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Explaining the importance of Euler's method in rigid-body dynamics

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Abstract

The purpose of this study is essentially pedagogical and aims to provide an additional argument in clarification of a question often raised by first-year undergraduate mechanical engineering students concerning the reason for using two frames of reference—one fixed in space and one fixed in the rigid-body—to describe its motion. The reasoning employed to illustrate the inappropriateness of using a single reference frame entails showing that the equations of motion, thus obtained, are far more complex than the equations resulting from application of the traditional Euler Method. This point is illustrated through the well-known frictionless symmetrical spinning top problem.

Keywords

Teaching techniques, Euler's equations, rotational dynamics, coordinate-dependent inertia matrix

Introduction

The teaching of Classical Dynamics in undergraduate mechanical engineering courses traditionally begins with the presentation of Newton's Laws and their application to the description of the motion of a free material particle subjected

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to an external force $\vec{F} = \vec{F}(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$ dependent on position, velocity and time, in the most general case. As lecturers of Theoretical Mechanics for first-year undergraduate engineering students who have attended one semester of Calculus and Linear Algebra classes, we realize that they have no difficulty in understanding the general method of solving this problem, which can be summarized by the following algorithm:

1. define an inertial frame of reference $Oxyz$;
2. associate to this frame of reference an orthonormal basis $\vec{i}\vec{j}\vec{k}$;
3. represent the particle in a generic position $\vec{r}(x, y, z)$ in $Oxyz$;
4. define an initial kinematic condition $\vec{r}_0(x_0, y_0, z_0)$ and $\dot{\vec{r}}_0(\dot{x}_0, \dot{y}_0, \dot{z}_0)$ for the particle;
5. write Newton's 2nd Law $(m\, d^2\vec{r}/dt^2 = \vec{F})$ for the particle;
6. integrate this differential equation between the instants of interest, assuming the initial conditions to be those established in step 4.

Naturally, the previous algorithm may raise many questions among students, such as: Does the differential equation have a unique solution, regardless of the function representing the applied force? How do I proceed if the obtained differential equation is not separable? What if the integral obtained is too complicated to be solved by any analytical method? And if a numerical integration method becomes necessary, which one is most recommended? It should be noted, however, that those doubts occur at some step of the process, but do not affect its general understanding.

The situation changes completely when moving on to the subsequent and natural problem in Dynamics: studying the motion of a constrained material system subjected to external forces. In order not to have to resort to the methods of Analytical Mechanics, this problem is usually simplified by adding that mutual distances between particles are invariant with time, that is, studying the motion of a rigid body subject to external forces. In such a case, it is not enough to determine the motion of the body centre of mass—a problem analogous to that of the motion of a single particle—but it is also necessary to cope with the changes of the orientation of the body—a problem in undergraduate engineering courses that is usually approached through the application of Euler's equations.[1–3]

At this step of the teaching process, we have observed that the required concepts in applying the Euler method for rigid-body dynamics give rise to doubts that may lead students to formulate incorrect mathematical models. As we know, this method requires the use of two frames of reference: one fixed in space, called *the fixed frame*, and another, firmly attached to the rigid body, called *the mobile frame*.

Students learn that the use of two frames of reference, rather than a single frame, provides the geometric elements for the definition of three angles—precession, nutation, and spin—that properly define the orientation of the body at every instant. Likewise they notice that the derivative of the angular momentum expression described in the fixed frame becomes more complex. At this moment, however, it is observed that students expect we were able to quantify the increase in complexity resulting from this approach.

The purpose of this article is, therefore, to address the issue raised in the previous paragraph. Further, although Euler's equations are applied in numerous articles devoted to teaching Engineering,[4,5] we found no references in the literature that have a similar focus. This article is organized as follows. In the second section, we review the angular momentum theorem, emphasizing the essential concepts of correct application when the material system is a rigid body. In the third section, this theorem is applied to deduce the motion equations of a symmetric top, a classical problem in Dynamics. In the fourth section, we explore the same problem but, rather than using the Eulerian approach, we derive the angular momentum vector with the aid of a symbolic mathematical tool. In the fifth section, numerical simulations are performed, and the results provided by the two methods are compared. The last section presents the conclusions of the study.

Euler's method for rigid-body dynamics

For a system of material particles $P_1, P_2, \ldots P_n$ subjected to forces $\vec{F}_1, \vec{F}_2, \ldots \vec{F}_n$ (see Figure 1), the angular momentum $\vec{H}_O$, in a pole O that moves with velocity $\vec{v}_O$, is given by

$$\vec{H}_O = \sum_i^n [(P_i - O) \wedge m_i \vec{v}_i] \tag{1}$$

in which the "$\wedge$" symbol stands for cross-product. Differentiating the above expression with respect to the fixed frame $\mathcal{F}(\mathcal{O}XYZ)$ results in:

$$\dot{\vec{H}}_O = -\vec{v}_O \wedge \sum_i^n m_i \vec{v}_i + \sum_i^n [(P_i - O) \wedge \vec{F}_i] = -\vec{v}_O \wedge m\vec{v}_G + \vec{M}_O \tag{2}$$

where $\vec{M}_O$ is the moment of the external forces at O.

