of M. Proceeding in a similar way with z, we get the result. □

This uniform bound is actually all that we need to prove the existence of the global attractor. We get that, if $\mathcal{V}$ is a bounded subset of $C((-r,0],H^1 \times H^1)$ then $\cup_{t\geq 0} U(t)\mathcal{V}$ is a bounded subset of $H^1 \times H^1$, where $U(t)$ is the semigroup generated by (1.5).

Therefore, since the semigroup is precompact, (1.5) has a compact global attractor $\mathcal{A}$ (see Hale [10]).

3 Synchronization

In this section we will use the a priori bounds of the previous section to get synchronization properties.

3.1 Closeness to diagonal. Once we get the uniform bounds, we can expect that the attractor is going to be as close to the diagonal as the coefficients permit, that is, we can expect **synchronization**. We follow some ideas from Rodrigues [21].

Let us consider a restricted set of parameters a, r and m. Let α, ρ, $\mu > 0$ and we define the following set:

$$\Lambda = \{(a, r, m) \in \mathbb{R}^3 : 0 \le a \le \alpha,\ 0 \le r \le \rho \text{ and } 0 \le m \le \mu\}.$$

From the continuity with respect to the parameters and the results in the previous section, one can see the following lemma holds.

Lemma 3.1 *There exists a bounded set $\mathcal{B} \subset C([-\rho, 0], H^1(0,1))$ such that, for all $(a, r, m) \in \Lambda$, bounded, $M > 0$ and $\mathcal{M} \subset C([-\rho, 0], H^1(0,1) \times H^1(0,1))$, if $a_1, a_2 \le a$, $\tau_1, \tau_2, \rho_1, \rho_2 \le r$, $\min\{M_1, M_2\} = M$ and $|M_1 - M_2| = m$, there exists $t_0 > 0$ such that $(u(t, u_0, v_0), v(t, u_0, v_0)) \in \mathcal{B} \times \mathcal{B}$, for every $t \ge t_0$ and $(u_0, v_0) \in \mathcal{M}$.*

With this we have the following theorem.

Theorem 3.2 *Let $(a, r, m) \in \Lambda$ and $\mathcal{M} \subset C([-\rho, 0], H^1(0,1) \times H^1(0,1))$. If $a_1, a_2 \le a$, $\tau_1, \tau_2, \rho_1, \rho_2 \le r$, $\min\{M_1, M_2\} = M$ and $|M_1 - M_2| = m$, there exists $t_0 \ge 0$, such that, for any $(u_0, v_0) \in \mathcal{M}$, the solution $(u(t, u_0, v_0), v(t, u_0, v_0))$ of (1.2) belongs to $\mathcal{B} \times \mathcal{B}$ and if M is big enough, we have*

$$\|v(t, u_0, v_0) - u(t, u_0, v_0)\| \le K e^{-\gamma(t - t_0)} \|v(t_0, u_0, v_0) - u(t_0, u_0, v_0)\|$$

$$+ L\left(|\tau_1 - \tau_2| + |\rho_1 - \rho_2| + |a_1 - a_2| + \left|\frac{1}{K_{21}} - \frac{1}{K_{11}}\right| - \left|\frac{1}{K_{22}} - \frac{1}{K_{12}}\right|\right)$$

for every $t \ge t_0$.

Proof Without loss of generality, we can assume that $t_0 = 0$. This can be justified by the change of variables, $t \mapsto t - t_0$. Therefore we will prove that, if $(u_0, v_0) \in \mathcal{B} \times \mathcal{B}$ is such that $(u(t, u_0, v_0), v(t, u_0, v_0)) \in \mathcal{B} \times \mathcal{B}$ for every $t \ge 0$, then

$$\begin{aligned}\|u(t, u_0, v_0) - v(t, u_0, v_0)\| &\le K e^{-\gamma t} \|v_0 - u_0\| \\ &+ L\left(|\tau_1 - \tau_2| + |\rho_1 - \rho_2| + |a_1 - a_2| + \left|\frac{1}{K_{21}} - \frac{1}{K_{11}}\right| - \left|\frac{1}{K_{22}} - \frac{1}{K_{12}}\right|\right),\end{aligned}$$

for every $t \ge 0$.

To prove the result, we will make, once again, the change of variables (2.7) and get that (w, z) satisfy (2.8).

Let us concentrate for a moment on the w equation. Adding and subtracting terms, we get

$$\begin{aligned}\dot{w} &= \Delta w - 2Mw - mw \\ &+\frac{1}{2}\left(a_2(1-\frac{v(t-\tau_2)}{K_{21}}-\frac{u(t-\rho_2)}{K_{22}})(v-u)\right. \\ &+a_2u(-v(t-\tau_2)(\frac{1}{K_{21}}-\frac{1}{K_{11}})-u(t-\rho_2)(\frac{1}{K_{22}}-\frac{1}{K_{12}})) \\ &+a_2u(-\frac{1}{K_{11}}(v(t-\tau_2)-v(t-\tau_1))-\frac{1}{K_{12}}(u(t-\rho_2)-u(t-\rho_1))) \\ &+a_2u((-\frac{1}{K_{11}}(v(t-\tau_1)-u(t-\tau_1))-\frac{1}{K_{12}}(u(t-\rho_1)-v(t-\rho_1))) \\ &\left.+(a_2-a_1)u(1-\frac{u(t-\tau_1)}{K_{11}}-\frac{v(t-\rho_1)}{K_{12}})\right).\end{aligned}$$

