MULTI-VALUED DYNAMICAL SYSTEMS ON TIME-DEPENDENT METRIC SPACES WITH APPLICATIONS TO NAVIER-STOKES EQUATIONS

HONGYONG CUI

Huazhong University of Science and Technology
School of Mathematics and Statistics, and Hubei Key Laboratory of Engineering Modeling and Scientific Computing
Wuhan 430074, P. R. China

RODIAK NICOLAI FIGUEROA LÓPEZ

Universidade Federal de São Carlos
Departamento de Matemática
13565-905 São Carlos - SP, Brazil

HERACLIO LEDGAR LÓPEZ-LÁZARO

Universidade de São Paulo
Instituto de Ciências Matemáticas e de Computação - ICMC
13566-590 - São Carlos - SP, Brazil

JACSON SIMSEN

Universidade Federal de Itajubá
Instituto de Matemática e Computação - IMC
37500-903 - Itajubá - MG, Brazil

ABSTRACT. In this paper, we develop an extension of the theoretical framework of multi-valued dynamical systems for families of time-dependent phase spaces, where special attention was paid to the relationship between the pullback attractors of homeomorphically equivalent dynamical systems. We apply this theory to show that the 3D Navier-Stokes equations defined on a non-cylindrical domain, satisfying certain hypotheses about the energy inequality, generate an upper-semicontinuous multi-valued dynamical system, and then, by means of the energy method, we show that this system is asymptotically compact and has a pullback attractor on a tempered universe. Using current techniques we also prove that pullback attractors associated with the single-valued dynamical systems that satisfy the smoothing property have finite fractal dimension. This latter result is applied to show that the 2D Navier-Stokes equation on a non-cylindrical domain has a pullback attractor with finite fractal dimension.

E-mail addresses: h.cui@outlook.com, rodiak@ufscar.br, heracliolopezl@icmc.usp.br, heracliolopezl@gmail.com, jacson@unifei.edu.br.

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CONTENTS

1. INTRODUCTION

In this paper we study the pullback attractors of the 2D- and 3D- Navier-Stokes equations defined on a non-cylindrical region Q_τ and with homogeneous Cauchy-Dirichlet boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla)u + \nabla \pi = f(t) & \text{in } \; Q_\tau, \\ \text{div } u = 0 & \text{in } \; Q_\tau, \\ u \equiv 0 & \text{on } \; \Sigma_\tau, \\ u(\tau) = u_\tau & \text{in } \; \mathcal{O}_\tau, \end{cases} \tag{NS}$$

where $u(x,t) = (u_1(x,t), \cdots, u_n(x,t))$ is the velocity field, u_τ is the initial velocity field, the real function $\pi(x,t)$ represents the pressure on the fluid, and $f(x,t) = (f_1(x,t), \cdots, f_n(x,t))$ is the external force. The non-cylindrical region is defined as follows: let $\{\mathcal{O}_t\}_{t\in\mathbb{R}}$ be a family of open bounded subset of $\mathbb{R}^n$ with $n \in \{2,3\}$, and for $\tau, T \in \mathbb{R}$ with $\tau \leqslant T$ let

$$Q_{\tau,T} := \bigcup_{t\in(\tau,T)} \mathcal{O}_t \times \{t\}, \qquad Q_\tau := \bigcup_{t\in(\tau,+\infty)} \mathcal{O}_t \times \{t\},$$

$$\Sigma_{\tau,T} := \bigcup_{t\in(\tau,T)} \partial\mathcal{O}_t \times \{t\} \qquad \text{and} \qquad \Sigma_\tau := \bigcup_{t\in(\tau,+\infty)} \partial\mathcal{O}_t \times \{t\}.$$

Clearly, the first equation (NS) is non-autonomous, and the non-autonomous feature is governed by the time-dependent forcing $f(t)$ as well as the time-varying domain. In the 2D case where the uniqueness of solutions holds (see [32, Theorem 2.8] or [38, Theorem 4.8]), it generates a single-valued dynamical system on a family of time-dependent phase spaces. The theoretical framework of this type of single-valued dynamical system for time-dependent phase spaces was initially studied in [19, 21] (see also [9, 12, 13]).

The concept of pullback attractors associated with single-valued dynamical systems (when the phase space is fixed e.g. [7, 16, 20, 29] and when it acts on time-dependent phase spaces e.g. [19, 21]), or multi-valued dynamical systems on fixed phase spaces, e.g. [4, 5, 25, 31, 36, 37], has been widely studied in last decades. In order to study the pullback dynamical behavior of (NS), inspired by the works mentioned above, we will study the pullback attractor theory for multi-valued dynamical systems on time-dependent phase spaces, and as expected, by adapting the concepts of dissipativity and asymptotic compactness we construct a pullback attractor associated with this type of dynamical systems, which is a main novelty of this manuscript.

We also introduce the concept of equivalent dynamical systems. This concept is motivated by the change of variables used to transform a system of partial differential equations defined over a non-cylindrical domain to a new system of partial differential equations defined over a cylindrical domain. Hence, the following question naturally arises: if both evolutionary systems have a pullback attractor, what is the relationship between such attractors? This question is answered in Theorem 3.20 and Corollary 3.21. On the other hand, it is currently known that pullback attractors associated with time-dependent evolution processes that satisfy the smoothing property have finite fractal dimension, cf. [12, Lemma 6.1], [13, Theorem 4.4] and [27, Theorem 4.11]. In this work, we present such a result, with a slight modification, and we analyze the consequences for equivalent time-dependent dynamical systems.

Applying the above theory to the Navier-Stokes equation (NS) we shall show that the time-dependent multi-valued dynamical system associated with the weak solutions of the 3D-Navier-Stokes equations that satisfy the energy inequality has a pullback attractor in some tempered universe, and particularly for the 2D case we prove that the pullback attractor has finite fractal dimension, which extends the analysis presented in [38].

The structure of the paper is as follows. Section 2 is devoted to establishing known results regarding the Navier-Stokes equations on some non-cylindrical domain. Furthermore, for the case of the 3D-Navier Stokes equations, on a hypothesis of the existence of weak solutions that satisfy an energy inequality when the associated initial data are regular, we can show that any weak solution also satisfies the energy inequality. Then, this will allow us to analyze the asymptotic behavior, in the pullback sense, of the weak solutions associated with the 3D-Navier Stokes equations. This idea was taken from [18] for the case where the domain is fixed. Section 3 is devoted to studying and developing the theoretical part on the existence of pullback attractors associated with time-dependent multi-valued dynamical systems that are upper-semicontinuous with closed values, also to introduce the concept of equivalent dynamical systems and study the consequences that this has concerning the existence of attractors. Then, we apply this theory to show the existence of a pullback attractor on a tempered universe associated with 3D-Navier-Stokes equations on some non-cylindrical domain. Section 4, in the same way as [27, Theorem 4.11], we state a method to estimate the fractal dimension of pullback attractors associated with time-dependent single-valued processes that satisfy the smoothing property, and we analyze the consequences of this method for equivalent single-valued dynamical systems. Finally, we apply this theory to prove the finiteness of the fractal dimension of the pullback attractor associated with 2D-Navier-Stokes equations on some non-cylindrical domain, given in [38, Theorem 6.4]. The latter was achieved by adjusting the techniques used in [23] for the case of fixed phase space and the external force satisfies the piecewise bounded integrability condition.

2. The 2D and 3D Navier-Stokes equations on a non-cylindrical domain

In this section, we present the Navier-Stokes equations on time-varying domains. It should be noted that the Navier-Stokes equations represent a physical model, which describes the flow of incompressible Newtonian fluids, cf. [17, 24, 33, 35, 38, 39, 40], for the case of incompressible non-Newtonian fluids cf. [15, 22, 28]. We are going to recall the existence of weak solutions of the n-dimensional Navier-Stokes equations, with $n = 2, 3$. We will also see some results of regularity for $n = 3$.

In this section, let us begin defining the non-cylindrical region where the incompressible Newtonian fluid will be concentrated. Let $\{\mathcal{O}_t\}_{t\in\mathbb{R}}$ be a family of open bounded subsets of $\mathbb{R}^n$. Given $\tau, T \in \mathbb{R}$ with $\tau \leqslant T$, denote by

$$Q_{\tau,T} := \bigcup_{t\in(\tau,T)} \mathcal{O}_t \times \{t\}, \qquad Q_\tau := \bigcup_{t\in(\tau,+\infty)} \mathcal{O}_t \times \{t\},$$
$$\Sigma_{\tau,T} := \bigcup_{t\in(\tau,T)} \partial\mathcal{O}_t \times \{t\} \quad \text{and} \quad \Sigma_\tau := \bigcup_{t\in(\tau,+\infty)} \partial\mathcal{O}_t \times \{t\}.$$

Now, the Navier-Stokes equations on a non-cylindrical domain with homogeneous Cauchy-Dirichlet boundary conditions, which we will indicate by (NS), is the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u\cdot\nabla)u + \nabla\pi = f(t) & \text{in } Q_\tau, \\ \text{div } u = 0 & \text{in } Q_\tau, \\ u \equiv 0 & \text{on } \Sigma_\tau, \\ u(\tau) = u_\tau & \text{in } \mathcal{O}_\tau, \end{cases} \tag{NS}$$

where $u(x,t) = (u_1(x,t), \cdots, u_n(x,t))$ is the velocity field, u_τ is the given initial velocity vector field, the function $\pi(x,t)$ represents the pressure on the fluid, and $f(x,t) = (f_1(x,t), \cdots, f_n(x,t))$ is the external force. Depending on the Euclidean dimension in which we are working ($n = 2$ or $n = 3$), we will indicate the system (NS) by 2D-(NS) and 3D-(NS), in dimension $n = 2$ and in dimension $n = 3$, respectively.

Now, we are going to assume some hypotheses of regularity on the family of open sets $\{\mathcal{O}_t\}_{t\in\mathbb{R}}$ within the framework of a suitable formulation of our problem. Let us denote by $\mathcal{O}$ a nonempty bounded open subset of $\mathbb{R}^n$, $n \in \{2,3\}$, with C^3 boundary $\partial\mathcal{O}$, and consider the map $r(\cdot,\cdot) = r(y,t)$ a vector function $r \in C^1(\overline{\mathcal{O}} \times \mathbb{R}; \mathbb{R}^n)$ such that

$$r(\cdot,t) : \mathcal{O} \to \mathcal{O}_t \text{ is a } C^3 - \text{diffeomorphism} \quad \forall\, t \in \mathbb{R},$$

and we denote the inverse by $\overline{r}(\cdot,t) := r^{-1}(\cdot,t)$.

