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ORIENTATION RAMSEY THRESHOLDS FOR CYCLES AND CLIQUES*

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Abstract. If G is a graph and $\vec{H}$ is an oriented graph, we write $G \to \vec{H}$ to say that every orientation of the edges of G contains $\vec{H}$ as a subdigraph. We consider the case in which G is the binomial random graph $G(n,p)$, establishing the threshold $p_{\vec{H}} = p_{\vec{H}}(n)$ for the property $G(n,p) \to \vec{H}$ for the cases in which $\vec{H}$ is an acyclic orientation of a complete graph or of a cycle.

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1. A Ramsey-type property.

For each (undirected) graph G and oriented graph $\vec{H}$, we write $G \to \vec{H}$ to mean that every orientation of G contains a copy of $\vec{H}$; the *orientation Ramsey number* $\vec{R}(\vec{H})$ is $\inf\{\, n : K_n \to \vec{H}\,\}$. Note that every graph G admits an acyclic orientation, and hence $G \not\to \vec{H}$ if $\vec{H}$ contains a directed cycle; moreover, since large tournaments contain large transitive subtournaments, $\vec{R}(\vec{H}) < \infty$ if $\vec{H}$ contains no directed cycles. The orientation Ramsey number has been investigated in a number of articles [31, 32, 14, 8, 34, 23, 33, 15, 24, 25, 26, 16, 13, 21, 22, 27, 12] among others, most of which concern a conjecture of Sumner [34]. *Sumner's universal tournament conjecture* states that $\vec{R}(\vec{T}) \le 2e(\vec{T})$ for every oriented tree $\vec{T}$; this has been confirmed for all sufficiently large trees by Kühn, Mycroft, and Osthus [22, 21]; see also [1, 28].

Thresholds. Thresholds for Ramsey-type properties are widely studied as well (see, e.g., [17, 29] and the many references therein). We call $p_{\vec{H}} = p_{\vec{H}}(n)$ a *threshold* for $G(n,p) \to \vec{H}$ if

$$\lim_{n\to\infty} \mathbb{P}\big[\, G(n,p) \to \vec{H}\,\big] = \begin{cases} 0 & \text{if } p \ll p_{\vec{H}}, \\ 1 & \text{if } p \gg p_{\vec{H}}, \end{cases}$$

where $a \ll b$ (or, equivalently, $b \gg a$) means $\lim_{n\to\infty} a_n/b_n = 0$. As is customary, we speak of "the threshold $p_{\vec{H}}$," since $p_{\vec{H}}$ is unique within constant factors. If $\vec{H}$ is acyclic, then the property $G(n,p) \to \vec{H}$ is nontrivial and monotone, and hence [3] it has a threshold $p_{\vec{H}} = p_{\vec{H}}(n)$. An upper bound for $p_{\vec{H}} = p_{\vec{H}}(n)$ involving the so-called

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maximum 2-density of $\vec{H}$ is proved in [7], following methods in [30]. For any graph or oriented graph G, the *maximum density* and (when $v(G) \geq 3$) the *maximum 2-density* of G are, respectively,

$$m(G) := \max_{\substack{J \subseteq G \\ v(J) \geq 1}} \frac{e(J)}{v(J)} \qquad \text{and} \qquad m_2(G) := \max_{\substack{J \subseteq G \\ v(J) \geq 3}} \frac{e(J)-1}{v(J)-2}.$$

THEOREM 1.1 (see [7]). *Let $\vec{H}$ be an acyclically oriented graph. There exists a constant $C = C(\vec{H})$ such that if $p \geq Cn^{-1/m_2(\vec{H})}$, then $\mathbb{P}\big[G(n,p) \to \vec{H}\big] \to 1$ as $n \to \infty$.*

We remark in passing that the regularity method can be used to give an alternative proof for Theorem 1.1 (such a proof would combine ideas from [17, section 8.5] and, say, [10]).

2. Contribution.

We establish the orientation Ramsey threshold for all acyclic orientations of the complete graph K_t and cycle C_t, for each $t \geq 3$, as well as for certain oriented bipartite graphs. We call a digraph $\vec{H}$ *anti-directed* if each vertex in $\vec{H}$ has either no inneighbours or no outneighbors (so $\vec{G}$ is bipartite and all arcs point to the same part). We call a graph H *strictly* 2*-balanced* if $v(H) \geq 3$ and the only subgraph of H attaining its maximum 2-density is H itself (i.e., $(e(J)-1)/(v(J)-2) < m_2(H)$ for each proper subgraph $J \subseteq H$ with $v(J) \geq 3$).

THEOREM 2.1. *If $\vec{H}$ is an acyclic orientation of K_t or C_t, then*

$$p_{\vec{H}}(n) = \begin{cases} n^{-1/m(K_4)} & \text{if } t = 3, \\ n^{-1/m_2(\vec{H})} & \text{if } t \geq 4 \end{cases}$$

is the threshold for $G(n,p) \to \vec{H}$. Moreover, if $\vec{H}$ is an anti-directed orientation of a strictly 2-balanced graph H such that $\delta(H) \geq 2$ and $m_2(H) - \lfloor m_2(H) \rfloor \leq 1/2$, then

$$p_{\vec{H}}(n) = n^{-1/m_2(\vec{H})}$$

is the threshold for $G(n,p) \to \vec{H}$.

The fact that the case $t = 3$ is "different" from the $t \geq 4$ cases suggests that the problem of determining $p_{\vec{H}} = p_{\vec{H}}(n)$ for general $\vec{H}$ may be hard. Indeed, the $t = 3$ case can be used to produce other $\vec{H}$ for which $p_{\vec{H}} \ll n^{-1/m_2(\vec{H})}$; see section 7 for further comments in this direction. A similar phenomenon has been observed in the context of a different Ramsey type property [18, 20].

In view of Theorem 1.1, to prove Theorem 2.1 (except for the case in which $\vec{H}$ is an orientation of K_3), it suffices to prove the so called 0-statement, that is, it is enough to show that if $p \ll n^{-1/m_2(\vec{H})}$, then $G(n,p) \to \vec{H}$ holds with vanishing probability. Our proof of this 0-statement uses recent advances in the study of Ramsey-type thresholds: a framework developed by Nenadov et al. [29] (outlined below) and structural results of Barros et al. [2].

We need only a simplified version of the results in [29] (see Definitions 10 and 11 in [29]). Let G and H be graphs, where $\delta(H) > 1$. An edge $e \in E(G)$ is *H-closed* if e belongs to at least two copies of H in G. A copy of H in G is *H-closed* if at least three of its edges are H-closed, and G is *H-closed* if all vertices and edges of G lie in H-closed copies of H. Finally, G is an *H-block* if G is H-closed and for each proper nonempty subset $E' \subsetneq E(G)$ there exists a copy H' of H in G such that $E(H') \cap E' \neq \emptyset$ and $E(H') \setminus E' \neq \emptyset$.