Now using the fact that the solution and its time derivative are uniformly bounded (since they are in $\mathcal{B}$), we have that

$$\begin{aligned}\dot{w} &\leq \Delta w - 2Mw - mw \\ &+\frac{1}{2}\left(\frac{a}{k}(k+\|u\|_\infty+\|v\|_\infty)|v-u|\right. \\ &+\frac{a}{k}\|u\|_\infty(|v(t-\tau_1)-u(t-\tau_1)|+|u(t-\rho_1)-v(t-\rho_1)|) \\ &+a\|u\|_\infty(\|v\|_\infty\left|\frac{1}{K_{21}}-\frac{1}{K_{11}}\right|-\|u\|_\infty\left|\frac{1}{K_{22}}-\frac{1}{K_{12}}\right|) \\ &+\frac{a}{k}(\|\dot{u}\|_\infty+\|\dot{v}\|_\infty)(|\tau_2-\tau_1|+|\rho_2-\rho_1|) \\ &\left.+|a_2-a_1|\frac{\|u\|_\infty}{k}(k+\|u\|_\infty+\|v\|_\infty)\right),\end{aligned}$$

where $k=\min\{K_{11},K_{12},K_{21},K_{22}\}$. And similarly if we denote $w=u-v$, thus as long as $u\neq v$ we can write, (if $w=|u-v|$),

$$\begin{aligned}\dot{w} &\leq \Delta w-(2M+m)w \\ &+\delta(w+w(t-\tau_1)+w(t-\tau_2)+w(t-\rho_1)+w(t-\rho_2)) \\ &+\sigma\left(|\tau_1-\tau_2|+|\rho_1-\rho_2|+|a_1-a_2|+\left|\frac{1}{K_{21}}-\frac{1}{K_{11}}\right|-\left|\frac{1}{K_{22}}-\frac{1}{K_{12}}\right|\right).\end{aligned}$$

Taking M big enough, we finish the proof. □

As mentioned before, the last theorem is a synchronization result in the sense that the solutions stay close to the diagonal. If the coefficients for each equation were the same (that is, $\tau_1=\tau_2$, $\rho_1=\rho_2$, $a_1=a_2$, $K_{11}=K_{21}$ and $K_{12}=K_{22}$), the solution asymptotically goes to the diagonal, and all the dynamical behavior occurs on the diagonal ($u=v$). If this is not the case, we will need stronger results.

3.2 Strong coupling. In this section we want to study the behavior of the solutions, and more precisely, of the attractors, as the coupling M approaches infinity. We know from the previous section that we will have synchronization, but what else can we say? To answer this question we will follow Carvalho, Rodrigues and Dlotko [2], which studies synchronization in parabolic equations without delays, and Carvalho and Oliveira [3] which studies upper semicontinuity of attractors in parabolic equations with delays.

Abstract results

We will state an abstract result which can be found in [3], and which we will use to get the behavior of the attractors.

Let $\nu > 0$ be a positive parameter, X_ν be a Banach space and $A_\nu : D(A_\nu) \subset X_\nu \to X_\nu$ be the generator of a semigroup. Let A_ν^α denote the fractional power of A_ν and X_ν^α the associated fractional power spaces.

Assume that the semigroup $\{T_\nu(t),\ t \geq 0\}$ generated by A_ν satisfies

$$\|T_\nu(t)w\|_{X_\nu^\alpha} \leq Ne^{-\beta(\nu)t}\|w\|_{X_\nu^\alpha},\ \ t \geq 0$$

$$\|T_\nu(t)w\|_{X_\nu^\alpha} \leq Nt^{-\alpha}e^{-\beta(\nu)t}\|w\|_{X_\nu},\ \ t > 0,$$

for any $w \in X_\nu^\alpha$, where $\beta(\nu)$ may depend upon the parameter ν and $N \geq 1$ is a constant.

Consider the weakly coupled system

$$\begin{cases} \dot{y}(t) = A_\nu\ y(t) + f_\nu(y_t, v_t) \\ \\ \dot{v}(t) = B\ v(t) + g_\nu(y_t, v_t) \end{cases} . \tag{3.1}$$

Let $g\ :\ C([-r,0], Y^\alpha) \to Y^\alpha$ be Lipschitz continuous in bounded sets of $C([-r,0], Y^\alpha)$ and assume that

$$\dot{v}(t) = B\ v(t) + g(v_t) \tag{3.2}$$

has a global attractor $\mathcal{A}$ in $C([-r,0], Y^\alpha)$. Suppose that there exists a constant $\mathcal{K} > 0$, independent of ν, such that the set

$$\mathcal{B} = \{u \in C([-r,0], X_\nu^\alpha \times Y^\alpha)\ :\ \|u\|_{C([-r,0],X_\nu^\alpha \times Y^\alpha)} \leq \mathcal{K}\} \tag{3.3}$$

attracts bounded sets of $C([-r,0], X_\nu^\alpha \times Y^\alpha)$ under the flow defined by (3.1).

