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Dynamics of a Dirac fermion in the presence of spin noncommutativity

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ABSTRACT

In this work, we gain further insight in the physical aspects of the spin noncommutativity, which mix spacetime and spin degrees of freedom in a noncommutative scenario, in a Lorentz invariant way. Due to the combination of noncommutativity and Lorentz invariance, time also becomes nonlocal so the modified Dirac equation in general contains an infinite tower of time derivatives. Nevertheless, we prove the existence of a conserved probability current in any order in the noncommutativity parameter and for any background field A_μ. Finally, we study the Landau problem in the presence of spin noncommutativity. In this case, we are able to derive a simple Hermitean noncommutative correction to the Hamiltonian operator. We show that the degeneracy of the excited states is lifted by the noncommutativity at the second order of perturbation theory. This is to be contrasted to the case of canonical noncommutativity, where these corrections are of the first order, imposing much stronger restrictions to the noncommutativity.

1. Introduction

The idea that spacetime may be noncommutative at very small scales has its roots in semiclassical arguments stating that the principles of quantum mechanics and general relativity together imply in an absolute limit in the localization of events near the Planck scale. One usually expects physical effects related to quantum gravity to appear only in very high-energy processes, where quantum field theory is the most adequate theoretical tool. However, the study of relativistic or even non-relativistic quantum mechanics with noncommutative coordinates has the advantage of exploring the noncommutativity of coordinates in a simpler setting, better clarifying its physical consequences.

In this context, various possibilities may arise, the simpler one defined by the commutation relations

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \tag{1}$$

usually called "canonical noncommutativity". Specific quantum mechanical potentials with canonical noncommutativity have been studied using standard perturbation theory [1–4] or $1/N$ expansion [5], for example. One shortcoming of this approach is that Lorentz – or rotational, in the non-relativistic case – symmetry is generally lost since the constant θ_{ij} may define a preferred direction in space (for other aspects of noncommutative quantum mechanics, see for example [6–12]).

One may find in the literature several alternative approaches which does not suffer from this symmetry loss – such as, for example, Snyder's work of 1947 [13], where the commutator of two coordinates is proportional to the Lorentz generator. Synder's algebra preserves Lorentz invariance as it involves only covariant objects (see [14–19] for some recents developments). For other ways to conciliate Lorentz Symmetry with noncommutativity of spacetime see for example [20–26]. Another point of view is to understand Eq. (1) as a first approximation to a more general setting where the commutator of coordinates may itself be a non-constant operator, a function of the coordinates themselves [27,7,9,28,29].

Recently, another idea was put forward in [30], involving a kind of noncommutativity with mixed spatial and spin degrees of freedom and a non-relativistic dynamics – to be hereafter referred as "spin noncommutativity". Such a mixture could be theoretically understood as a non-relativistic analog of the Snyder's proposal, where instead of the angular momentum, the commutator of coordinates is proportional to the spin. In [31], the spin noncommutativity was obtained by means of a consistent deformation of the Berezin–Marinov pseudoclassical model for the spinning particle [32]. Besides that, it was extended to the relativistic situation, and in this context the spin noncommutativity exhibits at least one advantage over the canonical one, which is the preservation of the Lorentz symmetry.

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The aim of the present work is to pursue further the study of the physical implications of this type of noncommutativity. This work is organized as follows: in Section 2, the action for our model presented and discussed, starting from the noncommutative Dirac equation. We discuss the existence of a conserved current in Section 3. Section 4 contains an investigation of the effects of the noncommutativity in a simple quantum mechanical problem, the Landau problem. Finally, Section 5 contains our conclusions and perspectives.

