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To cite this article: David R de Souza *et al* *J. Stat. Mech.* (2011) P03006

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# A new scale-invariant ratio and finite-size scaling for the stochastic susceptible–infected–recovered model

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Received 19 December 2010

Accepted 7 February 2011

Published 4 March 2011

Online at [stacks.iop.org/JSTAT/2011/P03006](http://stacks.iop.org/JSTAT/2011/P03006)

doi:[10.1088/1742-5468/2011/03/P03006](https://doi.org/10.1088/1742-5468/2011/03/P03006)

**Abstract.** The critical behavior of the stochastic susceptible–infected–recovered model on a square lattice is obtained by numerical simulations and finite-size scaling. The order parameter as well as the distribution in the number of recovered individuals is determined as a function of the infection rate for several values of the system size. The analysis around criticality is obtained by exploring the close relationship between the present model and standard percolation theory. The quantity  $UP$ , equal to the ratio  $U$  between the second moment and the squared first moment of the size distribution multiplied by the order parameter  $P$ , is shown to have, for a square system, a universal value 1.0167(1) that is the same for site and bond percolation, confirming further that the SIR model is also in the percolation class.

**Keywords:** critical exponents and amplitudes (theory), percolation problems (theory), epidemic modelling

**ArXiv ePrint:** [1012.4138](https://arxiv.org/abs/1012.4138)

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## 1. Introduction

The spread of an epidemic among a community of individuals has been described by several types of models, either deterministic or stochastic [1]–[7]. Among the latter we find models in which the space structure is explicitly taken into account [8]–[20]. These models are defined on a lattice which represents the space where the community lives, and in which each site is occupied by just one individual. In the susceptible–infected–recovered (SIR) stochastic lattice model [8, 9, 12]–[20], each individual can be either susceptible (S), infected (I) or recovered (R). A susceptible individual becomes infected ( $S \rightarrow I$ ) through an autocatalytic reaction, and an infected individual recovers ( $I \rightarrow R$ ) spontaneously. This model describes an epidemic in which the immunization is permanent; in other words, once an individual recovers, it becomes immune forever.

The main features of the SIR model are as follows. When the rate of infection of a susceptible individual by an infected individual is small compared to the immunization rate, there is no spreading of the disease. Increasing the infection rate, one reaches a critical value above which the infection spreads over the whole lattice. The transition from one regime to the other is regarded as a continuous phase transition whose critical behavior places the model into the dynamic percolation universality class, which corresponds to the standard percolation class with the addition of dynamical growth exponents [8, 9]. The SIR model can be described by just one parameter, either the reduced infection rate  $b$  or the reduced immunization rate  $c = 1 - b$ . The phase transition occurs at a critical value  $c = c_c$ , which has been estimated as 0.1765(5) on a square lattice by means of time-dependent numerical simulations [18]. A more accurate result  $c_c = 0.176\,5005(10)$  was later determined by extensive numerical simulation through the use of a technique borrowed from standard percolation theory [19].

A close relationship exists between the SIR model and dynamic (isotropic) percolation [4, 8], [19]–[23]. Starting from a single infected individual in a lattice full of susceptibles, a cluster composed of infected and recovered individuals grows, the infected individuals staying at the border of the cluster and the recovered individuals inside it. Eventually the cluster becomes composed of recovered individuals only due to the spontaneous immunization. For  $c < c_c$  (the spreading regime), an infinite cluster

of recovered individuals percolates the whole lattice. For  $c > c_c$  (the non-spreading regime), only finite clusters are present. It has been shown [19] that the cluster probability distribution obeys the same scaling laws as apply to standard percolation models.

Here we report numerical simulations and finite-size scaling analysis to obtain the critical behavior of the stochastic SIR model on a square lattice. To this end we determine the order parameter  $P$  (defined below), the mean number of recovered sites  $S$  and the mean value of the squared number of recovered sites  $M$ . These quantities are determined as a function of  $c$  and the linear system size  $L$ . We show that the ratio  $U = M/S^2$  between the second moment and the square of the first moment is not independent of the system size  $L$  at criticality, as occurs, for example, in the contact model [24, 25]. Instead, we show here that the quantity that is independent of the system size at criticality, and therefore universal, is the product  $UP$  between this ratio and the order parameter, consistent with the dynamical percolation universality class.

This paper is organized as follows. The stochastic SIR model on a regular lattice is defined in section 2, together with numerical simulations. In section 3 we explain how the model is related to percolation around criticality. The finite-size analysis is introduced in section 4. A conclusion is drawn in section 5.

## 2. Definition and simulations

The stochastic SIR model is defined on a regular lattice of  $N$  sites as follows. At each time step a site is chosen at random and the time is incremented by an amount equal to  $1/N$ . If the chosen site is in state S then it becomes I with probability  $b$  multiplied by the fraction of nearest-neighbor sites in state I. If the chosen site is in state I then it becomes R with probability  $c = 1 - b$ . If it is in state R it remains in this state. The number of individuals of type S, I and R are denoted by  $N_S$ ,  $N_I$  and  $N_R$ . The total number of individuals equals the total number of sites of the lattice,  $N_S + N_I + N_R = N$ .