THEOREM 2.2 (see [29, Corollary 13]). *Let H be a strictly 2-balanced graph with at least 3 edges such that H is not a matching. If $p \ll n^{-1/m_2(H)}$, then with high probability every H-block F of $G(n,p)$ satisfies $m(F) < m_2(H)$.*

Since complete graphs and cycles are strictly 2-balanced, Theorem 2.2 reduces the proof of the 0-statement in Theorem 2.1, except in the case $t = 3$, to showing that $G \not\to \vec{H}$ for every graph G whose H-blocks have maximum density strictly below $m_2(H)$. This is achieved for cycles using results from [2], whereas for tournaments and anti-directed graphs, as well as for the case $t = 3$ of Theorem 2.1 we use ad hoc methods (see Theorems 4.1, 5.1, and 5.2). Theorem 2.1 is proved in section 6.

Remark. Other Ramsey-type properties for directed graphs include requiring copies to be induced [9, 19, 4] and allowing colorings plus orientations [5, 6].

3. Auxiliary definitions and results.

We use standard notation (see, e.g., [17, 11]). A *k-path* is a path with k vertices; *k-cycles* are defined similarly. A *directed* k-path is an oriented path $v_1 \to \cdots \to v_k$. A *directed* k-cycle is oriented as $v_1 \to \cdots \to v_k \to v_1$. Let $\vec{G}$ be an oriented graph. A maximal directed path in $\vec{G}$ is called a *block*. A path or block is *long* if it has at least 3 edges. A graph G is *k-degenerate* if for all $J \subseteq G$ the minimum degree $\delta(J)$ is at most k. We use the following simple fact.

LEMMA 3.1. *Every graph G with at least one vertex is $2m(G)$-degenerate.*

Let G and H be graphs, and let $\vec{H}$ be an orientation of H. We denote by $\mathcal{C}_H(G)$ the *edge intersection graph* of H in G, whose vertices correspond to copies of H in G and whose edges join distinct copies which share a common edge in G. An *H-component* is a subgraph of G formed by the union of all copies of H in some connected component of $\mathcal{C}_H(G)$. Note that $G \not\to \vec{H}$ if and only if each H-component of G admits an $\vec{H}$-free orientation. Let G and H be graphs, and let C be an H-component of G. If H_1 is an arbitrary copy of H in C, then there exists a sequence $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_t = C$ with the following property. For each $i \in [t-1]$, there exists a copy H' of H such that $H' \nsubseteq H_i$, $E(H') \cap E(H_i) \neq \emptyset$ and $H_{i+1} = H_i \cup H'$. We say that $(H_1, \ldots, H_t)$ *constructs* C, and call $(H_1, \ldots, H_t)$ a *construction sequence* of C. For each $i \in [t-1]$, we say that a vertex or edge of H_{i+1} is *new* in H_{i+1} if it is not contained in H_i and say that $F \subseteq H_{i+1}$ is *new* (in H_{i+1}) if F contains a new edge in H_{i+1}. Moreover, if H_1' is a copy of H in H_i, then there exists a construction sequence $(H_1', \ldots, H_j')$ of H_i starting with H_1' and hence a construction sequence $(H_1', \ldots, H_j', H_{i+1}, \ldots, H_t)$ of C.

Let G be a graph, and suppose $E \subseteq E(G)$. We write $G[E]$ for the subgraph of G consisting of the edges in E and the vertices in G which are incident with those edges.

LEMMA 3.2 (see [29, Lemma 14]). *Let G and H be graphs. If G is H-closed, then $E(G)$ admits a partition $\{E_1, \ldots, E_k\}$ such that $G[E_1], \ldots, G[E_k]$ are H-blocks and each copy of H in G lies entirely in one of these H-blocks.*

Let $\vec{H}$ be an orientation of a graph H. We say $\vec{H}$ is *2-Ramsey-avoidable* if for all $e, f \in E(H)$, every orientation of e, f can be extended to an $\vec{H}$-free orientation of H.

Remark 3.3. Let $k \geq 4$. If $\vec{H}$ is either an orientation of C_k, an acyclic tournament of order k, or an anti-directed orientation of a graph H with $\delta(H) > 1$, then $\vec{H}$ is 2-Ramsey-avoidable.

Proof. Let H be the underlying graph of $\vec{H}$. In each of the following cases, let $e, f \in E(H)$ be chosen and oriented arbitrarily; it suffices to complete an $\vec{H}$-free orientation of H.

Suppose $\vec{H}$ is an orientation of C_k. Note that we can complete the orientation of e, f to orientations $\vec{C}_1, \vec{C}_2$ of C_k such that $\vec{C}_1$ has a block of length at least $k-1 \geq 3$ and $\vec{C}_2$ has no long block. If $\vec{H}$ has a block of length at least $k-1$, then we pick $\vec{C}_2$; else we pick $\vec{C}_1$.

If $\vec{H}$ is an acyclic orientation of K_k, we complete the orientation so that it contains a directed triangle (some triangle in H has at most one edge already oriented).

In the remaining case (anti-directed graph), we complete the orientation of H forming a directed 3-path (since $k \geq 4$, some $v \in V(H)$ is incident with precisely one of e, f, while $\delta(H) > 1$ implies some other edge incident with v has not been oriented). ∎

Remark 3.3 will be used with the next lemma and Theorem 2.2 to establish our main results.

LEMMA 3.4. *Let G be a graph, and let $\vec{H}$ be* 2-Ramsey-avoidable. *If $B \not\to \vec{H}$ for each H-block B of G, then $G \not\to \vec{H}$.*

Proof. Let H be the underlying graph of $\vec{H}$. To show that G admits an $\vec{H}$-free orientation, we may assume each edge of G lies in a copy of H (the orientation of other edges is irrelevant).

Let $G_0 = G$ and, for each $i = 1, 2, \dots$ proceed as follows. If G_{i-1} is H-closed, then stop, set $m := i-1$ and $F := G_m$. Otherwise, some copy F_i of H in G_{i-1} has at most two H-closed edges in G_{i-1}. Form G_i by deleting from G_{i-1} each non-H-closed edge of F_i and then each isolated vertex. Note that $G_{i-1} = G_i \cup F_i$ and that each $e \in E(G_i)$ lies in some copy of H.

Note that F is H-closed. By Lemma 3.2, F can be partitioned into a collection $\mathcal{B}$ of edge-disjoint H-blocks such that each copy of H in F lies entirely in some $B \in \mathcal{B}$. By assumption, $B \not\to \vec{H}$ for each $B \in \mathcal{B}$, so F admits an $\vec{H}$-free orientation $\vec{F}$ (the disjoint union of $\vec{H}$-free orientations of each $B \in \mathcal{B}$).

Finally, we extend $\vec{G}_m := \vec{F}$ to an $\vec{H}$-free orientation $\vec{G}_0$ of G. For each i, from m to 1, let $\vec{G}_{i-1}$ extend $\vec{G}_i$ by orienting the edges $E(F_i) \setminus E(G_i)$ so that F_i is $\vec{H}$-free (this is possible because $\vec{H}$ is 2-Ramsey-avoidable). Clearly, no copy of H in G induces $\vec{H}$ in $\vec{G}$, so $G \not\to \vec{H}$. ∎

4. Transitive triangles. Let TT_3 be the transitive triangle. In this section we show that the upper bound for $p_{\mathrm{TT}_3}(n)$ given in Theorem 1.1 is not tight.