(H1) There exists a level-preserving C^3-diffeomorphism $r(\cdot,\cdot) : \mathcal{O} \times \{t\} \to \mathcal{O}_t$, which satisfies

$$\text{Jac } \overline{r}(x,t) = \text{Jac}\left(\frac{\partial \overline{r}_i}{\partial x_j}(x,t)\right) \equiv \frac{1}{J(t)} > 0 \qquad i,j = 1,\cdots,n,$$

for all $(x,t) \in \overline{Q_{\tau,T}}$.

(H2) The inverse $\overline{r}(\cdot,t) := r^{-1}(\cdot,t)$ satisfies

$$\overline{r} \in C^{3,1}\left(\overline{Q_{\tau,T}}; \mathbb{R}^n\right) \quad \forall \tau < T,$$

i.e., $\overline{r}$, $\frac{\partial \overline{r}}{\partial t}$, $\frac{\partial \overline{r}}{\partial x_i}$, $\frac{\partial^2 \overline{r}}{\partial x_i \partial x_j}$, and $\frac{\partial^3 \overline{r}}{\partial x_i \partial x_j \partial x_k}$ belong to $C(\overline{Q_{\tau,T}}; \mathbb{R}^n)$ for all $1 \leqslant i,j,k \leqslant n$ and for any $\tau < T$.

(H3) The set

$$\Omega_t := \bigcup_{s \leqslant t} \mathcal{O}_s$$

is bounded in $\mathbb{R}^n$ for all $t \in \mathbb{R}$.

Moreover, for each subset $\Omega_t \subset \mathbb{R}^n$ we will identify the first eigenvalue of $-\Delta$ on $H_0^1(\Omega_t)^n$ as

$$\lambda_{1,t} := \min_{w \in H_0^1(\Omega_t)^n \setminus \{0\}} \frac{\|\nabla w\|^2_{L^2(\Omega_t)^n}}{\|w\|^2_{L^2(\Omega_t)^n}}. \tag{2.1}$$

Note that $\lambda_{1,(\cdot)} : \mathbb{R} \to (0, +\infty)$ with $\mathbb{R} \ni t \mapsto \lambda_{1,t}$ is a non-increasing function, e.g. [21, Section 7]. In fact, by definition of the family of bounded sets $\{\Omega_t\}_{t\in\mathbb{R}}$, given in the hypothesis **(H3)**, we have

$$H_0^1(\Omega_s)^n \subset H_0^1(\Omega_t)^n \quad \text{for any } \; s \leqslant t,$$

since $\Omega_s \subset \Omega_t$ for all $s \leqslant t$. Therefore, from (2.1), we obtain $\lambda_{1,t} \leqslant \lambda_{1,s}$ for all $s \leqslant t$.

2.1. **Functional spaces and coordinate transformations.** In this part we introduce the functional spaces that are necessary to study the non-cylindrical domain problem (NS). The readers are referred to [21, 38].

2.1.1. *Functional spaces evolving families of Banach spaces.* Let us denote the inner product in $L^2(\mathbb{R}^n)^n$ by $((\cdot,\cdot))$ with norm $|||\cdot||| = ((\cdot,\cdot))^{1/2}$, and the norm in $H^1(\mathbb{R}^n)^n$ by $\|\|\cdot\|\|$.

Let $\left\{(X_t, \|\cdot\|_{X_t})\right\}_{t\in\mathbb{R}}$ be a family of Banach spaces such that $X_t \subset L^1_{loc}(\mathcal{O}_t)^n$ for each $t \in \mathbb{R}$. Given $\tau, T \in \mathbb{R}$ with $\tau < T$ and $p \in [1,+\infty]$, let us denote by $L^p(\tau,T;X_t)$ the set of functions $u \in L^1_{loc}(Q_{\tau,T})^n$ such that $u(t) \in X_t$ for a.e. $t \in [\tau,T]$ and the function $\|u(\cdot)\|_{X_{(\cdot)}}$ defined by $t \mapsto \|u(t)\|_{X_t}$ belongs to $L^p(\tau,T)$. The space $L^p(\tau,T;X_t)$ is a Banach space with respect to the norm

$$\begin{cases} \|u\|^p_{L^p(\tau,T;X_t)} = \displaystyle\int_\tau^T \|u(t)\|^p_{X_t} dt, & \text{if } p \in [1,+\infty), \\ \|u\|_{L^\infty(\tau,T;X_t)} = \operatorname*{ess\,sup}_{t\in[\tau,T]} \|u(t)\|_{X_t}, & \text{if } p = +\infty. \end{cases}$$

Given $u \in L^1_{loc}(Q_{\tau,T})^n$, the trivial extension of u is a function $\widehat{u} : \mathbb{R}^n \times (\tau,T) \to \mathbb{R}^n$, such that

$$\widehat{u}(x,t) = \begin{cases} u(x,t), & \text{if } (x,t) \in Q_{\tau,T}, \\ 0, & \text{if } (x,t) \in \bigcup_{s\in(\tau,T)} \mathcal{O}_s^c \times \{s\}. \end{cases} \tag{2.2}$$

Definition 2.1. *Let $u \in L^1_{loc}(Q_{\tau,T})^n$ and $\widehat{u}$ be its trivial extension. Then*

(i) *we say that* $u \in C([\tau,T];L^2(\mathcal{O}_t)^n)$ *(resp.* $u \in C_w([\tau,T];L^2(\mathcal{O}_t)^n)$*) if* $\widehat{u} \in C([\tau,T];L^2(\mathbb{R}^n)^n)$ *(resp.* $\widehat{u} \in C_w([\tau,T];L^2(\mathbb{R}^n)^n)$*), and we say that a sequence* $\{u^m\}$ *converges to* u *in the space* $C([\tau,T];L^2(\mathcal{O}_t)^n)$ *(resp. in* $C_w([\tau,T];L^2(\mathcal{O}_t)^n)$*) as* $m \to \infty$*, if the trivial extensions* $\{\widehat{u}^m\}$ *converge to* $\widehat{u}$ *in* $C([\tau,T];L^2(\mathbb{R}^n)^n)$ *(resp. in* $C_w([\tau,T];L^2(\mathbb{R}^n)^n)$*) as* $m \to \infty$.

(ii) *we say that* $u \in C([\tau,T];H_0^1(\mathcal{O}_t)^n)$ *if* $\widehat{u} \in C([\tau,T];H^1(\mathbb{R}^n)^n)$*, and we say that a sequence* $\{u^m\}$ *converges to* u *in* $C([\tau,T];H_0^1(\mathcal{O}_t)^n)$ *as* $m \to \infty$*, if the trivial extensions* $\{\widehat{u}^m\}$ *converge to* $\widehat{u}$ *in* $C([\tau,T];H^1(\mathbb{R}^n)^n)$ *as* $m \to \infty$.

For $u \in L^1_{loc}(Q_{\tau,T})^n$, we denote by $\dfrac{\partial u}{\partial t} \in \mathcal{D}'(Q_{\tau,T})^n$ the partial derivative of u with respect to time t in the sense of distributions, that is,

$$\left\langle \frac{\partial u}{\partial t}, \psi \right\rangle = -\sum_{i=1}^{n} \int_{\tau}^{T} \int_{\mathcal{O}_t} u_i(x,t) \frac{\partial \psi_i}{\partial t}(x,t)\, dxdt,$$

for all $\psi = (\psi_1, \cdots, \psi_n) \in C_c^\infty(Q_{\tau,T})^n$. Throughout the text we will also use the notation u' for the derivative in time, i.e., $u' = \frac{\partial u}{\partial t}$.

Definition 2.2. *We denote by* $L^p(\tau,T;H^{-1}(\mathcal{O}_t)^n)$, $p \in [1,+\infty)$*, the set of all distributions* $w = (w_1, \cdots, w_n)$ *in* $\mathcal{D}'(Q_{\tau,T})^n$ *of the form*

$$w_i = f_{i_0} - \sum_{j=1}^{n} \frac{\partial f_{i_j}}{\partial x_j}, \quad \textit{with} \quad f_{i_j} \in L^p(\tau,T;L^2(\mathcal{O}_t)) \quad \textit{for all} \quad i = 1,\cdots,n \textit{ and } j = 0,\cdots,n.$$

that is,

$$\langle w, \psi \rangle = \sum_{i=1}^{n} \int_{Q_{\tau,T}} f_{i_0}(x,t)\psi_i(x,t)\, dxdt + \sum_{i,j=1}^{n} \int_{Q_{\tau,T}} f_{i_j}(x,t) \frac{\partial \psi_i}{\partial x_j}(x,t)\, dxdt,$$

for all $\psi \in C_c^\infty(Q_{\tau,T})^n$.

The space $L^p(\tau,T;H^{-1}(\mathcal{O}_t)^n)$ is a Banach space equipped with the norm

$$\|w\|_{L^p(\tau,T;H^{-1}(\mathcal{O}_t)^n)} := \left(\int_{\tau}^{T} |w(t)|^p_{-1,t} dt \right)^{1/p},$$

where by [14, Section 5.9] we have that

$$|w(t)|_{-1,t} = \left(\sum_{i=1}^{n} \sum_{j=0}^{n} \|f_{i_j}(t)\|^2_{L^2(\mathcal{O}_t)} \right)^{1/2},$$

where $|\cdot|_{-1,t}$ is the norm in $H^{-1}(\mathcal{O}_t)^n$ for any $t \in \mathbb{R}$.

2.1.2. *Coordinate transformations.* In order to handle the time-varying domain we now make some technical coordinate transformations by which the equation (NS) is transformed to a dynamically equivalent system defined on a fixed domain. The main idea is inspired by the spirit of [21, 38].