2. The noncommutative Dirac equation

The spin noncommutativity for a relativistic system may be implemented through the following deformation of the standard position and momentum operators,

$$x^\mu \to \hat{x}^\mu = x^\mu \mathbf{I} + \theta W^\mu, \qquad p^\mu \to \hat{p}^\mu = p^\mu, \tag{2}$$

where W^μ is the Pauli–Lubanski vector

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} p_\nu S_{\rho\sigma} = \frac{1}{2}\gamma^5 \sigma^{\mu\nu}\partial_\nu, \tag{3}$$

and $S_{\rho\sigma}$ is the spin operator.[1] A direct consequence of Eq. (2) is the noncommutativity among spacetime coordinates,

$$\left[\hat{x}^\mu, \hat{x}^\nu\right] = -i\theta\varepsilon^{\mu\nu\rho\sigma} S_{\rho\sigma} + i\frac{\theta^2}{2}\varepsilon^{\mu\nu\rho\sigma} W_\rho p_\sigma. \tag{4}$$

Only covariant objects appear in this last equation, so that Lorentz symmetry is preserved.

In standard quantum mechanics, $\hat{x}$ is an observable and therefore it should necessarily be a Hermitean operator. In our model the position operator has a non-trivial matrix structure in spinor space, and it satisfies

$$\left(\hat{x}^\mu\right)^\dagger = \gamma^0 \hat{x}^\mu \gamma^0. \tag{5}$$

The fact that $\hat{x}^\mu$ is not Hermitean poses a difficulty in its interpretation as an observable. Concerning this point, various interesting proposals may be found in the literature (see for instance [33]), but here we will adopt a more pragmatical standpoint by considering Eq. (5) as a natural requirement to construct a consistent theory. It will help us to obtain the conjugate Dirac equation and a real Lagrangian density for our model, for example.

The noncommutative Dirac equation for spin noncommutativity was introduced in [31] as

$$\mathcal{D}\psi(x) = \left\{i\gamma^\mu\left[\partial_\mu + ieA_\mu(\hat{x})\right] - m\right\}\psi(x) = 0, \tag{6}$$

where the operator $A_\mu(\hat{x})$ is constructed from $\hat{x}$ via the Weyl (symmetric) ordering,

$$f(\hat{x}) = \int \frac{d^4k}{(2\pi)^4}\tilde{f}(k) e^{-ik_\mu \hat{x}^\mu}. \tag{7}$$

It should be noted that the operator $A_\mu(x^\mu \mathbf{I} + \theta\gamma^5\sigma^{\mu\nu}\partial_\nu)$ has a non-trivial matrix structure which does not commute with γ^μ, so we face an ordering ambiguity in the noncommutative generalization for the matrix product $\gamma^\mu A_\mu$. We fix this ambiguity by requiring the noncommutative version of the Dirac operator satisfies

$$\mathcal{D}^+ = \gamma^0 \mathcal{D}\gamma^0, \tag{8}$$

similarly to the commutative case. With this requirement, we find the proper form of the noncommutative Dirac equation to be ordered symmetrically, i.e.,

$$\left[i\gamma^\mu\partial_\mu - m - \frac{e}{2}\left(\gamma^\mu A_\mu(\hat{x}) + A_\mu(\hat{x})\gamma^\mu\right)\right]\psi(x) = 0. \tag{9}$$

The symmetric ordering will ensure the reality of the Lagrangean density corresponding to Eq. (9); it also simplifies considerably the derivation of the noncommutative Hamiltonian we will discuss in Section 4.

An important feature of this model is that, in spite of the presence of noncommutativity and nonlocality, it is Lorentz invariant, in the sense that the deformed Dirac equation in Eq. (9) is Lorentz covariant, and the noncommutative parameter θ is a Lorentz scalar. Of course, the defining map in Eq. (2) was devised for this to happen, since it only contains covariant objects.

The action of Weyl ordered operator $f(x^\mu \mathbf{I} + \theta\gamma^5\sigma^{\mu\nu}\partial_\nu)$ on a spinor $\psi(x)$ can be represented by means of a "star operation" $\star$ as follows,

$$f\left(x^\mu \mathbf{I} + \theta\gamma^5\sigma^{\mu\nu}\partial_\nu\right)\psi = f\star\psi = f\exp\left(\theta\overleftarrow{\partial_\mu}\gamma^5\sigma^{\mu\nu}\overrightarrow{\partial_\nu}\right)\psi. \tag{10}$$

We note that the star operation defined above involves a regular (scalar) function f and a Dirac spinor (column vector): it is not a "star product" in the usual sense since it does not map two similar objects in the same class of objects, so it cannot be associative. We will shortly define what we mean by a "star operation" involving other objects such as conjugate spinors, and then discuss some of its properties.