Now, referring to Figure 2, when the material system is a rigid body B, the Euler method prescribes: (i) using two reference frames, namely, $\mathcal{F}(\mathcal{O}XYZ)$, fixed in space, and $\mathcal{M}(Oxyz)$, mobile relative to $\mathcal{F}$, but firmly attached to body B; (ii) adopting the system of coordinates of $\mathcal{M}$ to describe the following vectors: $(P_i - O)$ (position of particle P_i), $\vec{\omega} = [\omega_x \, \omega_y \, \omega_z]^T$ (angular velocity of B relative to $\mathcal{F}$) and $\vec{H}_O$(angular momentum of B with respect to O relative to $\mathcal{F}$). This is summarized in the following set of equations:

$$(P_i - O) = x_i\vec{i} + y_i\vec{j} + z_i\vec{k} \tag{3}$$

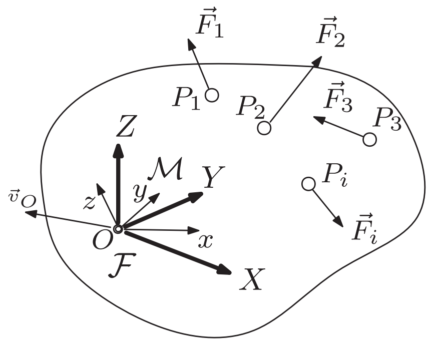

Figure 1. System of particles and fixed frame of reference $\mathcal{F}$ $(\mathcal{O}XYZ)$.

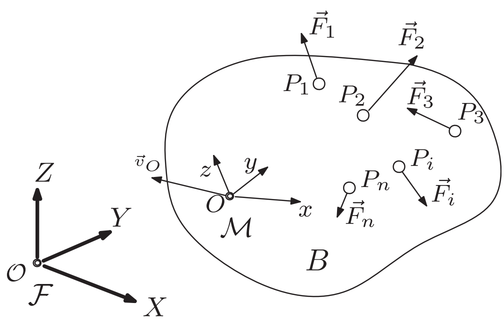

Figure 2. Rigid body *B* and frames of reference: fixed, $\mathcal{F}$ $(\mathcal{O}XYZ)$, and mobile, $\mathcal{M}$ $(Oxyz) \in B$.

$$\omega = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k} \tag{4}$$

$$\begin{aligned}
\vec{H}_O &= \sum_i^n [(P_i - O) \wedge m_i \vec{v}_i] \\
&= \sum_i^n [(P_i - O) \wedge m_i(\vec{v}_O + \vec{\omega} \wedge (P_i - O))] \\
&= (G - O) \wedge m\vec{v}_O + \sum_i^n m_i [(P_i - O) \wedge (\vec{\omega} \wedge (P_i - O))] \\
&= (G - O) \wedge m\vec{v}_O + \sum_i^n m_i \{[(P_i - O).(P_i - O)]\vec{\omega} - [(P_i - O).\vec{\omega}](P_i - O)\} \\
\vec{H}_O &= (G - O) \wedge m\vec{v}_O + [J_O]\vec{\omega}
\end{aligned} \tag{5}$$

where m is the mass of the rigid body B and

$$[J_o] = \begin{bmatrix} J_{Ox} & -J_{Oxy} & -J_{Oxz} \\ -J_{Oyx} & J_{Oy} & -J_{Oyz} \\ -J_{Ozx} & -J_{Ozy} & J_{Oz} \end{bmatrix} \tag{6}$$

is its inertia matrix with respect to pole O, written in coordinates of the $Oxyz$ system. The two terms on the right side of equation (5) can be interpreted as follows:

- $(G - O) \wedge m\vec{v}_O$: angular momentum with respect to pole O of an ideal particle of mass m, moving with instantaneous velocity $\vec{v}_O$ and situated at the centre of mass G of body B;
- $[J_O]\vec{\omega}$: angular momentum with respect to a hypothetically fixed pole O of rigid body B, moving around O with angular velocity $\vec{\omega}$.

Taking the derivative of the first term, relative to the fixed frame $\mathcal{F}$, results in

$$\frac{d}{dt}[(G - O) \wedge m\vec{v}_O] = -\vec{v}_O \wedge m\vec{v}_G + (G - O) \wedge m\vec{a}_O \tag{7}$$

To take derivative of the second term, relative to the fixed frame $\mathcal{F}$, we use the derivative formula of a generic vector function $\vec{W}$ described in a frame $\mathcal{M}$ that moves in relation to $\mathcal{F}$ with angular velocity $\vec{\omega}$,[2] that is:

$$\left.\frac{d\vec{W}}{dt}\right|_{\mathcal{F}} = \left.\frac{d\vec{W}}{dt}\right|_{\mathcal{M}} + \vec{\omega} \wedge \vec{W} \tag{8}$$