Let $R > 0$, $(\phi, \psi) \in C([-r,0], X_\nu^\alpha \times Y^\alpha)$, $\|(\phi,\psi)\|_{C([-r,0],X_\nu^\alpha \times Y^\alpha)} \leq R$. Suppose that there exist nonnegative constants M_f, L_f, depending only on R, such that

$$\|f_\nu(\phi,\psi)\|_{X_\nu} \leq L_f\|\phi\|_{C([-r,0],X_\nu^\alpha)} + M_f,$$

$$\text{and} \tag{3.4}$$

$$\|P_\nu(\phi,\psi)\|_Y \leq L_P(\nu)\|\phi\|_{C([-r,0],X_\nu^\alpha)} + M_P(\nu),$$

where $P_\nu(\phi,\psi) = g_\nu(\phi,\psi) - g(\psi)$, and $L_P(\nu)$, $M_P(\nu) \to 0$ as $\nu \to 0$.

Assume also that either of the following conditions is satisfied.

a) The flow defined by (3.1) is asymptotically smooth,

b) $M_f = 0$ and the flow defined by $\dot{v}(t) = B\ v(t) + g_\nu(0, v_t)$ is asymptotically smooth,

c) $M_f = 0$ and $M_P \equiv 0$.

Theorem 3.3 *Assume that A is a sectorial operator and that $\beta(\nu) \to \infty$ as $\nu \to 0$. Assume also that (3.3) and (3.4) are satisfied and that either a), b) or c) above is satisfied. Then there exists $\nu_0 > 0$ such that, for $0 < \nu \leq \nu_0$, the problem (3.1) has a global attractor $\mathcal{A}_\nu$ and the family of attractors $\{\mathcal{A}_\nu,\ 0 \leq \nu \leq \nu_0\}$ is* *upper semicontinuous at zero, where* $\mathcal{A}_0 := 0 \times \mathcal{A}$. *Furthermore, if c) is satisfied, there exists* ν_0, *such that* $\mathcal{A}_\nu = 0 \times \mathcal{A}$, $0 < \nu \leq \nu_0$.

Proof One can see [3] (Theorem 1.2) for the proof. □

Application

We are now able to apply the abstract result of Theorem 3.3 to system (1.2). We will fix from now on the parameters $K_{1j} = K_{2j}$, $a_1 = a_2$, $\tau_1 = \tau_2$, $\rho_1 = \rho_2$ and m, while letting the coupling parameter M change, and actually we wish it to tend to infinity. We already proved in the last sections that (1.2) has a global attractor $\mathcal{A}_M$ in $C([-r,0], H^1(0,1) \times H^1(0,1))$. We also obtained a priori bounds in $C([-r,0], L^\infty(0,1) \times L^\infty(0,1))$, independent of M. In order to apply Theorem 3.3, we will need once again to perform the change of variables (2.7) and get the new variables (w,z), which satisfy (2.8). Once again, (2.8) has a global attractor $\widetilde{\mathcal{A}}_M$.

In order to rewrite (2.8) in the form (3.1), let $X = L^2(0,1)$ and $A_\nu : D(A_\nu) \subset X \to X$ be the self adjoint operator defined by

$$D(A_\nu) = \left\{ \phi \in H^2(0,1) \ : \ \frac{\partial \phi}{\partial n} = 0 \right\}, \quad A_\nu \phi = \Delta \phi - \frac{2}{\nu}\phi, \ \forall \ \phi \in D(A_\nu). \tag{3.5}$$

Let X_ν^α denote the fractional power spaces associated to $-A_\nu$ endowed with the graph norm.

Let also $Y = L^2(0,1)$ and $B : D(B) \subset Y \to Y$ be the self adjoint operator defined by

$$D(B) = \left\{ \phi \in H^2(0,1) \ : \ \frac{\partial \phi}{\partial n} = 0 \right\}, \quad B\,\phi = \Delta \phi, \ \forall \ \phi \in D(B). \tag{3.6}$$

Let Y^α denote the fractional power spaces associated to $-B$ endowed with the graph norm. System (2.8) can now be rewritten in the abstract form

$$\begin{aligned} \frac{\partial w}{\partial t} &= A_\nu w + f_\nu(w_t, z_t) \\ \frac{\partial z}{\partial t} &= Bz + g_\nu(w_t, z_t), \end{aligned} \tag{3.7}$$

in the space $C([-r,0], X_\nu^{1/2} \times Y^{1/2})$, where, as in (1.5) and (1.6), $f_\nu : C([-r,0], X_\nu^{1/2} \times Y^{1/2}) \to X$ and $g_\nu : C([-r,0], X_\nu^{1/2} \times Y^{1/2}) \to Y$ are defined by

$$f_\nu(\phi,\psi)(x) = -m\phi(0) + \frac{1}{2}\left(h_2(\psi-\phi, \psi+\phi) - h_1(\psi-\phi,\psi+\phi)\right),$$

and

$$g_\nu(\phi,\psi)(x) = m\phi(0) + \frac{1}{2}\left(h_2(\psi-\phi, \psi+\phi) + h_1(\psi-\phi,\psi+\phi)\right)$$

which is in the form (3.1) and, moreover, we will consider $g(\psi) = g_\nu(0,\psi)$.