The relevant fact at this point is that the noncommutative Dirac equation (9) can be cast in terms of the star operation as

$$\left[-i\gamma^\mu\partial_\mu + m + \frac{e}{2}\left(A_\mu(x)\gamma^\mu\star + A_\mu(x)\star\gamma^\mu\right)\right]\psi(x) = 0, \tag{11}$$

or, more explictly,

$$\begin{aligned}\Big[&-i\gamma^\mu\partial_\mu + m + e\gamma^\mu A_\mu(x)\\ &+\frac{e\theta}{2}\partial_{\alpha_1}A_\mu\left(\gamma^\mu\gamma^5\sigma^{\alpha_1\beta_1} + \gamma^5\sigma^{\alpha_1\beta_1}\gamma^\mu\right)\partial_{\beta_1}\\ &+\frac{e\theta^2}{4}\partial_{\alpha_1}\partial_{\alpha_2}A_\mu\left(\gamma^\mu\sigma^{\alpha_1\beta_1}\sigma^{\alpha_2\beta_2} + \sigma^{\alpha_1\beta_1}\sigma^{\alpha_2\beta_2}\gamma^\mu\right)\partial_{\beta_1}\partial_{\beta_2}\\ &+\cdots\Big]\psi(x) = 0.\end{aligned} \tag{12}$$

The noncommutative Dirac equation has in general an infinite tower of time derivatives, so the usual Hamiltonian interpretation of quantum mechanics – based on a Hermitean Hamiltonian, which ensures unitarity of time evolution and conservation of probability – is not possible. Inspite of that, we shall demonstrate in Section 3 that a conserved charge current can be defined in general, and for a particular choice of A_μ, we shall be able to derive a consistent Hamiltonian formulation in Section 4.

The star operation and its properties are useful in deriving the noncommutative Dirac equation (9) from an action principle. We start by obtaining the conjugate Dirac equation, and for that end we define a star operation between the usual Dirac conjugate spinor $\bar{\psi} = \psi^\dagger\gamma^0$ and a function f by the rule $\bar{\psi}\star f \equiv (f\star\psi)^\dagger\gamma^0$, or, equivalently,

$$\bar{\psi}\star f = \bar{\psi}\exp\left(\theta\overleftarrow{\partial_\mu}\gamma^5\sigma^{\mu\nu}\overrightarrow{\partial_\nu}\right)f. \tag{13}$$

[1] Our conventions are the following: the flat spacetime metric satisfies $\eta^{00} = -\eta^{ii} = 1$, the Dirac gamma matrices are

$$\gamma^0 = \begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}; \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix},$$

in terms of the Pauli matrices σ^i; also, $\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Finally, we introduce a star operation between two spinors by the formula

$$\bar{\varphi} \star \psi = \bar{\varphi} \exp\big(\theta \overleftarrow{\partial_\mu} \gamma^5 \sigma^{\mu\nu} \overrightarrow{\partial_\nu}\big) \psi. \tag{14}$$

We may now quote some useful properties that can be proved regarding the star operation defined in Eqs. (10), (13) and (14). First, integration by parts and the antisymmetry of $\sigma^{\mu\nu}$ leads to

$$\int d^4x\, \bar{\varphi} \star \psi = \int d^4x\, \bar{\varphi}\psi + \text{surface terms}, \tag{15}$$

a property that is well known from the studies involving canonical noncommutativity and its associated star (Moyal) product.

