A SURVEY ON IRREGULAR DYNAMICS
PIECEWISE EXPANDING MAPS, TRANSFER OPERATORS, BESOV SPACES AND GRIDS.

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ABSTRACT. Deterministic dynamical systems often exhibit behaviors that appear random and unpredictable, blending order and chaos in intricate ways. Traditional methods for analyzing these systems struggle with systems featuring irregularities like discontinuities, singularities, or difficult-to-analyze invariant measures. This survey explores the application of transfer operator methods, Besov spaces and measure spaces with grids as tools for addressing these challenges. Focusing on piecewise expanding maps as a key example, we demonstrate how these methods provide a flexible framework for studying statistical properties of dynamical systems in irregular settings. Besov spaces capture localized irregularities, while measure spaces with grids facilitate systematic discretization and computational analysis. Together, these tools offer a powerful approach to understanding the intricate interplay between deterministic dynamics and statistical regularities.

CONTENTS

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1. INTRODUCTION

In some dynamical systems, deterministic rules give rise to behaviors that can appear random and unpredictable. These systems challenge our understanding of order and chaos, blending intricate patterns with statistical regularities. Over decades of research, tools from analysis and probability have been developed to study the long-term statistical behavior of these systems, uncovering profound connections between deterministic dynamics and randomness.

Yet, as the field has advanced, so too has the need for methods that handle increasingly complicated and irregular systems. Many real-world and theoretical dynamical systems defy the smoothness assumptions of classical approaches, with observables, phase spaces, or invariant measures that exhibit discontinuities, singularities, or even fractal-like structures. This is where *transfer operator methods*, *Besov spaces* and *measure spaces with grids* play a pivotal role.

Why Piecewise Expanding Maps? Piecewise expanding maps are a central and illustrative example of the challenges and opportunities in studying irregular dynamical systems. These maps arise naturally in many settings, from interval dynamics to models of population growth, symbolic dynamics, and hyperbolic systems. They are defined by deterministic rules that are smooth within individual regions but may exhibit discontinuities at the boundaries of these regions.

The irregular nature of piecewise expanding maps makes them an ideal test case for exploring statistical properties, as they often possess:

- **Invariant measures with intricate structures**, such as those that are absolutely continuous with respect to a reference measure but supported on fragmented or irregular sets.
- **Discontinuous dynamics**, where smoothness breaks down at well-defined boundaries, challenging classical analytical methods.
- **Sharp transitions between regions**, necessitating tools that can handle localized irregularities and fragmented dynamics.

Studying piecewise expanding maps has important implications. These systems are both theoretically rich and practically relevant, serving as a bridge between smooth dynamical systems and those with more complicated irregular behavior. They also highlight the power of Besov spaces, measure spaces with grids, and transfer operator methods in addressing questions about invariant measures, statistical properties, and long-term dynamics in settings where traditional tools often fall short.

Why Transfer Operators? Transfer operators are a central tool in the study of dynamical systems, particularly when investigating statistical properties. By encoding how measures are transformed under the dynamics, transfer operators allow us to study the invariant measures of a system and their statistical behavior. This approach has several advantages:

- They provide a direct link between the dynamics of the system and its statistical properties, such as decay of correlations, large deviations, and Central Limit Theorem.
- Transfer operators are well-suited for numerical computations, making them a powerful tool for studying irregular systems where explicit analytical results are difficult to obtain.

- They allow for a spectral perspective, connecting the asymptotic behavior of the system to the eigenvalues and eigenfunctions of the operator.

However, the effectiveness of transfer operators often depends on the choice of function spaces in which they are analyzed. Classical spaces have been extensively used, offering insights into the smoothness of observables and their interaction with invariant measures. While these spaces form the backbone of ergodic theory for smooth dynamical systems, they may not fully capture the irregular or fragmented nature of piecewise expanding maps and other similar systems. This motivates the use of Besov spaces and measure spaces with grids, which offer greater flexibility for analyzing such cases.

Why Besov Spaces? Besov spaces extend the classical Sobolev framework by incorporating a multiscale perspective on regularity. This makes them particularly effective for analyzing transfer operators associated with piecewise-defined or irregular dynamical systems. Besov spaces provide:

- A natural framework for studying functions or measures that exhibit localized irregularities or oscillations.
- Tools to analyze the statistical properties of systems, such as decay of correlations and the central limit theorem, even when the underlying dynamical systems are not smooth.
- A multiscale structure that aligns well with the spectral analysis of transfer operators.

By using Besov spaces, we gain a more refined understanding of the interplay between regularity and statistical properties, extending the reach of classical tools.

Why Measure Spaces with Grids? While Besov spaces provide a functional framework, *measure spaces with grids* offer a complementary perspective by enabling the systematic *discretization* of the phase space. This approach is particularly useful for studying transfer operators and irregular dynamical systems. Grids allow us to:

- Approximate the action of transfer operators on irregular invariant measures, dynamics, and phase spaces.
- Handle piecewise-defined or discontinuous dynamics in a structured way, without requiring global smoothness.
- Connect theoretical insights with computational approaches, enabling numerical studies of spectral properties and statistical behavior.

This discretized framework provides a practical and robust way to study systems and observables that might otherwise be intractable using other methods. For piecewise expanding maps, grids align naturally with the fragmented structure of the dynamics, facilitating analysis and computation in these challenging settings.

2. Piecewise Expanding maps

Let (I,d) be a metric space. A map

$$T\colon I \to I$$

is **piecewise expanding** if it "locally" expands distances. More precisely, one can find a countable partition

$$I = \cup_j I_j, \tag{2.1}$$

and $\lambda > 1$ such that for every j

$$d(T(x),T(y)) \geq \lambda d(x,y)$$

for all $x, y \in I_j$. Sometimes it is convenient to consider also subsets I_j that cover *almost all* I, that is, except for a subset with zero measure, for some given measure m on I.

Due to *local expansion*, orbits with close initial points tend to diverge. If I is a bounded metric space, such divergence can not continue forever. The combination of the local *exponential* divergence and the boundedness of the phase space often leads to a quite complicated, *chaotic* behaviour. Indeed, piecewise expanding maps are the simplest dynamical models with such chaotic behaviour, and they appear everywhere, from pure to applied mathematics.

3. EXAMPLES

There is a really long list of examples of piecewise expanding maps. The following list is not exhaustive, but certainly contains some of the most representative examples.

3.1. **One-dimensional maps.**

The first expanding maps to be studied were one- dimensional maps, that is, maps whose phase space are either an interval or a circle, and its partition as in (2.1) is made of intervals. For instance, consider a partition of $I = [0,1]$ by intervals $I_j = [a_j, b_j)$ in such way that

$$T\colon [a_j, b_j) \to [0,1]$$

extends to a C^1 map on $[a_j, b_j]$ satisfying

$$|T'(x)| \geq \lambda > 1$$

for every j and $x \in I_j$. On can say that the study of these maps have two main percursors. They appear for the first time in a letter of Gauss to Laplace [29]. Gauss wondered about the frequency of entries in the continuous fraction expansion of randomly chosen real numbers (uniform distribution). He says that he cannot tell the distribution of the nth term in the expansion, but he is able to tell *exactly* the limit of the distribution when n goes to infinity. The continuous fraction expansion is a system of numeration related with the now called Gauss map $G\colon (0,1] \to (0,1]$ defined by

$$G(x) = 1/x - [1/x].$$

Indeed G^2 is piecewise expanding. Here $[y]$ denotes the integer part of y.

The Borel's contribution on normals numbers [27] can be seen as an early study of ergodic properties of the piecewise expanding map $F\colon [0,1] \to [0,1]$ given by

$$F(x) = 10x - [10x].$$

due the connection of the dynamics of such maps with decimal expansions of real numbers. Gauss and Borel were not interested in specific continued fraction or decimal expansions, since they were aware of the infinite variety of behaviours of such expansions, but either in the *probability* of a behaviour (in Gauss's case) or the *frequency* of events in a *typical* expansion (in Borel's case). It is not a coincidence that both were pioneers of probability theory.

As far as I know, the recognization of such contributions as early examples of questions on *dynamical systems* and *ergodic theory* came much later in our history. We note that Birkhoff's ergodic theorem (G. D. Birkhoff, 1931) was originally proved just for invertible maps (indeed for flows on manifolds). The general theorem we learn today was stated by F. Riesz [66] in 1945, and indeed he was perhaps the first to realized that Borel's results (and the extension by Raikov [62] for arbitrary integrable observables) could be seen as a consequence of the ergodicity of F.

Indeed the work of Boltzman on ergodic theory is based on models from statistical mechanics that are *conservative* , that is, there is *a priori* natural invariant measure for the model. In the map $10x-[10x]$ this is also true, however for the Gauss map a natural invariant measure is far from obvious and, indeed, Gauss's main contribution was to see that a special absolutely continous invariant measure plays an essential role in his problem, that we now recognize as the natural invariant measure for the Gauss map. Finding such a natural measure (sometimes called *physical measure*) for a piecewise expanding maps is a problem that was the main driving force of most of the history of their study, and only had significant progress much later, in the end of 60's and 70's.

Indeed, those examples of piecewise expanding maps are *onto*, that is, for each branch we have $T(X_j) = X$. Those are particular examples for what is now called *Markovian* maps. A piecewise expanding map is Markovian if for each j the set $T(X_j)$ is a union of elements of the partition $\{X_j\}_j$. In this case $\{X_j\}_j$ is called a *Markov partition* of f.

For piecewise expanding maps that are non-Markovian, proving the existence of an absolutely continuous invariant measure is typically more challenging. Consider, for example, the β-transformation $g_\beta \colon [0,1] \to [0,1]$ defined by

$$g_\beta(x) = \beta x - \lfloor \beta x \rfloor,$$

where $\beta > 1$. If β is irrational, this map is non-Markovian. In 1957, Rényi [64] proved the existence of an ergodic, absolutely continuous invariant measure for this system. Later, Gel'fond [30] and Parry [57] provided an explicit formula for this invariant measure.

This is, as far as I know, the first such result for non-Markovian maps. Further progress on non-Markovian maps was pretty much non-existent until the seminal work of Lasota and Yorke [43] in the 70s, that proved the existence of absolutely continuous invariant probabilities for piecewise C^2 expanding maps on intervals. Indeed, much, if not all early work on ergodic theory of expanding maps is concentrated on the proof of the *existence* of absolutely continuous invariant probabilities.

While one-dimensional maps are the first to appear in literature, they were and continue to be a source of inspiration for new methods in smooth ergodic theory.

One may notice that all early contributions by Gauss, Borel, and later Gel'fond and Rényi [65], that gave conditions for the existence of absolutely continuous invariant probabilities for certain Markovian expanding maps on the interval, are all related with various *systems of numeration*. We can represent a real number as a sequence of integers in various ways (continued fraction, decimal expansions, binary expansion, β-expansion). This was much later connected with the more general *symbolic dynamics*, a way to study dynamical systems associating orbits with sequence of symbols. The next example is related with this method.

3.2. **Unilateral shift map.** Let $\mathbb{N}$ be the natural numbers (including 0). Consider a finite set $\mathscr{A}$ with more than one element. The *unilateral shift map*

$$\sigma \colon \mathscr{A}^{\mathbb{N}} \to \mathscr{A}^{\mathbb{N}}$$

is defined by

$$\sigma(x_0, x_1, \dots) = (x_1, x_2, \dots).$$

The metric

$$d(x,y) = \sum_{i=0}^{\infty} \frac{\delta_{x_i,y_i}}{2^i}$$

turns $\mathscr{A}^{\mathbb{N}}$ into a Cantor set. Here $x = (x_i)_{i\in\mathbb{N}}$, $y = (y_i)_{i\in\mathbb{N}}$ and

$$\delta_{a,b} = \begin{cases} 0, & if\ a = b, \\ 1, & if\ a \neq b. \end{cases}$$

We can define a partition of $\mathscr{A}^{\mathbb{N}}$ given by

$$X_a = \{(x_0, x_1, \ldots)\ such\ that\ x_0 = a\},$$

for every $a \in \mathscr{A}$. This partition is a Markov partition for σ, that is, $\sigma(X_a)$ is an union of elements of the partition (indeed, the whole X). Note that

$$d(\sigma(x), \sigma(y)) \geq 2d(x, y)$$

for every $x, y \in X_a$ and $a \in \mathscr{A}$. So σ is a (Markovian) piecewise expanding map.

Symbolic dynamics appears for the first time in the study of geodesic flow on surfaces of negative curvature by Hadamard. Its systematization in the study of dynamical systems by Morse and Hedlund in 1938 had a lasting impact. They see it as a part of a method to study recurrence and transitivity [51].