Thus, we have,

$$\frac{d}{dt}([J_O]\vec{\omega})|_{\mathcal{F}} = \frac{d}{dt}([J_O]\vec{\omega})|_{\mathcal{M}} + \vec{\omega} \wedge ([J_O]\vec{\omega}) \tag{9}$$

The above expression demonstrates the ingenuity of the Euler method: since the inertia matrix $[J_O]$, expressed in coordinates of the system $Oxyz$, is invariant for an observer firmly attached to frame$\mathcal{M}$, we arrive at the following result:

$$\frac{d}{dt}([J_O]\vec{\omega})|_{\mathcal{F}} = [J_O]\dot{\vec{\omega}} + \vec{\omega} \wedge ([J_O]\vec{\omega}) \tag{10}$$

Then, by combining equations (7) and (10) we see that the derivative takes the form

$$\left.\frac{d}{dt}\vec{H}_O\right|_{\mathcal{F}} = -\vec{v}_O \wedge m\vec{v}_G + (G - O) \wedge m\vec{a}_O + [J_O]\dot{\vec{\omega}} + \vec{\omega} \wedge ([J_O]\vec{\omega}) \tag{11}$$

Finally, comparing equations (11) and (2) results in the expression of the angular momentum theorem for a rigid body, *i*.e.:

$$(G - O) \wedge m\vec{a}_O + [J_O]\dot{\vec{\omega}} + \vec{\omega} \wedge ([J_O]\vec{\omega}) = \vec{M}_O \tag{12}$$

For the case where the pole O is a fixed point, the first term is zero and the resulting simplified version of equation (12) is called *Euler's equation*. It is important to emphasize that the many texts on Dynamics omit this first term on the left side of equation (12). The remaining equation, $[J_O]\dot{\vec{\omega}} + \vec{\omega} \wedge ([J_O]\vec{\omega}) = \vec{M}_O$, is valid only if O is a fixed point or $(G - O)$ is parallel to $\vec{a}_O$.

Dynamics of a symmetric top

In this section, we show how difficult it would be to solve a problem of Rigid Body Dynamics if, rather than applying Euler's method, we adopted the same approach—hereinafter referred to as" Newton's method"—used to describe the motion of a material point. In order to establish a basis for comparison between the application of the two methods, we have chosen the description of the dynamics of a balanced symmetric top moving around a frictionless, fixed point, under the action of gravity, as illustrated in Figure 3. The four frames of reference used to describe the orientation of the top are also displayed. The fixed frame of reference is $\mathcal{F}$ ($OXYZ$ axes, orthonormal basis $\vec{I}\vec{J}\vec{K}$). The first auxiliary frame, $Ox_1y_1z_1$, is obtained by applying to $OXYZ$ a precession ψ around OZ; the second auxiliary frame, $Ox_2y_2z_2$, results from a nutation θ of $Ox_1y_1z_1$ around Ox_1; and finally, the mobile frame $\mathcal{M}$ ($Oxyz$ axes, orthonormal basis $\vec{i}\vec{j}\vec{k}$), firmly attached to the top, is obtained through a rotation ϕ (spin) of $Ox_2y_2z_2$ around axis Oz_1. The transformation matrix— which converts the coordinates of the $\mathcal{M}$-frame to those of the $\mathcal{F}$-frame—is defined as

$$l = \begin{bmatrix} \vec{i}.\vec{I} & \vec{i}.\vec{J} & \vec{i}.\vec{K} \\ \vec{j}.\vec{I} & \vec{j}.\vec{J} & \vec{j}.\vec{K} \\ \vec{k}.\vec{I} & \vec{k}.\vec{J} & \vec{k}.\vec{K} \end{bmatrix} \tag{13}$$

Dynamics of a symmetric top using Euler's method: the easy way

As previously mentioned, Euler's method makes use of the mobile frame of reference $\mathcal{M}\,(Oxyz)$ to obtain the equations of motion. Such a task is well documented in textbooks on Mechanics[1,2]; thus, here we have chosen to straightforwardly present those equations. The reader interested in a thorough discussion may consult the cited references. Since O is a fixed point, equation (12) simplifies as

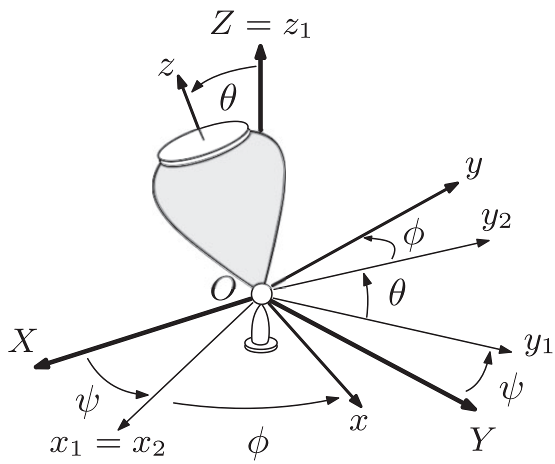

Figure 3. Spinning symmetric top.