Observing that $\|\phi\|_X \leq cte \cdot \nu^{1/2} \|\phi\|_{X_\nu^{1/2}}$, and using the uniform bounds in L^∞ that we got in the previous sections, we have that (3.4) is true with $M_f = 0$, and $L_P(\nu) \to 0$ as $\nu \to 0$. Therefore we can apply Theorem 3.3 to obtain the upper semicontinuity of the attractors $\left\{ \widetilde{\mathcal{A}}_{\frac{1}{\nu}}, \ 0 \leq \nu \leq \nu_0 \right\}$ at zero, or $\left\{ \widetilde{\mathcal{A}}_M, \ M \leq \infty \right\}$ at infinity, where $\widetilde{\mathcal{A}}_\infty = \{(0,z), \ z \in \widetilde{\mathcal{A}}\}$ and $\widetilde{\mathcal{A}}$ is the global attractor of the problem

$$\frac{\partial z}{\partial t} = \Delta z + \frac{1}{2}\left(h_2(z_t, z_t) + h_1(z_t, z_t)\right).$$

Thus, changing back to (u, v) we get the following.

Theorem 3.4 *The family of attractors $\{\mathcal{A}_M,\ M \leq \infty\}$ of (1.2) is upper semi-continuous at infinity where $\mathcal{A}_\infty = \{(u,u)\ :\ u \in \mathcal{A}_l\}$ and $\mathcal{A}_l$ is the global attractor of the limit problem*

$$\dot{u}(t) = \Delta u(t) + \frac{1}{2}\left(h_1(u_t, u_t) + h_2(u_t, u_t)\right).$$

4 Persistence of periodic orbits

As mentioned in the introduction, if (1.7) has a stable periodic orbit, then we will be able to show that (1.2) also has a periodic orbit. In order to simplify the notation, and therefore the presentation, we will consider the following simplified version of system (1.2), in which we can guarantee that (1.7) has a stable periodic orbit. For the general case the same results applies but we will not know that the limit equation has a stable periodic orbit. We consider the problem.

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = & \Delta u(t,x) + a_1 u(t,x)\left(1 - u(t-\tau,x)\right) - M(u(t,x) - v(t,x)) \\ \\ \dfrac{\partial v}{\partial t}(t,x) = & \Delta v(t,x) + a_2 v(t,x)\left(1 - v(t-\tau,x)\right) - M(v(t,x) - u(t,x)) \end{cases} \tag{4.1}$$

for all $t > 0$ and $x \in (0,1)$, where a_1, a_2, M and τ are positive constants, plus the following boundary condition

$$(\mathrm{D}) \begin{cases} u(t,0) = u(t,1) = 0 \\ \\ v(t,0) = v(t,1) = 0 \end{cases} \tag{4.2}$$

where $t \in (0, \infty)$. We will continue to consider only the following initial conditions

$$\left.\begin{array}{l} u(s,x) > 0 \\ v(s,x) > 0 \end{array}\right\} \text{ for all } (s,x) \in [-\tau, 0] \times [0,1]. \tag{4.3}$$

From the previous section, the problem is well posed in $\mathcal{Z} \times \mathcal{Z}$, where $\mathcal{Z} = C((-\tau,0), H_0^1(0,1))$. We will denote by $\|\cdot\|_\alpha$, $\|\cdot\|_1$ and $\|\cdot\|_0$ the norms in $\mathcal{Z}$, $H_0^1(0,1)$ and $L^2(0,1)$, respectively. Moreover, we have that there exists a bounded absorbing set $\mathcal{M} \in \mathcal{Z} \times \mathcal{Z}$, independent of M, and from now on we will suppose that the initial conditions are in this absorbing set, and also that $\mathcal{M}$ will indicate the bound in $\mathcal{Z} \times \mathcal{Z}$, uniformly in M. Once again, if we consider the change of variables

$$\begin{array}{lcl} w = \dfrac{u - v}{2} & & u = w + z \\ & \text{or} & \\ z = \dfrac{u + v}{2} & & v = z - w \end{array} \tag{4.4}$$

we get the following system

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \Delta w(t,x) - Mw(t,x) & +\frac{a_1}{2}(w+z)(t,x)\left(1 - (w+z)(t-\tau,x)\right) \\ & -\frac{a_2}{2}(z-w)(t,x)\left(1 - (z-w)(t-\tau,x)\right) \\ \\ \dfrac{\partial z}{\partial t}(t,x) = \Delta z(t,x) & +\frac{a_1}{2}(w+z)(t,x)\left(1 - (w+z)(t-\tau,x)\right) \\ & +\frac{a_2}{2}(z-w)(t,x)\left(1 - (z-w)(t-\tau,x)\right). \end{cases} \tag{4.5}$$

Denote by A_M the operator $-\Delta + MI$ in $H_0^1(0,1)$, and let λ_M be its principal eigenvalue. Also let

$$f(w,z) = \frac{a_1}{2}(w+z)(t,x)\,(1-(w+z)(t-\tau,x))$$

and

$$g(w,z) = \frac{a_2}{2}(z-w)(t,x)\,(1-(z-w)(t-\tau,x))\,.$$

In order to prove the persistence of periodic orbits, we will need some estimates.