3.3. **(Piecewise) Expanding maps on manifolds.** The first examples of multi-dimensional piecewise expanding maps are expanding maps on manifolds equipped with a volume form, introduced in Shub's PhD thesis [72] [73]. Consider a compact manifold I endowed with a Riemannian metric $||\cdot||_p$, $p \in I$. A local C^1 diffeomorphism $T\colon I \to I$ is an *expanding map* if $||D_pT||_p > 1$ for every $p \in I$. The simplest example is given by the map $F\colon \mathbb{S}^1 \to \mathbb{S}^1$ defined as $F(z) = z^2$.

They are quite interesting endomorphisms as Shub proved they are structural stable and yet have a quite chaotic dynamics: periodic points are dense and they have dense orbits. Soon after their introduction, Krzyzewski and Szlenk [42] proved the existence of absolutely continuous invariant probabilities for C^2 expanding maps on manifolds. This result is quite influential, in particular for the argument used there to control the (measure) distortion of high iterates of an expanding nonlinear map, a method used multiple times since then.

The study of non-Markovian piecewise expanding maps on manifolds with dimension large than one appears in the literature much later (1990's and 2000's).

3.4. **Expanding maps with singularities (Lorenz maps).** Those are unidimensional maps, however the derivative can blow up at some points. More precisely let

$$\{[a_j, b_j)\}_{j<n},$$

be a finite family of intervals that is a partition of $[0,1)$ and

$$T\colon \cup_j (a_j, b_j) \to [0,1]$$

be a map such that

$$T\colon (a_j, b_j) \to [0,1]$$

is C^1, it satisfies

$$|T'(x)| \geq \lambda > 1$$

and for every $p \in \{a_j, b_j\}$ there is $\beta_p \in (0,1]$ and a diffeomorphism ϕ_p defined close to p such that

$$T(x) = \phi_p(|x-p|^{\beta_p}).$$

These maps looks somewhat artificial, but they indeed appear naturally studying so called singular hyperbolic flows, and in particular the *Lorenz flow*, a quite simple polynomial vector field on $\mathbb{R}^3$ with a hyperbolic singularity at the origin, introduced by Lorenz in 1963 as a toy model in the study of atmospheric convection. Geometric Lorenz flows were introduced (Guckenheimer and Williams [34] and Williams [85]), which have many of the features conjectured for the Lorenz flow by Lorenz. Finally Tucker [82] gave a computer-assisted proof that original Lorenz flow is indeed a geometric Lorenz flow.

Lorenz maps appear in the study of geometric Lorenz flow in the following way: the hyperbolic singularity is smoothly linearizable (assuming some non-ressonant conditions on its eigenvalues). Considering an appropriate 2-dimensional Poincaré section, we see that its first return map has an invariant smooth stable foliation . Taking the quotient of the action of the first return map by this foliation we obtain a one-dimensional map that is an expanding map with singularities. The values of β_p depend on the eigenvalues of the singularity. See Araújo and Pacifico [1] for more details on singular hyperbolic flows.

3.5. **Hyperbolic Julia Sets.** One of the most fascinating examples of expanding dynamical systems is the Julia set of a hyperbolic rational map. Consider a rational map $T\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ where all critical points converge to a (hyperbolic) attracting periodic orbit. Such a map T is called a *hyperbolic rational map*. The complement of the set of points attracted to these periodic orbits is known as the *Julia set* $J(T)$ of T. This set is non-empty, compact, and perfect. Moreover, the map $T\colon J(T) \to J(T)$ is expanding.

There is a rich body of literature on the thermodynamic formalism and transfer operators for holomorphic dynamical systems. Notably, Bowen [11] made seminal contributions to this field. For comprehensive introductions, see the works of Zinsmeister [89] and Przytycki and Urbański [60].

4. CHAOTIC BEHAVIOUR

The dynamics of piecewise expanding maps are often quite complicated. For instance

A. Their periodic points can be dense in the phase space.
B. They may have dense orbits in the phase space.
C. They may have uncountably many possible dynamical behaviors for its orbits.

The list of piecewise expanding maps with those properties is long (for instance Markovian piecewise expanding maps, tent maps, expanding maps on manifolds). This results in the behavior of its orbits being too diverse to be fully understood. So from the beginning researchers started looking for *statistical properties* of its orbits, and in particular properties that hold for *most* of their orbits.

5. STATISTICAL PROPERTIES OF OBSERVABLES

Consider a measurable dynamical system $T\colon I \to I$ with an *ergodic invariant probability measure* μ, that is

$$\mu(T^{-1}A) = \mu(A)$$

for every measurable set A, and for every invariant set ($T^{-1}A = A$) we have either $\mu(A) = 0$ or $\mu(A) = 1$. Birkhoff's Ergodic Theorem (G. D. Birkhoff, 1931) tells us that for every $\phi \in L^1(\mu)$ and μ-almost every point x

$$\lim_N \frac{1}{N} \sum_{i<N} \phi(T^i(x)) = \int \phi \, d\mu.$$

A particulary illustrative case is when ϕ is the characteristic function of a measurable set, that is, $\phi = 1_A$. In this case, the left-hand side expresses the *frequency* with which a typical orbit visits the set A. It is remarkable one can say so much about the frequency of these events knowing so little about the dynamical system T.

This result likely reminds the reader of the strong law of large numbers for sequences of independent identically distributed random variables. One can asks if the sequence of functions

$$\phi, \phi \circ T, \dots \tag{5.2}$$

has more statistical similarities with this probabilistic setting.

Exponential decay of correlations. It is easy to see that in most cases ϕ and $\phi \circ T^n$ are *not* independent variables but one can ask if they are *nearly* independent when n is large. We say that a pair of functions $\phi_1, \phi_2 \in L^2(\mu)$ has *exponential decay of correlations* if there is $\lambda \in [0,1)$ and $C > 0$ such that

$$\left| \int \phi_1 \phi_2 \circ T^n \, d\mu - \int \phi_1 \, d\mu \int \phi_2 \, d\mu \right| \leq C\lambda^n.$$

Central Limit Theorem. Let $\phi \in L^1(\mu)$. We say that the sequence (5.2) satisfies the *Central Limit Theorem* if there is $\sigma > 0$ such that

$$\lim_n \mu\left(x : \frac{1}{\sqrt{N}} \sum_{n<N} (\phi \circ T^n(x) - \int \phi \, d\mu) < t \right) = \int_{-\infty}^{t} e^{-x^2/\sigma^2} \, dx.$$

Large Deviations. Let $\phi \in L^1(\mu)$. We say the sequence (5.2) has *Large Deviations* if for every $\epsilon > 0$ there is $\lambda \in [0,1)$ and $C > 0$ such that

$$\mu\left(x : \left| \frac{1}{N} \sum_{n<N} \phi \circ T^n(x) - \int \phi \, d\mu \right| > \epsilon \right) \leq C\lambda^n.$$

It is fair to ask why would we focus on such statistical properties? There is not a satisfactory answer for that, except that processes that we all agree are quite *random*, as sequences of flips of a fair coin, *do* have those properties, so we see them together as a *tell-tale sign* for strong randomness of the process. Perhaps one of the major achievements of the theory of dynamical systems in the 20th century is the discovery that deterministic dynamical system can behave in a quite random way.

A lot of dynamical systems do have invariant probabilities. Oxtoby and Ulam [56] tell us that every continuous map on a compact metric space has an invariant probability measure. Indeed many piecewise expanding maps have infinitely many periodic points. Every periodic orbit gives rise to an invariant probability measure supported on it, although this measure is rather uninteresting.

Moreover dynamical systems may have uncountably many invariant probabilities. Such a phenomenon may arise even in simple systems, such as those with at least two periodic orbits, since having just two ergodic measures already leads to uncountably many invariant measures via convex combinations.

Indeed, the *geometry* of the set of invariant probability measures can be quite intricate. For example, every compact metrizable Choquet simplex can be realized as the set of invariant probabilities supported on a minimal set of a logistic map (Cortez and Rivera-Letelier [23]).

It is worth noting that the set of *ergodic* invariant probability measures can be uncountable. A typical example is provided by the family of Gibbs equilibrium states for a $C^{1+\beta}$ expanding map on the circle.

So, do we see those strong statistical properties on every dynamical system with an invariant measure? The answer is *no*. Dynamical systems and invariant probabilities came in various shapes and sizes, and not all of them have those strong statistical properties. See for instance Dolgopyat, Dong, Kanigowski and Nándori [26], where they provide examples of dynamical systems that satisfy the Central Limit Theorem (CLT) but do not necessarily exhibit other desirable statistical properties.

Irrational rotations on the circle, for instance, admit a natural invariant measure (the arc-length), which is also the $unique$ invariant measure. However, the corresponding observables do not exhibit strong statistical properties. Indeed, under certain Diophantine conditions on the rotation number, every sufficiently regular observable ϕ with zero average is cohomologous to zero (Herman [37]); that is, there exists a continuous function ψ such that

$$\phi = \psi \circ T - \psi,$$

which implies that the Central Limit Theorem does not hold. For far more general examples, see Liardet and Volný [46].

However, *certain* invariant measures for piecewise expanding maps often *do* have very strong statistical properties, provided the observable ϕ is *regular enough.* Moreover those invariant measures are quite natural, since they are absolutely continuous with respect to the "ambient measure". For instance, they may be absolutely continuous with respect to the Lebesgue measure for one-dimensional maps, and with respect to the volume form for expanding maps acting on manifolds. Those invariant measures show that those dynamics systems have a behavior that is as *random* as one can get.

6. MONSTERS

The study of piecewise expanding maps is well developed. Many variations of these systems have been introduced over the years, and a wide range of examples is now available. In particular some of those have an “exotic” behavior, that is strikingly distinct from early examples.

For instance, there are one-dimensional, piecewise linear expanding maps on an interval, with an infinite partition, such that

A. They have orbits which are dense on the interval.
B. There is a point p such that almost every orbit converges to p.

The following example is presented by Nowicki and van Strien [54]. Let $\alpha \in (0,1)$, and define a map $T\colon [0,1) \to [0,1)$ as follows. The map T is affine on each interval $[\alpha^{n+1}, \alpha^n)$ for every $n \geq 0$, mapping:

- $[\alpha, 1)$ onto $[0,1)$, and
- each $[\alpha^{n+1}, \alpha^n)$ onto $[0, \alpha^{n-1})$ for $n \geq 1$.

Then the derivative satisfies the uniform lower bound

$$|DT(x)| \geq \frac{1}{1-\alpha}.$$

This is a Markov map, and one can show that it has dense orbits and that periodic points are dense in $[0,1)$. Moreover, if α is sufficiently close to 1, then almost every point converges to 0.

In particular those maps do not have an absolutely continuous invariant probability and the behavior of almost every orbit is really quite distinct from a classical expanding map on the interval with a *finite* Markov partition. That kind of example has an interesting application. It allows the construction of unimodal maps (smooth maps with just one turning point) that possess a *wild attractor* (see Milnor [50]). A map has a wild attractor if its attractor in the topological sense differs from its attractor in the measure-theoretical sense. See Bruin and Keller, Nowicki and van Strien [14] and Bruin, Keller and St. Pierre, Matthias[15] for results on the existence of wild attractions.

In higher dimensions, even piecewise expanding maps that have a *finite* partition may have exotic behavior.

Theorem 6.3 (Buzzi [18])**.** For $r < \infty$ there are piecewise C^r expanding maps on the square (with a finite partition) that do not have absolutely continuous invariant measures.

So that is why we are careful to say that a *lot* of examples of expanding maps has strong ergodic and statistical properties, but decisively not *all* of them have those. When we lower the regularity of the map, sometimes things get tricky. Quas [61] showed that a residual set of C^1 expanding maps acting on a compact manifold do not have an absolutely continuous invariant measure. Piecewise expanding maps on manifolds with dimension higher than one do not seem less regular than its one-dimensional cousins, however the partition of the iterated map can get far more irregular than its first iteration in higher dimensions. Even very smooth one-dimensional piecewise expanding maps with infinite many branches may have an exotic behaviour.

Another exotic example comes from the study of rotational subsets associated with the angle doubling map $T(z) = z^2$ on the unit circle $\mathbb{S}^1$ (see Bullett and Sentenac [16]). Given a closed semicircle $S \subset \mathbb{S}^1$, one can consider its maximal T-invariant subset $\Omega_S \subset S$. The restriction $T\colon \Omega_S \to \Omega_S$ is clearly expanding. However, for certain choices of S, the set Ω_S is a *minimal Cantor set* with *zero Hausdorff measure*. In particular, there are no periodic orbits.

This leads us to question the minimal regularity needed to be able to obtain the existence of absolutely continuous invariant probabilities with nice statistical properties. The wide variety of examples for which those properties do hold also raises the question of the existence of a unified approach to deal with those, even those with very low regularity.