THEOREM 4.1. *The threshold for $G(n,p) \to \mathrm{TT}_3$ is $p_{\mathrm{TT}_3}(n) = n^{-1/m(K_4)}$.*

Let W_5 be the *wheel graph* obtained from C_4 by adding a new vertex joined to all others. We prove the 0-statement of Theorem 4.1 using a structural property of K_3-components.

PROPOSITION 4.2. *If G is a K_3-component with $uw, vw \in E(G)$ and $uv \notin E(G)$, then there exists $J \subseteq G$ such that either $v(J) = 6$ and $e(J) = 9$ or $J + uv$ is isomorphic to K_4 or W_5.*

We write $F \simeq G$ to mean that the graphs F and G are isomorphic.

Proof. Let $F_1 \cdots F_s$ be a shortest path in $\mathcal{C}_{K_3}(G)$ such that $uw \in E(F_1)$ and $vw \in E(F_s)$. It suffices to show the following.

- If $s = 2$, then $J := F_1 \cup F_2$ satisfies $J + uv \simeq K_4$.
- If $s = 3$, then $J := F_1 \cup F_2 \cup F_3$ satisfies $J + uv \simeq W_5$.
- If $s \geq 4$, then $J := F_1 \cup F_2 \cup F_3 \cup F_4$ satisfies $v(J) = 6$ and $e(J) = 9$.

It is simple to check that the following hold by the choice of $F_1 \cdots F_s$.

(i) $|E(F_i) \cap E(F_{i+1})| = 1$ for all $i \in [s-1]$;

(ii) $\left|E(F_{i+1}) \cap \bigcup_{j\in[i]} E(F_j)\right| = 1$ for each $i \in [s-1]$; and

(iii) each $e \in E(G)$ belongs to at most two triangles in $F_1 \cup \cdots \cup F_s$.

The statement for $s = 2$ follows by item (i) since $F_1 \simeq K_3$. If $s = 3$, then $v(J) = 5$ (by items (i) and (ii)), so $J + uv \simeq W_5$. Again by item (i), for each $i \in [s-1]$ we have $|V(F_{i+1}) \setminus \bigcup_{j\in[i]} V(F_j)| \leq 1$, so $v(F_1 \cup \cdots \cup F_s) \leq s + 2$. Moreover, item (ii) implies $e(F_1 \cup \cdots \cup F_i) = 2i + 1$ for each $i \in [s]$. If $s \geq 4$, then $e(J) = 9$. Clearly $5 \leq v(J) \leq 6$; note that $v(J) \neq 5$ as otherwise there exists $e \in E(J)$ which belongs to three distinct triangles in J, contradicting item (iii). ∎

Let $(H_1, \ldots, H_t)$ be a K_3-component. For each $i \in [t-1]$, either (A) there are two new edges in H_{i+1} and one new vertex in H_{i+1}; (B) there are two new edges in H_{i+1} and $V(H_{i+1}) = V(H_i)$; or (C) there is exactly one new edge in H_{i+1} and $V(H_{i+1}) = V(H_i)$. A graph H is *AB-constructible* if no construction sequence of a K_3-component of H contains a step of type (C).

PROPOSITION 4.3. *If a graph H is AB-constructible, then $H \not\to \mathrm{TT}_3$.*

Proof. We may assume that H is itself a single K_3-component $(H_1, \ldots, H_t)$, as edges which do not belong to a copy of K_3 in H can be arbitrarily oriented and distinct K_3-components may be independently oriented. First note that, at each step, exactly one new copy of K_3 is added. This is clearly true for steps of type (A). Moreover, it is easy to see that if $H_{\alpha+1}$ is created by a step of type (B) and the new edges create two distinct copies of K_3 in $H_{\alpha+1}$, then H admits a construction sequence with a step of type (C), a contradiction. We orient H_1 forming a directed triangle and, for each $\alpha \in [t-1]$, orient the two new edges in $H_{\alpha+1}$ so as to form a new directed triangle. The resulting orientation is TT_3-free. ∎

Our final ingredient is the following classical result (see, e.g., [17]).

THEOREM 4.4 (see [17]). *Let H be a fixed graph. Then*

$$\lim_{n\to\infty} \mathbb{P}[H \subseteq G(n,p)] = \begin{cases} 1 & \text{if } p \gg n^{-1/m(H)}, \\ 0 & \text{if } p \ll n^{-1/m(H)}. \end{cases}$$

Proof of Theorem 4.1. If $p \gg n^{-2/3}$, then $K_4 \subseteq G(n,p)$ with high probability by Theorem 4.4; hence $G(n,p) \to \mathrm{TT}_3$ with high probability (as $K_4 \to \mathrm{TT}_3$). Now suppose that $p \ll n^{-2/3}$, and let $\mathcal{E}$ be the event "$G(n,p)$ is not AB-constructible." By Proposition 4.3, it suffices to show that $\mathbb{P}[\mathcal{E}] = \mathrm{o}(1)$. Let $\mathcal{J}$ be set of all non-isomorphic graphs of order 6 and size 9. By Proposition 4.2, every K_3-component of $G(n,p)$ which is not AB-constructible contains either K_4, W_5, or some $J \in \mathcal{J}$. Using Markov's inequality, we have

$$\begin{aligned}\mathbb{P}[\mathcal{E}] &\leq \mathbb{P}[K_4 \subseteq G(n,p)] + \mathbb{P}[W_5 \subseteq G(n,p)] + \sum_{J\in\mathcal{J}} \mathbb{P}[J \subseteq G(n,p)] \\ &\leq \sum_{J\in\{K_4,W_5\}\cup\mathcal{J}} \mathbb{E}\big[\,|\{J' \subseteq G(n,p) : J' \simeq J\}|\,\big] \;\leq\; n^4p^6 + n^5p^8 + |\mathcal{J}|n^6p^9.\end{aligned}$$

Since $p \ll n^{-2/3}$ and $|\mathcal{J}| = \Theta(1)$, we have $\mathbb{P}[\mathcal{E}] = \mathrm{o}(1)$. ∎

5. Graphs with low maximum 2-density. The following sections show that $G \not\to \vec{H}$ for some classes of oriented graphs when $\vec{H}$ has at least four vertices and $m(G) < m_2(\vec{H})$.

5.1. Transitive tournaments. We denote a tournament on k vertices by T_k, writing TT_k if it is transitive.

THEOREM 5.1. *If $k \geq 4$ and G is a graph with $m(G) < m_2(K_k)$, then $G \not\to \mathrm{TT}_k$.*

Proof. The proof is by induction on $n := v(G)$. The case $n < 4$ is trivial. Assume $n \geq 4$ and that $G' \not\to \mathrm{TT}_k$ whenever $m(G') < m_2(K_k)$ and $v(G') < n$. By Lemma 3.1, $\deg_G(u) \leq k$ for some $u \in V(G)$. Let $G' = G - u$, so $m(G') < m_2(K_k)$ and G' admits a TT_k-free orientation $\vec{G}$. We shall extend $\vec{G}$ to an orientation of G such that each T_k containing u has a directed cycle.