Given $\tau \in \mathbb{R}$, let us consider the interval $[\tau,+\infty) \subset \mathbb{R}$ and define the function $v(\cdot,\cdot) = (v_1(\cdot,\cdot), \cdots, v_n(\cdot,\cdot))$ as

$$v_k(y,t) = \sum_{i=1}^{n} \frac{\partial \overline{r}_k}{\partial x_i}(r(y,t),t) \cdot u_i(r(y,t),t), \quad y \in \mathcal{O},\ t \in [\tau,+\infty),\ k = 1,\cdots,n, \tag{2.3}$$

where $u(x,t) = (u_1(x,t), \cdots, u_n(x,t))$, with $x \in \mathcal{O}_t$ and $t \in [\tau,+\infty)$. Then we can express the coordinates of u in terms of the coordinates of v by direct calculation as

$$u_k(x,t) = \sum_{i=1}^{n} \frac{\partial r_k}{\partial y_i}(\overline{r}(x,t),t) \cdot v_i(\overline{r}(x,t),t), \quad x \in \mathcal{O}_t,\ t \in [\tau,+\infty),\ k = 1,\cdots,n. \tag{2.4}$$

Lemma 2.3. *Let* $n \in \mathbb{N}$ *and* $1 \leqslant p \leqslant +\infty$*. Then* *i)* $u(t) \in L^p(\mathcal{O}_t)^n$ *if and only if* $v(t) \in L^p(\mathcal{O})^n$*, with* $t \in [\tau, T]$*. Moreover, there are two positive constants* ℓ_1 *and* ℓ_2 *(which depend only on* n*,* p*,* τ*,* T*, and* $r(\cdot,\cdot)$*) such that*

$$\ell_1 \|u(t)\|_{L^p(\mathcal{O}_t)^n} \leqslant \|v(t)\|_{L^p(\mathcal{O})^n} \leqslant \ell_2 \|u(t)\|_{L^p(\mathcal{O}_t)^n}.$$

for all $u(t) \in L^p(\mathcal{O}_t)^n$*.*

ii) $u(t) \in H^1(\mathcal{O}_t)^n$ *if and only if* $v(t) \in H^1(\mathcal{O})^n$*, with* $t \in [\tau, T]$*. Moreover, there are two positive constants* $\widehat{\ell}_1$ *and* $\widehat{\ell}_2$ *(which depend only on* n*,* p*,* τ*,* T*, and* $r(\cdot,\cdot)$*) such that*

$$\widehat{\ell}_1 \|u(t)\|_{H^1(\mathcal{O}_t)^n} \leqslant \|v(t)\|_{H^1(\mathcal{O})^n} \leqslant \widehat{\ell}_2 \|u(t)\|_{H^1(\mathcal{O}_t)^n},$$

for all $u(t) \in H^1(\mathcal{O}_t)^n$*.*

Proof. Analogous as [38, Lemma 3.1 and Lemma 3.2]. □

Lemma 2.4. *Given* $p, q \in [1, +\infty]$ *and* $n \in \mathbb{N}$*. Under the assumptions given in (H1), let* $u(x,t)$ *and* $v(y,t)$ *be the functions satisfying the conditions given in* (2.3) *and* (2.4)*, respectively. Then*

$$\begin{aligned} u \in L^q(\tau, T; L^p(\mathcal{O}_t)^n) \quad &\Longleftrightarrow \quad v \in L^q(\tau, T; L^p(\mathcal{O})^n), \\ u \in L^2(\tau, T; H_0^1(\mathcal{O}_t)^n) \quad &\Longleftrightarrow \quad v \in L^2(\tau, T; H_0^1(\mathcal{O})^n). \end{aligned}$$

Proof. In the same way as [38, Lemmas 3.3 and 3.4] or [21] □

Definition 2.5. *Given* $w \in L^p\big(\tau, T; H^{-1}(\mathcal{O}_t)^n\big)$ *as in Definition 2.2. We say that* $w \in C\big([\tau, T]; H^{-1}(\mathcal{O}_t)^n\big)$*, if the trivial extension* $\widehat{f}_{i_j}$ *belongs to* $C\big([\tau, T]; L^2(\mathbb{R}^n)\big)$ *for* $i = 1, \cdots, n$ *and* $j = 0, \cdots, n$*. We say that a sequence* $\{w^m\}$ *converges to* w *in* $C\big([\tau, T]; H^{-1}(\mathcal{O}_t)^n\big)$ *as* $m \to \infty$*, if the sequence* $\{\widehat{f}^m_{i_j}\}$ *converges to* $\widehat{f}_{i_j}$ *in* $C\big([\tau, T]; L^2(\mathbb{R}^n)\big)$ *as* $m \to \infty$*, for* $i = 1, \cdots, n$ *and* $j = 0, \cdots, n$*.*

Proposition 2.6. *(cf.* [3, Proposition II.5.11.]*) Let* X *and* Y *be two Banach spaces such that* X *is continuously and densely embedded into* Y*. Let* $p, r \in [1, +\infty]$*. For* $T > \tau$*, we define*

$$E_{p,r} = \Big\{ u \in L^p(\tau, T; X) \mid u' \in L^r(\tau, T; Y) \Big\}.$$

Then, any element u *of* $E_{p,r}$ *possesses a continuous representation on* $[\tau, T]$ *with values in* Y*, and the embedding of* $E_{p,r}$ *into* $C([\tau, T]; Y)$ *is continuous. Moreover, for all* $s, t \in [\tau, T]$*, we have*

$$u(t) - u(s) = \int_s^t \frac{du}{dt}(\theta) d\theta,$$

where it is understood that we have identified u *and its continuous representation.*

Lemma 2.7. *Under the assumptions given in (H1), let* $u(x,t)$ *and* $v(y,t)$ *be the functions satisfying the conditions given in* (2.3) *and* (2.4)*, respectively. Then*

(a) $u \in C([\tau, T]; L^2(\mathcal{O}_t)^n) \quad \Longleftrightarrow \quad v \in C([\tau, T]; L^2(\mathcal{O})^n)$,
(b) $u \in C([\tau, T]; H_0^1(\mathcal{O}_t)^n) \quad \Longleftrightarrow \quad v \in C([\tau, T]; H_0^1(\mathcal{O})^n)$,
(c) $u \in C([\tau, T]; H^{-1}(\mathcal{O}_t)^n) \quad \Longleftrightarrow \quad v \in C([\tau, T]; H^{-1}(\mathcal{O})^n)$,
(d) *given* $p, r \in [1, +\infty)$*, we have*

$$\begin{cases} u \in L^p(\tau, T; H_0^1(\mathcal{O}_t)^n), \\ u' \in L^r(\tau, T; H^{-1}(\mathcal{O}_t)^n) \end{cases} \Longleftrightarrow \begin{cases} v \in L^p(\tau, T; H_0^1(\mathcal{O})^n), \\ v' \in L^r(\tau, T; H^{-1}(\mathcal{O})^n). \end{cases}$$

Proof. It follows from [38, Lemmas 3.8 and 3.9] or [21, Lemma 3.8, 3.9 and 3.11]. □

Remark 2.8. *Let* $u \in L^2(\tau, T; H_0^1(\mathcal{O}_t)^n)$ *and* $u' \in L^p(\tau, T; H^{-1}(\mathcal{O}_t)^n)$ *for some* $p \in [1, +\infty)$*. Then,* $u \in C([\tau, T]; H^{-1}(\mathcal{O}_t)^n)$*. Indeed, by item (d) in Lemma 2.7 we have that* $v \in L^2(\tau, T; H_0^1(\mathcal{O})^n)$ *and* $v' \in L^p(\tau, T; H^{-1}(\mathcal{O})^n)$*. Then, by applying Proposition 2.6, we have that* $v \in C([\tau, T]; H^{-1}(\mathcal{O})^n)$*. Then, again by Lemma 2.7, item (c), we conclude that* $u \in C([\tau, T]; H^{-1}(\mathcal{O}_t)^n)$*.*

Before continuing, let us take into account the following results.

Lemma 2.9. *(cf.* [3, Lemma II.5.9]*) Let X be a separable and reflexive Banach space, and let Y be a Banach space, such that $X \hookrightarrow Y$ (continuous embedding). Then*

$$L^\infty(0,T;X) \cap C_w([0,T],Y) = C_w([0,T];X).$$

Remark 2.10. *A consequence of Lemma 2.7, Remark 2.8 and Lemma 2.9 is that, if*

$$u \in L^\infty(\tau,T;L^2(\mathcal{O}_t)^n) \cap L^2(\tau,T;H_0^1(\mathcal{O}_t)^n) \quad \textit{and} \quad u' \in L^p(\tau,T;H^{-1}(\mathcal{O}_t)^n),$$

for some $p \in [1,+\infty)$, then $u \in C_w([\tau,T];L^2(\mathcal{O}_t)^n)$. In the particular case that $p=2$, by [38, Lemmas 2.6] *or* [21] *we have that u belongs to $C([\tau,T];L^2(\mathcal{O}_t)^n)$ and satisfies the energy equality*

$$\|u(t)\|^2_{L^2(\mathcal{O}_t)^n} - \|u(s)\|^2_{L^2(\mathcal{O}_s)^n} = 2\int_s^t (u'(r),u(r))_{-1,r}dr \quad \textit{for all } \tau \leqslant s \leqslant t \leqslant T.$$

Lemma 2.11. *Under the assumptions given in (H1), we have*

$$\sum_{i=1}^n \frac{\partial}{\partial y_i}\left(\frac{\partial \overline{r}_i}{\partial x_j}(r(y,t),t)\right) = 0 \quad \textit{and} \quad \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial r_i}{\partial y_j}(\overline{r}(x,t),t)\right) = 0,$$

for all $j = 1,\cdots,n$, where $x = r(y,t)$ with $y \in \mathcal{O}$ and $t \in [\tau,T]$.

Proof. We will only consider the three-dimensional case, since the two-dimensional case was done in [38, Lemma 3.5]. Let us denote by

$$A(x,t) := \frac{\partial(\overline{r}_1(x,t),\overline{r}_2(x,t),\overline{r}_3(x,t))}{\partial(x_1,x_2,x_3)} = \begin{pmatrix} \frac{\partial \overline{r}_1}{\partial x_1}(x,t) & \frac{\partial \overline{r}_1}{\partial x_2}(x,t) & \frac{\partial \overline{r}_1}{\partial x_3}(x,t) \\ \frac{\partial \overline{r}_2}{\partial x_1}(x,t) & \frac{\partial \overline{r}_2}{\partial x_2}(x,t) & \frac{\partial \overline{r}_2}{\partial x_3}(x,t) \\ \frac{\partial \overline{r}_3}{\partial x_1}(x,t) & \frac{\partial \overline{r}_3}{\partial x_2}(x,t) & \frac{\partial \overline{r}_3}{\partial x_3}(x,t) \end{pmatrix}.$$

Let $M_{ij}(x,t)$ be the determinant of the submatrix formed by deleting the i-th row and j-th column of the matrix $A(x,t)$. Let us denote by $C_{ij} = (-1)^{i+j}M_{ij}(x,t)$ with $i,j = 1,2,3$. Let $C(x,t) = \Big(C_{ij}(x,t)\Big)_{i,j=1}^3$ be the co-factor matrix of the matrix $A(x,t)$.