7. Transfer operators

7.1. **Transfer operator and invariant measures.** Given a finite measure μ on the phase space I, we can interpret it as a distribution of mass over I, where the mass assigned to a subset $A \subset I$ is given by $\mu(A)$. A natural question is how this mass distribution evolves under the action of a dynamical system $T\colon I \to I$. That is, if the mass located at a point x is transported to the point $T(x)$, what will the resulting distribution look like?

This evolution is described by the *push-forward* operator $T_\star$, which acts on measures on I. The new distribution is the finite measure $T_\star\mu$ defined by

$$T_\star\mu(A) := \mu(T^{-1}A),$$

for every measurable set $A \subset I$. Note that μ is T-invariant if and only if it is a fixed point of $T_\star$, i.e., $T_\star\mu = \mu$.

However, in the case of piecewise expanding maps, working directly with the operator $T_\star$ on the space of *all* finite measures is generally not desirable. As previously discussed, such systems can admit many invariant measures, while our primary interest lies in those that are absolutely continuous with respect to a reference measure m. To address this issue and fine-tune our search for invariant measures that are absolutely continuous with respect to the reference measure m, we need to introduce *transfer operators*.

Suppose that

$$T\colon \cup_{r\in\Lambda} I_r \to I$$

is a piecewise injective map with inverse branches

$$h_r\colon J_r \to I_r$$

indexed by a countable set Λ, and suppose that m is a reference measure such that for every finite measure $\mu \ll m$ with *density* ψ, that is

$$\mu(A) = \int_A \psi\, dm, \text{ with } \psi \in L^1(m), \psi \geq 0.$$

If we have $(h_r)^{-1}_\star m \ll m$ then the push-forward of the measure μ

$$T_\star\mu = \sum_r (h_r)^{-1}_\star \mu$$

can be written as

$$T_\star\mu(A) = \int_A \Phi(\psi)\, dm,$$

where the density $\Phi(\psi)$ is the function defined by

$$\Phi(\psi)(x) := \sum_r g_r(x)\psi(h_r(x)) \cdot 1_{J_r}(x), \tag{7.4}$$

and $g_r\colon J_r \to \mathbb{R}$ is an appropriate function, called the *Jacobian* of h_r with respect to m.

It is not difficult to see that Φ extends to a *bounded linear operator*

$$\Phi\colon L^1(m) \to L^1(m),$$

and that it is a *positive* operator; that is, if $f \geq 0$, then $\Phi(f) \geq 0$. Moreover

$$|\Phi|_{L^1(m)} = 1.$$

The operator Φ is known as the *transfer operator*, also referred to as the *Ruelle – Perron – Frobenius operator*, associated with the pair (T, m).

In short, the push-forward operator $T_\star$ preserves the class of measures that are absolutely continuous with respect to m—in other words, m is *quasi-invariant* under T. The transfer operator Φ describes the action of $T_\star$ on the densities of such measures. In particular

$$\mu(A) = \int_A \rho\, dm$$

is an invariant probability for T if and only if

$$\Phi\rho = \rho, \text{ with } \rho \geq 0 \text{ and } \int \rho\, dm = 1.$$

Thus, the problem of finding absolutely continuous invariant measures for T reduces to studying the existence of nontrivial, non-negative functions ρ in the 1-eigenspace of Φ acting on $L^1(m)$.

Rechard [63] introduced the transfer operator operator as a tool to study the existence of absolutely continuous invariant measures of one-dimensional maps $T\colon I \to I$, where I is an interval. In this case m is the Lebesgue measure on an interval and

$$g_r(x) = |h_r'(x)| = \frac{1}{|T'(h_r(x))|}.$$

As Ulam [83] (see also Stein and Ulam [79]) makes clear, those linear operators are analogous to positive matrices in an infinite-dimensional setting, and he asks whether an analogue of the Perron-Frobenius theorem holds in this context. This would allow not only a way to show the existence of such natural measures, but also potentially a practical, numerical method to find them.

Indeed the idea of approximating the dynamics of a system by a finite-state Markov process dates back to Ulam's pioneering vision in the 1960s. In his influential book [83], Ulam proposed discretizing the phase space into finitely many pieces and approximating the action of the transfer operator by tracking how mass is transferred between these pieces. This leads to the study of associated Perron–Frobenius matrices, laying the foundation for what is now known as the *Ulam method*. This approach gained significant traction in the late 1970s, particularly through the rigorous analysis of Li [45], who proved convergence of the method under suitable conditions. See Liverani [47] and the references therein for more information.

The first significant generalization of the Perron–Frobenius Theorem in the context of transfer operators was developed through the pioneering work of Ruelle [68] on symbolic dynamical systems, specifically shifts on a finite number of states.

7.2. **Transfer operator and statistical properties.** One of the most interesting applications of transfer operators is the study of the statistical properties of piecewise expanding maps. All the statistical properties we are interested in involve the Koopman operator

$$\gamma \mapsto \gamma \circ T,$$

which naturally raises the question of whether this operator should play a central role in our analysis. However, the Koopman operator does not exhibit good spectral properties when acting on function spaces such as $L^p(m)$, for $p \in [1,\infty]$.

For instance, in the case of well-behaved smooth expanding maps on the circle, the Koopman operator is nearly an isometry: there exists a constant $C_p \geq 1$ such that

$$\frac{1}{C_p}\|\gamma\|_{L^p(m)} \leq \|\gamma \circ T\|_{L^p(m)} \leq \|\gamma\|_{L^p(m)}.$$

Here, m denotes the Haar measure on the circle. As a consequence, the spectral radius of the Koopman operator on $L^p(m)$ is 1, *but* its spectrum near the unit circle *cannot* consist of isolated eigenvalues with finite-dimensional generalized eigenspaces near the circle of radius 1. In particular, its dynamical behavior may differ significantly from that of a linear operator on a finite-dimensional space.

Although the Koopman operator is often well defined on more regular function spaces that are continuously embedded in some $L^p(m)$, its spectral behavior remains similarly poor, and in some cases, even worse. Indeed, for expanding maps on the circle its spectral radius is greater than one when acting on spaces such as the space of functions of bounded variation or the space of Hölder continuous functions.

However its *adjoint* is the transfer operator. That means that

$$\int \gamma \circ T^n \psi \, dm = \int \gamma \Phi^n(\psi) \, dm \tag{7.5}$$

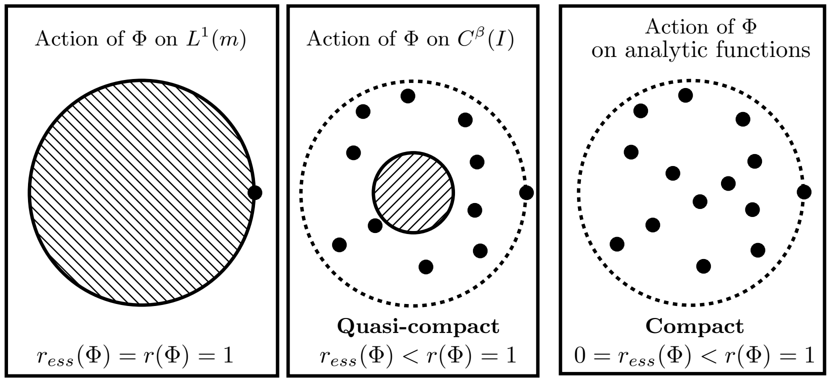

FIGURE 1. Schematic picture of transfer operator associated with a real-analytic expanding map of the circle, acting on various function spaces.

for every $\psi \in L^1(m)$ and $\gamma \in L^\infty(m)$. This allows us to rewrite important expressions in terms of the transfer operator. For instance, if we are interested in decay of correlations one must study

$$\int \gamma \circ T^n \psi \, dm,$$

that can be rewritten as the right-hand side of (7.5).

The primary advantage of working with the transfer operator rather than the Koopman operator lies in its often has more favorable behavior when acting on suitably regular function spaces. This distinction will become evident in Section 8.

8. QUASI-COMPACTNESS AND LASOTA-YORKE INEQUALITY

Both the Koopman operator and the transfer operator act on infinite-dimensional spaces. At first glance, the fact that these operators are *linear* operators may be a relief. However the dynamics of a general linear operator acting on infinite-dimensional Banach space can be very complicated, indeed it can be as complicated as an arbitrary non-linear map on a compact space (see Feldman [28] and Bayart and Matheron [7]).

However it turns out the spectral properties of the transfer operator are much nicer than the Koopman operator.

8.1. **Quasi-compactness.** Fortunately transfer operators often act as *quasi-compact* operators on properly chosen function spaces, what makes its dynamical behaviour easier to understand.

Let $L\colon B \to B$ be a bounded linear operator on a Banach space B. We say that L is *quasi-compact* if

$$r_{\text{ess}}(L, B) < r(L, B), \tag{8.6}$$

where $r(L, B)$ denotes the *spectral radius* of the operator L acting on the Banach space B, and $r_{\text{ess}}(L, B)$ denotes its *essential spectral radius*. These are, respectively, the radii of the spectrum and the essential spectrum of L.

There exist multiple, non-equivalent definitions of the *essential spectrum* in the literature. However, they all aim to capture the "wild" part of the spectrum—namely, the part that is stable under compact perturbations and typically lacks the nice spectral properties of isolated eigenvalues with finite multiplicity of linear operators acting in finite-dimensional spaces.

Remarkably, a result by Nussbaum [55] shows that, regardless of which reasonable definition of essential spectrum is adopted, the radius of this set remains the same. More precisely, he proved the formula

$$r_{\mathrm{ess}}(L,B) = \lim_{n\to\infty} \|L^n\|_{\mathrm{comp}}^{1/n}, \tag{8.7}$$

where

$$\|L\|_{\mathrm{comp}} := \inf\{\|L+K\| \colon K \text{ is a compact operator}\}.$$

Indeed, if L is quasi-compact on B, then the elements of its spectrum λ satisfying $|\lambda| > r_{\mathrm{ess}}(L,B)$—behave in a particularly nice way: they are isolated eigenvalues with finite-dimensional generalized eigenspaces.

More precisely, for such λ we have

$$\dim\left\{v \in B \colon (L-\lambda\,\mathrm{Id})^k v = 0 \text{ for some } k \geq 1\right\} < \infty.$$

In this sense, the spectrum of L near the outer spectral radius $|\lambda| = r(L,B)$ resembles that of a compact operator. That is, the "badly behaved" part of the spectrum—the essential spectrum—is strictly contained within the spectral circle, while the dominant part consists of isolated eigenvalues with finite multiplicity.

A practical way to verify that an operator L is quasi-compact on B is to find some $n_0 > 0$ and a compact operator K such that

$$\|L^{n_0} - K\| < r(L,B)^{n_0}. \tag{8.8}$$

This criterion follows from Nussbaum's formula for the essential spectral radius and the fact that the set of compact operators forms a two-sided ideal in the algebra of bounded operators.

The main feature of a quasi-compact operator is that it is easier to understand it dynamics. If L acting in B is *quasi-compact* one can write

$$L = A + K,$$

where K is a compact operator and A is a *finite rank* operator on B such that $r(A,B) < r(L,B)$ and $AK = KA = 0$. In particular $L^n = A^n + K^n$.

8.2. **Transfer Operators and Quasi-Compactness.** Transfer operators associated with piecewise expanding maps often act as quasi-compact operators on suitable function spaces. Consider, for instance, a real-analytic expanding map acting on the circle $\mathbb{S}^1$, with the Haar measure m as the reference measure. The transfer operator Φ has spectral radius equal to 1 when acting on the Lebesgue space $L^1(m)$, on the space of Hölder continuous functions $C^\beta(\mathbb{S}^1)$, and on certain spaces of analytic functions defined on annuli around $\mathbb{S}^1$. However, the spectral behavior of Φ on each of these spaces is distinctly different. (see Figure 1).

On the space of analytic functions, Φ is a compact operator: its spectrum is countable, and all nonzero spectral values are isolated eigenvalues with finite-dimensional generalized eigenspaces (Ruelle [69]).

On the other hand, the action of Φ on $L^1(m)$ is highly irregular. The spectrum fills the entire closed unit disk, and every complex number of modulus less than one is an eigenvalue with infinite multiplicity (Collet and Isola [22]).

The intermediate case is the space $C^\beta(\mathbb{S}^1)$. Here, Φ acts as a quasi-compact operator. While the essential spectrum is still nontrivial and exhibits behavior similar to that in

$L^1(m)$, it is strictly contained in a disk of radius less than one. Near the unit circle, the spectrum consists solely of isolated, well-behaved eigenvalues (again, see Collet and Isola [22]).