We may assume that u lies in some copy of K_k, say K; so $\deg(u) \geq k - 1$. If K is the only copy of K_k containing u, then choose two vertices $v, w \in V(K - u)$ and orient the edges uv and uw so that $\{u, v, w\}$ induces a directed triangle. Otherwise let K' be some K_k containing u other than K. Hence we must have $\deg(u) = k$. Let v be the unique vertex in $V(K) \setminus V(K')$, and w be the unique vertex in $V(K') \setminus V(K)$. Since $k - 2 \geq 2$, there are at least two vertices x and y in $V(K \cap K') \setminus \{u\}$. Orient the edges uv, ux, uw, and uy so that each of $\{u, v, x\}$ and $\{u, w, y\}$ induces a directed triangle. Since every K_k containing u has at least three vertices in $\{v, w, x, y\}$, the partial orientation of each K_k contains a directed cycle. Any remaining unoriented edge may be arbitrarily oriented. ∎

5.2. Anti-directed digraphs. We next consider anti-directed orientations of bipartite graphs such as $K_{t,t}$ and C_{2t}.

THEOREM 5.2. *Let G and H be graphs, where $\delta(H) \geq 2$. If $\vec{H}$ is an anti-directed orientation of H and $m(G) < \delta(H) - 1/2$, then $G \not\to \vec{H}$.*

Proof. We proceed by induction on $v(G)$. If $v(G) \leq 2$, then $G \not\to \vec{H}$. Let $\delta := \delta(H)$. By Lemma 3.1, there exists $v \in V(G)$ with $\deg(v) = \delta(G) \leq 2\delta - 2$. By induction, $G - v \not\to \vec{H}$. Fix an $\vec{H}$-free orientation of $G - v$, and orient $\lfloor \delta(G)/2 \rfloor$ edges incident with v toward v and the remaining $\lceil \delta(G)/2 \rceil$ edges away from v. Note that any copy of H in G containing v necessarily has two edges incident with v oriented in opposite directions, since $\delta(H) \geq 2$ and $\lceil \delta(G)/2 \rceil \leq \delta(H) - 1$. ∎

COROLLARY 5.3. *Let $\vec{H}$ be an anti-directed orientation of a graph H, where H is strictly 2-balanced, $\delta(H) \geq 2$ and $m_2(H) - \lfloor m_2(H) \rfloor \leq 1/2$. For every graph G, if $m(G) < m_2(H)$, then $G \not\to \vec{H}$.*

Proof. We have $(e(H - u) - 1)/(v(H - u) - 2) < (e(H) - 1)/(v(H) - 2)$ for all $u \in V(H)$, since H is strictly 2-balanced. It is easy to check that $m_2(H) < \delta(H)$, so $\lfloor m_2(H) \rfloor + 1 \leq \delta(H)$. Since $m(G) < m_2(H) \leq \lfloor m_2(H) \rfloor + 1/2 \leq \delta(H) - 1/2$, we can apply Theorem 5.2. ∎

5.3. Cycles. We now consider orientations of ℓ-cycles, where $\ell \geq 4$. The main results are Theorems 5.8 and 5.10, which deal with the cases $\ell = 4$ and $\ell \geq 5$, respectively. (We also include a simple proof for the case $\ell \geq 7$; see Theorem 5.5.) Recall that a block is *long* if it has at least 3 edges.

LEMMA 5.4. *Let $\vec{C}$ be an oriented cycle with a long block. If G is a graph and $m(G) < m_2(\vec{C})$, then $G \not\to \vec{C}$.*

Proof. Note that $v(\vec{C}) \geq 4$, so $m_2(\vec{C}) \leq m_2(C_4) = 3/2$. By Lemma 3.1, G is 2-degenerate; hence $\chi(G) \leq 3$. Fix a proper coloring $c\colon V(G) \to \{1, 2, 3\}$, and orient each edge towards its endvertex with the largest color. This orientation contains no long block, so $G \not\to \vec{C}$. ∎

While the next result is superseded by Theorem 5.10, its proof is much simpler.

THEOREM 5.5. *Let $\vec{C}$ be an orientation of C_ℓ, where $\ell \geq 7$. If G is a graph and $m(G) < m_2(\vec{C})$, then $G \not\to \vec{C}$.*

Proof. Let $\ell = e(\vec{C})$. If $\vec{C}$ contains a long block, then by Lemma 5.4 the theorem holds; hence, for the remainder of the proof, we may assume that the longest block of $\vec{C}$ has length at most 2.

Suppose, looking for a contradiction, that the statement is false. Without loss of generality, let G be a minimal counterexample (with respect to the subgraph relation). That is, $m(G) < m_2(\vec{C})$, and $G \to \vec{C}$, and $G' \not\to \vec{C}$ for each proper subgraph $G' \subseteq G$. Let W be the set of vertices in G with degree 2.

If there exists an edge uv joining vertices $u, v \in W$, then (since G is minimal) uv lies in an ℓ-cycle. Moreover, $G \setminus \{u, v\} \not\to \vec{C}$; so there exists an orientation $\vec{G}$ of $G \setminus \{u, v\}$ which avoids $\vec{C}$. Note that each ℓ-cycle in G is either completely oriented in $\vec{G}$ (while avoiding $\vec{C}$) or contains the three (not yet oriented) edges incident with either u or v. We extend $\vec{G}$ by orienting these edges so that they form a directed path or cycle. Since, by assumption, the length of any block of $\vec{C}$ is at most two, it follows that $\vec{G}$ is an orientation of G avoiding $\vec{C}$, a contradiction.

Hence no edge of G lies in W. By the minimality of G, every vertex $v \in V(G)$ lies an ℓ-cycle, so $\delta(G) \geq 2$. Let $n := v(G)$. Since each vertex of W has degree 2,

$$2|W| + 3(n - |W|) \leq \sum_{v \in V(G)} d(v) = 2e(G) \leq 2m(G)n < 2n\frac{\ell - 1}{\ell - 2}.$$

It follows that $|W| > n(1 - 2/(\ell - 2))$ and

$$2\left(1 - \frac{2}{\ell - 2}\right)n < 2|W| \leq e(G) \leq m(G)n = \left(1 + \frac{1}{\ell - 2}\right)n,$$

which is a contradiction for $\ell \geq 7$. ∎

The proofs for smaller cycles use a detailed characterization of the construction sequences of C_ℓ-components, which was obtained recently by Barros et al. [2]. We state the first of these results in a slightly modified form (the original has "$\ell \geq 5$" in place of "$\ell \geq 4$," but the same proof holds).

