ON WALSH CHARACTERISTIC FUNCTIONS

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ABSTRACT. The theory of characteristic functions is well known in Probability Theory and the characteristic function is a useful tool for the determination of moments of random variables and for the derivation of asymptotic distributions of sequences of random variables. In this paper we give the first steps in the theory of Walsh characteristic functions, connected with the concepts of Walsh functions and dyadic differentation.

RESUMO. *Sobre funções características de Walsh.* A teoria das funções características é bem conhecida na Teoria das Probabilidades e a função característica é uma ferramenta útil para se determinarem momentos de variáveis aleatórias e distribuições limites de seqüências de variáveis aleatórias. Neste artigo damos os primeiros passos para uma teoria de funções características de Walsh, relacionada com os conceitos de funções de Walsh e diferenciabilidade didática.

WALSH FUNCTIONS

In the last decade the Walsh functions have become increasingly important and many applications have been proposed, especially in the field of communications engineering. See Bass (1970) and Schreeber (1974), for example. These functions appear to be ideal to analyse linear, time-variable circuits based on binary digital components, as opposed to linear, time-invariant networks, where the complex exponentials are the natural tools. See Morettin (1973) and Pichler (1970) for applications of Walsh functions in linear system theory. According to Harmuth (1972), among the proposed uses, it seems that the areas of digital filtering and digital multiplex equipment are the most promising.

The Walsh functions were first introduced by Walsh (1923). Fine (1949), Chrestenson (1955) and Morgenthaler (1957) have developed a theory of Walsh-Fourier series and most of the results parallel those of the classical trigonometric series theory. We denote by $\{\psi(n,x), n = 0, 1, 2, \ldots, 0 \leq x < 1\}$ the orthonormal system of Walsh functions in the sense of Fine (1949). These functions are periodic, of period 1 and the first eight are illustrated in figure 1.

The generalized Walsh functions were introduced by Fine (1950). Let t and x non-negative real numbers and consider their dyadic expansions

$$t = \sum_{j=-N}^{\infty} t_j \cdot 2^{-j}, x = \sum_{j=-M}^{\infty} x_j \cdot 2^{-j},$$
$$t_j, x_j \in \{0,1\}. \tag{1}$$

Define the addition modulo 2 by

$$t \oplus x = \sum_{j=-L}^{\infty} (t_j \oplus x_j) 2^{-j}, \tag{2}$$

where $L = \max\{N,M\}$ and $t_j \oplus x_j$ is the usual (componentwise) addition modulo 2 for the integers 0 and 1, that is,

$$0 \oplus 0 = 1 \oplus 1 = 0, \quad 1 \oplus 0 = 0 \oplus 1 = 1.$$

Then, we define the generalized Walsh functions $\{\psi(t,x), t \geq 0, x \geq 0\}$, by

$$\psi(t,x) = \exp\{\pi i \sum_{k=-N}^{M+1} t_k x_{1+k}\} \tag{3}$$

where t and x have dyadic expansions given by (1). A useful property is that, for each y and almost all x,

$$\psi(t, x \oplus y) = \psi(t,x)\psi(t,y), \tag{4}$$

the exceptional x being those for wich $x \oplus y$ is a dyadic rational. For further details see the above mentioned papers of Fine.

DYADIC DERIVATIVES

The theory of dyadic differentiation was initiated by Gibbs and Millard (1969) for the discrete case and expanded by Pichler (1970) for real-valued functions of a continuous, non-negative real variable. If is such a function, we attach to f a function $f^{[1]}$ given by

$$f^{[1]}(t) = \sum_{k-\infty}^{\infty} [f(t) - f(t \oplus 2^{-k})]2^{k-2} \tag{5}$$