The quantities that we have measured in the Monte Carlo simulation are the following: the mean number of recovered individuals:

$$S = \langle N_R \rangle, \quad (1)$$

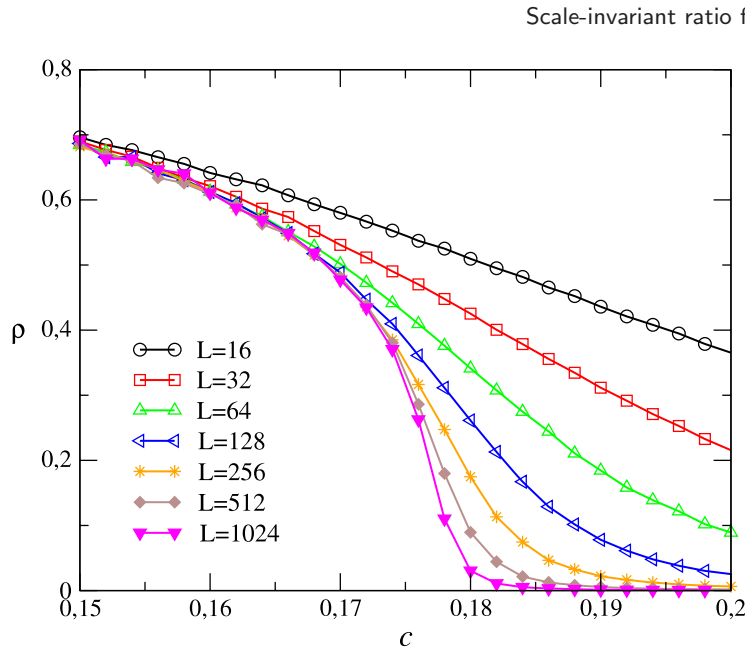
the mean value of the square of the number of recovered individuals:

$$M = \langle N_R^2 \rangle, \quad (2)$$

and the order parameter  $P$ , defined below. We also considered the ratio  $U$  between the second moment  $M$  and the square of the first moment  $S$  of the probability distribution of recovered individuals, that is

$$U = \frac{M}{S^2}. \quad (3)$$

The simulations were performed on a square lattice of  $N = L^2$  sites and periodic boundary conditions. We begin with an infected individual placed at the center of the lattice full of susceptible individuals. To speed up the simulations we keep a list of the I sites. At each step of the simulation we choose randomly an I site among the list of the  $N_I$  I sites. (If we were interested in the time, which is not the case here, we would increment it by an amount equal to  $1/N_I$ .) With probability  $c$  the chosen I site becomes



**Figure 1.** Density of recovered individuals  $\rho = S/N$  versus  $c$  for several values of the linear system size  $L$ .

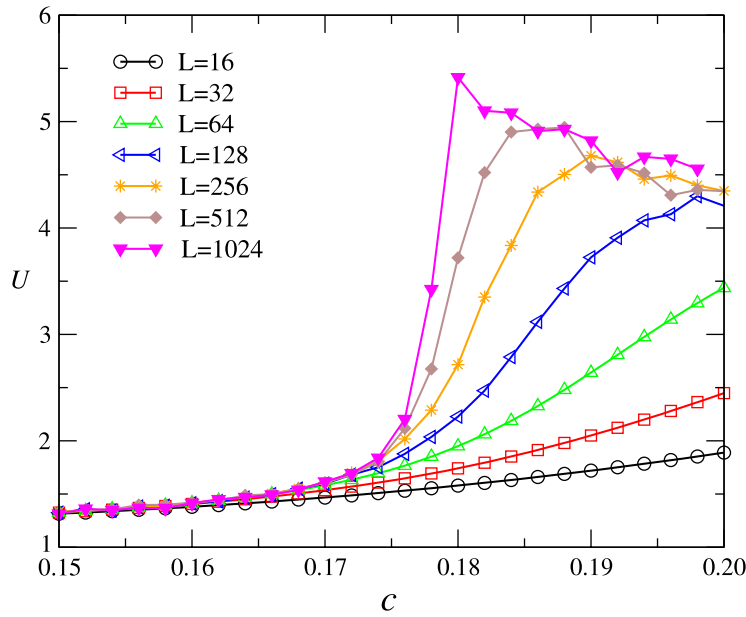
an R site. Otherwise (with the complementary probability  $b = 1 - c$ ), we choose one of its four nearest-neighbor sites; if the nearest-neighbor site is an S site, it becomes an I site, otherwise it remains unchanged. These rules are equivalent to the definition of the model given at the beginning of this section.

Here we are interested only in the stationary states, which are characterized by the absence of infected sites. Starting from a single infected site, the number of infected sites may increase but eventually decreases and vanishes. Without infected sites there is no activity and the dynamics stops. The stationary state is then an absorbing state constituted of S and R sites only. For each value of the parameter  $c$  and linear size  $L$  we performed a set of independent runs, ranging from  $10^7$  to  $10^8$ , and measured the quantities  $S$ ,  $M$  and  $P$  related to the final clusters of R sites. In figure 1 we show the density of recovered individuals  $\rho = S/N$  as a function of  $c$  for several values of the system size  $L$ . In figure 2 we show the ratio  $U = M/S^2$  as a function of  $c$  for several values of the system size  $L$ .