PROPOSITION 5.6 (see [2, Proposition 7]). *Let $\ell \geq 4$ be an integer, G be a graph with $m(G) < m_2(C_\ell)$, and $(H_1, \ldots, H_t)$ be a C_ℓ-component of G. The following holds for every $1 \leq i \leq t - 1$. If C is an ℓ-cycle added to H_i to form H_{i+1}, then there exists a labeling $C = u_1u_2 \cdots u_\ell u_1$ such that exactly one of the following occurs, where $2 \leq j \leq \ell$ and $3 \leq k \leq \ell - 1$.*

(A_j) *$u_1u_2 \cdots u_j$ is a j-path in H_i and $u_{j+1}, \ldots, u_\ell \notin V(H_i)$;*

(B_k) *$u_1u_2 \in E(H_i)$, $u_2u_3 \notin E(H_i)$, $\{u_3, \ldots, u_\ell\} \setminus \{u_k\} \subseteq V(H_{i+1}) \setminus V(H_i)$, and $u_k \in V(H_i)$.*

If $(H_1, \ldots, H_t)$ constructs a C_ℓ-component, then for each $i \in [t-1]$ the new edges in H_{i+1} form a path (by Proposition 5.6). We denote this path by Q_i and write x_i, z_i for its endvertices and y_i for the sole internal vertex of Q_i in H_i if it exists. (Again by Proposition 5.6, $V(Q_i) \cap V(H_i)$ is either $\{x_i, z_i\}$ or $\{x_i, y_i, z_i\}$.) We write type(i) to denote the operation ((A_j) or (B_k), where $2 \le j \le \ell$ and $3 \le k \le \ell - 1$) which constructs H_{i+1} from H_i.

PROPOSITION 5.7 (see [2]). *Let $\ell \ge 5$. If G is a C_ℓ-component $(H_1, \dots, H_t)$ and $m(G) < m_2(C_\ell)$, then for all distinct $i, j \in [t-1]$ and each $k \in \{3, \dots, \ell - 1\}$ we have the following.*

- *If* type$(i) = (A_\ell)$, *then every other step is of type (A_2) or (A_3).*
- *If* type$(i) = (A_{\ell-1})$, *then every other step is of type (A_2), (A_3), or (A_4).*
- *If* type$(i) =$ type$(j) = (A_{\ell-1})$, *then $\ell = 5$ and every other step is of type (A_2).*
- *If* type$(i) = (B_k)$, *then every other step is of type (A_2).*

5.3.1. Cycles of length 4. We first consider orientations of 4-cycles.

THEOREM 5.8. *If G is a graph such that $m(G) < m_2(C_4)$ and $\vec{C}$ is an orientation of C_4, then $G \not\to \vec{C}$.*

To prove Theorem 5.8 we use the following proposition.

PROPOSITION 5.9. *Let G be a C_4-component $(H_1, \dots, H_t)$ with $m(G) < m_2(C_4)$. If* type$(i) = (B_3)$ *for some i, then* type$(j) = (A_2)$ *for each $j \in [t-1] \setminus \{i\}$.*

Proof. For each $j \in [t-1]$, let v_j and e_j be, respectively, the number of new vertices and new edges in H_{j+1}. By Proposition 5.6 we have $e_j \ge 3v_j/2$ and $e_j > v_j$ for each $j \in [t-1]$. Suppose type$(i) = (B_3)$, and fix $j \in [t-1] \setminus \{i\}$. We have

$$\frac{3}{2} = m_2(C_4) > m(G) = \frac{4 + \sum_{\alpha \in [t-1]} e_\alpha}{4 + \sum_{\alpha \in [t-1]} v_\alpha} \ge \frac{4 + 3 + e_j + \sum_{\alpha \in [t-1] \setminus \{i,j\}} 3v_\alpha/2}{4 + 1 + v_j + \sum_{\alpha \in [t-1] \setminus \{i,j\}} v_\alpha},$$

so $v_j > 2(e_j - v_j) - 1$. Hence $v_j \ge 2$ (because $v_j < e_j$) and type$(j) = (A_2)$. ∎

Proof of Theorem 5.8. If $\vec{C}$ is anti-directed or contains a long block, then $G \not\to \vec{C}$ by Corollary 5.3 and Lemma 5.4, respectively. We may therefore assume $\vec{C}$ has precisely two blocks of length 2; we may also assume that G is a C_4-component with construction sequence $(H_1, \dots, H_t)$, because distinct C_4-components can be independently oriented and edges in no C_4-component can be arbitrarily oriented.

If there is no step of type (B_3), then G is bipartite. (Indeed, $H_1 \simeq C_4$ and steps of type (A_2), (A_3), or (A_4) preserve bipartiteness.) Fix a proper 2-coloring of G and orient every edge toward the same color class. This avoids directed paths with length 2, so $G \not\to \vec{C}$.

On the other hand, if type$(i) = (B_3)$, then every other step is of type (A_2) by Proposition 5.9. Let $u_1u_2u_3u_4u_1$ be the new cycle in H_{i+1}, where $u_1u_2 \in H_i$ (and $u_2u_3, u_3u_4, u_4u_1 \notin E(H_i)$, $u_1, u_2, u_3 \in V(H_i)$, $u_4 \notin V(H_i)$). We may assume that H_1 is a 4-cycle $u_1u_2abu_1$.

If there is a unique new 4-cycle in H_{i+1}, then it suffices to orient H_1 as a directed cycle and the new edges in each step as directed paths. Clearly H_1 has a long block and, for each $\alpha \in [t-1]$, every new 4-cycle in $H_{\alpha+1}$ contains a long block (formed by Q_α), so $G \not\to \vec{C}$.

Finally, if a 4-cycle in H_{i+1} contains u_2u_3 but avoids $u_3u_4u_1$, then we may replace the ith step (of type (B_3)) by one (A_4)-step (adding u_2u_3) and one (A_3)-step (adding $u_3u_4u_1$). This yields a construction sequence free from (B_3), which implies (as argued above) that G is bipartite and $G \not\to \vec{C}$. Similarly, if H_{i+1} contains a new 4-cycle which avoids u_2u_3, then we may replace the ith step by one (A_3)-step (adding $u_3u_4u_1$) and one (A_4)-step (adding u_2u_3) and also conclude that $G \not\to \vec{C}$. ∎

5.3.2. Cycles of length at least 5. We now generalize Theorem 5.5 for oriented cycles with at least 5 vertices.

THEOREM 5.10. *Let $\vec{C}$ be an orientation of C_ℓ, where $\ell \geq 5$. If G is a graph and $m(G) < m_2(C_\ell)$, then $G \not\to \vec{C}$.*

We use the following results.

Remark 5.11. Let G be a C_5-component. If G can be constructed solely by steps of type (A_2), then every cycle in G has length congruent to 2 (mod 3).