Therefore, from the property of determinants, it follows that

$$\frac{\partial(r_1(y,t),r_2(y,t),r_3(y,t))}{\partial(y_1,y_2,y_3)} = J(t)\begin{pmatrix} C_{11}(r(y,t),t) & C_{21}(r(y,t),t) & C_{31}(r(y,t),t) \\ C_{12}(r(y,t),t) & C_{22}(r(y,t),t) & C_{32}(r(y,t),t) \\ C_{13}(r(y,t),t) & C_{23}(r(y,t),t) & C_{33}(r(y,t),t) \end{pmatrix}. \tag{2.5}$$

Then, using the equality (2.5), we obtain

$$\begin{aligned} \sum_{i=1}^3 \frac{\partial}{\partial y_i}\left(\frac{\partial \overline{r}_i}{\partial x_j}(r(y,t),t)\right) &= \sum_{i=1}^3 \left(\frac{\partial^2 \overline{r}_i}{\partial x_1 \partial x_j}\frac{\partial r_1}{\partial y_i}(y,t) + \frac{\partial^2 \overline{r}_i}{\partial x_2 \partial x_j}\frac{\partial r_2}{\partial y_i}(y,t) + \frac{\partial^2 \overline{r}_i}{\partial x_3 \partial x_j}\frac{\partial r_3}{\partial y_i}(y,t)\right) \\ &= J(t)\sum_{i=1}^3 \left(\frac{\partial^2 \overline{r}_i}{\partial x_1 \partial x_j}C_{i1}(r(y,t),t) + \frac{\partial^2 \overline{r}_i}{\partial x_2 \partial x_j}C_{i2}(r(y,t),t) + \frac{\partial^2 \overline{r}_i}{\partial x_3 \partial x_j}C_{i3}(r(y,t),t)\right) \\ &= J(t)\frac{\partial}{\partial x_j}\left(\frac{1}{J(t)}\right) = 0. \end{aligned}$$

The case $\sum_{i=1}^3 \frac{\partial}{\partial x_i}\left(\frac{\partial r_i}{\partial y_j}(\overline{r}(x,t),t)\right) = 0$ is similar. □

Lemma 2.12. *Under the assumptions given in (H1), suppose that the functions $u(x,t)$ and $v(y,t)$ satisfy the conditions given in* (2.3) *and* (2.4)*, respectively. Then we have*

$$\mathrm{div}\ u = 0 \quad \textit{if and only if} \quad \mathrm{div}\ v = 0.$$

Proof. It is a consequence of Lemma 2.11. □

Now let us build the spaces with free-divergence.

Definition 2.13. *Let us define*

$$\begin{aligned}&\mathcal{V} := \{\varphi \in C_c^\infty(\mathcal{O})^n : \operatorname{div}\varphi = 0\};\\ &H := \textit{closure of } \mathcal{V} \textit{ in the } L^2(\mathcal{O})^n - \textit{norm};\\ &V := \textit{closure of } \mathcal{V} \textit{ in the } H_0^1(\mathcal{O})^n - \textit{norm};\\ &V^* := \textit{the dual space of } V,\end{aligned}$$

and, for each $t \in \mathbb{R}$*, let us define by*

$$\begin{aligned}&\mathcal{V}_t := \{\varphi \in C_c^\infty(\mathcal{O}_t)^n : \operatorname{div}\varphi = 0\};\\ &H_t := \textit{closure of } \mathcal{V}_t \textit{ in the } L^2(\mathcal{O}_t)^n - \textit{norm};\\ &V_t := \textit{closure of } \mathcal{V}_t \textit{ in the } H_0^1(\mathcal{O}_t)^n - \textit{norm};\\ &V_t^* := \textit{the dual space of } V_t.\end{aligned}$$

Now, for each $t \in \mathbb{R}$, let us denote the inner product in H_t by

$$(u, \widetilde{u})_t := \sum_{i=1}^n \int_{\mathcal{O}_t} u_i(x)\widetilde{u}_i(x)dx \quad \text{for any } u, \widetilde{u} \in H_t,$$

with norm $|\cdot|_t = (\cdot,\cdot)_t^{1/2}$. The inner product in V_t is defined by

$$(\nabla u, \nabla \widetilde{u})_t := \sum_{i,j=1}^n \int_{\mathcal{O}_t} \frac{\partial u_i}{\partial x_j}(x)\frac{\partial \widetilde{u}_i}{\partial x_j}(x)dx \quad \text{for any } u, \widetilde{u} \in V_t.$$

Let us denote the dual space of V_t by V_t^* and $(\cdot,\cdot)_{-1,t}$ the duality between V_t^* and V_t.

Take into account that

$$\begin{aligned}\frac{\partial u_i}{\partial x_j}(x) &= \sum_{k,p=1}^n \left[\frac{\partial^2 r_i}{\partial y_p \partial y_k}(\overline{r}(x,t),t)\frac{\partial \overline{r}_p}{\partial x_j}(x,t)v_k(\overline{r}(x,t)) + \frac{\partial r_i}{\partial y_k}(\overline{r}(x,t),t)\frac{\partial \overline{r}_p}{\partial x_j}(x,t)\frac{\partial v_k}{\partial y_p}(\overline{r}(x,t))\right],\\ \frac{\partial v_k}{\partial y_l}(y) &= \sum_{i,q=1}^n \left[\frac{\partial^2 \overline{r}_k}{\partial x_q \partial x_i}(r(y,t),t)\frac{\partial r_q}{\partial y_l}(y,t)u_i(r(y,t)) + \frac{\partial \overline{r}_k}{\partial x_i}(r(y,t),t)\frac{\partial r_q}{\partial y_l}(y,t)\frac{\partial u_i}{\partial x_q}(r(y,t))\right].\end{aligned}$$

Thus, for each $t \in \mathbb{R}$ let us denote an inner product in H by

$$\langle v, \widetilde{v}\rangle_t = \sum_{k,l=1}^n \int_{\mathcal{O}} g_{kl}(y,t)v_k(y)\widetilde{v}_l(y)\operatorname{Jac}(r,y,t)dy \quad \text{for any } v, \widetilde{v} \in H. \tag{2.6}$$

for any $v, \widetilde{v} \in V$, where $g_{kl}(y,t) = \sum_{i=1}^n \frac{\partial r_i}{\partial y_k}\frac{\partial r_i}{\partial y_l}$. Also, for each $t \in \mathbb{R}$ let us denote an inner product of V by

$$\langle \nabla_g v, \nabla_g \widetilde{v}\rangle_t = \sum_{i,k,l,p,q=1}^n \int_{\mathcal{O}} g^{pq}(y,t)\frac{\partial}{\partial y_p}\left(\frac{\partial r_i}{\partial y_k}(y,t)v_k(y)\right)\frac{\partial}{\partial y_q}\left(\frac{\partial r_i}{\partial y_l}(y,t)\widetilde{v}_l(y)\right)\operatorname{Jac}(r,y,t)dy, \tag{2.7}$$

for any $v, \widetilde{v} \in V$, where $g^{pq}(y,t) = \sum_{j=1}^n \frac{\partial \overline{r}_p}{\partial x_j}(r(y,t),t)\frac{\partial \overline{r}_q}{\partial x_j}(r(y,t),t)$ $(k,l,p,q = 1,\dots,n)$ and the identity $\overline{r}(r(y,t),t) = y$ implies that $\overline{r}_p(r(y,t),t) = y_p$, and $\operatorname{Jac}(r,y,t)$ denotes the absolute value of the determinat of the Jacobi matrix $\left(\frac{\partial r_i}{\partial y_j}(y,t)\right)_{n\times n}$. Similarly, we denote the dual space of V by V^*, and $\langle\cdot,\cdot\rangle_{-1,t}$ the duality between V and V^*.

Lemma 2.14. *Under the assumptions given in (H1), let* $u(x,t)$ *and* $v(y,t)$ *be the functions satisfying the conditions given in* (2.3) *and* (2.4)*, respectively. Then*

$$u \in C([\tau,T];H_t) \qquad \Longleftrightarrow \qquad v \in C([\tau,T];H),$$

and, given $p, r \in [1,+\infty)$*, we have*

$$\begin{cases} u \in L^r(\tau,T;V_t),\\ u' \in L^p(\tau,T;V_t^*)\end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} v \in L^r(\tau,T;V),\\ v' \in L^p(\tau,T;V^*).\end{cases}$$

Proof. It is a consequence of applying the Lemma 2.7 and Lemma 2.12. □

2.2. **Existence of weak solutions.** Inspired by [32, Section 1] and [38, Subsection 3.4], we consider $v \in L^1_{loc}\big(\mathcal{O}\times(\tau,T)\big)^n$ given in (2.3). If $u \in L^1_{loc}(Q_{\tau,T})^n$ is a solution of (NS) in the sense that u satisfies each one of the equations given in (NS) (strong solution), then the function v satisfies the following equation

$$
\begin{cases}
\sum_{k=1}^{n} \frac{\partial r_i}{\partial y_k}(y,t)\Big\{ v_k'(y,t) + \frac{\partial \overline{r}_k}{\partial x_i}(r(y,t),t)\Big[\frac{\partial^2 r_i}{\partial t\partial y_k}(y,t)v_k(y,t) \\
+ \sum_{p=1}^{n} \frac{\partial}{\partial y_p}\Big(\frac{\partial r_i}{\partial y_k}(y,t)v_k(y,t)\Big)\frac{\partial \overline{r}_p}{\partial t}(r(y,t),t) \\
- \sum_{l,p=1}^{n} \frac{\partial^2}{\partial y_p\partial y_l}\Big(\frac{\partial r_i}{\partial y_k}(y,t)v_k(y,t)\Big)g^{lp}(y,t) \\
- \sum_{p=1}^{n} \frac{\partial}{\partial y_p}\Big(\frac{\partial r_i}{\partial y_k}(y,t)v_k(y,t)\Big)\Delta_x\overline{r}_p(r(y,t),t) \\
+ \sum_{j,l,p=1}^{n} \frac{\partial r_j}{\partial y_l}(y,t)\frac{\partial \overline{r}_p}{\partial x_j}(r(y,t),t)v_l(y,t)\frac{\partial}{\partial y_p}\Big(\frac{\partial r_i}{\partial y_k}(y,t)v_k(y,t)\Big)\Big]\Big\} \\
= \sum_{k=1}^{n} \frac{\partial r_i}{\partial y_k}(y,t)\Big[(f_g)_k(y,t) - (\nabla_x\pi_g)_k(y,t)\Big], \qquad i=1,\cdots,n,\ y\in\mathcal{O},\ t>\tau, \\
\operatorname{div} v(y,t) = 0, \qquad y\in\mathcal{O},\ t\geqslant\tau, \\
v(y,t) = 0, \qquad y\in\partial\mathcal{O},\ t>\tau, \\
v(y,\tau) = v_\tau(y), \qquad y\in\mathcal{O},
\end{cases}
\tag{2.8}
$$