Estimate on $\|w_t\|_\alpha$

We have that $\|w_t\|_\alpha = \sup_{\theta\in(-\tau,0)} \|w(t+\theta,x)\|_1$, and thus using the variation of constant formula, we get

$$\begin{aligned}
\|w(t,x)\|_\alpha \leq\ & e^{-\lambda_M(t-t_0)}\|w(t_0,x)\|_1 \\
& + \textstyle\int_{t_0}^t e^{-\lambda_M(t-s)}(t-s)^{-\frac{1}{2}}\|f(w,z)-g(w,z)\|_0 ds \\
\leq\ & e^{-\lambda_M(t-t_0)}\mathcal{M} + K_1(a_1,a_2,\mathcal{M})\int_{t_0}^t e^{-\lambda_M(t-s)}(t-s)^{-\frac{1}{2}}ds \\
=\ & e^{-\lambda_M(t-t_0)}\mathcal{M} + K_1(a_1,a_2,\mathcal{M})\int_0^{\lambda_M(t-t_0)} e^{-s}\lambda_M^{-\frac{1}{2}}s^{-\frac{1}{2}}ds \\
\leq\ & e^{-\lambda_M(t-t_0)}\mathcal{M} + K_1(a_1,a_2,\mathcal{M})\int_0^{\infty} e^{-s}\lambda_M^{-\frac{1}{2}}s^{-\frac{1}{2}}ds \\
=\ & e^{-\lambda_M(t-t_0)}\mathcal{M} + K_1(a_1,a_2,\mathcal{M})\lambda_M^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right),
\end{aligned}$$

where Γ denotes the gamma function.

Therefore, for $t \geq \tau$, and assuming that $t_0 = 0$, we have

$$\|w_t\|_\alpha \leq e^{-\lambda_M(t-\tau)}\mathcal{M} + K_1(a_1,a_2,\mathcal{M})\lambda_M^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right). \tag{4.6}$$

Estimate on $\|z_t\|_\alpha$

In order to estimate z, we will introduce the limit equation

$$\frac{\partial y}{\partial t}(t,x) = \Delta y(t,x) + \frac{a_1+a_2}{2}y(t,x)\,(1-y(t-\tau,x))\,, \tag{4.7}$$

obtained by taking $w = 0$ in the second equation.

Let us compare z and y in $\mathcal{Z}$, that is let us compute $\|z_t(z_0) - y_t(y_0)\|_\alpha$, where z_0 and y_0 are initial conditions in $\mathcal{M}$ (regarded as an absorbing set in $\mathcal{Z}$). We will drop the dependence on the initial condition in the notation bellow. But first note that

$$f(w,z) = f(w,0) + f(0,z) - \frac{a_1}{2}\left(w(t)z(t-\tau) + z(t)w(t-\tau)\right)$$

and

$$g(w,z) = g(w,0) + g(0,z) - \frac{a_2}{2}\left(w(t)z(t-\tau) + z(t)w(t-\tau)\right).$$

Thus, for any $t \geq t_0$, we have

$$
\begin{aligned}
&\|z(t,x) - y(t,x)\|_1 \\
&= \|e^{A(t-t_0)}(z_0(0) - y_0(0)) \\
&\quad + \int_{t_0}^{t} e^{A(t-s)} \left(f(w,z) + g(w,z) - f(0,y) - g(0,y)\right) ds\|_1 \\
&\leq e^{-\lambda_0(t-t_0)}\|z_0(0) - y_0(0)\|_1 \\
&\quad + \left(\frac{a_1+a_2}{2}\right) \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} \|z(s)w(s-\tau)\|_0 ds \\
&+ \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} \|f(w,0) + g(w,0) - \frac{a_1+a_2}{2} w(s)z(s-\tau)\|_0 ds \\
&\quad + \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} \|f(0,z) - f(0,y)\|_0 ds \\
&\quad + \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} \|g(0,z) - g(0,y)\|_0 ds \\
&\leq e^{-\lambda_0(t-t_0)}\|z_0(0) - y_0(0)\|_1 \\
&\quad + \left(\frac{a_1+a_2}{2}\right) \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} \|z(s)w(s-\tau)\|_0 ds \\
&\quad + \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} K_2(a_1,a_2,\mathcal{M}) \|w(s)\|_1 ds \\
&\quad + \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} \|f(0,z) - f(0,y)\|_0 ds \\
&\quad + \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} \|g(0,z) - g(0,y)\|_0 ds,
\end{aligned}
$$

and finally using the estimate on $\|w(s)\|_1$, we get

$$
\begin{aligned}
&\|z(t,x) - y(t,x)\|_1 \\
&\leq e^{-\lambda_0(t-t_0)}\|z_0(0) - y_0(0)\|_1 \\
&\quad + \left(\frac{a_1+a_2}{2}\right) \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} \|z(s)w(s-\tau)\|_0 ds \\
&\quad + K_2\mathcal{M} \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} e^{-\lambda_M(s-t_0)} ds + K_1 K_2 \lambda_M^{-\frac{1}{2}} \lambda_0^{-\frac{1}{2}} \Gamma^2(1/2) \\
&\quad + \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} \|f(0,z) - f(0,y)\|_0 ds \\
&\quad + \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} \|g(0,z) - g(0,y)\|_0 ds.
\end{aligned}
\tag{4.8}
$$