For more irregular dynamics, the space of Hölder continuous functions may not even be Φ-invariant. In such cases, one must identify a function space B, dense in $L^1(m)$ and continuously embedded in it, on which Φ acts as a quasi-compact operator. Transfer operators are positive linear operators with the additional properties

$$|\Phi(|\psi|)|_{L^1} = |\psi|_{L^1} \text{ and}$$
$$|\Phi(\psi)|_{L^1} \leq |f|_{L^1},$$

that implies $r(\Phi, L^1) = 1$ and the *peripherical* spectrum of Φ on B is even more simple, that is, there are a finite number of complex numbers λ_i, with $i = 0, \ldots, n$, $|\lambda_i| = 1$, and $\lambda_0 = 1$, and linear projections P_i with finite rank and an operator A, with $r(A, B) < 1$ such that $P_i P_j = P_j A = AP_j = 0$ for $i \neq j$ and

$$\Phi = \sum_i \lambda_i P_i + A. \tag{8.9}$$

8.3. **Lasota-Yorke inequality.** How do you show that the transfer operator is quasicompact acting on a space of functions B? Lasota and Yorke [43] introduced the now called *Lasota-Yorke inequality* in the study of transfers operators. We say that the pair of function spaces (B, L^1) satisfies the Lasota-Yorke inequality if there is $\lambda \in (0,1)$ and $C > 0$ such that

$$|\Phi(\psi)|_B \leq \lambda |\psi|_B + C|\psi|_{L^1} \tag{8.10}$$

for every $\psi \in B$ and

- B is continuously embedded in L^1.
- The unit ball of B is compact in L^1.
- B is dense in L^1.

Under these conditions, the operator Φ acts quasi-compactly on B (see Ionescu and Marinescu [39]). Indeed, it implies even more: It follows not only that absolutely continuous invariant probabilities do exist , but that *all those probabilities have densities in* B.

Note that there are many slightly different versions of the Lasota–Yorke inequality in the literature. To be fair, in practice, rather than proving (8.10) for Φ, we usually show that Φ is a bounded operator on a Banach space B, and that there exist $n_0 \geq 1$, $\beta \in (0,1)$, and $C > 0$ such that

$$|\Phi^{n_0}(\psi)|_B \leq \beta|\psi|_B + C|\psi|_{L^1}.$$

A standard iteration argument then implies (see for instance Viana [84, Proof of Proposition 3.1]) that there exists $C > 0$ such that, for every $n \geq 1$,

$$|\Phi^n(\psi)|_B \leq C\lambda^n|\psi|_B + C|\psi|_{L^1},$$

for some $\lambda \in (0,1)$. This estimate is sufficient to conclude that Φ is quasi-compact on B, and that every absolutely continuous invariant probability measure belongs to B.

8.4. **Statistical properties and Lasota-Yorke inequality.** If transfer operators are quasi-compact and/or satisfies Lasota-Yorke when acting on some space B, then one can often obtain nice statistical properties for the associated dynamics, such as ergodicity, exponential decay of correlations and Central Limit Theorem for observables in B. Some of these consequences as quite straightforward, others need some hard work.

8.4.1. *Ergodicity and exponential mixing.* Let $sp_B(\Phi)$ be the spectrum of the operator $\Phi\colon B \to B$. Suppose that

A. the transfer operator Φ satisfies the Lasota-Yorke inequality for a pair (B, L^1),
B. $r_{L^1}(\Phi) = 1$,
C. $\mathrm{sp}_B(\Phi) \cap \mathbb{S}^1 = \{1\}$ and
D. the 1-eigenspace in B is one-dimensional.

Then it is easy to show that T has a unique physical measure that is ergodic and every observable in B has exponential decay of correlations. If the transfer operator satisfies A-D then Φ has *spectral gap*, that is, 1 is a simple eigenvalue, and the rest of the spectrum belongs to a ball around 0 with radius smaller than 1.

Usually properties C. and D. are obtained through *topological properties* of T. For instance, if T is one-dimensional and B is the space of bounded variation functions then a density of a physical measure of T needs to have bounded variation, and in particular must be positive on a nonempty open set. If T is *topological mixing*, that would imply C. and D.

Conditions $A-D$ implies exponential decay of correlations for all observables in B.

8.4.2. *Central Limit Theorem.* We do not know if under the same conditions above one can obtain the Central Limit Theorem for observables in B. Indeed the method to obtain CLT for dynamical systems is more convoluted and its origins traces back to Rousseau-Egele [67] and Guivarc'h and Hardy [35], influenced by the early works of Nagaev [52] on Markov chains and Le Page [44] on random matrices. Suppose that we want to obtain CLT to an observable ψ. By Levy's continuity theorem it is enough to show that the characteristic function of

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\psi\circ T^k$$

converges to the characteristic function of a normal distribution, that is, there is $\sigma > 0$ such that

$$\lim_n \int e^{it\frac{1}{\sigma\sqrt{n}}\sum_{i=0}^{n-1}\psi\circ T^i}\,\rho\, dm = e^{-\frac{t^2}{2}}.$$

The first step to prove this limit is to consider the function $e^{it\psi}$, with $t\in\mathbb{R}$, and to check if it defines a *multiplier* acting on B, that is, define

$$M_{e^{it\psi}}(f) = e^{it\psi}f.$$

Then $M_{e^{it\psi}}$ is a bounded operator on B. When B is a (quasi) Banach algebra, using the Taylor expansion of $\exp(it\psi)$, one can easily check that every real-valued function for $\psi\in B$ defines a multiplier. The space of bounded variation functions or the space of Hölder functions with a given exponent provide standard examples for B. However, providing interesting examples of multipliers for more general classes of functions (such as, for example, Sobolev and Besov spaces) can be tricky. The second step is to consider perturbations of the transfer operator of the form

$$\Phi_t = \Phi\circ M_{e^{it\psi}}$$

and see how its leading eigenvalue γ_t moves analytically with t along this family (note that $\gamma_0 = 1$). Using the fact that Φ is the transfer operator and $\rho\, dm$ is invariant, we get

$$\begin{aligned}\int e^{it\frac{1}{\sigma\sqrt{n}}\sum_{i=0}^{n-1}\psi\circ T^i}\,\rho\, dm &= \int \Phi^n(e^{it\frac{1}{\sigma\sqrt{n}}\sum_{i=0}^{n-1}\psi\circ T^i}\,\rho)\, dm\\ &= \int \Phi^n_{\frac{t}{\sqrt{n}}}(\rho)\, dm.\end{aligned}$$

Since Φ_t are also quasicompact, one can obtain a decomposition of Φ_t similar to (8.9) and, after a few manipulations, obtain the CLT.

See Baladi [4] and Broise [13] for superb presentations of transfer operators for piecewise C^2 expanding maps on an interval. More recently Goüzel [33] gave us a very general application of this method that allows us just to worry mainly with step one.

8.5. **How regular must functions on B be?** Once you get an ergodic measure μ for a map T, we can use Birkhoff's ergodic theorem to get a "strong law of large numbers" for all observables $\psi \in L^1(\mu)$. One need to stress that is, even in optimal regularity assumptions, as when T is an C^∞ expanding map on the circle, not all observables in $L^p(\mu)$, for some $p \geq 1$ and $C^0(\mathbb{S}^1)$, have exponential decay of correlations and satisfy CLT.

This is indeed related with the bad spectral properties of the transfer operator acting on $L^p(m)$. Collet and Isola [22] showed that for a transfer operator of Markov expanding map acting in an interval the essential spectral radius and the spectral radius of the transfer operator in C^0 coincide, so in particular we do not have quasi-compactness.

8.6. **Short history and a question.** There is a long list of piecewise expanding maps for which one can find a Banach space of functions on which the action of the transfer operator is quasi-compact. The most classical examples are subshifts of finite type with a Hölder potential. In this case the a space of Hölder functions fits the bill. Parry and Pollicott [58] is a good reference on this development. This is an important example since the discovery that Markov partitions are ubiquos for hyperbolic dynamical systems by Sinai [74] allows the use of such result for a wide variety of dynamics (See Bowen [10]).

However, dealing with non-Markovian expanding maps was out of reach until the pioneering work of Lasota and Yorke [43] for piecewise C^2 expanding maps. Hofbauer and Keller [38] generalized this to piecewise expanding maps with bounded variation derivatives and also obtained many statistical properties of bounded variation observables. See also Rychlik [70] for maps with infinitely many branches. Keller [41] obtained similar results for piecewise $C^{1+\alpha}$ one-dimensional expanding maps, introducing a new class of function spaces. Results for certain C^2 expanding maps on $\mathbb{R}^n$ was obtained by Góra and Boyarsky [32], for bounded variation observables, and Saussol [71] generalized [43] to the same setting, using a function space similar to that used by Keller [41]. Cowieson [25] obtained a result for *generic* piecewise expanding maps on $\mathbb{R}^n$.

We can ask for wider class of observables for which one can get the quasi-compactness of the transfer operator of piecewise expanding maps. Thomine [80] (see also Baladi [5]) obtained quasi-compactness for certain piecewise expanding maps of $\mathbb{R}^n$ acting on certain Sobolev spaces. Nakano and Sakamoto [53] have results for smooth expanding maps on compact manifolds acting on Besov spaces. These results are limited to dynamical systems acting on manifolds, using techniques of Fourier analysis.

However there are many interesting examples of piecewise dynamical systems acting on a wider class of phase spaces, such as symbolic spaces, fractals (as for instance Julia sets), real trees (geodesic metric spaces in which all triangles are tripods), etc, for which classical harmonic analysis is often not available.

We ask if there is a minimal structure that all these phase spaces share in such way that we can develop a rich and unified theory for transfer operators for piecewise expanding maps?

Arbieto and S. [2] and S. [75] established the quasi-compactness of the transfer operators associated with piecewise expanding maps when the phase space exhibits a simple structure: a probability space equipped with a sequence of finite partitions, referred to as a *grid*. This framework allows the application of transfer operator methods to a broad range of piecewise expanding maps, enabling the derivation of statistical properties for a wide class of observables belonging to *Besov spaces* with fractional regularity. In the following sections, we provide an overview of these results.

9. MEASURE SPACES WITH GOOD GRIDS

A structure that is shared for all these phase spaces is that they are *measure spaces with a good grid*, introduced in S. [77]. Let X be a measure space with a finite measure m. For a measurable subset A we denote $|A| = m(A)$. We say that a *good grid* on X is a sequence of finite partitions $\mathscr{P}^n$, $n = 0,1,2,3,\ldots$, formed by measurable sets such that $\mathscr{P}^{n+1}$ is finer than $\mathscr{P}^n$, and there are $\lambda_1, \lambda_2 \in (0,1)$ satisfying

$$\lambda_1 \leq \frac{|Q|}{|P|} \leq \lambda_2,$$

for every $Q \in \mathscr{P}^{n+1}$, $P \in \mathscr{P}^n$ satisfying $Q \subset P$.

Measure spaces with good grids are a fairly general structure. One can find them in many settings, as compact manifolds, real trees, homogeneous spaces (a measure space with a doubling measure), symbolic spaces and fractals.

The simplest example is the unit interval endowed with the sequence of dyadic partitions

$$\mathscr{D}^k = \{[\frac{j}{2^k}, \frac{j+1}{2^k}],\ j < 2^k\},$$

and the Lebesgue measure. This is an importante example and the most used grid to study piecewise expanding maps acting on the interval $[0,1]$.

Another simple example is a smooth compact manifold M with a volume form and a smooth triangulation. We can choose succesive refinements of this triangulation to generate a good grid. This good grid can be used to study transfer operator of piecewise expanding maps acting on M.

For Markovian maps it sometimes convenient to choose good grids whose partitions are Markov partitions. For a full shift endowed with a Gibbs measure of a given potential, it is interesting to use the good grid generated by the cylinders in the symbolic space.

There is a simple and general way to construct nice good grids for a compact doubling space, and in particular for an Alhfors regular compact metric space, using a quite famous result by Christ [21]. That can be used to obtain good grids on self-similar conformal fractals as Julia sets of hyperbolic polynomials, for instance. If such a fractal has dimension d, then its d-dimensional Hausdorff measure is Ahlfors regular. Another approach in the same setting is to use Markov partitions.

10. BESOV SPACES

Let X be a measure space with a finite measure m and a good grid $\mathscr{P} = (\mathscr{P}^n)_n$. The good grid structure is enough to study transfer operators because this quite weak structure allows us to do a lot of harmonic analysis on it, such as to define Besov spaces, and in particular Sobolev spaces, provided we keep ourselves in the low, positive regularity setting (that is, we work with function spaces rather than spaces of distributions). Let

$s > 0$ and $p \in [1,\infty)$ be such that $s < 1/p$. A *canonical* (s,p)*-Souza's atom* with support in $Q \in \cup_n \mathscr{P}^n$ is the function $a_Q \colon X \to \mathbb{R}$ given by

$$a_Q(x) = \begin{cases} |Q|^{s-1/p}, & if\ x \in Q, \\ 0, & if\ x \notin Q. \end{cases}$$

We must think Souza's atoms as simple *wavelets*. Usually to define wavelets we need a phase space on which one can talk about scalings and translations of functions (as in $\mathbb{R}^n$). We have a mother wave and we obtain an orthogonal basis to $L^2(m)$ considering scalings and translations of the mother wave. In an abstract measure space with a good grid we do not have such homogeneous structure. However the grid gives us a way to define distinct scales, and we can view a_Q and a_J, with Q, J in the same partition $\mathscr{P}^n$, as *roughly* translated versions of each other.