We will also need to work with expressions of the general form $\int d^4x\, \bar{\varphi}(f \star \psi)$, involving two arbitrary spinors φ and ψ and a function f. Starting with

$$\begin{aligned} &\int d^4x\, \bar{\varphi}(f \star \psi) \\ &\quad = \int d^4x \bigg(\bar{\varphi} f \psi + \theta \bar{\varphi} \partial_\mu f \gamma^5 \sigma^{\mu\nu} \partial_\nu \psi \\ &\qquad + \frac{\theta^2}{2} \bar{\varphi} \partial_{\mu_1} \partial_{\mu_2} f \gamma^5 \sigma^{\mu_1\nu_1} \gamma^5 \sigma^{\mu_2\nu_2} \partial_{\nu_1} \partial_{\nu_2} \psi + \cdots \bigg), \end{aligned} \tag{16}$$

one integrates by parts all derivatives acting on ψ, taking care of the antisymmetry of $\sigma^{\alpha\beta}$, thus obtaining

$$\int d^4x\, \bar{\varphi}(f \star \psi) = \int d^4x \big[(\bar{\varphi} \star f)\psi + \partial_\mu E^\mu \big], \tag{17}$$

where

$$\begin{aligned} E^\mu &= \theta \bar{\varphi} \partial_\nu f \gamma^5 \sigma^{\nu\mu} \psi + \frac{\theta^2}{2} \bar{\varphi} \partial_{\mu_1} \partial_{\mu_2} f \gamma^5 \sigma^{\mu_1\nu_1} \gamma^5 \sigma^{\mu_2\mu} \partial_{\nu_1} \psi \\ &\quad - \frac{\theta^2}{2} \partial_{\nu_2} \bar{\varphi} \partial_{\mu_1} \partial_{\mu_2} f \gamma^5 \sigma^{\mu_1\mu} \gamma^5 \sigma^{\mu_2\nu_2} \psi + \mathcal{O}(\theta^3). \end{aligned} \tag{18}$$

It should be stressed that, while we have only explicitly written E^μ up to the second order in θ, the fact that Eq. (17) holds (i.e., the difference between the two integrals is a surface term) actually is true for any order of θ.

These definitions are examples of the general procedure of implementing noncommutativity of spacetime via a star product of "classical functions"; for some general comments see [31]. Due to the spin indices, however, the associativity of this "product" actually holds up to a surface term, as we have shown.

In particular, expressions like the one in Eq. (17) will appear in which f is the electromagnetic potential A_μ, which always appears contracted with a γ^μ. In this case, one should be careful with the order of the star operation and the γ^μ since they do not commute. In any case, it can be shown that,

$$\int d^4x\, \bar{\varphi}\big(A_\mu \star \gamma^\mu \psi\big) = \int d^4x \big[\big(\bar{\varphi} \star A_\mu \gamma^\mu\big)\psi + \partial_\mu F^\mu \big], \tag{19a}$$

$$\int d^4x\, \bar{\varphi}\big(A_\mu \gamma^\mu \star \psi\big) = \int d^4x \big[\big(\bar{\varphi} \gamma^\mu \star A_\mu\big)\psi + \partial_\mu G^\mu \big], \tag{19b}$$

where

$$G^\mu = \theta \bar{\varphi} \partial_\alpha A_\nu \gamma^5 \sigma^{\alpha\mu} \gamma^\nu \psi + \mathcal{O}(\theta^2), \tag{20a}$$

$$H^\mu = \theta \bar{\varphi} \partial_\alpha A_\nu \gamma^\nu \gamma^5 \sigma^{\alpha\mu} \psi + \mathcal{O}(\theta^2). \tag{20b}$$

Finally, we can write an action describing the interaction of a Dirac fermion with an electromagnetic potential A_μ in a spacetime with spin noncommutativity,

$$\begin{aligned} S[\psi, A] = \int d^4x\, \bar{\psi}(x) \bigg\{ &-i\gamma^\mu \partial_\mu \psi(x) + m\psi(x) \\ &+ \frac{e}{2} \big[A_\mu(x) \gamma^\mu \star + A_\mu(x) \star \gamma^\mu \big] \psi(x) \bigg\}. \end{aligned} \tag{21}$$

Clearly, Eq. (11) is obtained from $\delta S/\delta\bar{\psi}(x) = 0$. We split, as usual, this action in free and interaction part, $S = S_0 + S_I$. The usual free Dirac action can be written in a symmetrical form involving the star operation due to Eq. (15),

$$S_0 = \int d^4x \bigg(-\frac{i}{2} \bar{\psi} \star \gamma^\mu \partial_\mu \psi + \frac{i}{2} \partial_\mu \bar{\psi} \gamma^\mu \star \psi + m \bar{\psi} \star \psi \bigg). \tag{22}$$