Proof. The proof is by induction on $i \in [t]$ where $(H_1, \dots, H_t)$ is the construction sequence of G. The base holds because H_1 is a 5-cycle. Now suppose every cycle in H_i has length congruent to 2 (mod 3), where $i \geq 1$. We form H_{i+1} by a step of type (A_2), i.e., by adding a 5-path P joining the endvertices of an edge uv of H_i. Any new cycle C is formed by an uv-path P' in H_i, together with P. If $P' = uv$, then C has length 5, and the claim holds. On the other hand, if $uv \notin E(P')$, then $C = P' \cup P$, but since $C' := P' + uv$ is a cycle in H_i, it follows that $e(C') \equiv 2 \pmod 3$, so $e(C) = e(P') + e(P) = e(C') - 1 + e(P) \equiv 2 \pmod 3$. ∎

Remark 5.12. Let G be a C_5-component. If G is constructed solely by steps of the types (A_2) and (A_3), then G contains no C_3 and no C_4.

Proof. Let $(H_1, \dots, H_t)$ be a construction sequence of G. Note that $C_3 \nsubseteq G$: indeed, $C_3 \nsubseteq H_1$ since H_1 is a C_5; moreover, for each $i \in [t-1]$ we have $H_{i+1} = H_i \cup Q_i$, and Q_i is a path of length at least 3 which is internally disjoint from H_i, so $C_3 \nsubseteq H_{i+1}$.

Similarly $C_4 \nsubseteq H_1$, and if $H_i \cup Q_i$ contains a C_4, then type$(i) = (A_3)$, so x_i and z_i are connected by a path $x_i w z_i$ in H_i (which, together with Q_i, creates a C_5). But then $x_i w z_i x_i$ is a C_3 in G, a contradiction. ∎

We are now in position to prove the main result of this section.

Proof of Theorem 5.10. Let G be a graph with $m(G) < (\ell - 1)/(\ell - 2)$, where $\ell \geq 5$, and let $\vec{C}$ be an oriented ℓ-cycle. By Lemma 5.4, if $\vec{C}$ contains a long block, then $G \not\to \vec{C}$, so we may assume that every block of $\vec{C}$ has length at most two. We will show that the C_ℓ-components of G admit an orientation in which every ℓ-cycle has a long block. It suffices to consider one such component F, as C_ℓ-components can be independently oriented (they do not share edges) and remaining edges can be arbitrarily oriented (each ℓ-cycle in G lies in some C_ℓ-component).

Let F be a C_ℓ-component $(H_1, \dots, H_t)$ of G. Hence, for all $i \in [t-1]$, each ℓ-cycle $C \subseteq H_{i+1}$ which did not exist in H_i contains either the subpath $x_i Q_i y_i$ or $y_i Q_i z_i$ (if Q_i intersects H_i in three vertices) or the whole new path Q_i. (We write aPb to denote the subpath of P whose endvertices are a and b.)

Case 0. For each $i \in [t-1]$ we have type$(i) \in \{A_j : 2 \leq j < \ell - 1\}$.

For each $i \in [t-1]$, every new cycle in H_{i+1} contains Q_i and $e(Q_i) \geq 3$. We construct an orientation of F which avoids $\vec{C}$ as follows. Fix a directed orientation of H_1, and for each $i \in [t-1]$ fix a directed orientation of Q_i. Clearly H_1 does not contain $\vec{C}$, and for each $i \in [t-1]$ every new ℓ-cycle in H_{i+1} contains a long block (since Q_i is directed), so $F \not\to \vec{C}$.

Case 1. There is precisely one index $i \in [t-1]$ such that type$(i) = (A_{\ell-1})$.

Let $Q_i = x_i v z_i$, and let C be an ℓ-cycle in H_i containing z_i. We may assume that $H_1 = C$. Note that $e(Q_j) \geq 3$ for each $j \in [t-1] \setminus \{i\}$ since (by Proposition 5.7) type$(j) \in \{(A_2), (A_3), (A_4)\}$ and also $\ell \geq 6$ if type$(j) = (A_4)$ for some j. We orient F as follows.

Firstly, orient H_1 so that z_i is the origin of a long block, and so that z_i has no inneighbors in H_1. Secondly, for each $j \in [i-1]$, orient Q_j forming a directed path, while ensuring that z_i has no inneighbors in H_{j+1}. (This is possible since, if Q_j contains z_i, then z_i is an endvertex of Q_j.) Orient Q_i as a directed path from x_i to z_i. Finally, for each $j \in [t-1] \setminus [i]$ orient Q_j so as to form a directed path.

Clearly, the orientation of H_1 avoids $\vec{C}$. Since $e(Q_j) \geq 3$ for each $j \in [t-1] \setminus \{i\}$, each new ℓ-cycle in H_{j+1} has a long block (as it contains Q_j). Finally, every new cycle C in H_{i+1} must contain Q_i as well as some edge $z_i z \in E(H_i)$. As z_i has no inneighbors in H_i, the edge $z_i z$ extends the directed path $x_i \to v \to z_i$, forming a long block in C. This shows that every ℓ-cycle has a long block, so $F \not\to \vec{C}$.

Case 2. There exists $i \in [t-1]$ such that type$(i) = (A_\ell)$.

Let $\alpha \in [t-1]$. By Proposition 5.7, if $\alpha \neq i$, then type$(\alpha) \in \{(A_2), (A_3)\}$, so $e(Q_\alpha) \geq 3$. We may assume that H_1 is an ℓ-cycle in H_i containing z_i. We orient the edges of F as follows. Let N be the set of neighbors of z_i in H_i.

First orient H_1 with two blocks of length at least 2 and origin z_i (see Figure 1). Next, for each $j \in [i-1]$, we do the following. *If no endvertex of* Q_j *lies in* $\{z_i\} \cup N$, fix an arbitrary directed orientation of Q_j. *If a single endvertex* q *of* Q_j *lies in* $\{z_i\} \cup N$, then orient Q_j to form a directed path with origin q. *If both endvertices* q, r *of* Q_j *lie in* $\{z_i\} \cup N$, where we assume $r \neq z_i$, then orient Q_j so that it has precisely two blocks, starting from q and r, and so that the latter has precisely one arc. Finally, orient $x_i \to z_i$, and for each $j \in [t-1] \setminus [i]$ fix a directed orientation of Q_j (see Figure 1).

Let us check that every ℓ-cycle in F has a long block. This is clearly true in H_1. Now suppose $\alpha \in [t-1] \setminus \{i\}$. Note that each new cycle in $H_{\alpha+1}$ contains Q_α and that $e(Q_\alpha) \geq 3$ since type$(\alpha) \in \{(A_2), (A_3)\}$. Moreover, Q_α has a block of length at least $e(Q_\alpha) - 1$ if $\alpha < i$, and a block of length at least $e(Q_\alpha)$ if $\alpha > i$. Hence, if $e(Q_\alpha) \geq 4$ or if $\alpha > i$, then Q_α has a long block. So we may suppose that $\ell = 5$, $e(Q_\alpha) = 3$, and $\alpha \in [i-1]$. Hence type$(\alpha) = (A_3)$, and there is precisely one new 5-cycle C in $H_{\alpha+1}$ (as otherwise two 3-paths joining x_α and z_α, would form a 4-cycle in H_α, contradicting Remark 5.12). If $\left|\{x_\alpha, z_\alpha\} \cap (\{z_i\} \cup N)\right| \leq 1$, then C has a long block containing Q_α. Otherwise, $\{x_\alpha, z_\alpha\} \subseteq \{z_i\} \cup N$. Note that $x_\alpha z z_\alpha \subseteq H_\alpha$ for some $z \in V(H_\alpha)$ since $C \subseteq H_{\alpha+1}$; if $z_i \in \{x_\alpha, z_\alpha\}$, then $x_\alpha z_\alpha \in E(H_i)$, so $x_\alpha z_\alpha z x_\alpha$ is a triangle in H_i, contradicting Remark 5.12. Therefore $z_i \notin \{x_\alpha, z_\alpha\}$, so $C = Q_\alpha \cup x_\alpha z_i z_\alpha$ (since $z \neq z_i$ implies $x_\alpha z z_\alpha z_i x_\alpha$ is a 4-cycle in H_i, which contradicts Remark 5.12). Since Q_α has a directed 3-path from either x_α or z_α to a vertex $w \in V(Q_\alpha) \setminus V(H_\alpha)$, and both x_α and z_α are outneighbors of z_i, it follows that C has a long block.