Let $q \in [1,\infty]$. A function $\psi \in L^p(\mu)$ belongs to the *Besov space* $\mathscr{B}^s_{p,q}$ if there are $c_Q \in \mathbb{C}$ such that

$$\psi = \sum_{n=0}^{\infty} \sum_{Q \in \mathscr{P}^n} c_Q a_Q, \tag{10.11}$$

and with *finite* (s,p,q)*-cost*

$$\Big(\sum_{n=0}^{\infty} \Big(\sum_{Q \in \mathscr{P}^n} |c_Q|^p \Big)^{q/p} \Big)^{1/q} < \infty. \tag{10.12}$$

Note that the (s,p,p)-cost has a much simpler expression. The $(s,1,1)$-cost is even simpler

$$\sum_{n=0}^{\infty} \sum_{Q \in \mathscr{P}^n} |c_Q| < \infty.$$

A representation of ψ as in (10.11) is called an *atomic representation* of ψ by (s,p)-Souza's atoms. If it exists, it is *not* unique. (10.12) gives the (s,p,q)-cost of this representation. The norm $|\cdot|_{\mathscr{B}^s_{p,q}}$ on $\mathscr{B}^s_{p,q}$ is defined as the infimum of the cost of all possible such representations. We have

Proposition 10.13 (S. [77])**.** *There is* $C_1 \in (0,1)$ *such that*

$$C_1 \le ||a_P||_{\mathscr{B}^s_{1,1}} \le 1$$

for every $P \in \mathscr{P}$.

It turn out that

Theorem A (S. [77])**.** *Let* $0 < s < 1/p$, $p \in [1,\infty)$ *and* $q \in [1,\infty]$. *We have that* $(\mathscr{B}^s_{p,q}, |\cdot|_{\mathscr{B}^s_{p,q}})$ *is a Banach space. Moreover* $\mathscr{B}^s_{p,q} \subset L^1(m)$ *and it is dense in* $L^1(m)$*, the inclusion is continuous and the unit ball of* $\mathscr{B}^s_{p,q}$ *is compact in* $L^1(m)$.

We call those spaces of functions Besov spaces since they are generalisations of Besov spaces in more classical settings. For instance if we choose the sequence of dyadic partitions on $[0,1]^d$ endowed with the Lebesgue measure we obtain the classical Besov space in this setting. For a general compact homogeneous space we can consider a sequence of partitions as defined by Christ [21], and using these partitions the corresponding Besov spaces coincide with classical Besov spaces as defined by Han, Lu and Yang [36]. See S. [76] for details.

We stress that there are many alternative ways to define such Besov spaces. See Peetre [59], S. [77] and the references therein. The definition using Souza's atoms is the most elementary, low-level way to do it.

The main idea of a measure space with grids and Besov spaces in this setting is *pixelization*. We approximate functions by very simple functions that are constant on elements of $\mathscr{P}^k$.

Besov spaces $\mathscr{B}^s_{p,q}$, with $s \in (0,1/p)$, $p \in [1,\infty)$ and $q \in [1,\infty]$ contain very general functions. For instance consider the cube $[0,1]^D$ with the sequence of dyadic partitions and the Lebesgue measure m. Then βD-Hölder functions belong to $\mathscr{B}^s_{1,1}$ provided $0 < s < \beta < 1$. For the case $D = 1$, functions with β-bounded variation ($\beta > 1$) belong to $\mathscr{B}^s_{1,1}$ with $0 < s < 1/\beta$ as well functions with singularities of the form $|x|^{-\gamma}$, with $\gamma \in (0,1)$ sufficiently small. See S. [77] for details.

Using measure spaces with grids, multiresolution analysis, and Besov spaces opens up exciting possibilities for studying dynamical systems, especially those with irregular behavior . Grids make it easy to break the phase space into smaller, manageable pieces, which is particularly helpful for systems with sharp transitions, fragmented phase spaces, or piecewise dynamics. This localized approach not only enhances theoretical insights but also aligns naturally with elementary numerical methods, like finite element methods or wavelet-based approximations, which simplify implementation and computational analysis. When combined with the flexibility of Besov spaces —designed to handle oscillations and irregularities— and associated multiresolution analysis, these tools create a robust framework for tackling problems that traditional methods often find challenging.

10.1. **On the parameters s, p, q.** It's easy to feel overwhelmed by the number of parameters in the definition of Besov spaces. To help make sense of them, let's take a closer look at the classical Besov spaces $B^s_{p,q}$ on $[0,1]$ to try to understand what each parameter really means. Each of the three parameters — s, p, and q — captures a different aspect of how the functions in the space behave:

The **smoothness** parameter $s \in \mathbb{R}$ quantifies the degree of regularity, extending the classical notion of differentiability to fractional orders. Notably, the space of s-Hölder continuous functions $C^s([0,1])$, $s \in (0,1)$, coincides with the Besov space $B^s_{\infty,\infty}$.

The line $s = 1/p$ is known as the critical line. For $s - 1/p > 0$, the embedding $B^s_{p,q} \subset C^{s-1/p}$ holds, implying that such Besov spaces consist of continuous functions. However, in order to allow discontinuities—essential, for example, when studying the quasi-compactness of transfer operators associated with piecewise expanding maps—it is necessary to consider the regime $s < 1/p$. Piecewise s-Hölder functions belong to $B^\beta_{p,q}$ for $\beta < s$.

The **integrability** parameter $p \in [1,\infty]$ governs the L^p integrability of the function or its derivatives. For $0 < s < 1/p$ we have $B^s_{p,q} \subset L^t$, where

$$\frac{1}{t} = \frac{1}{p} - s.$$

In particular $B^s_{p,q} \subset L^p$. This inclusion also holds for our Besov spaces $\mathscr{B}^s_{p,q}$.

The **summability** parameter $q \in [1,\infty]$ is perhaps the least intuitive parameter. It tells us how the function's smoothness is distributed across different frequency scales. in multiscale decompositions (e.g., wavelets or Littlewood-Paley blocks).

Another noteworthy example is the space $B^s_{2,2}$, which coincides with the Sobolev space H^s.

Together, the triple (s,p,q) allows for a refined classification of function spaces, interpolating between Sobolev, Hölder, and other classical spaces.

10.2. **On the Origins and Development of Besov Spaces.** Besov spaces were originally introduced by Oleg Besov in the 1960s [9] and have since become a central object of study in modern analysis. One of the most appealing features of the Besov scale is its inclusiveness: many classical spaces, such as Sobolev and Hölder spaces, emerge as particular cases within this broader framework. Another remarkable aspect is the rich variety of equivalent definitions for $B^s_{p,q}(\mathbb{R}^n)$—ranging from Fourier-analytic and interpolation-based to atomic and wavelet constructions.

For foundational treatments of Besov spaces on $\mathbb{R}^n$, we recommend the classical texts by Stein [78], Peetre [59], and Triebel [81]. Readers interested in the historical development and broader context will find valuable insights in Triebel's comprehensive account [81], as well as informative surveys by Jaffard [40], Yuan, Sickel, and Yang [88], and Besov and Kalyabin [8].

11. EXISTENCE AND STRUCTURE OF INVARIANT ABSOLUTELY CONTINUOUS MEASURES

Consider the following cases of dynamical systems where irregularities arise.

C1. Let $T\colon \cup_{r\in\Lambda} I_r \to I$ be a piecewise $C^{1+\beta}$ expanding map, where I, I_r are intervals. Let $\mathscr{P}$ be the sequence of dyadic partitions on I and $g_r(x) = |Dh_r(x)|$. Let m be the Lebesgue measure.

C2. Let $T\colon \cup_{r\in\Lambda} I_r \to I$ be a piecewise $C^{1+BV_{1/\beta}}$ expanding map, where I, I_r are intervals. Here $BV_{1/\beta}$ denote the space of bounded $1/\beta$ variation. Let $\mathscr{P}$ be the sequence of dyadic partitions on I and $g_r(x) = |Dh_r(x)|$. Let m be the Lebesgue measure.

C3. Let $T\colon \cup_{r\in\Lambda} I_r \to I$ be a Lorenz map with non-flat singularities, where I, I_r are intervals. Let $\mathscr{P}$ be the sequence of dyadic partitions on I and $g_r(x) = |Dh_r(x)|$. Let m be the Lebesgue measure.

C4. Let $T\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic rational map acting on its Julia set $J(T)$. Let $\mathscr{P}^n$ be the a sequence of Markov partitions of $J(T)$ for T^n and $g_r(x) = |Dh_r(x)|^d$, where d is the Hausdoff dimension of $J(T)$. Let m be the d-dimensional Hausdoff measure on $J(T)$.

C5. Let $\sigma\colon \mathscr{A}^{\mathbb{N}} \to \mathscr{A}^{\mathbb{N}}$ be the unilateral shift with some finite alphabet $\mathscr{A}$. Let $\mathscr{P}$ be the a sequence of partitions by cylinders. Let ϕ is β-Hölder function with zero topological pressure and $g_r(x) = e^{\phi(h_r(x))}$. Let m be an eigenmeasure of the transfer operator associated to the potential ϕ.

C6. Let I be a compact Riemannian manifold. Let $T\colon \bigcup_{r\in\Lambda} I_r \to I$ be a $C^{1+\beta}$ piecewise expanding map, defined on a suitable residual set, such that the complexity of the partition $\mathscr{P}^n$ induced by T^n does not grow too rapidly (see Cowieson [24] and S. [75] for details) and $g_r(x) = (\mathrm{Jac}\, Df(h_r(x)))^{-1}$. Let m be the measure generated by a volume form on I.

Theorem B (Lasota-Yorke Inequality and Quasi-compactness-Arbieto and S. [2] and S. [75])**.** *On the assumption of each one of the cases C1-C6 there are* $0 < s < 1/p$*, and* $p \in [1,\infty)$*,* $q \in [1,\infty)$ *such that pair* $(\mathscr{B}^s_{p,q}, L^1(m))$ *satisfies the Lasota-Yorke inequality. In particular* Φ *acts as a quasi-compact operator on* $\mathscr{B}^s_{p,q}$

Theorem C (Arbieto and S. [2] and S. [75])**.** *On each one of the cases C1-C6 there are* $0 < s < 1/p$, *and* $p \in [1,\infty)$, $q \in [1,\infty)$ *such that* T *has an invariant probability that absolutely continuous with respect to m whose density* ρ *belongs to* $\mathscr{B}^s_{p,q}$, *and*

$$S = \{x \in I:\ \rho(x) > 0\}$$

is, up to a set of zero m-measure, a union of elements of $\mathscr{P}$. *Indeed, one can write*

$$\rho = \sum_{n=0}^{\infty} \sum_{Q\in\mathscr{P}^n} c_Q(\rho) a_Q,$$

where $c_Q(\rho) \geq 0$ *for every* Q *and*

$$\Big(\sum_{n=0}^{\infty} \Big(\sum_{Q\in\mathscr{P}^n} |c_Q(\rho)|^p\Big)^{q/p}\Big)^{1/q} < \infty.$$

Theorem C generalizes the known result for C^2 piecewise expanding maps on an interval, namely that their physical measure is supported on a union of intervals. In these case the density is a bounded variation function, so its support is a countable union of intervals and in certain cases a *finite* union of intervals. See Boyarsky and Góra [12].

The choice of parameters s, p, q in Theorems B and C depends on the regularity of the map T. For instance, for $C^{1+\beta}$ expanding maps on the circle, one can choose $s \in (0,\beta)$ and arbitrary parameters $p \in [1,\infty)$ and $q \in [1,\infty)$. However, when T is non-Markovian, we usually choose $p = q = 1$.

Moreover, if Theorems B and C hold for some choice of parameters s, p, q associated to a given map T, then one can replace s by any $\hat{s} \in (0, s)$ and obtain analogous results for the same map T.

There are many other examples to which our methods can be applied. See Arbieto and S. [2] and S. [75] for more applications.

12. Why the transfer operator is quasi-compact on $\mathscr{B}^s_{1,1}$

In [75], we present a broad range of examples of piecewise expanding maps and potentials for which the transfer operator satisfies the Lasota–Yorke inequality and acts as a quasi-compact operator on the Besov space $\mathscr{B}^s_{p,q}$, provided that s, p, and q are chosen appropriately. This includes the results stated in Theorem B. We now briefly outline how these results are proven.