Separating terms, let us compute each term separately. First, we have for any $t_0 < t_1 < t$,

$$\begin{aligned}
&\int_{t_1}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}e^{-\lambda_M(s-t_0)}ds \\
&= \int_{t_1}^{\frac{t+t_1}{2}} + \int_{\frac{t+t_1}{2}}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}e^{-\lambda_M(s-t_0)}ds \\
&\leq e^{-\lambda_0\left(\frac{t-t_1}{2}\right)}\left(\frac{t-t_1}{2}\right)^{-\frac{1}{2}}\int_{t_1}^{\frac{t+t_1}{2}} e^{-\lambda_M(s-t_0)}ds \\
&\quad + e^{-\lambda_M\left(\frac{t+t_1}{2}-t_0\right)}\int_{\frac{t+t_1}{2}}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}ds \\
&\leq \Phi(t,t_0,t_1,M) + \Psi(t,t_0,t_1,M),
\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
\Phi(t,t_0,t_1,M) &= \frac{e^{-\lambda_M(t_1-t_0)}e^{-\lambda_0(t-t_1)/2}((t-t_1)/2)^{-1/2}}{\lambda_M}\left(1-e^{-\lambda_M(t-t_1)/2}\right) \\
\Psi(t,t_0,t_1,M) &= e^{-\lambda_M\left(\frac{t+t_1}{2}-t_0\right)}\int_{\frac{t+t_1}{2}}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}ds.
\end{aligned} \tag{4.10}$$

Remark 4.1 Observe that for any $t_0 \leq t_1 \leq t$, we have that

1. Φ and Ψ are continuous for $t \in [t_1, \infty)$ and continuously differentiable for $t \in (t_1, \infty)$;
2. $\Phi(t_1, t_0, t_1, M) = \Psi(t_1, t_0, t_1, M) = 0$;
3. $\Phi(t, t_0, t_1, M)$ and $\Psi(t, t_0, t_1, M)$ are positive and bounded, for any $t \geq t_1$;
4. $\lim_{t\to\infty} \Phi(t, t_0, t_1, M) = \lim_{t\to\infty} \Psi(t, t_0, t_1, M) = 0$;
5. There exists a $t^* = t^*(t_0, t_1, M)$, with $t^* > t_1$, such that $\Phi(t^*, t_0, t_1, M) = \max_{t\geq t_1} \Phi(t, t_0, t_1, M)$;
6. There exists a $t^{**} = t^{**}(t_0, t_1, M)$, with $t^{**} > t_1$, such that $\Psi(t^{**}, t_0, t_1, M) = \max_{t\geq t_1} \Psi(t, t_0, t_1, M)$;
7. $\lim_{M\to\infty} t^* = \lim_{M\to\infty} \Phi(t^*, t_0, t_1, M) = 0$;
8. $\lim_{M\to\infty} t^{**} = \lim_{M\to\infty} \Psi(t^{**}, t_0, t_1, M) = 0$.

With this, we can take care of the next term. In order to do that let us suppose for a moment that $0 \leq t - t_0 \leq \tau$. Then

$$\begin{aligned}
&\left(\frac{a_1+a_2}{2}\right)\int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}\|z(s)w(s-\tau)\|_0 ds \\
&\leq K_3(a_1, a_2, \mathcal{M})\|w_0\|_\alpha \lambda_0^{-\frac{1}{2}}\Gamma(1/2).
\end{aligned} \tag{4.11}$$

On the other hand, if $t - t_0 \geq \tau$, we get

$$
\begin{aligned}
&\left(\frac{a_1+a_2}{2}\right)\int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}\|z(s)w(s-\tau)\|_0 ds \\
&\leq \left(\frac{a_1+a_2}{2}\right)\left(\int_{t_0}^{t_0+\tau}+\int_{t_0+\tau}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}\|z(s)w(s-\tau)\|_0 ds\right) \\
&\leq K_3\|w_0\|_\alpha \lambda_0^{-\frac{1}{2}}\Gamma(1/2) \\
&\quad + K_3\int_{t_0+\tau}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}\left(e^{-\lambda_M(s-t_0)}\mathcal{M} + K_1\lambda_M^{-\frac{1}{2}}\Gamma(1/2)\right) ds \\
&\leq K_3\|w_0\|_\alpha \lambda_0^{-\frac{1}{2}}\Gamma(1/2) + K_3K_1\lambda_M^{-\frac{1}{2}}\Gamma^2(1/2) \qquad (4.12) \\
&\quad + K_3\mathcal{M}\int_{t_0+\tau}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}e^{-\lambda_M(s-t_0)} ds \\
&\leq K_3\|w_0\|_\alpha \lambda_0^{-\frac{1}{2}}\Gamma(1/2) + K_3K_1\lambda_M^{-\frac{1}{2}}\Gamma^2(1/2) \\
&\quad + K_3\mathcal{M}\big(\Phi(t^*(t_0,t_0+\tau,M),t_0,t_0+\tau,M) \\
&\quad + \Psi(t^{**}(t_0,t_0+\tau,M),t_0,t_0+\tau,M)\big).
\end{aligned}
$$

Thus combining (4.11) and (4.12) we can write, for all $t \geq t_0$,

$$
\begin{aligned}
&\frac{a_1+a_2}{2}\int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}\|z(s)w(s-\tau)\|_0 ds \\
&\leq K_3\|w_0\|_\alpha \lambda_0^{-\frac{1}{2}}\Gamma(1/2) + \chi_{(t_0+\tau,+\infty)}(t)[K_3K_1\lambda_M^{-\frac{1}{2}}\Gamma^2(1/2) \qquad (4.13) \\
&\quad + K_3\mathcal{M}(\Phi(t^*,t_0,t_0+\tau,M)+\Psi(t^{**},t_0,t_0+\tau,M))],
\end{aligned}
$$

where χ_I denotes the characteristic function on I.