To conclude Case 2, we consider the new ℓ-cycles in H_{i+1}. Each of these cycles contains the arc $x_i \to z_i$, so it suffices to show that every 3-path $z_i z w$ in H_i (where

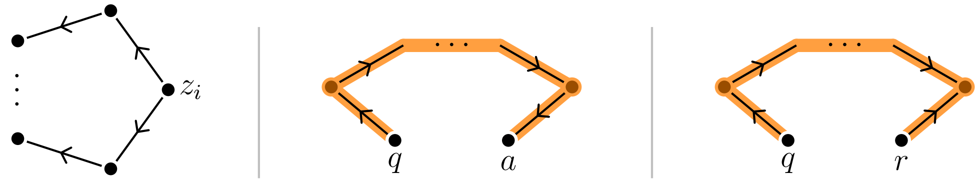

FIG. 1. *Orientations in Case 2. Left: orientation of H_1; note H_1 has a long block starting from z_i (since $\ell \geq 5$). Center and right: orientations of Q_α (where $\alpha \neq i$); in the figure, $a \notin \{z_i\} \cup N$, $r \in N$, and $q \in \{z_i\} \cup N$, where $N := N_{H_i}(z_i)$.*

$x_i \notin \{z, w\}$) is directed from z_i to w. Note that for each $j \in [i-1]$ and each pair of distinct new edges e_1, e_2 in H_{j+1}, there exist distinct new vertices $v_1 \in e_1, v_2 \in e_2$ in H_{j+1}. It follows that either $z_i zw \subseteq H_1$ (if no edge of $z_i zw$ lies in some Q_j with $j \in [i-1]$); or $z_i z \in H_\alpha$ and $zw \subseteq Q_\alpha$ and z is an endvertex of Q_α for some $\alpha \in [i-1]$ (if each Q_j contains at most one edge of $z_i zw$); or $z_i zw \subseteq Q_\beta$ and z_i is an endvertex of Q_β for some $\beta \in [i-1]$ (if some Q_β contains all edges of $z_i zw$). In each of these cases $z_i zw$ has the required orientation.

Hence every ℓ-cycle of F has a long block and $F \nrightarrow \vec{C}$.

Case 3. There exist $i, j \in [t-1]$ such that $\mathrm{type}(i) = \mathrm{type}(j) = (A_{\ell-1})$.

By Proposition 5.7 we have $\ell = 5$ and $\mathrm{type}(\alpha) = (A_2)$ for each $\alpha \in [t-1] \setminus \{i, j\}$. We may suppose $i < j$. Let $P = x_i u_2 u_3 z_i \subseteq H_i$ and $Q = x_j v_2 v_3 z_j \subseteq H_j$, and let $Q_i = x_i u_5 z_i$ and $Q_j = x_j v_5 z_j$. By Remark 5.11, every cycle in H_i has length congruent to 2 modulo 3, so H_i contains no C_3, no C_4, and no C_6. In particular, since the union of two distinct 4-paths with common ends contains a cycle with one of these lengths (see Figure 2), we conclude that P is the unique 4-path between x_i and z_i in H_i and hence the unique such path in H_{i+1}. The argument splits into three cases according to how the 5-cycles in H_i intersect P.

Case (a). There exists a 5-cycle C in H_i containing P.

We may assume that $H_1 = C = x_i u_2 u_3 z_i x x_i$ and that $H_2 = H_{i+1} = C \cup x_i u_5 z_i$, so $i = 1$. We first prove that

$$H_j \text{ contains no } C_3, \text{ no } C_6, \text{ and precisely one } C_4. \tag{5.1}$$

Crucially, note that a step of type (A_2) cannot create a C_3 or a C_4. Therefore, since $H_1 \simeq C_5$, each C_3 and each C_4 in H_j were created in the ith step resulting in H_{i+1}. Since H_{i+1} is the union of C and $z_i u_5 x_i$, we conclude that $C_3 \nsubseteq H_{i+1}$, so $C_3 \nsubseteq H_j$; moreover, the unique $C_4 \subseteq H_j$ is $x_i x z_i u_5 x_i$. It remains to show that H_j contains no C_6. Suppose, looking for a contradiction, that $\alpha \in [j-1]$ is the smallest index such that $H_{\alpha+1}$ has a 6-cycle C'. Note that H_{i+1} contains no C_6, so $\alpha > i$. Since $\mathrm{type}(\alpha) = (A_2)$, it follows that C' contains a path $abcde$ whose edges are new in $H_{\alpha+1}$, so $C' = abcdefa$ for some $f \in V(H_\alpha)$. Moreover, $abcdea$ is a (new) 5-cycle in $H_{\alpha+1}$. We conclude that $aefa$ is a 3-cycle in H_α, which is a contradiction since $C_3 \nsubseteq H_j$. This proves (5.1).

CLAIM 5.13. *There exists $e \in E(H_j)$ with $e \cap \{x_j, z_j\} \neq \emptyset$ which lies in every 4-path from x_j to z_j in H_j.*

Proof. By (5.1), every 4-path between x_j and z_j in H_j other than Q intersects $x_j v_2 v_3 z_j$ (i.e., Q) in precisely one edge h; moreover, $h \neq v_2 v_3$ (as $C_3 \nsubseteq H_j$; see Figure 2). If Claim 5.13 is false, then there are paths $x_j x v_3 z_j$ and $x_j v_2 y z_j$ in H_j with $x \neq v_2$ and $y \neq v_3$. But this contradicts (5.1), because then either H_j has a 3-cycle $x_j x v_2 x_j$ (if $x = y$) or H_j contains distinct 4-cycles $x_j x v_3 v_2 x_j$ and $v_2 v_3 z_j y v_2$ (if $x \neq y$). ∎

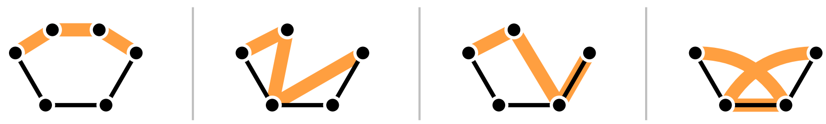

FIG. 2. *Unions of distinct 4-paths with common endvertices.*

We now return to the proof of Case (a), describing the orientation of F. Let e be the edge common to all 4-paths between x_j and z_j in H_j (as per Claim 5.13). Orient H_1 so that it is a directed cycle. For every $\alpha \in [t-1] \setminus \{j\}$, orient the new edges to form a directed path. Finally, orient $x_j v_5 z_j$ so that the path it forms with e is directed.