To keep this survey as straightforward as possible, we aim to outline the proof in the specific case where the partition I_r is *finite* and $p = q = 1$. This case encapsulates the core ideas of the argument.

12.1. On the Action of the Transfer Operator.

The action of the transfer operator Φ on a function $\psi \in \mathscr{B}^s_{1,1}$ is intricate and can be written as

$$\Phi(\psi)(x) = \sum_r g_r(x)\psi(h_r(x)) \cdot 1_{I_r}(h_r(x)).$$

To analyze this action, we will decompose it into a sum of terms involving *composition operators* and *pointwise multipliers*. Here we explain why Φ is a *bounded operator* on $\mathscr{B}^s_{1,1}$. In the next section we explain why Φ is quasi-compact.

Restriction. Given $\psi \in \mathscr{B}^s_{1,1}$, we must *restrict* ψ to each domain I_r in the partition $\{I_r\}_r$. This is achieved via the pointwise multipliers

$$\psi \mapsto M_{1_{I_r}}(\psi) = \psi \cdot 1_{I_r}.$$

Thus, we must ensure that $M_{1_{I_r}}$ is a bounded operator on $\mathcal{B}^s_{1,1}$. This is not guaranteed if the domain I_r is too irregular. For this reason, we require that I_r be a *strongly regular domain* (see Section 13).

We refer to the decomposition of ψ into functions supported within each I_r as *dynamical slicing (Step II)*. To carry this out, we first decompose ψ into a combination of atoms with *small* support and atoms with *large* support. Atoms with small support intersect fewer elements of the partition, allowing us to obtain better estimates for their contributions during the dynamical slicing process.

Composition with the Inverse Branches. Once the support of $\psi \cdot 1_{I_r}$ is contained in I_r, we can compose it with h_r:

$$\psi \cdot 1_{I_r} \mapsto (\psi \cdot 1_{I_r}) \circ h_r.$$

This composition operator must also be bounded. In particular, if P is an element of the grid inside I_r, then

$$1_P \circ h_r = 1_{h_r^{-1}(P)}$$

must belong to $\mathcal{B}^s_{1,1}$. This requirement means that h_r^{-1} must not *distort* the "geometry" of grid elements too severely. Specifically, we must assume that $h_r^{-1}(P)$ is a *regular domain* (see Section 14.1).

Thanks to the atomic decomposition of functions in $\mathcal{B}^s_{1,1}$ into sums of atoms, this regularity condition is sufficient to guarantee the boundedness of the composition operator.

Multiplication by the Jacobian as a Pointwise Multiplier. The final step is to multiply by the Jacobian of h_r

$$1_P \circ h_r = 1_{h_r^{-1}(P)} \mapsto M_{g_r}(1_P \circ h_r) = g_r(x)\psi(h_r(x)) \cdot 1_{I_r}(h_r(x)).$$

Thus, we need to show that M_{g_r} is a bounded operator. If g_r is constant, this is straightforward. We will see that in many relevant cases—such as when g_r is Hölder continuous—this operator acts as a bounded multiplier. See Section 14.4 for details.

12.2. **Strategy to show quasi-compactness.** Note that T^n has also a similar structure with a dynamical partition $\{I^n_r\}_{r\in\Lambda_n}$, inverse branches

$$h^n_r : J^n_r \to I^n_r$$

and corresponding induced potential

$$g^n_r(x) = \Pi^n_{i=1} g_{r_i}(T^i(h^n_r(x)).$$

where $T^{i-1}(h^n_r(x)) \in I_{r_i}$.

The proof of the Lasota-Yorke inequality and quasi-compactness of the transfer operator Φ (Theorem B) is divided into four steps, with Step II (Dynamical Slicing) and Step III (Contraction of Atoms with Small Support) being the most crucial. These steps are discussed in detail in the following sections. Be aware that several constants appear throughout the argument—some of which depend on n, while others do not. Controlling these constants is an important part of the analysis. We will denote them by C_1, C_2, ..., but at times we will also use C to represent a generic constant, which may vary from line to line in the argument.

Step I. There are functionals $c_P \in (L^1(m))^\star$ and C_2 such that

$$\psi = \sum_k \sum_{P \in \mathscr{P}^k} c_P(\psi) a_P$$

and

$$\sum_k \sum_{P \in \mathscr{P}^k} |c_P(\psi)| \leq C_2 ||\psi||_{\mathscr{B}^s_{1,1}}$$

for every $\psi \in \mathscr{B}^s_{1,1}$.

Since the norm $||\cdot||_{\mathscr{B}^s_{1,1}}$ on $\mathscr{B}^s_{1,1}$ is defined as the infimum of the $(s,1,1)$-cost over all such representations, the inequality above shows that these functionals effectively select a *(nearly) optimal representation* of ψ. This follows from the use of a *Haar basis* in our setting and, as this is not related with the dynamical problems we are dealing here, we will skip the details. See S. [77].

Given $C_3(n) \in \mathbb{N}$ we can decompose ψ as the sum of two functions

$$\psi_{\text{small}} = \sum_{k \geq C_3(n)} \sum_{P \in \mathscr{P}^k} c_P(\psi) a_P,$$

$$\psi_{\text{large}} = \sum_{k \leq C_3(n)} \sum_{P \in \mathscr{P}^k} c_P(\psi) a_P.$$

Here, ψ_{small} denotes the contribution of atoms with "small" support in the representation of ψ, and ψ_{large} denotes the contribution of those with "large" support. Note that there are only a *finite* number of atoms with large support.

The choice of $C_3(n)$ depend on the geometry and topology of the partition $\{I^n_r\}_r$. In many cases we want to choose $C_3(n)$ so that each element of $\mathscr{P}^{C_3(n)}$ intersects a small number of elements of $\{I^n_r\}_r$. For instance, if T is an interval map such that $\{I_r\}_r$ is a finite partition on intervals then we can choose $C_3(n)$ so large that every element of $\mathscr{P}^{C_3(n)}$ intersects at most *two* intervals in $\{I_r\}_r$.

We have the natural continuous linear projections

$$\pi_{\text{small},C_3(n)}(\psi) = \psi_{\text{small}},\ \pi_{\text{large},C_3(n)}(\psi) = \psi_{\text{large}}.$$

Note that $||\pi_{\text{small},C_3(n)}|| \leq C_2$, $||\pi_{\text{large},C_3(n)}|| \leq C_2$ and C_2 *does not depend on* n and $C_3(n)$.

We can decompose Φ^n as the sum of two operators on $\mathscr{B}^s_{1,1}$

$$\Phi^n = \Phi^n \circ \pi_{\text{large},C_3(n)} + \Phi^n \circ \pi_{\text{small},C_3(n)}.$$

In particular,

$$\psi \mapsto \Phi^n(\psi_{\text{large}}) = \Phi^n(\pi_{\text{large}}(\psi))$$

is a bounded operator of *finite rank*, and hence compact. Therefore, it has no impact on the essential spectral radius of Φ^n. Moreover, since $c_P \in (L^1)^\star$, we have

$$||\Phi^n \circ \pi_{\text{large},C_3(n)}(\psi)||_{\mathscr{B}^s_{1,1}} = ||\Phi^n(\psi_{\text{large}})||_{\mathscr{B}^s_{1,1}} \leq C_4(n)||\psi||_{L^1}. \tag{12.14}$$

Indeed, the constant $C_4(n)$ may be very large, but we are not concerned with that. On the other hand, if we can choose n and $C_3(n)$ such that

$$\beta_{n,C_3(n)} := ||\Phi^n \circ \pi_{\text{small},C_3(n)}|| < 1, \tag{12.15}$$

then, by the characterization of quasi-compactness (8.8), it follows that Φ is quasi-compact.

Additionally, using (12.14), we obtain the Lasota-Yorke inequality for Φ^n:

$$||\Phi^n(\psi)||_{\mathscr{B}^s_{1,1}} \leq C_4(n)||\psi||_{L^1} + \beta_{n,C_3(n)}||\psi||_{\mathscr{B}^s_{1,1}}.$$

Furthermore, if one can show that there exist constants $C_6 > 0$ and $\hat{\beta} \in (0,1)$ such that, for every n, there exists $C_3(n)$ satisfying

$$\beta_{n,C_3(n)} \leq C_6 \hat{\beta}^n, \tag{12.16}$$

then Nussbaum's formula (8.7) implies that

$$r_{\text{ess}}(\Phi, \mathcal{B}^s_{1,1}) \leq \hat{\beta}.$$

Thus, the remaining task is to establish the bound in (12.16). This will be carried out in Steps II, III, and IV described below.

Step II (Dynamical slicing). If the *geometry and topology* of the partition $\{I^n_r\}_r$ is regular enough then we can decompose

$$\psi_{\text{small}} = \sum_r \psi_{\text{small}} 1_{I^n_r},$$

with

$$\psi_{\text{small}} 1_{I^n_r} = \sum_k \sum_{\substack{Q \in \mathcal{P}^k \\ Q \subset I^n_r}} c^r_Q a_Q, \tag{12.17}$$

such that

$$\sum_r \sum_k \sum_{\substack{Q \in \mathcal{P}^k \\ Q \subset I^n_r}} |c^r_Q| \leq C_5(n) ||\psi||_{\mathcal{B}^s_{1,1}}$$

There is the issue of controlling the (possible) growth of $C_5(n)$ with respect to n. In the simplest cases—such as interval maps with a finite partition or Markovian maps—we indeed have

$$\sup_n C_5(n) < \infty. \tag{12.18}$$

In some cases, the situation can be more subtle. There exist piecewise expanding maps on $\mathbb{R}^n$ that do not satisfy (12.18), as the topology of the partition $\{I^n_r\}_n$ can become very complicated as n increases. However, generic maps of this kind do satisfy the condition.

We will show how Step II is carried out with precise control over the geometry and topology of the partitions $\{I^n_r\}_r$.

Step III (Contraction of atoms with small support). The representation (12.17) is convenient since the support P of every atom a_P in this representation *is inside some* I^n_r, so they do not *see* the discontinuities of T^n. So this step depends more on the regularity of every branch and potential on T^n. Note that

$$\Phi^n(a_P) = g^n_r a_P \circ h^n_r = |P|^{s-1} g^n_r 1_{T^n P},$$

where $h^n_r \colon J_r \to I_r$ is an inverse branch of T^n and g^n_r is the corresponding induced potential. We then show that there is $\beta \in (0,1)$ and C_6 such that

$$||\Phi^n(a_P)||_{\mathcal{B}^s_{1,1}} \leq C_6 \beta^n ||a_P||_{\mathcal{B}^s_{1,1}},$$

for every $P \in \mathcal{P}$ with $P \subset I^n_r$.

Step IV. Now we can put all together

$$\begin{aligned}||\Phi^n(\psi)||_{\mathscr{B}^s_{1,1}} &\leq ||\Phi^n(\psi_{\text{large}})||_{\mathscr{B}^s_{1,1}} + ||\Phi^n(\psi_{\text{small}})||_{\mathscr{B}^s_{1,1}} \\ &\leq C_4(n)||\psi||_{L^1} + ||\Phi^n(\psi_{\text{small}})||_{\mathscr{B}^s_{1,1}} \\ &\leq C_4(n)||\psi||_{L^1} + \sum_r \sum_k \sum_{\substack{Q\in\mathscr{P}^k \\ Q\subset I^n_r}} |c^r_Q| ||\Phi^n(a_Q)||_{\mathscr{B}^s_{1,1}} \\ &\leq C_4(n)||\psi||_{L^1} + C_6\beta^n \sum_r \sum_k \sum_{\substack{Q\in\mathscr{P}^k \\ Q\subset I^n_r}} |c^r_Q| ||a_Q||_{\mathscr{B}^s_{1,1}} \\ &\leq C_4(n)||\psi||_{L^1} + C_6 C_5(n)\beta^n ||\psi||_{\mathscr{B}^s_{1,1}}.\end{aligned}$$

Provided we can show that

$$C_5(n)\,\beta^n < C\,\hat{\beta}^n$$

for some $\hat{\beta} \in (0,1)$, we can then apply a classical argument to eliminate the dependence of the constants $C_4(n)$ on n and obtain the Lasota–Yorke inequality (see for instance Viana [84, Proof of Proposition 3.1]).

13. Nice partitions and Dynamical Slicing (Step II)

Here we give more details on Step III. A crucial step in proving the quasi-compactness of the transfer operator Φ on the Besov space $\mathscr{B}^s_{1,1}$ is to understand how Φ acts on functions whose support is not confined to a single partition element of $\{I_r\}_r$. As discussed previously, this action involves restricting a function ψ to the elements of a dynamical partition $\{I^n_r\}_r$ and composing the resulting pieces with inverse branches h_r of the dynamics. For this process to define a bounded operator on $\mathscr{B}^s_{1,1}$, the domains I^n_r must have a sufficiently regular geometric structure.