For the last terms, once again, let us first assume that $0 \leq t - t_0 \leq \tau$. Then

$$\int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}\|f(0,z) - f(0,y)\|_0 ds$$

$$+ \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}\|g(0,z) - g(0,y)\|_0 ds$$

$$= \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} \frac{a_1 + a_2}{2}$$

$$\cdot \|z(s)(1 - z(s-\tau)) - y(s)(1 - y(s-\tau))\|_0 \, ds$$

$$= \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} \frac{a_1 + a_2}{2}$$

$$\cdot \|(z(s) - y(s)) - z(s)z(s-\tau) + z(s)y(s-\tau) \tag{4.14}$$

$$-z(s)y(s-\tau) + y(s)y(s-\tau)\|_0 ds$$

$$\leq \left(\frac{a_1 + a_2}{2}\right) \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}} \big(\|z(s) - y(s)\|_1$$

$$+\mathcal{M}\|z_0 - y_0\|_\alpha + \mathcal{M}\|z(s) - y(s)\|_1\big) ds$$

$$\leq K_4(a_1, a_2, \mathcal{M})\lambda_0^{-\frac{1}{2}}\Gamma(1/2)\|z_0 - y_0\|_\alpha$$

$$+K_5(a_1, a_2, \mathcal{M}) \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}\|z(s) - y(s)\|_1 ds.$$

If $t - t_0 \geq \tau$, then we just use the Lipschitz continuity of f and g to get

$$\int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}\|f(0,z) - f(0,y)\|_0 ds$$

$$+ \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}\|g(0,z) - g(0,y)\|_0 ds \tag{4.15}$$

$$\leq K_6(a_1, a_2, \mathcal{M}) \int_{t_0}^{t} e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}\|z_s - y_s\|_\alpha ds.$$

Thus putting these inequalities all together, we get

$$
\begin{aligned}
&\|z(t,x)-y(t,x)\|_1 \\
&\quad\leq\; e^{-\lambda_0(t-t_0)}\|z_0(0)-y_0(0)\|_1+K_3\|w_0\|_\alpha\lambda_0^{-\frac{1}{2}}\Gamma(1/2) \\
&\qquad+\chi_{(t_0+\tau,+\infty)}(t)\big[K_3K_1\lambda_M^{-\frac{1}{2}}\Gamma^2(1/2) \\
&\qquad+K_3\mathcal{M}\left(\Phi(t^*,t_0,t_0+\tau,M)+\Psi(t^{**},t_0,t_0+\tau,M)\right)\big] \\
&\qquad+K_2\mathcal{M}\left(\Phi(t^*(t_0,t_0,M),t_0,t_0,M)+\Psi(t^{**}(t_0,t_0,M),t_0,t_0,M)\right) \\
&\qquad+K_1K_2\lambda_M^{-\frac{1}{2}}\lambda_0^{-\frac{1}{2}}\Gamma^2(1/2) \qquad\qquad (4.16)\\
&\qquad+\chi_{[t_0,t_0+\tau]}\big(K_4\lambda_0^{-\frac{1}{2}}\Gamma(1/2)\|z_0-y_0\|_\alpha \\
&\qquad+K_5\int_{t_0}^{t}e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}\|z(s)-y(s)\|_1 ds\big) \\
&\qquad+\chi_{(t_0+\tau,+\infty)}K_6\int_{t_0}^{t}e^{-\lambda_0(t-s)}(t-s)^{-\frac{1}{2}}\|z_s-y_s\|_\alpha ds.
\end{aligned}
$$

In order to simplify the notation above, let us denote by:

$$
\begin{aligned}
M_0 &= e^{\lambda_0 t_0}+K_4\lambda_0^{-\frac{1}{2}}\Gamma(1/2); \\
M_1(M,\|w_0\|_\alpha) &= K_3K_1\lambda_M^{-\frac{1}{2}}\Gamma^2(1/2)+K_3\|w_0\|_\alpha\lambda_0^{-\frac{1}{2}}\Gamma(1/2) \\
&\quad+K_3\mathcal{M}\left(\Phi(t^*,t_0,t_0+\tau,M)+\Psi(t^{**},t_0,t_0+\tau,M)\right) \\
&\quad+K_2\mathcal{M}\left(\Phi(t^*,t_0,t_0,M)+\Psi(t^{**},t_0,t_0,M)\right) \qquad (4.17)\\
&\quad+K_1K_2\lambda_M^{-\frac{1}{2}}\lambda_0^{-\frac{1}{2}}\Gamma^2(1/2); \\
M_2 &= K_5+K_6.
\end{aligned}
$$