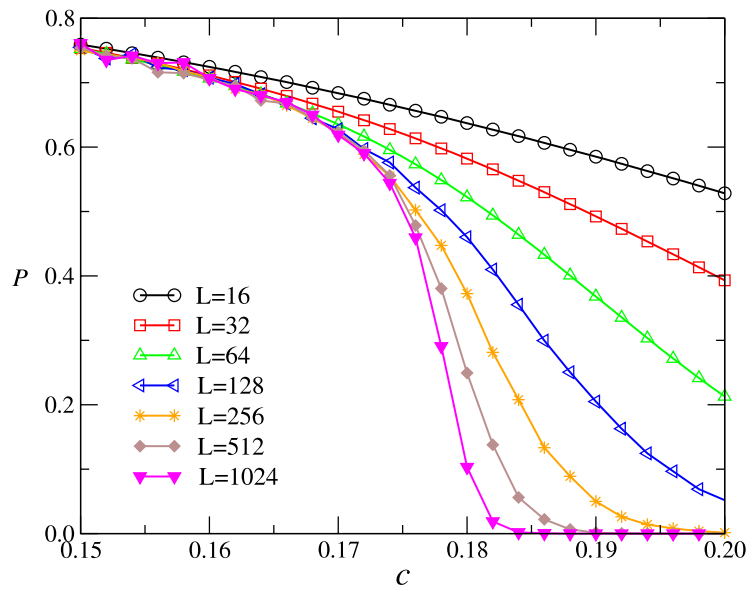
To find the order parameter  $P$ , we checked whether the growing cluster of infected individuals reached the border of the lattice, and the fraction of runs for which this happens is defined as  $P$ . In the thermodynamic limit, it becomes the probability that the central site belongs to the infinite cluster in accordance with the definition of order parameter in standard percolation theory. In figure 3 we show  $P$  versus  $c$  for several values of the linear size  $L$ .

We note that in percolation one often uses the fraction of sites belonging to the largest cluster in a lattice fully occupied with clusters as the order parameter. However, this method cannot be used here because, by definition, there is only one cluster in each epidemic sample. Note also that in our definition of  $P$ , one can think of the system as having open boundary conditions and we are finding if the cluster hits that boundary;

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**Figure 2.** Ratio  $U = M/S^2$  versus  $c$  for several values of the linear system size  $L$ .



**Figure 3.** Order parameter  $P$  (the fraction of epidemics that hit the lattice boundary) versus  $c$  for several values of the linear system size  $L$ .

this is related of the idea of midpoint percolation considered recently [26]. An equivalent interpretation is that we have an infinite lattice, and we are seeing if the epidemic starting from the origin reaches the  $L \times L$  boundary.

### 3. Relation to percolation

Here we summarize some results of percolation theory [27]–[31] that will be useful for showing the relation to the SIR model. In standard percolation theory the probability that an occupied site belongs to a cluster of size  $s$  is

$$P_s = sn_s, \quad (4)$$

where  $n_s$  is the mean number of clusters of size  $s$  per lattice site. (For site percolation, one should also technically divide this by  $p$ , the probability that the site is occupied, but we will suppress this factor.) From this quantity one obtains the mean epidemic size  $S$ :

$$S = \sum_s sP_s = \sum_s s^2n_s, \quad (5)$$

and the mean-square epidemic cluster size  $M$ :

$$M = \sum_s s^2P_s = \sum_s s^3n_s. \quad (6)$$

The order parameter  $P$  is the probability that a site belongs to the infinite percolating cluster.

Around the critical point  $p = p_c$  for an infinite system, these quantities behave as

$$P \sim \varepsilon^\beta \quad \varepsilon \geq 0, \quad (7)$$

$$S \sim |\varepsilon|^{-\gamma}, \quad (8)$$

and

$$M \sim |\varepsilon|^{-\beta-2\gamma}, \quad (9)$$

where  $\varepsilon = p - p_c$ , and  $\beta$  and  $\gamma$  are critical exponents associated with the order parameter and with the mean cluster size, respectively. The quantity  $p$  is the parameter associated with the percolation problem and  $p_c$  is its critical value. In site (bond) percolation,  $p$  is the probability that a site (bond) is occupied.

It has been argued that the clusters of recovered individuals generated by the dynamics of the stochastic SIR model follow the statistics of the cluster size distribution of the standard percolation theory summarized here [8]. The mean cluster size in percolation theory is then identified as the mean number of recovered individuals. This leads us to identify the quantities  $P$ ,  $S$  and  $M$  defined in section 2 for the SIR model with the quantities  $P$ ,  $S$  and  $M$  defined in this section. As a consequence, their critical behavior is given by equations (7)–(9) with

$$\varepsilon = c - c_c. \quad (10)$$

An implication of the critical behavior (8) and (9) of  $S$  and  $M$  is that the ratio  $U = M/S^2$  should behave as

$$U \sim |\varepsilon|^{-\beta}. \quad (11)$$

#### 4. Finite-size scaling

We start from the assumption that a standard finite-size scaling analysis [27, 32] can be performed to obtain the critical behavior of this model. We assume that the phase transition in the SIR model is characterized by a correlation length  $\xi$  which diverges in the limit where the system is infinite as

$$\xi \sim |\varepsilon|^{-\nu_{\perp}}. \quad (12)$$

Here we use  $\nu_{\perp}$  rather than the usual  $\nu$  of standard percolation because we consider the SIR cluster growth as a dynamical process. The linear size of the system  $L$  scales as  $\xi$  and a finite-system quantity  $A_L$  will behave according to the finite-size scaling as