This section introduces the concept of *strongly regular domains*, which provides a quantitative framework to control the complexity of the partition elements. These domains allow one to decompose any function ψ supported in a grid element Q into a sum of atoms supported inside individual I^n_r's—this is the core of the *dynamical slicing* argument described in Step II.

The geometry of the partition $\{I^n_r\}_r$ is an important information. If the "boundary" of I^n_r is too complicate then it is unlikely that the transfer operator will preserve a nice space of functions as $\mathscr{B}^s_{1,1}$. Indeed the first step to apply the transfer operator Φ to a function ψ is to consider the restriction of ψ to each element of this partition, that is, to consider the map

$$M_{1_{I^n_r}}(\psi) = \psi 1_{I^n_r}$$

for each $r \in \Lambda$. Since the support of $M_{1_{I^n_r}}(\psi)$ is inside I^n_r, it will be easier to deal with the composition $M_{1_{I^n_r}}(\psi)\circ h_r$. So if we want Φ to act as a continuous operator on $\mathscr{B}^s_{1,1}$ we want that

$$M_{1_{I^n_r}} : \mathscr{B}^s_{1,1} \to \mathscr{B}^s_{1,1}$$

to be well-defined and bounded, that is, the multiplication by 1_{I_r} is a pointwise multiplier if the space $\mathscr{B}^s_{1,1}$.

We can express this "nice" geometry purely in terms of the structure of the good grids.

Definition 13.19. A measurable set $\Omega \subset I$ is a (α, C_7, C_3)-**strongly regular domain** if for every $Q \in \mathscr{P}^j$, with $j \geq C_3$, there is a family $\mathscr{F}^k(Q\cap\Omega) \subset \mathscr{P}^k$ such that

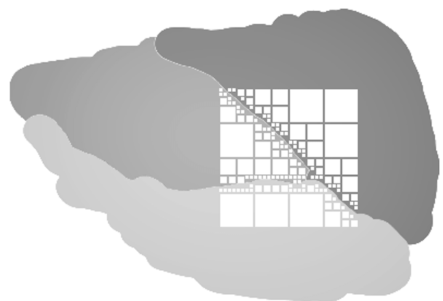

FIGURE 2. Dynamical slicing: Each atom a_Q has a support Q (represented by the large white square) that may not be entirely contained within a single element of the partition $\{I_r\}_r$ (each element of the partition is represented by a shaded region). Therefore, we must decompose it into smaller atoms, each of which has its support entirely contained within a single partition element (their supports are represented by the smaller squares).

i. We have $Q \cap \Omega = \cup_k \cup_{P \in \mathscr{F}^k(Q\cap\Omega)} P$.
ii. If $P, W \in \cup_k \mathscr{F}^k(Q \cap \Omega)$ and $P \neq W$ then $P \cap W = \emptyset$.
iii. We have

$$\sum_{P \in \mathscr{F}^k(Q\cap\Omega)} |P|^\alpha \leq C_7 |Q|^\alpha. \tag{13.20}$$

Remark 13.21. The simplest examples of strongly regular domains are intervals. If we endows $I = [0,1]$ with the dyadic grid and the Lebesgue measure them every interval $[a,b] \subset I$ is a $(0,2,0)$-strongly regular domain.

Proposition 13.22 (Dynamical Slicing-see Arbieto and S. [2])**.** *Given (α, C_7, C_3), with $\alpha < 1-s$, there is C_8 such that the following holds: If $\{I_r\}_r$ is a partition of I such that*

- *Every set I_r is a (α, C_7, C_3)-strongly regular domain,*
- *There is K such that for every $Q \in \mathscr{P}^{C_3}$ we have*

$$\#\{r:\ Q \cap I_r \neq \emptyset\} \leq K, \tag{13.23}$$

then for every representation of ψ

$$\psi = \sum_{k \geq C_3} \sum_{Q \in \mathscr{P}^k} c_Q a_Q,$$

where a_Q is an atom on Q, there are representations

$$\psi 1_{I_r} = \sum_{k \geq C_3} \sum_{Q \in \mathscr{P}^k} c_Q^r a_Q,$$

such that

$$\sum_r \sum_k \sum_{Q \in \mathscr{P}^k} |c_Q^r| \leq C_8 K \sum_k \sum_{Q \in \mathscr{P}^k} |c_Q^r|,$$

and moreover $c_Q^r \neq 0$ implies $Q \subset I_r$. In particular

$$\sum_r ||\psi 1_{I_r}||_{\mathscr{B}^s_{1,1}} \leq C_8 K ||\psi||_{\mathscr{B}^s_{1,1}}.$$

Figure 2 illustrates this proposition with a function ψ consisting of a single atom a_Q.

If we restrict ourselves to the setting of an Ahlfors-regular metric space, there is a straightforward way to construct strongly regular domains. A metric space X, equipped with a Borel measure m, is called *Ahlfors-regular* if there exist constants $r_0 > 0$, $d \geq 0$, and $C > 1$ such that

$$\frac{1}{C} r^d \leq m(B(x,r)) \leq C r^d$$

for all $0 < r \leq r_0$. That definition is slightly more general than the usual one that takes m to be the d-dimensional Hausdorff measure m_d. However, note that m and m_d are absolutely continuous with respect to each other, and their corresponding densities are bounded.

The *dimension* of X is defined as d. For instance, $\mathbb{R}^d$, equipped with the Lebesgue measure, is the simplest example, but there are many others, such as compact manifolds with a volume form or hyperbolic Julia sets endowed with an appropriate Hausdorff measure.

An open subset Ω of an Ahlfors-regular metric space X is an example of a *strongly regular domain* if its boundary $\partial\Omega$ is *also* an Ahlfors-regular metric space, equipped with another measure m', and has dimension $d' < d$. The simplest example would be an open set $\Omega \subset \mathbb{R}^d$ where its boundary is a bi-Lipschitz image of the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$.

However, there are numerous examples where the dimension of the boundary exceeds $d-1$. One way to construct such examples is by considering domains bounded by self-similar fractals. For instance, a domain bounded by the *Rauzy fractal* (see Messaoudi [49]) serves as an example.

Partitions consisting of strongly regular domains are essential for ensuring that transfer operators preserve well-behaved spaces of functions.

14. APPLYING THE BRANCHES TO $\psi 1_{I_r^n}$ (STEP III)

14.1. **Bounded distortion of geometry (Step III).** Here we give details on Step III. To control the essential spectral radius of the transfer operator, it is not enough to understand how functions decompose along the dynamical partition (Step II); we must also control how the dynamics itself transforms geometric structures. In particular, we need to ensure that the inverse branches of T^n do not excessively *distort* the shapes of small sets in the reference grid. This requirement is crucial to guarantee that the composition operators involved in Φ^n act in a bounded way on the Besov space $\mathscr{B}^s_{1,1}$.

This step draws inspiration from the classical Whitney decomposition of open sets in Euclidean spaces (see Stein [78] and Figure 3), which allows one to cover a regular domain by a disjoint family of cubes from a dyadic grid, with precise control on sizes and overlaps. We adapt this idea to our setting by introducing the notion of *regular domains*, which can be written as a disjoint union of grid elements with a quantified control.

We then define the notion of *geometric bounded distortion*: an inverse branch T^n satisfies this property if it maps each grid element $P \subset I_r^n$ to a regular domain. This allows us to estimate the effect of composing an atom a_P with h_r—the inverse branch of T^n—in terms of the ratio between the volume of $T^n P$ and P. This control is critical in Step III, where we estimate how Φ^n acts on atoms with small support.

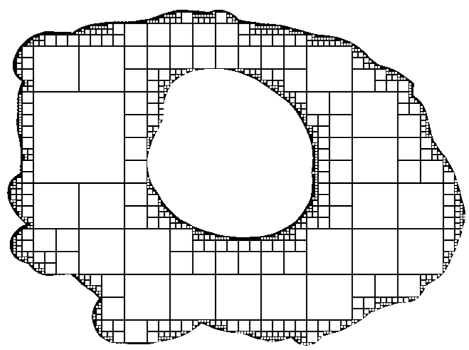

FIGURE 3. Example of a decomposition of a regular domain in elements of the grid

The control of distortion of the *shape* of sets in the grid is a given for one-dimensional maps, since intervals are mapped to intervals, however even for piecewise expanding maps on $\mathbb{R}^n$ this is not that easy to understand.

If $\Omega \subset X$ is a union of elements of the grid, we define

$$k_0(\Omega) := \min\{n\colon \text{there exists } P \in \mathscr{P}^n \text{ such that } P \subset \Omega\}.$$

Definition 14.24. A subset $\Omega \subset X$ is called a (α, C_9, λ_1)-*regular domain* if one can find families $\mathscr{F}^k(\Omega) \subset \mathscr{P}^k$, $k \geq k_0(\Omega)$ satisfying

A. We have $\Omega = \cup_{k\geq k_0(\Omega)} \cup_{Q\in\mathscr{F}^k(\Omega)} Q$.

B. For $P, Q \in \cup_{k\geq k_0(\Omega)} \mathscr{F}^k(\Omega)$ and $P \neq Q$ we have $P \cap Q = \emptyset$.

C. We have

$$\sum_{Q\in\mathscr{F}^k(\Omega)} |Q|^\alpha \leq C_9 \lambda_1^{k-k_0(\Omega)} |\Omega|^\alpha. \tag{14.25}$$

Proposition 14.26. *If Ω is a (α, C_9, λ_1)-domain and $\alpha < 1-s$ then $1_\Omega \in \mathscr{B}^s_{1,1}$,*

$$\lim_{N\to\infty} \sum_{k=k_0(\Omega)}^{N} \sum_{Q\in\mathscr{F}^k(\Omega)} 1_Q = 1_\Omega$$

on $\mathscr{B}^s_{1,1}$, and

$$\begin{aligned} |||\Omega|^{s-1} 1_\Omega||_{\mathscr{B}^s_{1,1}} &\leq |\Omega|^{s-1} \sum_k \sum_{Q\in\mathscr{F}^k(\Omega)} ||1_Q||_{\mathscr{B}^s_{1,1}} \\ &\leq |\Omega|^{s-1} \sum_k \sum_{Q\in\mathscr{F}^k(\Omega)} |Q|^{1-s} \leq C(\alpha, C_9, \lambda_1, s). \end{aligned} \tag{14.27}$$

Remark 14.28. Remember that the building blocks of functions in $\mathscr{B}^s_{1,1}$ are the Souza's atoms a_P, where $P \in \mathscr{P}$. These atoms satisfies $||a_P||_{\mathscr{B}^s_{1,1}} \sim 1$. Functions as in the left-hand side of (14.27) have a similar form, replacing P by a a regular domain, and Proposition 14.26 shows they have a similar property.

Definition 14.29. We say that a branch $T^n\colon I^n_r \to J^n_r$ has $(\alpha, C_9(n), \lambda_1)$-*geometric bounded distortion* if for every $P \in \mathscr{P}$ satisfying $P \subset I^n_r$ we have that $T^n P$ is a $(\alpha, C_9(n), \lambda_1)$-regular domain.

A simple consequence is

Proposition 14.30. *Let $\alpha < 1-s$ and suppose that $T^n\colon I^n_r \to J^n_r$ has $(\alpha, C_9(n), \lambda_1)$-geometric bounded distortion. Then there is $C_{10}(n)$ such that for every $P \in \mathscr{P}^k$, with $P \subset I^n_r$ we have*

$$||a_P \circ h_r||_{\mathscr{B}^s_{1,1}} \leq C_{10}(n)\Big(\frac{|T^n P|}{|P|}\Big)^{1-s}||a_P||_{\mathscr{B}^s_{1,1}}.$$

14.2. **Branches expand the measure m with bounded distortion.** *Up to this point,* no assumptions regarding the *expansion* of the measure have been made or used. To complete the proof of quasi-compactness, however, we now require additional assumptions on the behavior of the inverse branches of the dynamics. While the previous sections focused on geometric slicing and the distortion of the shape of grid elements, we now introduce analytic control over how the branches of T^n affect the reference measure m. Specifically, we assume that the branches expand the measure at a controlled exponential rate and with bounded distortion.

The first assumption is that each branch T^n expands the measure m exponentially. More precisely, we require the existence of constants $C_{11} > 0$ and $\beta_1 \in (0,1)$ such that for every grid element $P \subset I^n_r$, with $r \in \Lambda^n$,

$$\frac{|P|}{|T^n P|} \leq C_{11}\beta_1^n. \tag{14.31}$$

This inequality is expressed as a contraction in the ratio of volumes and plays a key role in showing that the transfer operator Φ^n acts as a contraction when applied to atoms with small support.