where $g^{lp}(y,t) = \sum_{j=1}^n \frac{\partial \overline{r}_l}{\partial x_j}(r(y,t),t)\frac{\partial \overline{r}_p}{\partial x_j}(r(y,t),t)$, and $\Delta_x\overline{r}_p(r(y,t),t) = \sum_{j=1}^n \frac{\partial^2 \overline{r}_p}{\partial x_j^2}(r(y,t),t)$, $j,l,p = 1,2,3$. Note that, if v satisfies the (2.8), then u satisfies the system (NS). For ease of notation, the system (2.8), which we will indicate by (CC-NS) "Change Coordinates of Navier-Stokes equations", is rewritten in a more compact way as

$$
\begin{cases}
\frac{\partial v}{\partial t} + Mv(y,t) - Lv(y,t) + Nv(y,t) = f_g(y,t) - \nabla_x\pi_g & y\in\mathcal{O},\ t>\tau, \\
\operatorname{div}\ v(y,t) = 0 & y\in\mathcal{O},\ t\geqslant\tau, \\
v(y,t) = 0 & y\in\partial\mathcal{O},\ t>\tau, \\
v(y,\tau) = v_\tau(y) & y\in\mathcal{O},
\end{cases}
\tag{CC-NS}
$$

where $Mv := \big((Mv)_1,\cdots,(Mv)_n\big)$, $Lv := \big((Lv)_1,\cdots,(Lv)_n\big)$, $Nv := \big((Nv)_1,\cdots,(Nv)_n\big)$, $f_g := \big((f_g)_1,\cdots,(f_g)_n\big)$ and $\nabla_x\pi_g := \big((\nabla_x\pi_g)_1,\cdots,(\nabla_x\pi_g)_n\big)$ are defined as:

$$(Mv)_k(y,t) := \sum_{i=1}^{n}\frac{\partial \overline{r}_k}{\partial x_i}(r(y,t),t)\left[\frac{\partial^2 r_i}{\partial t\partial y_k}(y,t)v_k(y,t) + \sum_{p=1}^{n}\frac{\partial}{\partial y_p}\Big(\frac{\partial r_i}{\partial y_k}(y,t)v_k(y,t)\Big)\frac{\partial \overline{r}_p}{\partial t}(r(y,t),t)\right];$$

$$(Lv)_k(y,t) := \sum_{i=1}^{n}\frac{\partial \overline{r}_k}{\partial x_i}(r(y,t),t)\left[\sum_{l,p=1}^{n}\frac{\partial^2}{\partial y_p\partial y_l}\Big(\frac{\partial r_i}{\partial y_k}(y,t)v_k(y,t)\Big)g^{lp}(y,t) + \sum_{p=1}^{n}\frac{\partial}{\partial y_p}\Big(\frac{\partial r_i}{\partial y_k}(y,t)v_k(y,t)\Big)\Delta_x\overline{r}_p(r(y,t),t)\right];$$

$$(Nv)_k(y,t) := \sum_{i=1}^{n}\frac{\partial \overline{r}_k}{\partial x_i}(r(y,t),t)\left[\sum_{j,l,p=1}^{n}\frac{\partial r_j}{\partial y_l}(y,t)\frac{\partial \overline{r}_p}{\partial x_j}(r(y,t),t)v_l(y,t)\frac{\partial}{\partial y_p}\Big(\frac{\partial r_i}{\partial y_k}(y,t)v_k(y,t)\Big)\right];$$

$$(f_g)_k(y,t) := \sum_{i=1}^{n}\frac{\partial \overline{r}_k}{\partial x_i}(r(y,t),t)f_i(r(y,t),t);$$

$$\big(\nabla_x\pi_g\big)_k(y,t) := \sum_{i=1}^{n}\frac{\partial \overline{r}_k}{\partial x_i}(r(y,t),t)\big(\nabla\pi\big)_i(r(y,t),t).$$

The Navier-Stokes equations are associated with the trilinear function $b_t(u,v,w)$ (convective term of (NS)) defined as

$$b_t(u,v,w)=\sum_{i,j=1}^{n}\int_{\mathcal{O}_t}u_j(x)\frac{\partial v_i}{\partial x_j}(x)w_i(x)dx.$$

For simplicity, we indicate $b_t(u,v,w)$ as $b_t(u,v,w)=\int_{\mathcal{O}_t}u_j\frac{\partial v_i}{\partial x_j}w_i dx$. Then, cf. [35, Chapter 9, pp. 243] or [39, Chapter III], we have the following inequalities on $b_t(\cdot,\cdot,\cdot)$.

Lemma 2.15. *In (2D) and (3D) cases, we have*

(1) $b_t(u,v,w)=-b_t(u,w,v)$ *for all* $u\in H_t, v,w\in V_t$*;*
(2) $b_t(u,v,v)=0$ *for all* $u\in H_t, v,w\in V_t$*;*
(3)

$$|b_t(u,v,w)|\leqslant c_t\begin{cases}\|u\|_{L^6(\mathcal{O}_t)^n}|\nabla v|_t|w|_t^{1/2}\|w\|_{L^6(\mathcal{O}_t)^n}^{1/2}, & \text{if } n\in\{2,3\},\ u,v,w\in V_t,\\ |u|_t^{1/2}|\nabla u|_t^{1/2}|\nabla v|_t|w|_t^{1/2}|\nabla w|_t^{1/2}, & \text{if } n=2,\ u,v,w\in V_t,\\ |u|_t^{1/4}|\nabla u|^{3/4}|\nabla v|_t|w|_t^{1/4}|\nabla w|_t^{3/4}, & \text{if } n=3,\ \forall u,v,w\in V_t;\end{cases} \tag{2.9}$$

(4) *for all* $u\in V_t, v\in D_t(A), w\in H_t$*,*

$$|b_t(u,v,w)|\leqslant c_t\begin{cases}|u|_t^{1/2}|\nabla u|_t^{1/2}|\nabla v|^{1/2}|Av|_t^{1/2}|w|_t, & \text{if } n=2,\\ |\nabla u|_t|\nabla v|_t^{1/2}|Av|^{1/2}|w|_t, & \text{if } n=3;\end{cases} \tag{2.10}$$

where $A=-\mathbb{P}\Delta$ *is the Stokes operator with domain* $D_t(A)=H^2(\mathcal{O}_t)^n\cap V_t$ *(note that* $\mathbb{P}$ *is the Leray's operator e.g.* [28, Appendix A.4]*);*

(5) *by* [40, Inequality (3.79)], *for* $n=3$ *we have*

$$b_t(w,w,u)\leqslant c_t|w|_t^{1/4}|\nabla w|_t^{7/4}\|u\|_{L^4(\mathcal{O}_t)^3}. \tag{2.11}$$

Definition 2.16. *Let* $n\in\{2,3\}$*,* $u_\tau\in H_\tau$*,* $f\in L^2_{loc}(\mathbb{R};V_t^*)$*. We say that a function* $u(\cdot)=u(\cdot;\tau,u_\tau)\in L^1_{loc}(Q_\tau)^n$ *is a weak global solution of* (NS) *if for any* $T>\tau$ *the function* $u(\cdot)$ *belong to the class*

$$u\in L^\infty(\tau,T;H_t)\cap L^2(\tau,T;V_t)\qquad \textit{with}\qquad \frac{\partial u}{\partial t}\in L^p(\tau,T;V_t^*) \tag{2.12}$$

where $p=2$ *if* $n=2$ *and* $p=4/3$ *if* $n=3$*, and satisfies the following weak formulation*

$$\begin{aligned}-\int_\tau^T(u(t),\varphi'(t))_t dt+\int_\tau^T(\nabla u(t),\nabla\varphi(t))_t dt+\int_\tau^T b_t(u(t),u(t),\phi(t))dt\\ =(u_\tau,\varphi(\tau))_\tau+\int_\tau^T(f(t),\varphi(t))_{-1,t}dt,\end{aligned} \tag{2.13}$$

for all $\varphi\in\mathcal{U}_{\tau,T}:=\{\varphi\in C([\tau,T];V_t):\varphi_k(x,t)=h(t)\sum_{i=1}^n\frac{\partial r_k}{\partial y_i}(\overline{r}(x,t),t)\cdot v_i(\overline{r}(x,t)),\ k=1,\cdots,n,\ v\in V, h\in C^1([\tau,T];\mathbb{R}),\ h(T)=0\}$*, and also satisfies the initial condition* $u(\tau)=u_\tau$*.*

Remark 2.17. *As is known, the way to show the existence of at least one weak global solution to the system* (NS) *is to first show that the system* (CC-NS) *has at least one weak global solution, i.e., for any* $T>\tau$ *the function* $v(\cdot)$ *belong to the class*

$$v\in L^\infty(\tau,T;H)\cap L^2(\tau,T;V)\qquad \textit{with}\qquad \frac{\partial v}{\partial t}\in L^p(\tau,T;V^*) \tag{2.14}$$

where $p=2$ *if* $n=2$ *and* $p=4/3$ *if* $n=3$*, and satisfies the following weak formulation*

$$\begin{aligned}-\int_\tau^T\langle v(t),\varphi'(t)\rangle_t dt+\int_\tau^T\langle Mv(t)-Lv(t)+Nv(t),\varphi(t)\rangle_t dt\\ =\langle v_\tau,\varphi(\tau)\rangle_\tau+\int_\tau^T\langle f_g(t),\varphi(t)\rangle_{-1,t}dt,\end{aligned} \tag{2.15}$$

for all $\varphi(t)=h(t)v$*, with* $v\in V$*,* $h\in C^1([\tau,T];\mathbb{R})$ *such that* $h(T)=0$*, and also satisfies the initial condition* $v(\tau)=v_\tau$*. Therefore, if* v *belonging to class* (2.14) *is a weak solution of* (CC-NS)*, then by Lemmas 2.12 and*

2.14 the function u (defined in (2.4)) belongs to class (2.12) and is a weak solution of (NS). Then all this is summarized in the following result.