$$[J_O]\dot{\vec{\omega}} + \vec{\omega} \wedge ([J_O]\vec{\omega}) = \vec{M}_O \tag{14}$$

In adopting the simplified notation sin(•)= s and cos(•)= c, the absolute angular velocity vector and the inertia matrix of the top, both described in the mobile frame of reference $\mathcal{M}\,(Oxyz)$ are, respectively,

$$\vec{\omega} = (\dot{\psi} s\theta s\phi + \dot{\theta} c\phi)\vec{i} + (\dot{\psi} s\theta c\phi - \dot{\theta} s\phi)\vec{j} + (\dot{\psi} c\theta + \dot{\phi})\vec{k} \tag{15}$$

$$\text{and} \quad [J_o] = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & J \end{bmatrix} \tag{16}$$

By differentiating the angular velocity vector $\vec{\omega}$ in relation to time and with respect to the fixed frame $\mathcal{F}$, and noticing that the only moment about O is provided by the weight $m||\vec{g}||$ at G, such that $(G - O) = z\vec{k}$, the equations of motion are,

$$\ddot{\psi} = \left(\frac{J - 2I}{I}\right)\left(\frac{c\theta}{s\theta}\right)\dot{\theta}\dot{\psi} + \left(\frac{J}{I}\right)\left(\frac{1}{s\theta}\right)\dot{\phi}\dot{\theta} \tag{17}$$

$$\ddot{\theta} = \left(\frac{I - J}{I}\right)s\theta c\theta\dot{\psi}^2 - \left(\frac{J}{I}\right)\dot{\psi}\dot{\phi}s\theta + mgz\left(\frac{s\theta}{I}\right) \tag{18}$$

$$\ddot{\phi} = \dot{\theta}\dot{\psi}s\theta - c\theta\left[\left(\frac{J-2I}{I}\right)\left(\frac{c\theta}{s\theta}\right)\dot{\psi}\dot{\theta} + \left(\frac{J}{I}\right)\left(\frac{1}{s\theta}\right)\dot{\theta}\dot{\phi}\right] \tag{19}$$

Those 2nd order non-linear ordinary differential equations can be integrated either numerically or analytically to provide the motion of the top. Although some algebraic work was necessary to uncouple the terms containing the 2nd derivatives in relation to time, the simplicity of Euler's method justifies the title of this section.

Dynamics of a symmetric top using "newton's method": the hard way

We will now show how the spinning top equations of motion become intricate when "Newton's method", suitable for describing the motion of a single material particle, is applied to describe the rotation motion of a rigid body. In this case, the top's angular momentum $\vec{H}_O$ is written in the fixed frame of reference. First, we use the transformation matrix of equation (13) to convert the coordinates from the $\mathcal{M}$ to the $\mathcal{F}$-frame. The result of computing the inner products for each of the components of the l matrix is

$$\begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = \begin{bmatrix} c\psi c\phi - s\psi c\theta s\phi & s\psi c\phi + c\psi c\theta s\phi & s\theta s\phi \\ -c\psi s\phi - s\psi c\theta c\phi & s\psi s\phi + c\psi c\theta c\phi & s\theta c\phi \\ s\psi s\theta & -c\psi s\theta & c\theta \end{bmatrix} \begin{bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{bmatrix} \tag{20}$$

Equation (20) is used to obtain the angular velocity vector and the inertia matrix in the $\mathcal{F}$-frame. The angular velocity is

$$\vec{\Omega} = (\dot{\theta}c\psi + \dot{\phi}s\psi s\theta)\vec{I} + (\dot{\theta}s\psi - \dot{\phi}c\psi s\theta)\vec{J} + (\dot{\psi} + \dot{\phi}c\theta)\vec{K} \tag{21}$$

In order to describe the inertia matrix, $[\mathbb{J}_O]$, in the $\mathcal{F}$-frame, we apply the following transformation:

$$[\mathbb{J}_O] = l^T.[J_o].l \tag{22}$$

Matrix $[\mathbb{J}_O]$ has components whose expressions are cumbersome. Therefore, it is presented as a partitioned matrix,

$$[\mathbb{J}_O] = [\,\mathbb{J}_1 \quad \mathbb{J}_2 \quad \mathbb{J}_3\,] \tag{23}$$

with columns

$$\mathbb{J}_1 = \begin{bmatrix} J(c\psi)^2(c\theta)^2 - I(c\psi)^2(c\theta)^2 - J(c\psi)^2 - (c\theta)^2 J + I(c\psi)^2 + I(c\theta)^2 + J \\ c\psi s\psi\Big((c\theta)^2 J - I(c\theta)^2 - J + I\Big) \\ s\psi s\theta c\theta(-I+J) \end{bmatrix} \tag{24}$$

$$\mathbb{J}_2 = \begin{bmatrix} c\psi s\psi\Big((c\theta)^2 J - I(c\theta)^2 - J + I\Big) \\ -J(c\psi)^2(c\theta)^2 + I(c\psi)^2(c\theta)^2 + J(c\psi)^2 - I(c\psi)^2 + I \\ -c\psi s\theta c\theta(-I+J) \end{bmatrix} \tag{25}$$

$$\mathbb{J}_3 = \begin{bmatrix} s\psi s\theta c\theta(-I+J) \\ -c\psi s\theta c\theta(-I+J) \\ (c\theta)^2 J - I(c\theta)^2 + I \end{bmatrix} \tag{26}$$