Therefore, for any $t \geq t_0$, we have that

$$\|z_t-y_t\|_\alpha \leq M_0\|z_0-y_0\|_\alpha+M_1(M,\|w_0\|_\alpha)+M_2\int_{t_0}^{t}(t-s)^{-\frac{1}{2}}\|z_s-y_s\|_\alpha ds. \quad (4.18)$$

Now, applying Lemma 7.1.1, page 188, from [12], we get that, given $T \geq 0$, and for any $t_0 \leq t < T$, we have

$$\|z_t-y_t\|_\alpha \leq \left(M_0\|z_0-y_0\|_\alpha+M_1(M,\|w_0\|_\alpha)\right)E_{1/2}((M_2\Gamma(1/2))^2 t), \quad (4.19)$$

where

$$E_{1/2}(z)=\sum_{n=0}^{\infty}\frac{z^{n/2}}{\Gamma(n/2+1)}.$$

Estimate on $D_\xi(w,z)(t)$

Let us consider an initial condition $\xi=(w_0,z_0)$ for (4.5), and the corresponding solution $w(t,t_0,\xi)$. We want to estimate

$$\|D_\xi(w,z)(t,t_0,\xi)(\omega_1,\omega_2)-(0,D_{z_0}y(t,t_0,z_0)\omega_2\|_{Z\times Z}.$$

We know that for each $\omega_i \in \mathcal{Z}$, $(U,V) = D_\xi(w,z)(t,t_0,\xi)(\omega_1,\omega_2)$ is a solution of

$$\begin{cases} \dfrac{d}{dt}U + A_M U & = \left\{\dfrac{a_1}{2}(1-(w+z)(t-\tau)) + \dfrac{a_2}{2}(1-(z-w)(t-\tau))\right\}\cdot \\ & \cdot U(t) \\ & -\left\{\dfrac{a_1}{2}((w+z)(t)) + \dfrac{a_2}{2}((z-w)(t))\right\}U(t-\tau) \\ & +\left\{\dfrac{a_1}{2}(1-(w+z)(t-\tau)) - \dfrac{a_2}{2}(1-(z-w)(t-\tau))\right\}\cdot \\ & \cdot V(t) \\ & -\left\{\dfrac{a_1}{2}((w+z)(t)) - \dfrac{a_2}{2}((z-w)(t-\tau))\right\}V(t-\tau) \\ \dfrac{d}{dt}V + A_0 V & = \left\{\dfrac{a_1}{2}(1-(w+z)(t-\tau)) - \dfrac{a_2}{2}(1-(z-w)(t-\tau))\right\}\cdot \\ & \cdot U(t) \\ & -\left\{\dfrac{a_1}{2}((w+z)(t)) - \dfrac{a_2}{2}((z-w)(t))\right\}U(t-\tau) \\ & +\left\{\dfrac{a_1}{2}(1-(w+z)(t-\tau)) + \dfrac{a_2}{2}(1-(z-w)(t-\tau))\right\}\cdot \\ & \cdot V(t) \\ & -\left\{\dfrac{a_1}{2}((w+z)(t)) + \dfrac{a_2}{2}((z-w)(t))\right\}V(t-\tau) \end{cases} \tag{4.20}$$

plus the initial conditions

$$\begin{cases} U(t_0) = \omega_1 \\ V(t_0) = \omega_2 \end{cases}.$$

Similarly, $Y = D_{z_0}y(t,t_0,z_0)\omega_2$ is a solution of

$$\frac{d}{dt}Y + A_0 Y = \frac{a_1+a_2}{2}(1-y(t-\tau))Y(t) - \frac{a_1+a_2}{2}y(t)Y(t-\tau) \tag{4.21}$$

plus the initial conditions

$$Y(t_0) = \omega_2.$$

Once again, proceeding as in the last subsection, we can make

$$\|D_\xi(w,z)(t,t_0,\xi)(\omega_1,\omega_2) - (0, D_{z_0}y(t,t_0,z_0)\omega_2)\|_{\mathcal{Z}\times\mathcal{Z}}$$

small, uniformly in t and (ω_1,ω_2), making $\|w_0\|$ small, and M large.

The Poincaré map

Now let us suppose that the limit equation, that is (4.7), has a periodic orbit $p(t)$ with period T, and let us suppose that it is stable. Then, following [12] (page 259), we can define a "surface section" S and a Poincaré map Φ. With this we can extend this section S to the product space and get a new section (for a given $\varepsilon > 0$) $\mathfrak{S} = B_\varepsilon(0) \times S$. Since we can take ε small (that is $\|w_0\|$ small) and M large, we have that (foliating this neighbourhood) this is still a surface section, and we have a Poincaré map well defined for (1.2). Using the fact that the derivatives are closed and that Φ is a contraction (p is stable), we get that this new Poincaré map is a contraction and thus has a periodic orbit.

Therefore, we had just proved the following theorem.

Theorem 4.2 *Let us suppose that (4.7) has a stable periodic orbit. Then for M sufficiently large, (4.1) has a periodic orbit which is close to the diagonal, and to the periodic orbit of the limit equation.*

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