$$A_L = L^{\theta/\nu_{\perp}} \hat{A}(L^{1/\nu_{\perp}} \varepsilon), \quad (13)$$

where  $\hat{A}(X)$  is a universal function. The exponent  $\theta$  describes the behavior of  $A_L$  in the limit of the infinite system, that is

$$A_{\infty} \sim |\varepsilon|^{-\theta}. \quad (14)$$

Using the finite-size scaling, we may write the following relations for the quantities  $P$ ,  $S$ ,  $M$  and  $U$  [27, 32]:

$$P = L^{-\beta/\nu_{\perp}} \hat{P}(L^{1/\nu_{\perp}} \varepsilon), \quad (15)$$

$$S = L^{\gamma/\nu_{\perp}} \hat{S}(L^{1/\nu_{\perp}} \varepsilon), \quad (16)$$

$$M = L^{(\beta+2\gamma)/\nu_{\perp}} \hat{M}(L^{1/\nu_{\perp}} \varepsilon), \quad (17)$$

$$U = L^{\beta/\nu_{\perp}} \hat{U}(L^{1/\nu_{\perp}} \varepsilon), \quad (18)$$

and at the critical point  $\varepsilon = 0$  we have

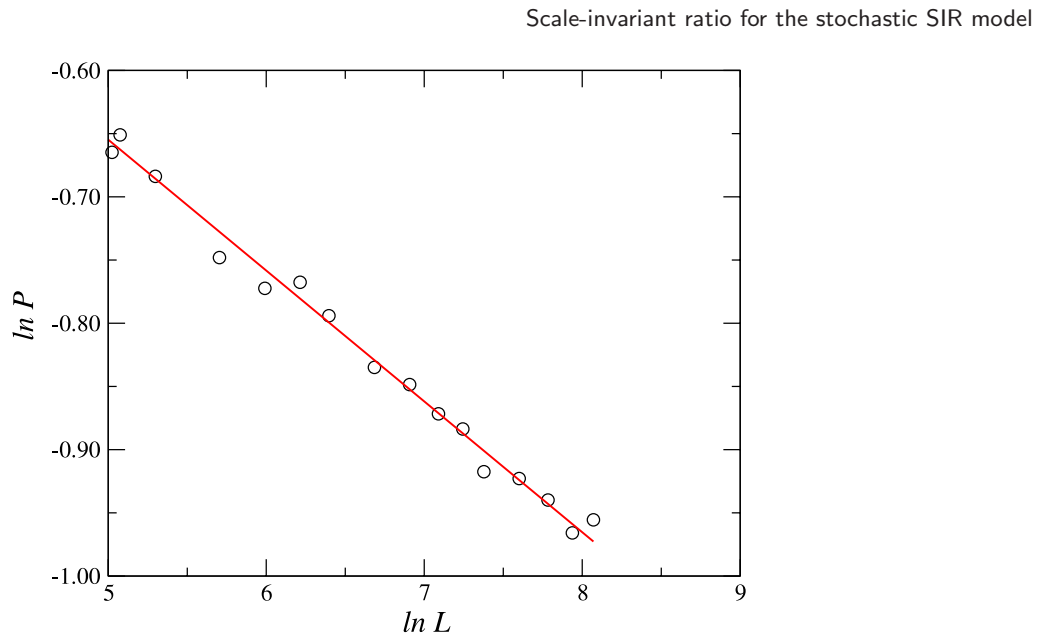
$$P \sim L^{-\beta/\nu_{\perp}}, \quad (19)$$

$$S \sim L^{\gamma/\nu_{\perp}}, \quad (20)$$

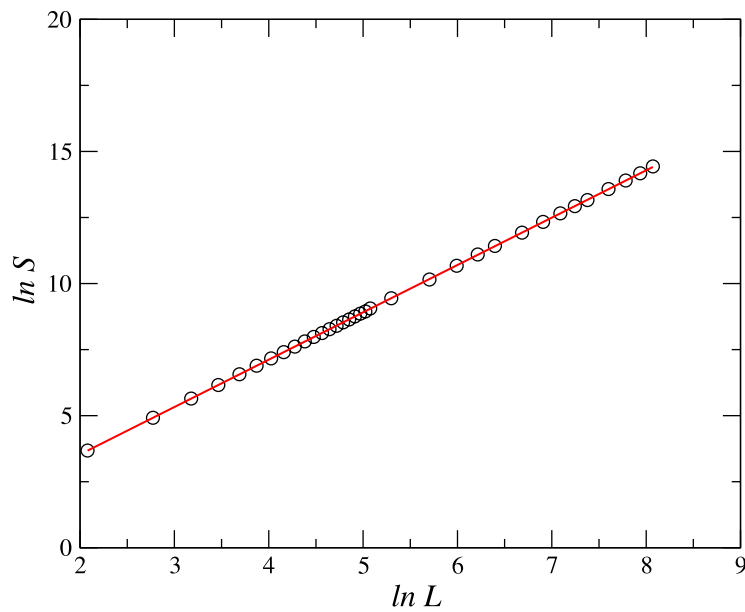
$$U \sim L^{\beta/\nu_{\perp}}. \quad (21)$$

The quantities  $P$  and  $S$  are plotted as a function of  $L$  in figures 4 and 5. Each curve was obtained by performing a number of runs, of the order of  $10^7$ . From the log-log plots we may estimate the critical exponents. From the slope of a straight line fitted to the data points of figure 4 we get the value  $\beta/\nu_{\perp} = 0.1048$  and from figure 5 we get the exponent  $\gamma/\nu_{\perp} = 1.7923$ . These results should be compared with the exact results  $\beta/\nu_{\perp} = 5/48 = 0.1042$  and  $\gamma/\nu_{\perp} = 43/24 = 1.792$  coming from the exact values  $\beta = 5/36$ ,  $\gamma = 43/18$  and  $\nu_{\perp} = 4/3$  of percolation in two dimensions.