Therefore, the absolute angular momentum $\vec{H}_O$ of the top, with respect to the fixed pole O, described in $\mathcal{F}$-frame is

$$\vec{H}_O = [\mathbb{J}_O]\vec{\Omega} = H_X\vec{I} + H_Y\vec{J} + H_Z\vec{K}, \tag{27}$$

with

$$\begin{aligned} H_X = &\Big((J-I)(c\psi)^2(c\theta)^2 + (I-J)\Big((c\psi)^2 + (c\theta)^2\Big) + J\Big)(\dot{\phi}c\psi + \dot{\phi}s\psi c\theta) \\ &+ c\psi s\psi\Big((J-I)(c\theta)^2 - J + I\Big)(\dot{\theta} - \dot{\phi}c\psi s\theta) \\ &+ s\theta s\psi c\theta(J-I)(\dot{\psi} + \dot{\phi}c\theta) \end{aligned} \tag{28}$$

$$\begin{aligned} H_Y = &\, c\psi s\psi\Big((J-I)(c\theta)^2 - J + I\Big)(\dot{\theta}c\psi + \dot{\phi}s\psi c\theta) \\ &+ \Big((I-J)(c\psi)^2(c\theta)^2 + (J-I)(c\psi)^2 + I\Big)(\dot{\theta}s\psi - \dot{\phi}c\psi s\theta) \\ &- s\theta c\psi c\theta(J-I)(\dot{\psi} + \dot{\phi}c\theta) \end{aligned} \tag{29}$$

$$\begin{aligned} H_Z = &\, s\theta s\psi c\theta(J-I)(\dot{\theta}c\psi + \dot{\phi}s\psi s\theta) \\ &- s\theta c\psi c\theta(J-I)(\dot{\theta}s\psi - \dot{\phi}c\psi s\theta) \\ &+ \Big(I + (J-I)(c\theta)^2\Big)(\dot{\psi} + \dot{\phi}c\theta) \end{aligned} \tag{30}$$

The next step, the application of angular momentum theorem, equation (12), requires the differentiation of equation (27) in relation to time and with respect to the $\mathcal{F}$-frame, seen as

$$\dot{\vec{H}}_O = \dot{H}_X\vec{I} + \dot{H}_Y\vec{J} + \dot{H}_Z\vec{K} \tag{31}$$

The time derivatives of the scalar components of $\vec{H}_O$ are:

$$\begin{aligned}
\dot{H}_X = {} & \Big(2(I-J)c\psi(c\theta)^2\dot{\psi}s\psi + 2(I-J)(\psi)^2c\theta\dot{\theta}s\theta \\
& + 2(J-I)c\psi\dot{\psi}s\psi + 2(J-I)c\theta\dot{\theta}s\theta\Big)(\dot{\theta}c\psi + \dot{\phi}s\psi s\theta) \\
& + \Big((J-I)(c\psi)^2(c\theta)^2 + (I-J)(c\psi)^2 + (I-J)(c\theta)^2 + J\Big). \\
& (\ddot{\theta}c\psi - \dot{\theta}\dot{\psi}s\psi + \ddot{\phi}s\psi s\theta + \dot{\phi}\dot{\psi}c\psi s\theta + \dot{\phi}\dot{\theta}s\psi c\theta) \\
& - \dot{\psi}(s\psi)^2\Big((J-I)(c\theta)^2 - J + I\Big)(\dot{\theta}s\psi - \dot{\phi}c\psi s\theta) \\
& + (c\psi)^2\dot{\psi}\Big((J-I)(c\theta)^2 + I - J\Big)(\dot{\theta}s\psi - \dot{\phi}c\psi\theta) \\
& + c\psi s\psi\Big(2(I-J)\dot{\theta}c\theta s\theta\Big)(\dot{\theta}s\psi - \dot{\phi}c\psi s\theta) \\
& + c\psi s\theta\Big((J-I)((c\theta)^2 - 1)\Big)(\ddot{\theta}s\psi + \dot{\theta}\dot{\psi}c\psi - \ddot{\phi}c\psi s\theta + \dot{\phi}\dot{\psi}s\psi s\theta - \dot{\phi}\dot{\theta}c\psi c\theta) \\
& + \dot{\theta}(c\theta)^2 s\theta(J-I)(\dot{\psi} + \dot{\phi}c\theta) + s\theta\dot{\psi}c\psi c\theta(J-I)(\dot{\psi} + \dot{\phi}c\theta) \\
& - (s\theta)^2 s\psi\dot{\theta}(J-I)(\dot{\psi} + \dot{\phi}c\theta) + s\theta s\psi c\theta(J-I)(\ddot{\psi} + \ddot{\phi}c\theta - \dot{\phi}\dot{\theta}s\theta)
\end{aligned} \tag{32}$$