On the other hand, for the interaction part we write

$$S_I = \frac{e}{2} \int d^4x\, \bar{\psi} \big(\gamma^\mu A_\mu \star \psi + A_\mu \star \gamma^\mu \psi \big), \tag{23}$$

which can also be cast as

$$\begin{aligned} S_I = \frac{e}{4} \int d^4x \big[&\bar{\psi} \star \big(\gamma^\mu A_\mu \star \psi\big) + \big(\bar{\psi} \gamma^\mu \star A_\mu\big) \star \psi \\ &+ \bar{\psi} \star \big(A_\mu \star \gamma^\mu \psi\big) + \big(\bar{\psi} \star \gamma^\mu A_\mu\big) \star \psi \big], \end{aligned} \tag{24}$$

after using Eq. (15). One can verify that the action presented here is real: this property is a consequence of the symmetric ordering adopted in Eq. (9).

One might also introduce a Yukawa interaction in this model by adding a term proportional to $\int d^4x\, \bar{\psi} \star \phi \star \psi$ in the action, where ϕ is an external scalar field. An interesting question, which however is not the subject of this Letter, would be how to describe the interaction of the fields A_μ and ϕ in the model of spin noncommutativity. To answer this question one should define a more general star operation, using the deformed coordinate operators (2), to be able to correctly map the desired noncommutativity in the interactions involving the scalar field.

3. Conservation of the electrical current

In this section, we want to find an expression for the conserved electric current j^μ in our theory, since the existence of such a current is crucial for the physical meaning of the model. The action (21) has global phase invariance, so Noether's theorem provides a general formula for the associated conserved current. Due to the appearance of arbitrary high-order derivatives in ψ, one would need to generalize the well-known formula for the Noether current (see for example [34]). Expanding Eq. (23) in the first order of θ, however, one finds

$$\begin{aligned} S_I = e \int d^4x\, A_\mu \bar{\psi} \gamma^\mu \psi - \frac{e\theta}{2} \int d^4x\, \bar{\psi} \gamma^5 \big[\sigma^{\mu\nu}, \gamma^\alpha\big] \partial_\mu A_\alpha \partial_\nu \psi \\ + \mathcal{O}\big(\theta^2\big), \end{aligned} \tag{25}$$

which, with the help of the identities

$$\gamma^\mu \sigma^{\rho\sigma} = +\big(\eta^{\rho\mu} \gamma^\sigma - \eta^{\sigma\mu} \gamma^\rho\big) - i\varepsilon^{\mu\nu\rho\sigma} \gamma^5 \gamma_\nu, \tag{26a}$$

$$\sigma^{\rho\sigma} \gamma^\mu = -\big(\eta^{\rho\mu} \gamma^\sigma - \eta^{\sigma\mu} \gamma^\rho\big) - i\varepsilon^{\mu\nu\rho\sigma} \gamma^5 \gamma_\nu, \tag{26b}$$

can be cast as

$$\begin{aligned} S_I &= \int d^4x\, e A_\mu \bar{\psi} \gamma^\mu \psi \\ &\quad - ie\theta \int d^4x\, \bar{\psi} \gamma^5 \big(\partial_\mu A^\nu \gamma^\mu \partial_\nu \psi - \partial_\mu A^\mu \gamma^\nu \partial_\nu \psi \big) \\ &\quad + \mathcal{O}\big(\theta^2\big). \end{aligned} \tag{27}$$

Since, in this approximation, there are no higher-order derivatives acting on ψ, on may use the standard formula for the Noether current associated to the phase symmetry $\delta\psi = -i\alpha\psi$,

$$j^\mu = -i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\psi + \mathcal{O}(\theta^2) \tag{28}$$

$$= \bar\psi\gamma^\mu\psi + e\theta\bar\psi\gamma^5(\partial_\nu A^\nu\gamma^\mu - \partial_\nu A^\mu\gamma^\nu)\psi + \mathcal{O}(\theta^2). \tag{29}$$