**Figure 4.** Log-log plot of the order parameter  $P$  versus  $L$  for  $c = 0.1765$ . The slope of the data points gives the value  $\beta/\nu_{\perp} = 0.1048$ .

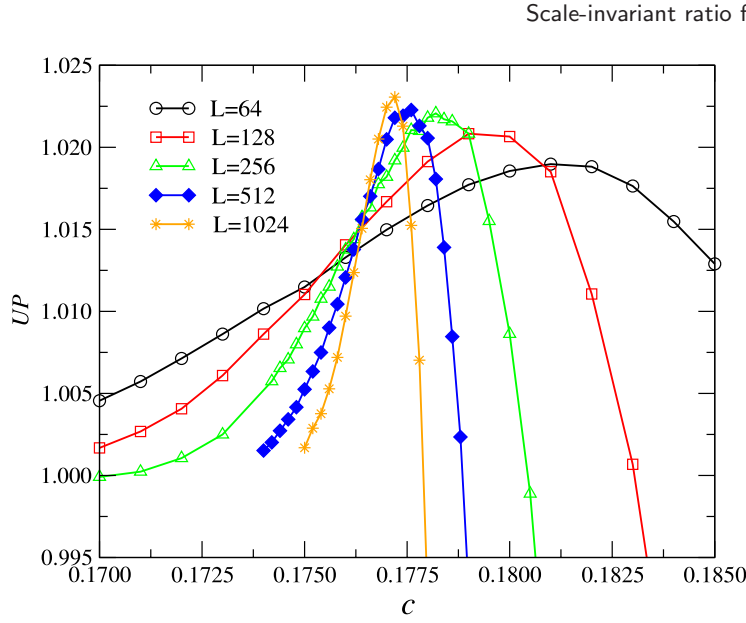


**Figure 5.** Log-log plot of the mean number of recovered  $S$  versus  $L$  for  $c = 0.1765$ . The slope of the data points gives the value  $\gamma/\nu_{\perp} = 1.7923$ .

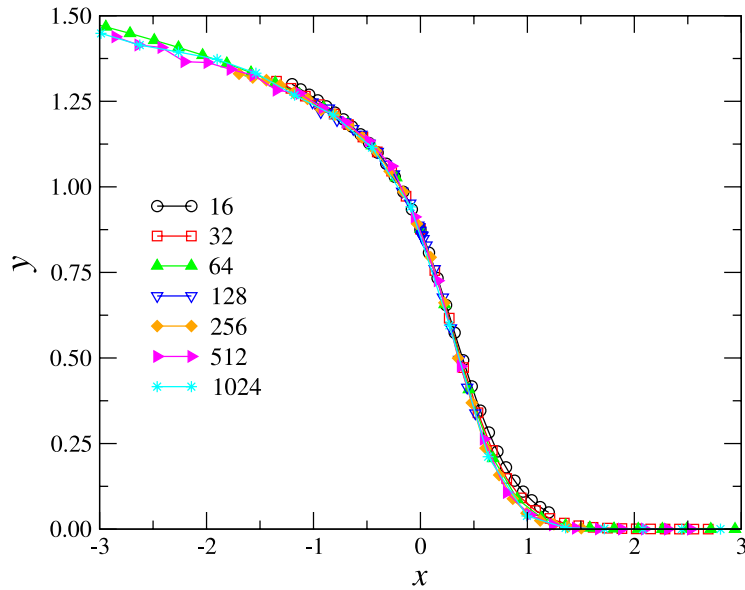
If we multiply equations (15) and (21) for the order parameter  $P$  and the ratio  $U$  we get

$$UP = \hat{F}(L^{1/\nu_{\perp}}\varepsilon). \quad (22)$$

At the critical point  $\varepsilon = 0$  the product  $UP$  is then a quantity independent of  $L$  and may thus be used to locate the critical point. Figure 6 shows a plot of the quantity  $UP$  versus