The second assumption guarantees that this expansion is uniformly distributed across each partition element. That is, there exists a constant $C_{12} > 1$ such that for every $J \in \mathscr{P}$ with $J \subset J^n_r$, with $r \in \Lambda^n$, we have

$$\frac{1}{C_{12}}\frac{|I^n_r|}{|J^n_r|} \leq \frac{|h_r(J)|}{|J|} \leq C_{12}\frac{|I^n_r|}{|J^n_r|}.$$

This bounded distortion condition ensures that the inverse branches do not introduce wild variations in the measure density across different regions, allowing us to estimate compositions and multiplications in the operator in a uniform way.

Together, these assumptions form the final analytic ingredient in establishing the Lasota–Yorke inequality for Φ^n and completing the proof of quasi-compactness.

14.3. **Simpler Case: Piecewise Constant Potential.** To illustrate the core ideas of Step III in a more tractable setting, we now consider the special case where the Jacobian g_r of the inverse branches h_r with respect to the reference measure m is *constant*. This simplification is not only instructive but also relevant in the study of invariant measures of maximal entropy, where such potentials naturally arise.

More precisely, we assume that for every $x \in J^n_r$ and every $P \subset I^n_r$, we have

$$g_r(x) = \frac{|P|}{|T^n P|}.$$

Under this assumption, Proposition 14.30 implies the following estimate:

$$||\Phi^n(a_P)||_{\mathscr{B}^s_{1,1}} = ||g_r \cdot a_P \circ h_r||_{\mathscr{B}^s_{1,1}} \leq C_{10}(n)\left(\frac{|P|}{|T^n P|}\right)^s ||a_P||_{\mathscr{B}^s_{1,1}}.$$

This estimate reveals a crucial mechanism: if $C_{10}(n) \leq 1$ and the branches of T^n expand the measure m, then the transfer operator Φ^n strictly *contracts* each atom a_P whose support P lies entirely within a single partition element I^n_r. This contraction mechanism is one of the fundamental reasons why the transfer operator acts as a quasi-compact operator on $\mathscr{B}^s_{1,1}$.

Unfortunately, in general, the constant $C_{10}(n)$ may grow with n, potentially undermining this contraction. However, we can still control the operator in many relevant cases. Since the branches of T expand the measure m, we know that

$$\left(\frac{|P|}{|T^k P|}\right)^s \le C_{11}^s \beta_1^{ks},$$

which decays exponentially fast with k. On the other hand, in many classical settings—such as piecewise $C^{1+\beta}$ expanding maps on intervals, conformal expanding maps on subsets of $\mathbb{C}$, or Markovian maps on more general domains—we can verify that the geometric distortion constants satisfy

$$\sup_n C_{10}(n) < \infty.$$

In such cases, we recover the desired conclusion: for sufficiently large n, the operator Φ^n contracts every atom a_P whose support P lies within some I^n_r. This completes the key estimate required for Step III in the piecewise constant potential case.

14.4. **Potential with Bounded Distortion.** We now extend the contraction estimates of Step III to the case of non-constant potentials. To handle this setting, we need to control how much the Jacobian g^n_r varies across each branch of the dynamics.

Let $P \in \mathscr{P}$ with $P \subset J^n_r$. Observe that

$$\int_P g^n_r \, dm = |h^n_r(P)|,$$

so the average value of g^n_r over P is

$$m(g^n_r, P) = \frac{|h^n_r(P)|}{|P|}.$$

In the case of constant potentials, g^n_r would coincide with this average on each P. It turns out that in various important settings—such as when g^n_r is Hölder continuous or of bounded variation (in the one-dimensional case)—this equality holds approximately (See S. [75]), and there exists a constant C_{13} such that

$$\left\| g^n_r 1_P - \frac{|h^n_r(P)|}{|P|} \cdot 1_P \right\|_{\mathscr{B}^s_{1,1}} \le C_{13} \frac{|h^n_r(P)|}{|P|^s},$$

and in particular,

$$\| g^n_r 1_P \|_{\mathscr{B}^s_{1,1}} \le (C_{13}+1) \frac{|h^n_r(P)|}{|P|^s}.$$

Consequently, if $P \subset I_r^n$, then by Proposition 14.30 we obtain:

$$
\begin{aligned}
||\Phi^n(a_P)||_{\mathscr{B}^s_{1,1}} &= \left\| g_r^n \cdot a_P \circ h_r \right\|_{\mathscr{B}^s_{1,1}} = \left\| \frac{1}{|P|^{1-s}} g_r^n \cdot 1_{T^n P} \right\|_{\mathscr{B}^s_{1,1}} \\
&\le \frac{1}{|P|^{1-s}} \sum_{k \ge k_0(T^n P)} \sum_{J \in \mathscr{F}^k(T^n P)} \left\| g_r^n \cdot 1_J \right\|_{\mathscr{B}^s_{1,1}} \\
&\le (C_{13}+1) \frac{1}{|P|^{1-s}} \sum_{k \ge k_0(T^n P)} \sum_{J \in \mathscr{F}^k(T^n P)} \frac{|h_r(J)|}{|J|^s} \\
&\le (C_{13}+1) \left(\frac{|I_r^n|}{|T^n I_r^n|} \right) \frac{1}{|P|^{1-s}} \sum_{k \ge k_0(T^n P)} \sum_{J \in \mathscr{F}^k(T^n P)} |J|^{1-s} \\
&\le (C_{13}+1) C_{10}(n) \cdot \left(\frac{|I_r^n|}{|T^n I_r^n|} \right) \left(\frac{|T^n P|}{|P|} \right)^{1-s} \\
&\le (C_{13}+1) C_{12}^{2-s} \cdot C_{10}(n) \left(\frac{|I_r^n|}{|T^n I_r^n|} \right)^s \cdot ||a_P||_{\mathscr{B}^s_{1,1}}.
\end{aligned}
$$

Here, the constant $C_{10}(n)$ captures the geometric distortion introduced by the branch T^n on the domain I_r^n. For conformal maps—including one-dimensional piecewise expanding maps—and for Markovian maps (by selecting $\mathscr{P}^k$ as the k-step Markov partition), we have

$$\sup_n C_{10}(n) < \infty,$$

and we recover exponential contraction of atoms supported inside I_r^n.

However, if T is not conformal, $C_{10}(n)$ can grow exponentially with n, and more care is needed. Let us consider the case in which T^n is a piecewise $C^{1+\beta}$ diffeomorphism on $\mathbb{R}^D$. After refining the partition $\{I_r^n\}_r$, one can show that there exists a constant C such that for every $x \in I_r^n$, letting $0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_D$ denote the singular values of $DT^n(x)$, we have

$$C_{10}(n) \le C \cdot \prod_{i \neq 1} \frac{\alpha_i^s}{\alpha_1^s},$$

as the right-hand side reflects how far $DT^n(x)$ is from being a scalar multiple of an isometry (See S. [75]). and

$$\left(\frac{|I_r^n|}{|T^n I_r^n|} \right)^s \le C \cdot \prod_i \alpha_i^{-s},$$

since $\prod_i \alpha_i = |\det\, DT^n(x)|$. Therefore,

$$C_{10}(n) \cdot \left(\frac{|I_r^n|}{|T^n I_r^n|} \right)^s \le C^2 \cdot \alpha_1^{-sD}.$$

Since T is assumed to be *metrically expanding*, there exist constants $C > 0$ and $\gamma > 1$ such that

$$\alpha_1 \ge C \cdot \gamma^n,$$

and we again obtain exponential contraction of atoms whose support lies inside I_r^n.

15. Open questions

15.1. **Natural Spaces of Functions.** Most Banach function spaces on $\mathbb{R}$ discussed in the literature have norms defined as sums of pseudo-norms that behave nicely under *scaling* and *translation*.

We say that a pseudo-norm $n(\cdot)$ (i.e., a function that satisfies the norm axioms except that it may assign zero to nonzero vectors) on a function space over an interval I is *purely natural* if there exists a constant $C>0$ such that:

Almost invariance under affine transformations: There exist $t\in\mathbb{R}$ and $C>0$ such that for any affine and invertible transformation $v\colon\mathbb{R}\to\mathbb{R}$ and any function $\phi\colon I\to\mathbb{C}$ satisfying

$$v^{-1}(\text{supp }\phi)\subset I,$$

we have $\phi\circ v\in B$ and

$$\frac{1}{C}|v'|^t n(\phi)\le n(\phi\circ v)\le C|v'|^t n(\phi).$$

We call t the *degree of homogeneity of* n.

A pseudo-norm n that is a finite sum of purely natural pseudo-norms is called *natural*. Specifically, there exist purely natural pseudo-norms n_i, with $i\le j$, and degrees of homogeneity t_i such that:

$$n(\phi)=\sum_{i\le j} n_i(\phi).$$

We can ask if the Besov spaces $\mathscr{B}^s_{p,q}$ are the largest "natural" spaces of functions with good statistical properties. By natural we mean functions space whose norm behaves nicely with respect to scalings, rotations and translations. To be more precise, lets state the question for expanding maps on the internal.

Question 15.1. *Let f be an $C^{1+\beta}$ expanding map acting on an interval I with $\beta>0$. Let $(B,|\cdot|_B)$ be a Banach space of functions on I such that*

- *B is a dense subspace of $L^1(m)$ and the unit ball of B is compact in L^1.*
- *The pair (B,L^1) satisfies the Lasota-Yorke inequality*

$$|\phi|_B\le\lambda|\phi|_B+C|\phi|_{L^1}$$

  for some $\lambda\in(0,1)$ and $C>0$.
- *The Banach space B is natural.*

One can asks if there is $s>0$, $p\in[1,\infty)$ and $q\in[1,\infty]$ with $s<1/p$ such that $B\subset B^s_{p,q}$ (this is the classical Besov space on I).

15.2. **Towers and Induced Maps.** Towers and induced maps are classical and powerful tools in dynamical systems theory. They simplify the analysis of complicated systems—such as those with critical points, singularities, or discontinuities—by reducing them to more tractable piecewise expanding maps, and in some cases, even to Markovian dynamics. Young towers [87] are tailored to Hölder continuous observables, while Baladi–Viana towers [6] are designed for observables of bounded variation. Both approaches rely on constructing a (sometimes ad hoc) dynamical partition of the (extended) phase space.

Besov spaces with fractional smoothness $s<1/p$ naturally arise in this context due to their robust handling of discontinuities.

Question 15.2. *How can classical ergodic theory tools, such as towers and induced maps, be employed to broaden the study of transfer operators acting on Besov spaces of observables?*

A promising starting point is the result by Chazottes, Collet, and Schmitt [20], which investigates the Besov regularity of invariant probability densities for unimodal maps with critical points.

15.3. **Obstruction to a Small Essential Spectral Radius.** Recently, Butterley, Canestrari, and Jain [17] demonstrated that for *non-Markovian* piecewise expanding maps on the interval, it is impossible to achieve an *arbitrarily small essential spectral radius* in function spaces B that are continuously embedded in L^∞.

Question 15.3. *Can results similar to those in* [17] *be extended to all natural function spaces $B \subset L^1$ that satisfy the Lasota-Yorke inequality? Furthermore, could Besov spaces yield the smallest essential spectral radius among non-Markovian piecewise expanding dynamical systems?*

15.4. **Computational approach.** Numerical studies on the statistical properties of dynamical systems have seen remarkable growth in recent years. Notable contributions include the works of Li [45], and Liverani [47]. The *linear response problem* has also been explored in various settings by Bahsoun, Galatolo, Nisoli, and Niu [3], Wormell and Gottwald [86], Chandramoorthy and Jézéquel [19].

The intersection of measure spaces with grid structures and Besov spaces presents a compelling framework for advancing the numerical/computing assisted multiscale analysis of transfer operators. This approach not only enriches the theoretical landscape but also paves the way for innovative computational methods and practical applications.

Besov spaces allows us to capture function regularity through wavelet decompositions, and offer a highly flexible setting for multiresolution analysis. By employing unbalanced Haar wavelets (see Girardi and Sweldens [31]), functions within Besov spaces can be efficiently decomposed using fast wavelet transform algorithms (Mallat [48]).

The synergy between grid structures in measure spaces and the sophisticated analytical tools provided by Besov spaces enables the discretization of transfer operators, even in highly irregular phases spaces and dynamics, significantly enhancing their accessibility for numerical approximation. This discretization is particularly instrumental in the computational verification of Lasota-Yorke inequalities, which are crucial for understanding the statistical properties of dynamical systems.

Building on this foundation, future research can develop robust algorithms for simulating and analyzing transfer operators, potentially uncovering new insights into the stability, mixing behavior, and statistical laws governing dynamical systems. The fusion of theoretical rigor and computational innovation in this context holds tremendous promise for advancing both mathematical theory and its practical applications.

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