To see the existence of a conserved current j^μ at arbitrary order in θ, we shall employ the following trick: using Eq. (9) and its conjugate, one writes the identity,

$$\left(i\partial_\mu\bar\psi\gamma^\mu + m\bar\psi + \frac{e}{2}\bar\psi\star\gamma^\mu A_\mu + \frac{e}{2}\bar\psi\gamma^\mu\star A_\mu\right)\psi - \bar\psi\left(-i\gamma^\mu\partial_\mu\psi + m\psi + \frac{e}{2}\gamma^\mu A_\mu\star\psi + \frac{e}{2}A_\mu\star\gamma^\mu\psi\right) = 0. \tag{30}$$

In the usual case (without the star operation), all that would remain would be $i\partial_\mu\bar\psi\gamma^\mu\psi + i\bar\psi\gamma^\mu\partial_\mu\psi = \partial_\mu(i\bar\psi\gamma^\mu\psi) = 0$, giving the conservation of the usual electric current. In our case, Eq. (30) can be written as

$$\partial_\mu(\bar\psi\gamma^\mu\psi) + i\frac{e}{2}\left[(\bar\psi\star\gamma^\mu A_\mu)\psi - \bar\psi(A_\mu\star\gamma^\mu\psi)\right] + i\frac{e}{2}\left[(\bar\psi\gamma^\mu\star A_\mu)\psi - \bar\psi(\gamma^\mu A_\mu\star\psi)\right] = 0, \tag{31}$$

or, by virtue of Eqs. (19), (20),

$$\partial_\mu\left[\bar\psi\gamma^\mu\psi + i\frac{e\theta}{2}\bar\varphi\partial_\alpha A_\nu(\gamma^5\sigma^{\alpha\mu}\gamma^\nu + \gamma^\nu\gamma^5\sigma^{\alpha\mu})\psi + \mathcal{O}(\theta^2)\right] = 0. \tag{32}$$

Finally, from Eqs. (26),

$$\partial_\mu\left[\bar\psi\gamma^\mu\psi + e\theta\bar\psi\gamma^5(\partial_\nu A^\nu\gamma^\mu - \partial_\nu A^\mu\gamma^\nu)\psi + \mathcal{O}(\theta^2)\right] = 0. \tag{33}$$

As commented in the paragraph containing Eq. (17), this last equation holds for any order of θ, which ensures the existence of the conserved current j^μ on general grounds.

It is noteworthy that the conserved current depends, already at first order in θ, on the electromagnetic potential, which we have treated as a fixed background field. The same feature appears in canonical noncommutativity [6], and it makes interesting the problem of incorporating a dynamical potential A_μ in a consistent way.

4. Landau problem in the presence of spin noncommutativity

Having further explored the formal aspects of the spin noncommutativity, in this section we want to gain some insight into its possible observable consequences in a particular physical problem. We consider the bound state problem for a charged particle subject to a constant magnetic field, known as the Landau problem.

When the noncommutativity is not present, the Landau problem is described in many textbooks such as [35]. The gauge potential corresponding to a constant magnetic field perpendicular to the xy plane can be chosen as $A^\mu = Bx^1\delta_2^\mu$ and the Dirac's Hamiltonian

$$H_0 = -i\gamma^0\gamma^i\partial_i + m\gamma^0 + eBx_1\gamma^0\gamma^2, \tag{34}$$

has the energy levels $E_{n,\alpha} = \sqrt{p_3^2 + m^2 + eB(2n+1-\alpha)}$, with $\alpha = \pm 1$ for spin up and down, respectively. All energy levels exhibit an infinite-degeneracy relative to p_1 and p_2. Besides that, except for the (unique) ground-state $|0\rangle = |0,+1\rangle$, the excited energy levels are two-fold degenerate, since $|n,+1\rangle$ and $|n-1,-1\rangle$ have the same energy.