If $f^{[1]}$ exists, we call it the *first dyadic derivative of* f. In general, define

$$f^{[k]}(t) = f^{[1]}(f^{[k-1]}(t)), \tag{6}$$

$k = 2,3,4, \ldots$ It follow in particular that

$$\psi^{[1]}(t,x) = x \cdot \psi(t,x), \tag{7}$$

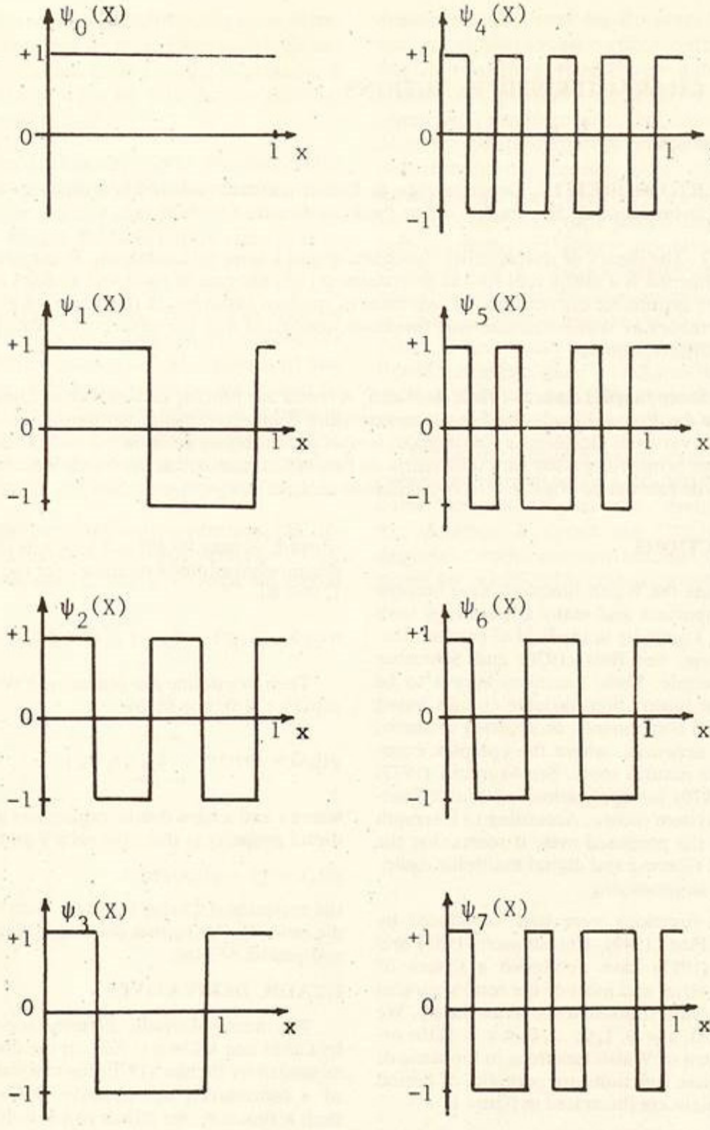

Figure 1. The first eigh Walsh functions.

for all t,x $\geq$ 0. This is similar to

$$\frac{d}{dt} e^{itx} = ix\, e^{itx} \tag{8}$$

Relation (7) shows that the Walsh functions are eigenfunctions of the dyadic differential operator, satisfying

$$f^{[1]} - xf = 0.$$

WALSH CHARACTERISTIC FUNCTIONS

Let X be a random variable (r.v.) defined on a probability space Ω,A,P) with values in $(\mathcal{R}_+, \mathbb{B}_+, P_x)$, where $\mathcal{R}_+$ is the set of non-negative real numbers, β_+ is the Borel field on $\mathcal{R}$ and P_x is the probability measure induced by X.

We define the *Walsh characteristic function* (WCF) of the r.v. X, or of the probability measure

P_X, or of the corresponding distribution function (d.f.) F_X of X, to be

$$f(t) = E\{\psi(t,X)\} = \int \psi(t,X(\omega)dP(\omega) =$$

$$= \int_{\mathbb{R}_+} \psi(t,x)dP_X(x) = \int_0^\infty \psi(t,x)dF_X(x). \qquad (10)$$

We see immediately that:

$$\text{(i) } f(0) = \int_\Omega dP(\omega) = 1, \ |f(t)| \leq 1 = f(0); \qquad (11)$$