Theorem 2.18. *(Existence of weak solutions) Under the hypotheses* **(H1)**-**(H2)** *and $n \in \{2,3\}$. Let us consider τ, T with $\tau < T$. Then, for any $u_\tau \in H_\tau$, and $f \in L^2(\tau,T;V_t^*)$, there exists at least one weak solution of the problem (NS). In the case that the dimension $n = 2$ the weak solutions to (NS) are unique in the class of the weak solutions.*

Proof. The proof of the existence of at least one weak solution to (NS) is given in [32, Theorem 2.2] or [38, Theorem 4.5], where is proved that there exists a function $v : \mathcal{O} \times (\tau,T) \to \mathbb{R}^n$ belonging to the class $v \in L^\infty(\tau,T;H) \cap L^2(\tau,T;V)$ such that satisfies the weak formulation (2.15). Thus, thanks to the previous results the function $u : Q_{\tau,T} \to \mathbb{R}^n$ given in (2.4) belong to the class $L^\infty(\tau,T;H_t) \cap L^2(\tau,T;V_t)$ and satisfies the weak formulation (2.13).

It follows from [38, Lemma 4.3] that $u' \in L^2(\tau,T;V_t^*)$ in the case $n = 2$. Now, we will focus on showing that $u' \in L^{4/3}(\tau,T;V_t^*)$ in the case $n = 3$. For each $t \in \mathbb{R}$, let $\{w_k(t) = w_k(y,t)\}_{k=1}^\infty$ be the eigenfunctions of $-\Delta$ on V, with respect the inner products given in (2.6) and (2.7), satisfying homogeneous Dirichlet boundary conditions, i.e.,

$$\langle \nabla_g w_k(t), \nabla_g v\rangle_t = \lambda_k(t)\langle w_k(t), v\rangle_t \quad \text{for all } v \in V,\ t \in \mathbb{R}.$$

Thus, we can assume that $\{w_k(t)\}_{k=1}^\infty$ is an orthogonal basis of V and orthonormal basis of H. Now, for each $k \in \mathbb{N}$, let us consider $\overline{w}_k = (\overline{w}_k^1, \overline{w}_k^2, \overline{w}_k^3)$

$$\overline{w}_k^i(x,t) = \sum_{j=1}^{3} \frac{\partial r_i}{\partial y_j}(\overline{r}(x,t),t) w_k^j(\overline{r}(x,t)) \quad \text{for } i = 1,2,3.$$

Then, for each $t \in \mathbb{R}$ it is easy see that $\{\overline{w}_k(t)\}_{k=1}^\infty$ is an orthogonal basis of V_t and H_t.

On the other hand, let us remember the approximate Galerkin solutions of (CC-NS) which are written as follows

$$v_m(y,t) = \sum_{k=1}^{m} h_k^m(t) w_k(t),$$

where the coefficients $h_1^m, \cdots, h_m^m$ with $m \in \mathbb{N}$ are solutions of the following ordinary differential system

$$\begin{cases} \langle v_m'(t), w_k(t)\rangle_t = \langle Lv_m(t) - Mv_m(t) - Nv_m(t), w_k(t)\rangle_t + \langle f_g(t), w_k(t)\rangle_{-1,t}, \\ h_k^m(\tau) = \langle v_\tau, w_k(\tau)\rangle_\tau, \end{cases}$$

where $k = 1, \cdots, m$. In the same way as [38, Lemma 4.2] or [32, Theorem 2.2], the sequence $\{v_m\}$ is bounded in $L^\infty(\tau,T;H) \cap L^2(\tau,T;V)$. Now, thanks to (2.4) we can consider the sequence $\{u_m\}$ given by

$$u_m(x,t) = \sum_{k=1}^{m} h_k^m(t)\overline{w}_k(x,t), \quad \text{with } x \in \mathcal{O}_t,\ t \geqslant \tau.$$

Then, we have that the sequence $\{u_m\}$ is bounded in $L^\infty(\tau,T;H_t) \cap L^2(\tau,T;V_t)$, and for each $m \in \mathbb{N}$, u_m satisfies

$$\overline{P}_m\big(u_m'(t)\big) = \overline{P}_m\big(\Delta u_m(t)\big) - \overline{P}_m\big((u_m(t)\cdot\nabla)u_m(t)\big) + \overline{P}_m\big(f(t)\big),$$

where $\overline{P}_m : H_t \to \operatorname{Span}\{\overline{w}_1, \cdots, \overline{w}_m\}$ is defined as

$$\overline{P}_m g = \sum_{k=1}^{m} (g, \overline{w}_k)_t \overline{w}_k.$$

Indeed, by using (2.9), for any $v \in V_t$, we have that

$$\big((u_m\cdot\nabla)u_m, v\big)_{-1,t} \leqslant \big|b_t(u_m, v, u_m)\big| \leqslant c_t|u_m|_t^{1/2}|\nabla u_m|_t^{3/2}|\nabla v|_t$$

for all $v \in V_t$. Then we obtain

$$\big|(u_m\cdot\nabla)u_m\big|_{-1,t} = \sup_{v\in V_t,\ |\nabla v|_t=1} \big|\big((u_m\cdot\nabla)u_m, v\big)_{-1,t}\big| \leqslant c_t|u_m|_t^{1/2}|\nabla u_m|_t^{3/2}.$$

Note that,

$$\int_\tau^T \big|(u_m\cdot\nabla)u_m\big|_{-1,t}^{4/3}dt \leqslant \hat{c}_t\|u_m\|^{2/3}_{L^\infty(\tau,T;H_t)}\int_\tau^T |\nabla u_m|_t^2 dt < +\infty.$$

Therefore, $\overline{P}_m\big((u_m\cdot\nabla)u_m\big)$ is bounded in $L^{4/3}(\tau,T;V_t^*)$. Then, we conclude that the sequence $\{u_m'\}$ is bounded in $L^{4/3}(\tau,T;V_t^*)$. □

Remark 2.19. *(Energy, e.g.* [3, Theorem V.1.4]*) Under conditions of Theorem 2.18 and $f\in L^2_{loc}([\tau,+\infty),V_t^*)$, we have that*
• For $n=2$, the weak solution constructed from the Galerkin method is unique and also $u\in C([\tau,+\infty);H_t)$. Moreover, any weak solution satisfies the following energy equality

$$|u(t)|_t^2 + 2\int_s^t |\nabla u(r)|_r^2 dr = |u(s)|_s^2 + 2\int_s^t (f(r),u(r))_{-1,r}dr \quad \forall \tau\leqslant s\leqslant t, \tag{2.16}$$

• For $n=3$, the weak solution constructed from the Galerkin method is weakly continuous, that is, $u\in C_w([\tau,+\infty);H_t)$, see Remark 2.10. Moreover, it satisfies the following energy inequality

$$|u(t)|_t^2 + 2\int_s^t |\nabla u(r)|_r^2 dr \leqslant |u(s)|_s^2 + 2\int_s^t (f(r),u(r))_{-1,r}dr \quad \forall \tau\leqslant s\leqslant t. \tag{2.17}$$

It should be noted that, in the case of dimension $n=3$, we cannot guarantee that any weak solution satisfies the energy inequality given in (2.17)*, cf.* [3, Chapter V].

2.3. **A conditional existence result for 3D-Navier-Stokes equations.** Since our objective, in the remainder of the manuscript, is to demonstrate the existence of a pullback attractor associated with the dynamical system defined from the weak solutions of the system (NS), in the same way as [18] we need an additional hypothesis about the existence of our weak solutions, that is

(HF) Let $n=3$ and $f\in L^2_{loc}(\tau,+\infty;V_t^*)$. For any $u_\tau\in V_\tau$ assume the existence of a globally defined weak solution such that for any $T>\tau$

$$\|u(t)\|_{L^4(\mathcal{O}_t)^3} \leqslant F\big(|\nabla u_\tau|_\tau,\tau,T\big) \quad \text{for all } t\in[\tau,T], \tag{2.18}$$

where F is continuous and non-decreasing with respect to the first variable, and non-increasing with respect to the second variable.

This new hypothesis will allow us to show that our weak solutions have a representative in the space of continuous functions, i.e., if **(HF)** is valid, any weak solution u of 3D-(NS) belongs to $C([\tau,T];H)$ and satisfies the energy inequality (2.17). Then, this additional regularity will allow us to prove asymptotic compactness by employing the energy method, which is an important ingredient in the construction of the pullback attractor.

Now, in the same way as [18], we will demonstrate a series of results, which under hypothesis (HF) will allow us to obtain weak solutions, with continuous representatives, that satisfy the energy inequality given in (2.17).

Lemma 2.20. *Under conditions of the Theorem 2.18 and the hypothesis* **(HF)**. *Then for any $u_\tau\in V_\tau$ the weak solution given in* **(HF)** *satisfies*

$$u\in C([\tau,+\infty);H_t), \quad u'\in L^2_{loc}(\tau,+\infty;V_t^*) \tag{2.19}$$

and

$$|u(t)|_t^2 + 2\int_s^t |\nabla u(r)|_r^2 dr \leqslant |u(s)|_s^2 + 2\int_s^t (f(r),u(r))_{-1,r}dr, \quad \forall \tau\leqslant s\leqslant t. \tag{2.20}$$

Also, u is unique in the class of weak solutions such that $u\in L^8_{loc}(\tau,+\infty;L^4(\mathcal{O}_t)^3)$. Moreover, u is unique in the class of weak solutions satisfying (2.20).

Proof. Since, by hypothesis **(HF)**, $u\in L^\infty_{loc}(\tau,+\infty;L^4(\mathcal{O}_t)^3)\subset L^8_{loc}(\tau,+\infty;L^4(\mathcal{O}_t)^3)$, in the same way as [3, Theorem II.5.16] and [40, Theorem 3.4] we have that $u\in C([\tau,+\infty);H_t)$, $u'\in L^2_{loc}(\tau,+\infty;V_t^*)$. Also, the energy inequality is proved in a standard way since it is possible to use $u(t)$ as a test function. Regarding the uniqueness, let us take into account the following: let $u(t)$, $v(t)$ be two weak solutions of 3D-(NS) associated with the initial conditions u_0 and v_0, respectively. Now, denote by $w := u - v$ and take w as a test function in the weak formulation, thus we have

$$\frac{1}{2}\frac{d}{dt}|w|_t^2 + |\nabla w|_t^2 = -2b_t(w, w, u).$$

From (2.11) we have that

$$b_t(w, w, u) \leqslant c_t|w|_t^{1/4}|\nabla w|_t^{7/4}\|u\|_{L^4(\mathcal{O}_t)^3}.$$

By applying Young's inequality we get

$$\frac{d}{dt}|w|_t^2 \leqslant \tilde{c}_t\|u\|^8_{L^4(\mathcal{O}_t)^3}|w|_t^2.$$

Then the uniqueness is immediate. □

Under hypothesis **(HF)** the following result will prove the existence of weak continuous global solutions for initial data in H_τ with any $\tau \in \mathbb{R}$.