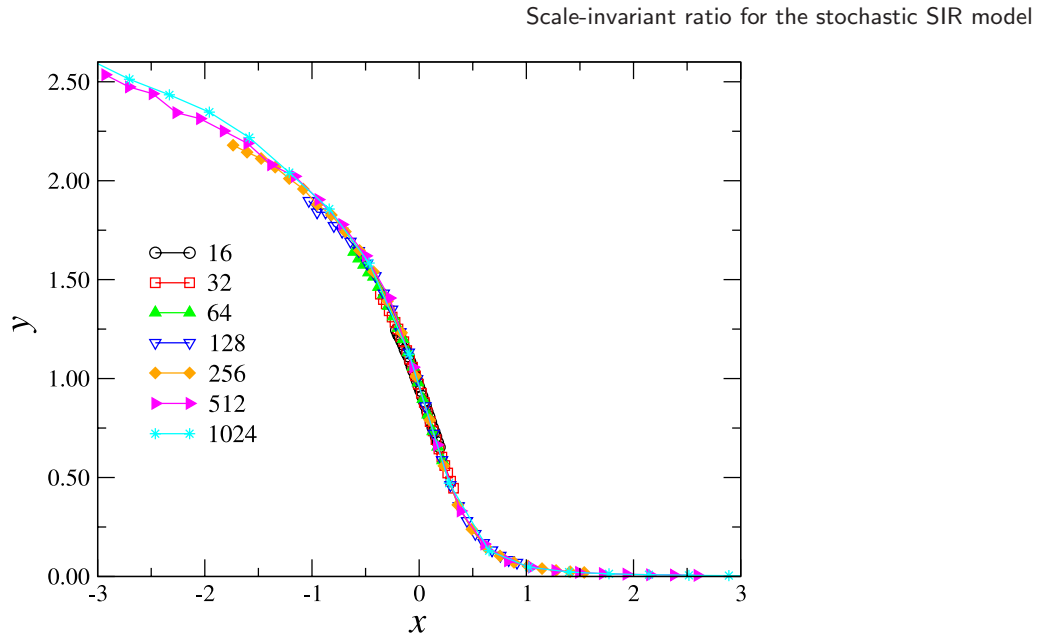
**Figure 6.** Product  $UP$  as a function of  $c$  for several values of the size  $L$ .



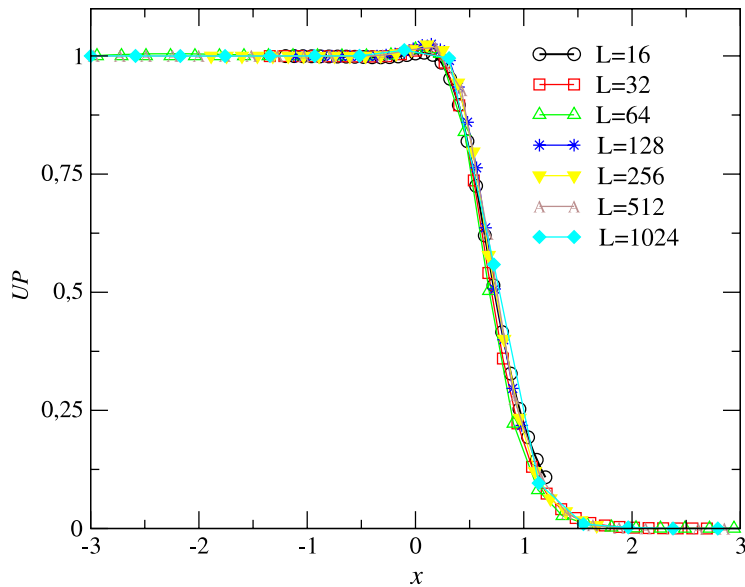
**Figure 7.** Data collapse of the order parameter  $P$  versus  $c$  and  $L$ . The quantities  $y$  and  $x$  are defined by  $y = PL^{\beta/\nu_{\perp}}$  and  $x = \varepsilon L^{1/\nu_{\perp}}$ , where  $\varepsilon = c - c_c$ . The critical values used are  $\beta/\nu_{\perp} = 0.1048$ ,  $\nu_{\perp} = 1.333$  and  $c = 0.1765$ . Values of  $L$  are given in the legend.

$c$  for several values of the system size  $L$ . We see that the curves indeed cross each other, for sufficiently large values of  $L$ , at a point identified as the critical point. From the plot we get the value  $c_c = 0.176\,50(2)$  in agreement with the result  $c_c = 0.176\,5005(10)$  [19], and the value  $\approx 1.016$  for  $UP$  at the critical point.

Using the critical points and the critical exponents we have done a data collapse for the quantities  $P$ ,  $S$  and  $UP$ , shown in figures 7, 8 and 9, respectively. These plots confirm



**Figure 8.** Data collapse of the mean number of recovered  $S$  versus  $c$  and  $L$ . The quantities  $y$  and  $x$  are defined by  $y = SL^{-\gamma/\nu_{\perp}}$  and  $x = \varepsilon L^{1/\nu_{\perp}}$ , where  $\varepsilon = c - c_c$ . The critical values used are  $\gamma/\nu_{\perp} = 1.792$ ,  $\nu_{\perp} = 1.333$  and  $c = 0.1765$ . Values of  $L$  are given in the legend.

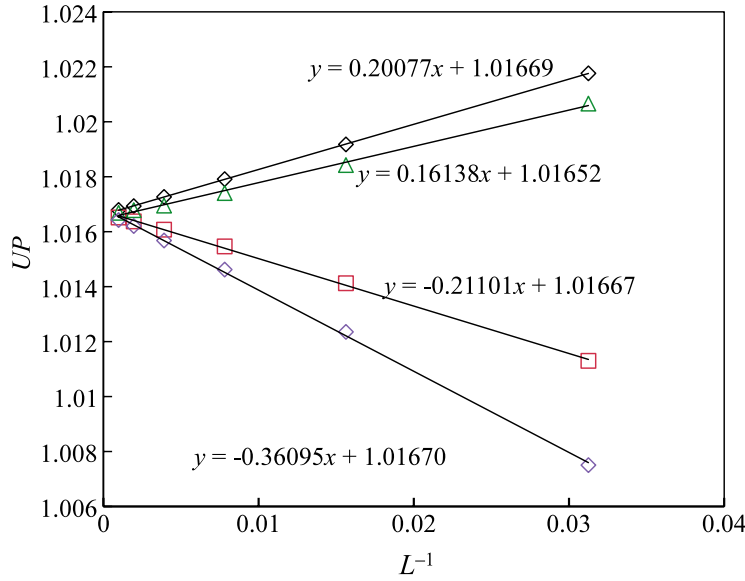


**Figure 9.** Data collapse of the ratio  $UP$  versus  $c$  and  $L$ . The quantity  $x$  is defined by  $x = \varepsilon L^{1/\nu_{\perp}}$ , where  $\varepsilon = c - c_c$ . The critical values used are  $\nu_{\perp} = 1.333$  and  $c_c = 0.1765$ .