(ii) f is uniformly W-continuous*, since

$$|f(t \oplus h) - f(t)| \leq \int_0^\infty |\psi(t \oplus h,x) - \psi(t,x)|dF_X(x)$$

(4) $\int_0^\infty |\psi(t,x)||\psi(h,x) - 1|\, dF_X(x) \to 0$, as $h \to 0$, independent of t, since $|\psi(t,x)| = 1$ and $|\psi(t,x) - 1| \leq 2$, using the Lebesgue dominated convergence theorem;

(iii) If $Y = aX \oplus b$, then $f_Y(t) = \psi(t,b)\, f_{aX}(t)$.

Let X_1, X_2 be r.v.'s with d.f.'s F_1, F_2, respectively. Define their *dyadic convolution* by

$$F(x) = (F_1 \oplus F_2)(x) = \int_0^\infty F_1(x \oplus y)\, dF_2(y). \qquad (12)$$

With an argument similar to that of Chung, 1968, page 134, we can prove the following

Theorem 1 — Let $X_1 \oplus X_2$ be the dyadic sum of X_1 and X_2. If X_1 and X_2 are independent, with d.f.'s F_1 and F_2, respectively, then the d.f. of $X_1 \oplus X_2$ is $F_1 \oplus F_2$.

We have also the result that follows.

Theorem 2 — If X_1 and X_2 are independent r.v.'s, the WCF of $X_1 \oplus X_2$ is the product of the individual WCF of X_1 and X_2.

Proof — Immediate, since

$f_{X_1 \oplus X_2}(t) = E\{\psi(t, X_1 \oplus X_2)\} = E\{\psi(t,X_1)\psi(t,X_2)\}$
$= E\{\psi(t,X_1)\}\, E\{\psi(t,X_2)\} = f_{X_1}(t) f_{X_2}(t)$. The second equality follow by (4) and the third by the independence of X_1 and X_2.

We now show that $f^{[1]}(t)$ can be computed in a similar way of

$$f'(t) = \int_{-\infty}^\infty e^{itx}(ix)dF_X(x),$$

for the usual characteristic function.

Theorem 3 — If X is a r.v. with WCF f(t), then

$$f^{[1]}(t) = \int_0^\infty x \cdot \psi(t,x)\, dF_X(x). \qquad (13)$$

Proof — By definition (5),

$$f^{[1]}(t) = \sum_{k=-8}^\infty [\int_0^\infty \psi(t,x)\, dF_X(x) - \int_0^\infty \psi(t \oplus 2^{-k}, x)\, dF_X(x)]2^{k-2} =$$

$$= \sum_{k=-\infty}^\infty [\int_0^\infty \psi(t,x)\ dF_X(x) - \int_0^\infty \psi(t,x)\psi(2^{-k},x)\, dF_X(x)]2^{k-2}$$

$$= \sum_{k=-\infty}^\infty [\int_0^\infty \psi(t,x)\{1 - \psi(2^{-k},x)\}\, dF_X(x)]2^{k-2}.$$

Now, by (3),

$$\psi(2^{-k},x) = \exp\{\pi i\, x_{1-k}\},$$

and

$$2^{-1}[1 - \psi(2^{-k},x)] = x_{1-k}.$$

Therefore,

$$f^{[1]}(t) = \sum_{k=-\infty}^\infty [\int_0^\infty \psi(t,x)\, x_{1-k}\, dF_X(x)]2^{k-1},$$

or

$$f^{[1]}(t) = \int_0^\infty \sum_{k=-\infty}^\infty x_{1-k}\, 2^{k-1}\psi(t,x)\, dF_X(x).$$

By the substitution $1 - k = j$, we obtain (13):

$$f^{[1]}(t) = \int_0^\infty \sum_{k=-\infty}^\infty x_j 2^{-j}\, \psi(t,x)\, dF_X(x) = \int_0^\infty x \cdot \psi(t,x)\, dF_X(x).$$