Theorem 2.21. *Under conditions of the Theorem 2.18 and the hypothesis* **(HF)***, we have that for any $u_\tau \in H_\tau$ there exists at least one weak solution such that*

$$u \in C([\tau, +\infty); H_t), \ \ \textit{and} \ \ u \in L^\infty(s, T; L^4(\mathcal{O}_t)^3) \ \textit{for all} \ \tau < s < T, \tag{2.21}$$

and

$$\|u\|_{L^\infty(\tau+\delta,T;L^4(\mathcal{O}_t)^3)} \leqslant G(|u_\tau|_\tau, \tau, T, \delta), \tag{2.22}$$

for all $T > \tau$, $0 < \delta < T - \tau$, where $\theta \mapsto G(\theta, \tau, T, \delta)$ is non-decreasing and continuous. Also, $\tau \mapsto G(x, \tau, T, \delta)$ is non-decreasing. Moreover, u satisfies the energy inequality (2.20).

Proof. Let $T > \tau$ be fixed, and let us consider a sequence $\{u_\tau^m\} \subset V_\tau$ such that $u_\tau^m \to u_\tau$ in H_τ. Thus, it follows from Lemma 2.20 that there exists a sequence of weak solutions $\{u_m\}$ of 3D-(NS) satisfying, the inequality (2.18) and, by (2.20), the following energy inequality

$$|u_m(t)|_t^2 + \int_\tau^t |\nabla u_m(r)|_r^2 dr \leqslant |u_\tau^m|_\tau^2 + \int_\tau^T |f(r)|^2_{-1,r} dr. \tag{2.23}$$

Thus, taking into account $\|b_t(u_m, u_m, \cdot)\|_{V_t^*} \leqslant c_t|u_m|_t^{1/2}\|u_m\|_t^{3/2}$ and Lemma 2.14, we obtain, up to a subsequence, that

$$\begin{cases} u_m \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(\tau, T; H_t), \\ u_m \rightharpoonup u & \text{weakly in } L^2(\tau, T; V_t), \\ \dfrac{\partial u_m}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} & \text{weakly in } L^{4/3}(\tau, T; V_t^*), \\ u_m \to u & \text{strongly in } L^2(\tau, T; H_t). \end{cases} \tag{2.24}$$

Note that, with the convergences given in (2.24), it is possible to show that u is a weak solution of 3D-(NS) corresponding to the initial condition $u_\tau \in H_\tau$, and such that $u \in C_w([\tau, T]; H_t)$, see Remark 2.10.

On the other hand, let us fix $t > \tau$. By using (2.23) we have that

$$\int_\tau^{\frac{t+\tau}{2}} |\nabla u_m(r)|_r^2 dr \leqslant \int_\tau^{\frac{t+\tau}{2}} \frac{2}{t-\tau}\left(|u_\tau^m|_\tau^2 + \int_\tau^T |f(s)|^2_{-1,s} ds\right) dr.$$

Then, for each m there is $t_m^* \in (\tau, \frac{\tau+t}{2})$ such that

$$|\nabla u_m(t_m^*)|^2_{t_m^*} \leqslant \frac{2}{t-\tau}\left(|u_\tau^m|_\tau^2 + \int_\tau^T |f(s)|^2_{-1,s} ds\right). \tag{2.25}$$

Thus, it follows from Lemma (2.20) that $u_m(t)$ is the unique weak solution to 3D-(NS) on $[t_m^*, T]$ with $u_{t_m^*} = u_m(t_m^*)$ and satisfying the energy inequality (2.17). Therefore, by applying the hypothesis **(HF)** on $[t_m^*, T]$ we get

$$\|u_m(s)\|_{L^4(\mathcal{O}_s)^3} \leqslant F\left(\left(\frac{M}{t-\tau}\right)^{1/2}(1 + |u_\tau|_\tau^2)^{1/2}, \tau, T\right) =: G(|u_\tau|_\tau, \tau, T, t-\tau),$$

for all $s \in [t_m^*, T]$, where $M := 2\max\left\{1, \int_\tau^T |f(s)|_{-1,s}^2 ds\right\}$.

Now, let $\delta \in (0, T-\tau)$, and let us take $t = \tau + \delta$. Then we obtain that

$$\|u_m(s)\|_{L^4(\mathcal{O}_s)^3} \leqslant G(|u_\tau|_\tau, \tau, T, \delta), \tag{2.26}$$

for all $\tau + \delta \leqslant s \leqslant T$ (note that $\tau + \delta \geqslant t_m^*$ for all m, since $t_m^* \in (\tau, \frac{\tau+t}{2})$ for all m).

By using Hölder's inequality it is easy to see that

$$|b_t(u,v,w)| \leqslant c_t \|u\|_{L^4(\mathcal{O}_t)^3} |\nabla w|_t \|v\|_{L^4(\mathcal{O}_t)^3} \quad \text{for all } u, v, w \in V_t,\ t \in \mathbb{R}.$$

Thus, using (2.26) we obtain that u_m' is bounded in $L^2(\tau+\delta, T; V_t^*)$. Then, up to a subsequence, we have that

$$\begin{cases} u_m \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(\tau+\delta, T; L^4(\mathcal{O}_t)^3), \\ u_m' \rightharpoonup u' & \text{weakly in } L^2(\tau+\delta, T; V_t^*). \end{cases} \tag{2.27}$$

Thus, since $u \in L^2(\tau, T; V_t)$, $u' \in L^2(\tau+\delta, T; V_t^*)$, we have that $u \in C([\tau+\delta, T]; H_t)$, see Remark 2.10. In a standard way, it is proven that u satisfies the energy inequality

$$|u(t)|_t^2 + 2\int_s^t |\nabla u(r)|_r^2 dr - 2\int_\tau^t (f(r), u(r))_{-1,r} dr \leqslant |u(\tau)|_\tau^2. \tag{2.28}$$

Note that, it follows from (2.28) that

$$\limsup_{t\to\tau^+} |u(t)|_t^2 \leqslant |u(\tau)|_\tau^2.$$

Since $u \in C_w([\tau, T]; H)$, thus we will deduce that $|u(t)|_t \to |u(\tau)|_\tau$ as $t \to \tau^+$, so that, by Definition 2.1, we have that

$$\begin{aligned} \limsup_{t\to\tau^+} |||\widehat{u}(t) - \widehat{u}(\tau)|||^2 &= \limsup_{t\to\tau^+} |||\widehat{u}(t)|||^2 - 2\liminf_{t\to\tau^+}((\widehat{u}(t), \widehat{u}(\tau))) + \limsup_{t\to\tau^+} |||\widehat{u}(\tau)|||^2 \\ &\leqslant |||\widehat{u}(\tau)||| - 2((\widehat{u}(\tau), \widehat{u}(\tau))) + |||\widehat{u}(\tau)|||^2 = 0. \end{aligned}$$

Therefore, we deduce that $\widehat{u}(t) \to \widehat{u}(\tau)$ in $L^2(\mathbb{R}^3)^3$ also. Hence, $u \in C([\tau, T]; H)$.

On the other hand, it follows from (2.26) and (2.27), we have that

$$\|u\|_{L^\infty(\tau+\delta, T; L^4(\mathcal{O}_t)^3)} \leqslant G(|u_\tau|_\tau, \tau, T, \delta).$$

Then, (2.22) hold and therefore (2.21) is proved. □

Corollary 2.22. *Under conditions of Theorem 2.18 and the hypothesis* **(HF)**, $u_\tau \in H_\tau$ *and* $f \in L^2_{loc}(\tau, +\infty; V_t^*)$, *for every globally defined weak solution satisfying* (2.20) *we have that*

$$u \in C([\tau, +\infty); H_t), \quad u \in L^\infty_{loc}\left(s, +\infty; L^4(\mathcal{O}_t)^3\right), \quad \textit{for all } s > \tau. \tag{2.29}$$

Proof. We know that $u \in L^2_{loc}(\tau, +\infty; V_t)$. Thus, for any $t > \tau$ there exists $s \in (\tau, t)$ such that $u(s) \in V_s$. Then, it follows from Lemma (2.20) that u is the unique weak solution to 3D-(NS) on $[s, +\infty)$ with initial data $u(s) \in V_s$ and satisfying (2.20). Therefore, we have $u \in C([t, +\infty); H_t)$ for any $t > \tau$, then $u \in C((\tau, +\infty); H_t)$. Now we prove the continuity of $u(\cdot)$ at $t = \tau$. We know that $u \in C_w([\tau, +\infty), H_t)$, so that

$$|u(\tau)|_\tau \leqslant \liminf_{t\to\tau^+} |u(t)|_t.$$

On the other hand, by energy inequality (2.17), we obtain

$$\limsup_{t\to\tau^+} |u(t)|_t \leqslant |u(\tau)|_\tau.$$

Then, we have that $\lim_{t\to\tau} |u(t)|_t = |u(\tau)|_\tau$ and u is continuous on $[\tau, +\infty)$. Now condition **(HF)** implies that $u \in L^\infty_{loc}(s, +\infty; L^4(\mathcal{O}_t)^3)$ for all $s > \tau$. Therefore, the result is proved. □

3. Dynamics of 3D-Navier-Stokes equations on a non-cylindrical domain

In this section, we extend some well-known concepts and results concerning non-autonomous dynamical systems to multi-valued processes that act on families of metric spaces parameterized in time, and we will establish some results about the existence and relationships of minimal pullback attractors. This theory will be applied to analyse the asymptotic behavior, in the pullback sense, of the weak solutions of the problem 3D-(NS), on the families of Hilbert spaces $\{H_t\}_{t\in\mathbb{R}}$.

3.1. **Multi-valued processes on time-varying phase spaces: theoretical results.** The results presented here generalize, for instance, [31] (for the multi-valued autonomous case) and [5, Section 2] or [1, 4, 6, 30] among many others (for single-valued non-autonomous case), and [11] (for multi-valued non-autonomous systems on a fixed domain). When the uniqueness holds, our theory would reduce to the standard pullback attractor theory for single-valued process, e.g. [16].

Given a family of complete metric spaces $\{(X_t, d_{X_t})\}_{t\in\mathbb{R}}$, and denoting $\mathbb{R}^2_d = \{(t,s) \in \mathbb{R}^2 : t \geqslant s\}$ and $\mathcal{P}(X_t)$ the family of nonempty subsets of X_t for each $t \in \mathbb{R}$, we recall the notion of multi-valued process.