Let us check that every 5-cycle in F has a long block. Clearly, the two 5-cycles in H_2 have each a long block. For each $\alpha \in [t-1] \setminus \{i, j\}$, each new 5-cycle in $H_{\alpha+1}$ contains Q_α and hence has a long block ($e(Q_\alpha) \geq 3$ since type$(\alpha) = (A_2)$). Finally, every new 5-cycle in H_{j+1} contains the directed path formed by e and $x_j v_5 z_j$. We conclude that $F \not\to \vec{C}$.

Case (b). There exists a 5-cycle C in H_i containing precisely two edges of P.

We may assume that no 5-cycle in H_i contains all edges of P; otherwise we would be done by Case (a). Note that C cannot avoid u_2u_3, since $C_3 \nsubseteq H_i$. We may therefore assume that C is a 5-cycle in H_i with $z_i u_3 u_2 \subseteq C$ and that $H_1 = C$.

Let $\alpha \in [t-1]$ be such that $u_2 x_i$ is new in $H_{\alpha+1}$, and let C_α be a new 5-cycle in $H_{\alpha+1}$ containing $u_2 x_i$. Note that type$(\alpha) = (A_2)$, so $Q_\alpha = u_2 x_i x y v$, where $u_2, v \in V(H_\alpha)$ and $x_i, x, y \notin V(H_\alpha)$. We modify the construction sequence of F to a construction sequence of F where the ith step is omitted and the αth step is replaced by consecutive steps adding, in this order, $u_2 x_i u_5 z_i$ and $x_i x y v$. In the new sequence, type(α) = type$(\alpha+1) = (A_3)$, type$(j) = (A_4) = (A_{\ell-1})$, and each other step remains of type (A_2). By the argument in Case 2, $F \not\to \vec{C}$.

Case (c). Every 5-cycle in H_i contains at most one edge of P.

This is similar to the preceding case. Let $C = H_1$ be a 5-cycle containing $z_i u_3$. We first show that if $u_2 u_3$ is new in $H_{\alpha+1}$ and $u_2 x_i$ is new in $H_{\beta+1}$, then $\alpha < \beta < i$. (Note that for all $j \in [i-1]$ every edge in H_j lies in a 5-cycle.) Indeed, $\alpha, \beta < i$ by definition, and $\alpha \neq \beta$ as otherwise a cycle in $H_{\alpha+1}$ would contain two edges of P. Moreover, type(α) = type$(\beta) = (A_2)$ by Proposition 5.7, so each new edge in $H_{\alpha+1}$ and $H_{\beta+1}$ contains at least one new endvertex. Hence $\alpha < \beta$.

Let $Q_\beta = u_2 x_i x y v$, where $u_2, v \in V(H_\beta)$ and $x_i, x, y \notin V(H_\beta)$. As in Case (b), we define an alternative construction sequence of F, where the ith step is omitted and the βth step is replaced by consecutive steps adding $u_2 x_i u_5 z_i$ and $x_i x y v$ (in this order). By Case 2, $F \not\to \vec{C}$.

Case 4. There exists $i \in [t-1]$ such that type$(i) = (B_j)$, where $3 \leq j \leq \ell - 1$.

By Proposition 5.7, for each $\alpha \in [t-1] \setminus \{i\}$ we have type$(\alpha) = (A_2)$, and thus $e(Q_\alpha) \geq 3$. Recall that $y_i \in V(Q_i) \cap H_i$. Note that no new cycle in H_{i+1} avoids both $x_i Q_i y_i$ and $y_i Q_i z_i$.

If a new ℓ-cycle in H_{i+1} contains $x_i Q_i y_i$ but not $y_i Q_i z_i$, then there exists an alternative construction sequence of F which satisfies the hypothesis of one of the previous cases, and hence and $F \not\to \vec{C}$: indeed, we may replace the ith step in $(H_1, \ldots, H_t)$ by consecutive steps adding $x_i Q_i y_i$ and $y_i Q_i z_i$ (a similar exchange appears at the end of the proof of Theorem 5.8). We argue similarly if a new ℓ-cycle in H_{i+1} avoids $x_i Q_i y_i$.

If every new ℓ-cycle in H_{i+1} contains all of Q_i, then for each $\alpha \in [t-1]$ every new cycle in $H_{\alpha+1}$ contains Q_α. We fix a directed orientation of H_1 and orient Q_α as a directed path for each $\alpha \in [t-1]$. Then H_1 has a long block, and for all $\alpha \in [t-1]$ the new ℓ-cycles in $H_{\alpha+1}$ have a long block as well (since $e(Q_\alpha) \geq 3$). Hence, $F \not\to \vec{C}$. ◻

6. Proof of the main theorem (Theorem 2.1). Theorem 4.1 establishes the case $t = 3$ of Theorem 2.1. We may therefore suppose $\vec{H}$ is either an acyclic orientation of $H \in \{K_t, C_t\}$, with $t \geq 4$, or that $\vec{H}$ is an anti-directed orientation of a strictly 2-balanced graph H with $\delta(H) \geq 2$. In each one of these cases $\vec{H}$ is 2-Ramsey-avoidable (by Remark 3.3), so (by Theorems 1.1 and 2.2 together with Lemmas 3.2 and 3.4) it suffices to show that $G \not\to \vec{H}$ whenever $m(G) < m_2(H)$. Indeed, this follows by Theorem 5.1 (when H is complete), by Theorems 5.8 and 5.10 (when H is a cycle), and by Corollary 5.3 otherwise.

7. Concluding remarks. We have shown that if $\vec{H}$ is an oriented clique or cycle, then the threshold for $G(n,p) \to \vec{H}$ is $n^{-1/m_2(\vec{H})}$ except if $\vec{H} \simeq \mathrm{TT}_3$. Interestingly, TT_3 is not the only oriented graph whose threshold is *not* of the form $n^{-1/m_2(\vec{H})}$. For instance, fix $\varepsilon > 0$, and construct $\vec{G}$ from an oriented tree $\vec{T}$ of order $n^{1/2-\varepsilon}$ by identifying with each $v \in V(\vec{T})$ the source of a distinct copy $\vec{H}_v$ of TT_3. It can be shown that $p_{\vec{G}} \ll n^{-1/m_2(\vec{G})} = n^{-1/m_2(\mathrm{TT}_3)}$. In a forthcoming paper, the authors describe a richer class of oriented graphs $\vec{G}$ for which $p_{\vec{G}} \ll n^{-1/m_2(\vec{G})}$.

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