$$\begin{aligned}
\dot{H}_Y = {} & -\dot{\psi}(s\theta)^2\Big((J-I)((c\theta)^2 - 1)\Big)(\dot{\theta}c\psi + \dot{\phi}s\psi s\theta) \\
& + \dot{\psi}(c\psi)^2\Big((J-I)((c\theta)^2 - 1)\Big)(\dot{\theta}c\psi + \dot{\phi}s\psi s\theta) \\
& + c\psi s\psi\Big(2(I-J)\dot{\theta}s\theta c\theta\Big)(\dot{\theta}c\psi + \dot{\phi}s\psi s\theta) \\
& + c\psi s\psi\Big((J-I)((c\theta)^2 - 1)\Big)(\ddot{\theta}c\psi - \dot{\theta}\dot{\psi}s\psi + \ddot{\phi}s\psi s\theta + \dot{\phi}\dot{\psi}c\psi s\theta + \dot{\phi}s\psi\dot{\theta}c\theta) \\
& + \Big(2(J-I)\dot{\psi}s\psi c\psi(c\theta)^2 + 2(J-I)\dot{\theta}s\theta c\theta(c\psi)^2 + 2(I-J)\dot{\psi}s\psi c\psi\Big)(\dot{\theta}s\psi - \dot{\phi}c\psi s\theta)\Big) \\
& + \Big((I-J)(c\psi)^2(c\theta)^2 + (J-I)((c\psi)^2 + 1)\Big) \\
& \times (\ddot{\theta}s\psi + \dot{\theta}\dot{\psi}c\psi - \ddot{\phi}c\psi s\theta + \dot{\phi}\dot{\psi}s\psi s\theta - \dot{\phi}\dot{\theta}c\psi c\theta) \\
& - (s\theta)^2 c\psi\dot{\theta}(J-I)(\dot{\psi} + \dot{\phi}c\theta) + s\theta\dot{\psi}s\psi c\theta(J-I)(\dot{\psi} + \dot{\phi}c\theta) \\
& + (s\theta)^2 c\psi\dot{\theta}(J-I)(\dot{\psi} + \dot{\phi}c\theta) - s\theta c\psi c\theta(J-I)(\ddot{\psi} + \ddot{\phi}c\theta - \dot{\phi}\dot{\theta}s\theta)
\end{aligned} \tag{33}$$

$$
\begin{aligned}
\dot{H}_Z = {} & \dot{\theta}(c\theta)^2 s\theta(J-I)(\dot{\theta}c\psi + \dot{\phi}s\psi s\theta) + s\theta\dot{\psi}c\psi c\theta(J-I)(\dot{\theta}c\theta + \dot{\phi}s\psi s\theta) \\
& - (s\theta)^2 s\psi\dot{\theta}(J-I)(\dot{\theta}c\psi + \dot{\phi}s\psi s\theta) + s\theta s\psi c\theta(J-I)(\ddot{\theta}c\psi - \dot{\theta}\dot{\psi}s\psi + \ddot{\phi}s\psi s\theta \\
& + \dot{\phi}\dot{\psi}c\psi s\theta + \dot{\phi}s\psi\dot{\theta}c\theta) - \dot{\theta}(c\theta)^2 c\psi(J-I)(\dot{\theta}s\psi - \dot{\phi}c\psi s\theta) \\
& + s\theta\dot{\psi}s\psi c\theta(J-I)(\dot{\theta}s\psi - \dot{\phi}c\psi s\theta) + (s\theta)^2 c\psi\dot{\theta}(J-I)(\dot{\theta}s\psi - \dot{\phi}c\psi s\theta) \\
& - s\theta c\psi c\theta(J-I)(\ddot{\theta}s\psi + \dot{\theta}\dot{\psi}c\psi - \ddot{\phi}c\psi s\theta + \dot{\phi}\dot{\psi}s\psi s\theta - \dot{\phi}c\psi\dot{\theta}c\theta) \\
& + 2(I-J)\dot{\theta}c\theta s\theta(\dot{\psi} + \dot{\phi}c\theta) + \Big((J-I)(c\theta)2 + I\Big)(\ddot{\psi} + \ddot{\phi}c\theta - \dot{\phi}\dot{\theta}s\theta)
\end{aligned}
\tag{34}
$$

At this point, we must stress that the computation of the derivatives presented in equations (32) to (34) was only possible with the help of a symbolic math software (*Maple*); moreover, if all operations among parentheses were expanded, the whole expression for $\dot{\vec{H}}_0$ would have around 300 summation terms, many of them containing 4 or 5 multiplications among sines and cosines of the arguments θ, ϕ and ψ, as well as their first and/or second derivatives. Comparatively, using Euler's approach, according to equations (17) to (19) the total number of summation terms is 8, with 1 to 5 internal multiplications.