The energy eigenfunctions of H_0 can be cast in terms of the two-components eigenvectors χ_α of the Pauli matrix σ^3 as follows,

$$|n,\alpha\rangle = c_{n,\alpha}\begin{pmatrix} |n\rangle\chi_\alpha \\ \frac{\vec\sigma\cdot\vec\pi}{E_{n,\alpha}+m}|n\rangle\chi_\alpha \end{pmatrix}. \tag{35}$$

Here, $|n\rangle$ are essentially eigenstates of the harmonic oscillator,

$$\varphi_n(x) = \langle x|n\rangle = e^{i(p_2x_2+ip_3x_3)}e^{-\xi^2/2}H_n(\xi), \tag{36}$$

where $\xi = \sqrt{eB}(x_2 - \frac{p_2}{eB})$ and $c_{n,\alpha}$ is a normalization factor,

$$c_{n,\alpha} = \frac{(eB)^{1/4}}{2\pi}\sqrt{\frac{E_{n,\alpha}+m}{E_{n,\alpha}}}\frac{1}{\sqrt{\sqrt{\pi}2^{n+1}n!}}, \tag{37}$$

chosen such that $\langle n',\alpha'|n,\alpha\rangle = \delta_{n',n}\delta_{\alpha',\alpha}$. The canonical momentum $\vec\pi$ is

$$\vec\pi \equiv (-i\partial_1, p_2 - eBx_1, p_3) = \sqrt{eB}\left(-i\partial_\xi, -\xi, \frac{p_3}{\sqrt{eB}}\right). \tag{38}$$

The fact that the noncommutative Dirac equation (11) has in general higher orders in time derivatives precludes the definition of a Dirac Hamiltonian in the standard way. It is actually a consequence of Lorentz invariance that the nonlocality in space introduced by noncommutativity should also extend to the time variable. This difficulty is circumvented in the particular problem studied in this section because the linearity of the electromagnetic potential makes the noncommutativity modification of the Dirac equation local both in time and space. We end up with the simple Hamiltonian $H = H_0 + H_I$ where

$$H_I = \frac{i}{2}\theta eBp_2\gamma^2\gamma^3 = -\theta\frac{eB}{2}p_2\begin{pmatrix}\sigma^1 & 0 \\ 0 & \sigma^1\end{pmatrix}. \tag{39}$$

It should be stressed that Eq. (39) contains the exact modification of the Hamiltonian for the present problem. This observation is necessary since we will use the $\mathcal{O}(\theta)$ correction in Eq. (39) to calculate the corrections to the energy levels up to $\mathcal{O}(\theta^2)$ in the sequel. Another remark is that the symmetric ordering adopted in Eq. (9) is also essential in keeping the noncommutative modification to the Hamiltonian exactly of first order in θ: if we had chosen another ordering, the calculation of the noncommutative Hamiltonian would involve a multiplicative factor $[\gamma^0 - \frac{i}{2}a\theta eB\gamma^3]^{-1}$ involving a nonvanishing constant a, which would introduce higher-orders corrections. Finally, we note that Eq. (39) is Hermitean, so it maintains the reality of the energy spectrum.

One may readily see that the first order corrections in the energy levels vanish exactly: the spin noncommutativity does not change the spectrum of the Landau problem in the first order in θ. We found non-trivial corrections to the energy levels in the second order of perturbation theory. For the ground-state energy, one has to calculate

$$\delta E_0^{(2)} = \sum_{n,i}\frac{|W_{n,i}|^2}{E_0 - E_n}, \quad n \geqslant 1 \text{ and } i = 1,2, \tag{40}$$

where $W_{n,i}$ are matrix elements of H_I between the ground-state and the excited state $|n,i\rangle$. The only nonvanishing of these matrix elements are

$$W_{1,1} = \frac{i\theta(Be)^{3/2}p_2p_3}{2(E_0+m)(E_1+m)}\frac{c_{0,+1}}{c_{1,+1}}, \tag{41a}$$

$$W_{1,2} = \frac{\theta eBp_2}{2}\frac{[c_{1,-1}c_{1,+1}(2eB+p_3^2) + c_{0,-1}c_{0,+1}eB]}{c_{1,-1}c_{1,+1}(E_0+m)(E_1+m)}, \tag{41b}$$

$$W_{3,2} = \frac{3\theta(eB)^2 p_2}{(E_0+m)(E_3+m)} \frac{c_{0,+1}c_{2,-1}}{c_{1,-1}c_{1,+1}}, \tag{41c}$$