It is easy to see that, in general, we have

$$f^{[k]}(t) = \int_0^\infty x^k \psi(t,x)\, dF_X(x), \qquad (14)$$

$k = 1, 2, 3, \ldots$

Putting $t = 0$ in (14), for $k = 1$ and $k = 2$ we have

$$f^{[1]}(0) = \int_0^\infty x\psi(0,x)\, dF_X(x) = \int_0^\infty x dF_X(x) = E(X),$$

$$f^{[2]}(0) = \int_0^\infty x^2 dF_X(x) = E(X^2),$$

from what follows that the variance of X is

$$Var(X) = f^{[2]}(0) - [f^{[1]}(0)]^2. \qquad (15)$$

From (14) we obtain the k-th moment of X as

$$E(X^k) = f^{[k]}(0). \qquad (16)$$

$k = 1, 2, 3, \ldots$

COMMENTS

These are the first steps in this theory. To be operational, delicate theorems as the uniquess and continuity theorems have to be established. A topic for further study is to use the concept of strong derivative, introduced by Butzer and Wagner (1973), which generalizes the concepts of dyadic derivative presented here.

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* The function $f(\cdot)$ of period 1 is *W-continuous* at $x \in [0,1]$ if given $\varepsilon > 0$, there is $\delta(\varepsilon, x) > 0$ such that $|f(x \oplus y) - f(x)| < \varepsilon$, whenever $0 \leq y < \delta(\varepsilon, x)$. See Morgenthaler (1957).

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COMPORTAMIENTO REPRODUCTOR ABERRANTE EN *Columba livia*

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RESUMEN. Se describe una sucuencia de comportamientos interpretados como ofrenda nupcial y comportamiento constructor en una pareja de palomas *Columba livia.* Se trata de un aporte de elementos metálicos (clavos, tornillos y alambres) efectuado por el macho, recibidos y tirados fuera del nido por la hembra, lo que significó un aporte de 551 objetos metálicos con un peso de 1kg, 343g. Durante el período de nidificación la pareja contruyó parte de su nido con ese mismo tipo de materiales metálicos, cumpliendo puestas e incubación normal.

ABSTRACT. *Aberrant reproductor behavior in* Columba livia. A sequence of behaviors interpreted as nuptial gift and constructive behavior in a couple of *Columba livia* is described. It deals with the carrying of metallic objects (nails, screws, wires) by the male, to the nest, and received and thrown by the female, totalling 551 metallic objects with a weight of 1.343kg.

During the nesting period, the couple built part of its nest with the same kind of metallic materials, performing normal egg-laying and incubation.

RESUMO. *Comportamento reprodutor aberrante em* Columba livia. Descreve-se uma seqüência de comportamentos interpretados como oferenda nupcial e comportamento construtor num casal de pombas *Columba livia.* Trata-se de uma contribuição de elementos metálicos (pregos, parafusos e arames) efetuada pelo macho, recebidos e atirados fora do ninho pela fêmea, o que significou uma contribuição de 551 objetos metálicos com um peso de 1 kg, 343 g. Durante o período de nidificação o casal construiu parte de seu ninho com esse mesmo tipo de materiais metálicos cumprindo as posturas e a incubação normalmente.

I. INTRODUCCIÓN

En la presente comunicación exponemos un caso particular de comportamiento, en relación con el sexual y reproductor, observado por azar en una pareja de palomas, en el 2º piso de la Facultad de Humanidades y Ciencias.

II. OBSERVACIONES

La pareja en cuestión — durante el otoño — eligió, para nidificar, la minúscula plataforma protegida situada en una de las columnas metálicas de 5mts. 56 de altura, que soportan el techo del corredor del 2º piso. Los cuatro corredores de cada piso encuadran los patios abiertos del edificio. Tal ubicación del nido nos fué advertida por las deyecciones caídas al piso desde la columna.

Observamos asi, que ambos integrantes de la pareja, que permanecian largo rato en la indicada reducida plataforma efectuaban frecuentes viajes en espacio abierto, especialmente hasta la azotea del 3er. piso. Al regresar se posaban sobre dicha plataforma, seguramente en tales condiciones que se cum-