Numerical simulation

In the previous section, we obtained the equations of motion for the spinning top using "Newton's method". With the aid of the *Maple* symbolic math software, we managed to differentiate the angular momentum vector and decouple the resulting equations. The expressions obtained are not presented in the article, for they spread throughout several pages of $A4$-format paper. Integrating a system of second order differential equations with such a huge number of terms is not justifiable, since the desired results can be easily achieved through Euler's method. According to this rationale, we did not integrate those awkward equations.

Although this finding illustrates the significant difficulty of applying Newton's method, rather than Euler's method, in rigid-body dynamics, we consider that it is still worthwhile presenting the behaviour of the inertia matrix and of the angular momentum vector in the fixed frame, which is possible once the time evolution of the Euler angles is available as input to equations (23) to (30).

To this end, we employed Euler's method to numerically simulate a specific instance of the spinning top motion, namely, regular precession, thus obtaining the necessary time history of the Euler angles. In such a motion, there is a strict relationship among the initial values of nutation angle, precession, and rotation (spin) angular velocities given by[6]

$$
\dot{\phi} = \frac{I}{2J\dot{\psi}}\left[2\dot{\psi}^2\left(1 - \frac{J}{I}\right)\cos\theta + \left(\frac{2mgz}{I}\right)\right] \tag{35}
$$

Table 1. Spinning top parameters.

$m\,(kg)$	$I\,(kg.m^2)$	$J\,(kg.m^2)$	$z\,(m)$
$85e^{-3}$	$1.72e^{-4}$	$3.75e^{-5}$	$3.75e^{-2}$

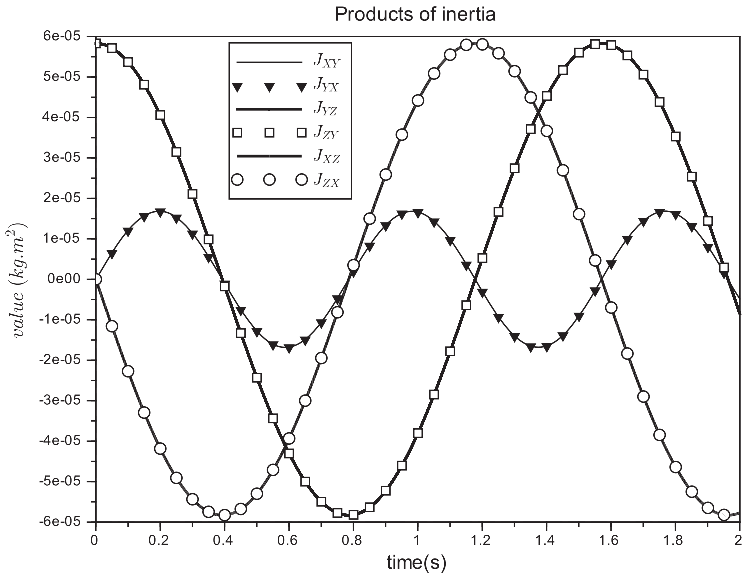

Figure 4. Products of inertia described in coordinates of the $\mathcal{F}$-frame.

In adopting precession velocity $\dot{\psi} = 4\,rad/s$ and nutation angle $\theta = \pi/6\,rad$, the rotation velocity computed according to equation (35) is approximately $221\,rad/s$. The spinning top parameters used in the simulation are given in Table 1.

Before proceeding, we must point out that regular precession is only a theoretical situation, due to the frictionless hypothesis of the mathematical model. In a real scenario, even if point O is modelled as a smooth semi-spherical surface, the action of a frictional moment should be considered. The discussion on the general dynamics of spinning tops goes beyond the scope of this work and can be found in the literature.[6,7]

The results of the simulation are now presented. As seen in Figure 4, the time evolution of the products of inertia exhibit a periodic behaviour in which the oscillation period is $T = 2\pi/\dot{\psi} \simeq 1.57\,s$, in accordance with the prescribed regular precession. Moreover, the phase difference of $\pi/2$ between J_{XZ} and $J_{Y\,Z}$, and of $\pi/4$ between J_{XY} and J_{XZ} & $J_{Y\,Z}$, is in agreement with the description of the