from which the final (nonvanishing) expression for $\delta E_0^{(2)}$ can be calculated,

$$\begin{aligned}\delta E_0^{(2)} = &-\frac{\theta^2(eB)^2 p_2^2}{4(E_0+m)^2}\Bigg[\frac{eBp_3^2}{(E_1-E_0)(E_1+m)^2}\left(\frac{c_{0,+1}}{c_{1,+1}}\right)^2 \\ &+ \frac{1}{(E_1-E_0)(E_1+m)^2}\left(eB\left(\frac{c_{0,+1}c_{0,-1}}{c_{1,-1}c_{1,+1}}-2\right)+p_3\right)^2 \\ &+ \frac{(eB)^2 p_3^2}{(E_3-E_0)(E_3+m)^2}\left(\frac{c_{0,+1}c_{2,-1}}{c_{1,+1}c_{1,-1}}\right)^2\Bigg]. \end{aligned}\tag{42}$$

More interesting is the calculation of the second order energy corrections to the degenerate levels, since we can investigate whether the degeneracy is broken by the noncommutativity. Physically, when the perturbation breaks the degeneracy, that means some symmetry is lost; in our problem, it is the constant magnetic field which breaks part of the rotational symmetry. In the commutative case, one still has the two-fold degeneracy of the excited levels $|n,i\rangle$. Since the noncommutative correction to the Hamiltonian H_I depends on the magnetic field, it might be that this degeneracy is broken, even if the noncommutativity itself does not break further symmetries.

Second order corrections to the energy of degenerate levels are found by solving the secular equation [36]

$$\det\left(W_{ij} + \sum \frac{W_{n,i;m,\ell}W_{m,\ell;n,j}}{E_n - E_m} - \delta E_n^{(2)}\delta_{ij}\right) = 0, \tag{43}$$

where the sum is for $m \geqslant 1$ and $m \neq n$, and $\ell = 1,2$, and $W_{n,i;m,\ell}$ is the matrix element of H_I between two degenerate states. This calculation is straightforward but quite involved, so it was done using a Computer Algebra System (CAS) [37]. The resulting expressions are complicated and not particularly informative to be quoted here, but the relevant fact is that Eq. (43) usually has *two* different solutions $\delta E_n^{(2)}$, what means degeneracy is indeed broken at the second order.

5. Conclusions and perspectives

In this work, we gained further insight into the spin noncommutativity proposed in [31]. We have shown that the noncommutative Dirac equation can be derived from an action principle, involving a Lagrangean which is real and has global phase invariance. This implies, by Noether's theorem, the existence of a conserved current. The existence of this current is encouraging because it is important for the physical interpretation of the model.

We also investigated a very simple quantum mechanical system – the Landau problem – and verified the physical effects of the introduction of the spin noncommutativity. In this simple setting, it was possible to derive a Hermitean Hamiltonian from the noncommutative Dirac equation, which consisted on the standard Dirac Hamiltonian plus a noncommutative correction of order θ. By using standard perturbation theory, we shown that the corrections to the energy spectrum appear at the second order in θ, and they break the degeneracy of the excited states, despite the fact that the noncommutativity does not introduce further preferred directions in the problem. These results are potentially interesting from the phenomenological point of view. In most treatments of similar problems in noncommutative quantum mechanics, both in relativistic and non-relativistic regimes, corrections to the spectra are found already at the first order in θ [4,3,1,38], which can pose very stringent constraints on the noncommutativity parameters. In our relativistic model, the noncommutativity parameters could be less constrained by existing experimental bounds.

Many questions are still open, however, regarding further developments in this line of research. Instead of a fixed background field, the dynamics of the electromagnetic field should be consistently incorporated in this scenario. More complicated potentials could be investigated, such as the Coulomb potential, and a particular interesting question is whether the physical effects of the noncommutativity appear only at order θ^2, as in the Landau problem. Finally, since noncommutativity is expected to be a very high-energy effect, one might investigate whether a quantum field theory could be defined based on this type of noncommutativity. The definition of a novel type of noncommutative quantum field theories, which preserves Lorentz invariance by construction, would certainly be a very interesting problem.

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