that the critical behavior of the stochastic SIR model obeys the finite-size scaling defined above.

In figure 10 we plot  $UP$  at the critical point  $c_c = 0.1765$  for systems of sizes  $L = 32, 64, 128, 256$  and  $1024$ . Here we did at least  $10^8$  samples for each size. As seen in that

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**Figure 10.** This plots critical values of  $y = UP$  for bond percolation (top), site percolation counting only occupied sites (second to top), the SIR model (second to bottom) and site percolation counting both occupied and surrounding vacant sites (bottom), as a function of  $x = L^{-1}$ . The equations represent a linear fit to the simulation data; the error bars are smaller than the size of the symbols.

figure, we find a very good fit assuming the finite-size corrections are proportional to  $1/L$ . The data extrapolate to a value of  $UP = 1.0167(1)$  for an infinite system.

We also ran similar simulations of standard site and bond percolation on a square lattice at their critical points  $p_c = 0.592\,746$  and  $p_c = 0.5$ , respectively, using an epidemic growth algorithm to generate the percolation clusters, and also show those results in figure 10. We defined the order parameter for percolation in exactly the same way as in the SIR model: whether the cluster growing from the center of the lattice reaches the boundaries (ignoring the periodic boundary conditions). The definition of the cluster mass depended upon the model. For bond percolation, we characterized the cluster mass by the number of sites that are visited or ‘wetted’. For site percolation, we considered two definitions of the cluster mass: the first is the number of occupied sites of the cluster, which conforms to the standard definition in percolation, while for the second we used the number of both occupied and vacant sites surrounding the clusters (the so-called perimeter sites) to characterize the cluster size. The latter definition corresponds to the site percolation limit of the SIR model, in which an I site simultaneously infects all its S neighbors with probability  $p$ , and then recovers, so that the R sites correspond to both occupied and vacant sites of the percolation cluster. These systems show similar correction-to-scaling  $\approx L^{-1}$ , but with different coefficients—positive for bond and regular site percolation, negative for the SIR model and the occupied + vacant form of site percolation. The extrapolation of all three systems for  $L \rightarrow \infty$  is to a common value  $1.0167(1)$ , showing that the SIR model is equivalent to percolation not just for the critical exponents but for this amplitude ratio as well.

Note that a correction-to-scaling behavior of  $L^{-1}$  is often seen in percolation problems when there are boundaries or lattice effects present [33, 34]; however, the precise source

of the corrections is not clear here. It is not certain that the exponent is exactly  $-1$ ; the data in figure 10 is well fitted for exponents in the range of  $-1$  to  $-1.1$  (ignoring higher-order corrections), depending somewhat upon the system. Nevertheless, the value of the extrapolation for  $L \rightarrow \infty$  is not changed substantially by such small changes in the scaling exponent.

While the universality of amplitude ratios in percolation has been studied for many combinations of quantities (e.g. [35]–[40]), it seems that the quantity  $UP$  has not been examined previously for percolation, either away from, or exactly at, the critical point as done here. For scaling away from  $p_c$ ,  $UP$  corresponds to the critical amplitude  $R_3$  or  $v_3$ , which has been studied for the Ising model [41]–[43] but evidently not for percolation. For percolation at the critical point, some studies have been carried out on other universal amplitude ratios [39, 40], but not for  $UP$ .

The value of  $UP$  is very close to 1. This behavior can be understood easily for standard percolation as follows: at the critical point, the largest cluster  $s_{\max}$  is of the order of the size of the system and much bigger than the other clusters, implying  $S \approx s_{\max}^2/L^2$ ,  $M \approx s_{\max}^3/L^2$ , and (by definition)  $P = s_{\max}/L^2$ , thus yielding  $UP \approx 1$ . By universality, this should also apply to the SIR model. As can be seen in figure 9, for  $c < c_c$  (which corresponds to  $p > p_c$  for regular percolation),  $UP$  goes to the value of exactly 1, as would be expected by these arguments.

## 5. Conclusion

We have used numerical simulations to investigate the critical behavior of the stochastic SIR model on finite square lattices. We have determined the order parameter  $P$ , the mean number of recovered individuals  $S$  and the mean squared number of recovered individuals  $M$ . These quantities obey the same scaling laws used in percolation theory. The cluster size distribution of percolation theory is identified with the cluster distribution of recovered individuals generated by the SIR dynamics. By studying lattices of different sizes we obtain the critical behavior by means of a finite-size scaling borrowed from standard percolation theory. The values of the critical exponents are in agreement with those of the isotropic percolation as one would expect. We have shown that the ratio  $U = M/S^2$  is not universal at the critical point, as is the case of the models belonging to the directed percolation universality class, and diverges as  $L^{\beta/\nu_{\perp}}$ . Instead, we have shown that the quantity that is universal is the product  $UP$ . The value of the critical point  $c_c$ , found from the fact that  $UP$  is independent of the system size, is found to be in excellent agreement with previous calculations. The value of  $UP$  at the critical point, 1.0167(1) is shown to be consistent with measurements of standard site and bond percolation, thus confirming that the corresponding amplitude ratios of the two models are the same, and showing a deeper level of agreement between the SIR and percolation models than just having common critical exponents.