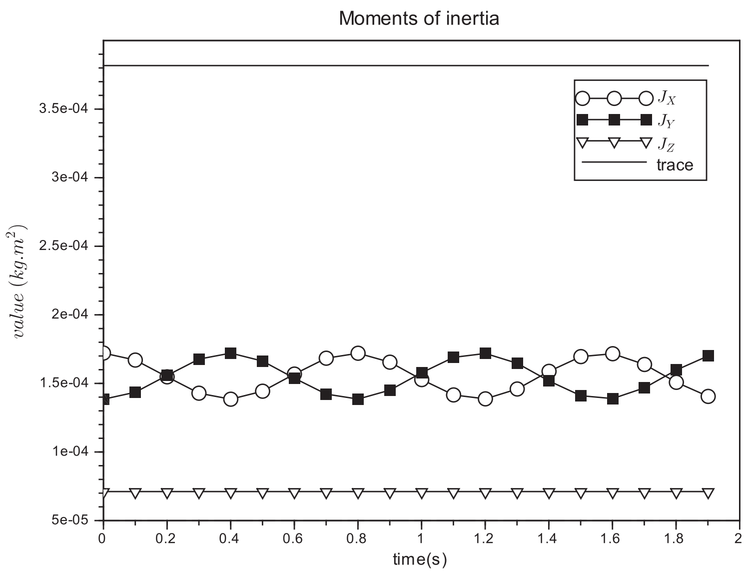

Figure 5. Moments of inertia described in coordinates of the $\mathcal{F}$-frame.

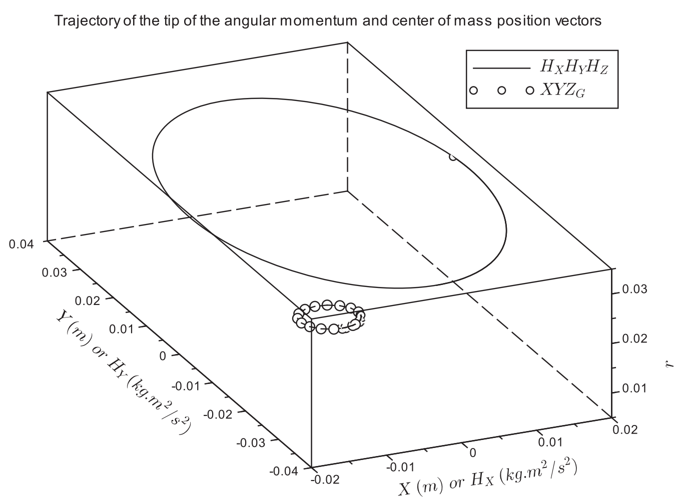

Figure 6. Trajectory of the tip of the angular momentum vector $\vec{H}_O$ and of the centre of mass G in the $\mathcal{F}$-frame.

motion in a fixed frame of reference, since the inertia ellipsoid, associated to the similarity transformation given by equation (23), rotates around the O_Z axis.

Another consistency check of the results presented is given by the invariance of the trace of the inertia matrix, depicted in Figure 5, along with the individual components of the main diagonal. The projection of the angular momentum $\vec{H}_O$ in the O_Z axis, H_Z, is an invariant of the spinning top motion. In the particular case of a regular precession, invariance of H_Z implies that the moment of inertia about O_Z, J_Z, is also invariant; the values of J_X and J_Y must, as a consequence, be in opposition of phase, in order to keep the trace invariant, as shown in this same figure.

The trajectory of the tip of the angular momentum vector $\vec{H}_O$ and of the centre of mass G are presented in Figure 6. The projection of $\vec{H}_O$ onto O_Z is constant as well as the Z coordinate of the centre of mass, thus corroborating the assertion of the previous paragraph.

Conclusion

In this paper, we propose to answer a common question posed by undergraduate mechanical engineering students who are faced, for the first time, with learning Euler's method for rigid-body dynamics:" Why is it necessary to describe the angular momentum of the rigid body in the body-fixed frame of reference? Why can we not adopt the procedure used when applying Newton's 2nd Law, i.e., describing the angular momentum in a fixed frame of reference?"

Our approach to highlighting the inappropriateness of adopting such a method consisted of describing the angular momentum of the body in a fixed frame of reference and showing that the resulting differential equations of motion are so intricate that, even with the aid of symbolic and numeric mathematical tools, integrating those equations is extremely difficult.

Then, in order to stress the ingenuity of Euler's method in rigid-body dynamics, we solved the problem of a spinning top constrained to moving around a frictionless pivot and showed that the solution to this problem was far more straightforward when using Euler's method. Furthermore, we used a simple transformation of the results from the moving to the fixed frame in the special case of regular precession to validate Euler's approach—a result that could not be achieved by straightforwardly integrating the corresponding differential equations obtained with the Newton method, as emphasized before.

It might be argued that the paper could be more convincing if there were more numerical simulation examples with comparisons. The investigation of more complex problems of rigid-body movement around a fixed point (e.g.: assuming that the body mass is not homogeneously distributed, or that there are friction forces at the contact) would only hinder the process of building and validating of dynamic models based either on Euler's or Newton's methods, but would not bring any advantage in return.

Of course, other examples of more complex mechanical systems could still have been examined. Nevertheless, attempts to solve those kind of problems using Newton's method would be inadequate. The dynamic modelling, based on Euler's approach, of any mechanism made up of bodies connected to each other by means of hinges and linear guides in open or closed kinematic chains already produces highly complex nonlinear differential equations.

In conclusion, we feel that the arguments and examples presented in this article will be useful in convincing students of the overwhelming practical challenges in applying Newton's method to problems in rigid-body dynamics and the wisdom of using Euler's approach.

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