## Acknowledgments

We wish to acknowledge the financial support of the Brazilian agency CNPq, and RMZ acknowledges partial support from the US National Science Foundation grant no. DMS-0553487.

## References

- [1] Kermack W O and McKendrick A G, 1927 *Proc. R. Soc. London A* **115** 700
- [2] Bailey N T J, 1953 *Biometrika* **40** 177
- [3] Bailey N T J, 1957 *The Mathematical Theory of Epidemics* (New York: Hafner)
- [4] Mollison D, 1977 *J. R. Stat. Soc. B* **39** 283
- [5] Renshaw E, 1991 *Modelling Biological Populations in Space and Time* (Cambridge: Cambridge University Press)
- [6] Hastings A, 1996 *Population Biology: Concepts and Models* (New York: Springer)
- [7] Keeling M J and Rohani P, 2008 *Modeling Infectious Diseases in Human and Animals* (Princeton, NJ: Princeton University Press)
- [8] Grassberger P, 1983 *Math. Biosci.* **63** 157
- [9] Cardy J and Grassberger P, 1985 *J. Phys. A: Math. Gen.* **18** L267
- [10] Satulovsky J and Tomé T, 1994 *Phys. Rev. E* **49** 5073
- [11] Satulovsky J and Tomé T, 1997 *J. Math. Biol.* **35** 344
- [12] Antal T, Droz M, Lipowski A and Ódor G, 2001 *Phys. Rev. E* **64** 036118
- [13] Dammer S M and Hinrichsen H, 2003 *Phys. Rev. E* **68** 016114
- [14] Dammer S M and Hinrichsen H, 2004 *J. Stat. Mech.* P07011
- [15] Arashiro E and Tomé T, 2007 *J. Phys. A: Math. Theor.* **40** 887
- [16] Schütz G M, Brandau M and Trimper S, 2008 *Phys. Rev. E* **78** 061132
- [17] Assis V R V and Copelli M, 2009 *Phys. Rev. E* **80** 061105
- [18] de Souza D and Tomé T, 2010 *Physica A* **389** 1142
- [19] Tomé T and Ziff R M, 2010 *Phys. Rev. E* **82** 051921
- [20] Tomé T and de Oliveira M J, 2011 *J. Phys. A: Math. Theor.* **44** 095005
- [21] Neri F M N, Pérez-Reche F J, Taraskin S N and Gilligan C A, 2011 *J. R. Soc. Interface* **8** 201
- [22] Kuulasmaa K, 1982 *J. Appl. Probab.* **19** 745
- [23] Newman M E J, 2002 *Phys. Rev. E* **66** 016128
- [24] de Oliveira M M and Dickman R, 2005 *Phys. Rev. E* **71** 016129
- [25] Tomé T and de Oliveira M J, 2005 *Phys. Rev. E* **72** 026130
- [26] Baek S K, Minnhagen P and Kim B J, 2010 *Phys. Rev. E* **81** 041108
- [27] Stauffer D and Aharony A, 1994 *Introduction to Percolation Theory* (London: Taylor and Francis)
- [28] Lorenz C D and Ziff R M, 1998 *Phys. Rev. E* **57** 230
- [29] Newman M E J and Ziff R M, 2001 *Phys. Rev. E* **64** 016706
- [30] Lee M J, 2008 *Phys. Rev. E* **78** 031131
- [31] Feng X, Deng Y and Blöte H W J, 2008 *Phys. Rev. E* **78** 031136
- [32] Stauffer D, 1979 *Phys. Rep.* **54** 1
- [33] Ziff R M, 1992 *Phys. Rev. Lett.* **69** 2670
- [34] Hovi J-P and Aharony A, 1996 *Phys. Rev. E* **53** 235
- [35] Aharony A, 1980 *Phys. Rev. B* **22** 400
- [36] Privman V, Hohenberg P C and Aharony A, 1991 *Phase Transitions and Critical Phenomena* vol 14, ed C Domb and J L Lebowitz (New York: Academic)
- [37] Saleur H and Derrida B, 1985 *J. Physique* **46** 1043
- [38] Adler J, Aharony A, Meir Y and Harris A B, 1987 *J. Phys. A: Math. Gen.* **19** 3631
- [39] Aharony A and Stauffer D, 1997 *J. Phys. A: Math. Gen.* **30** L301
- [40] Daboul D, Aharony A and Stauffer D, 2000 *J. Phys. A: Math. Gen.* **33** 1113
- [41] Watson P G, 1969 *J. Phys. C: Solid State Phys.* **2** 1883
- [42] Zinn S-Y, Lai S-N and Fisher M E, 1996 *Phys. Rev. E* **54** 1176
- [43] Pelissetto A and Vicari E, 2002 *Phys. Rep.* **368** 549