Definition 3.1. *A multi-valued map $\mathcal{U} : \{(t,\tau) \times X_\tau : (t,\tau) \in \mathbb{R}^2_d\} \to \{\mathcal{P}(X_t) : t \in \mathbb{R}\}$ is a multi-valued process on the family $\{X_t\}_{t\in\mathbb{R}}$ if*

(i) *$\mathcal{U}(t,\tau) : X_\tau \to \mathcal{P}(X_t)$ for each $(t,\tau) \in \mathbb{R}^2_d$;*
(ii) *$\mathcal{U}(\tau,\tau)x = \{x\}$ for any $\tau \in \mathbb{R}$ and all $x \in X_\tau$;*
(iii) *$\mathcal{U}(t,\tau)x \subset \mathcal{U}(t,s)(\mathcal{U}(s,\tau)x)$ for any $\tau \leqslant s \leqslant t$ and all $x \in X_\tau$, where $\mathcal{U}(t,\tau)B := \cup_{z\in B}\mathcal{U}(t,\tau)z$, for any $B \subset X_\tau$.*

The multi-valued process $\mathcal{U} : \{(t,\tau) \times X_\tau : (t,\tau) \in \mathbb{R}^2_d\} \to \{\mathcal{P}(X_t) : t \in \mathbb{R}\}$ is said to be strict if the inclusion in (iii) is an equality.

For each $t \in \mathbb{R}$, we denote by $\mathrm{dist}_{X_t}(\mathcal{O}_1, \mathcal{O}_2)$ the Hausdorff semi-distance in X_t between two sets $\mathcal{O}_1$ and $\mathcal{O}_2$, defined as

$$\mathrm{dist}_{X_t}(\mathcal{O}_1, \mathcal{O}_2) = \sup_{x\in\mathcal{O}_1} \inf_{y\in\mathcal{O}_2} d_{X_t}(x,y) \quad \text{for } \mathcal{O}_1, \mathcal{O}_2 \subset X_t.$$

As a convenient shorthand, we will refer to the process $\mathcal{U}(\cdot,\cdot)$ rather than the process $\mathcal{U} : \{(t,\tau) \times X_\tau : (t,\tau) \in \mathbb{R}^2_d\} \to \{\mathcal{P}(X_t) : t \in \mathbb{R}\}$ in all that follows. Though for nonlinear processes the notation $\mathcal{U}(t,\tau,x)$ is also often employed in the literature, we keep here the notation $\mathcal{U}(t,\tau)x$ for any $(t,\tau) \in \mathbb{R}^2_d$ and $x \in X_\tau$.

Definition 3.2. *A multi-valued process $\mathcal{U}(\cdot,\cdot)$ on the family $\{X_t\}_{t\in\mathbb{R}}$ is upper-semicontinuous if for any pair $(t,\tau) \in \mathbb{R}^2_d$, the mapping $\mathcal{U}(t,\tau) : X_\tau \to \mathcal{P}(X_t)$ satisfies that, given a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_τ and $x \in X_\tau$ with $x_n \to x$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \mathrm{dist}_{X_t}(\mathcal{U}(t,\tau)x_n, \mathcal{U}(t,\tau)x) = 0.$$

Remark 3.3. *A multi-valued process $\mathcal{U}(\cdot,\cdot)$ on the family $\{X_t\}_{t\in\mathbb{R}}$ is upper-semicontinuous if for any pair $(t,\tau) \in \mathbb{R}^2_d$, the mapping $\mathcal{U}(t,\tau) : X_\tau \to \mathcal{P}(X_t)$ satisfies for each $x \in X_\tau$ and neighborhood $V_t(\mathcal{U}(t,\tau)x) \subset X_t$ of $\mathcal{U}(t,\tau)x$ that there exists a neighborhood $V_\tau(x) \subset X_\tau$ of x such that $\mathcal{U}(t,\tau)y \subset V_t(\mathcal{U}(t,\tau)x)$ for any $y \in V_\tau(x)$.*

In the context of non-autonomous dynamical systems it is more natural to consider not only fixed sets but also families of sets indexed in time. This is due to the fact that at what time the data was put often affects the dynamical behavior of a non-autonomous dynamical system, as we saw in Section 2 (this is originally motivated by the random dynamical system theory; it also provides extra information about attractors even in the fixed bounded sets setting).

Let $\mathcal{D}_X$ be a universe of some families of nonempty sets, that is, each element $\widehat{D}$ of $\mathcal{D}_X$ is a family $\widehat{D} = \{D(t) \subset X_t : D(t) \neq \varnothing,\ t \in \mathbb{R}\}$ satisfying certain conditions. For ease of analysis we assume that the universe $\mathcal{D}_X$ is *inclusion-closed*, i.e., if $\widehat{D} \in \mathcal{D}_X$ and $\widehat{D}' = \{D'(t) \subset X_t : D'(t) \neq \varnothing,\ t \in \mathbb{R}\}$ satisfies $D'(t) \subset D(t)$ for all $t \in \mathbb{R}$, then $\widehat{D}' \in \mathcal{D}_X$.

The key ingredients for the establishment of an attraction object are absorption and asymptotic compactness.

Definition 3.4. *A family $\widehat{B}_0 = \{B_0(t) \subset X_t : t \in \mathbb{R}\}$ is pullback $\mathcal{D}_X$-absorbing for a multi-valued process $\mathcal{U}(\cdot,\cdot)$ if for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_X$, there exists $\tau(\widehat{D}, t) \leqslant t$ such that $\mathcal{U}(t,\tau)D(\tau) \subset B_0(t)$ for all $\tau \leqslant \tau(\widehat{D}, t)$.*

For the absorption property we do not require that the absorbing family above belongs to the universe.

Definition 3.5. • *Given a family $\widehat{D}_0 = \{D_0(t) \subset X_t : t \in \mathbb{R}\}$, multi-valued process $\mathcal{U}(\cdot,\cdot)$ is called pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\}$ with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, it holds that any sequence $\{y_n\}$ with $y_n \in \mathcal{U}(t,\tau_n)x_n$ is relatively compact in X_t.*
• *A multi-valued process $\mathcal{U}(\cdot,\cdot)$ is said pullback $\mathcal{D}_X$-asymptotically compact if it is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}_X$.*

Next, we present the definition of the minimal pullback attractor for a multi-valued process in time-varying spaces.

Definition 3.6. *A family $\mathcal{A} = \{\mathcal{A}(t) \subset X_t : t \in \mathbb{R}\}$ is called the minimal pullback $\mathcal{D}_X$-attractor for a multi-valued process $\mathcal{U}(\cdot,\cdot)$ if the following properties are fulfilled:*

(i) *$\mathcal{A}(t)$ is a nonempty compact subset of X_t for each $t \in \mathbb{R}$;*
(ii) *$\mathcal{A}$ is pullback $\mathcal{D}_X$-attracting, i.e.*

$$\lim_{\tau\to-\infty} \operatorname{dist}_{X_t}(\mathcal{U}(t,\tau)D(\tau), \mathcal{A}(t)) = 0$$

for any $\widehat{D} \in \mathcal{D}_X$ and all $t \in \mathbb{R}$;
(iii) *$\mathcal{A}$ is negatively invariant under the process $\mathcal{U}(\cdot,\cdot)$, i.e.*

$$\mathcal{A}(t) \subset \mathcal{U}(t,\tau)\mathcal{A}(\tau) \quad \textit{for any} \quad t \geqslant \tau;$$

(iv) *if $\widehat{C} = \{C(t) : t \in \mathbb{R}\}$ is a family of compact sets which pullback $\mathcal{D}_X$-attracts under $\mathcal{U}(\cdot,\cdot)$, then $\mathcal{A}(t) \subset C(t)$ for all $t \in \mathbb{R}$.*

A pullback attractor fulfilling only conditions (i)-(iii) above (which is a common meaningful definition in the literature) does not need to be unique (cf. [29]). Condition (iv) of minimality gives uniqueness. On the other hand, when the attractor $\mathcal{A} \in \mathcal{D}_X$, then it is also the unique family of closed subsets in $\mathcal{D}_X$ satisfying (ii)-(iii).

Note that in the literature the minimality of the pullback attractor is often referred as the minimal among pullback attracting *closed* sets, rather than only among pullback attracting *compact* sets as in Definition 3.6. The following lemma indicates that the two definitions of the minimality are equivalents.

Lemma 3.7. *Suppose that $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ and $\mathcal{B} = \{B(t) : t \in \mathbb{R}\}$ are two families of closed sets and are pullback $\mathcal{D}_X$-attracting under multi-valued process $\mathcal{U}$, then $\mathcal{A} \cap \mathcal{B} := \{A(t) \cap B(t) : t \in \mathbb{R}\}$ is pullback $\mathcal{D}_X$-attracting.*

Proof. First we show that $A(t) \cap B(t)$ is nonempty. If $A(t) \cap B(t) = \varnothing$ for some $t \in \mathbb{R}$, then by the closedness of $A(t)$ and $B(t)$ we know there exists a small $r > 0$ such that the open r-neighborhoods of $A(t)$ and $B(t)$, $N_r(A(t))$ and $N_r(B(t))$, are disjoint. This contradicts the pullback $\mathcal{D}_X$-attracting property of $A(t)$ and $B(t)$, because for any $\hat{D} \in \mathcal{D}_X$ we have

$$\mathcal{U}(t,-\tau)D(-\tau) \subset N_r(A(t)) \quad \text{and} \quad \mathcal{U}(t,-\tau)D(-\tau) \subset N_r(B(t)) \tag{3.1}$$

for τ large enough, indicating the nonempty of $N_r(A(t)) \cap N_r(B(t))$. Hence, $A(t) \cap B(t) \neq \varnothing$.

We now prove by contradiction that $\mathcal{A} \cap \mathcal{B}$ is pullback $\mathcal{D}_X$-attracting. If not, there would be a $\hat{D} \in \mathcal{D}_X$ not pullback attracted by $\mathcal{A} \cap \mathcal{B}$, that is, there are sequences $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ such that

$$\operatorname{dist}_{X_t}(\mathcal{U}(t,\tau_n,x_n), A(t) \cap B(t)) > 2\delta, \quad \forall n \in \mathbb{N},$$

for some $t \in \mathbb{R}$ and $\delta > 0$; in other words, $\mathcal{U}(t,\tau_n,x_n) \notin N_{2\delta}(A(t) \cap B(t))$ for all n. On the other hand, since $\hat{D}$ is pullback attracted by $\mathcal{A}$, $dist(\mathcal{U}(t,\tau_n,x_n), A(t)) \to 0$, so

$$\operatorname{dist}_{X_t}(\mathcal{U}(t,\tau_n,x_n), A(t)\backslash N_\delta(A(t) \cap B(t)) \to 0, \quad \text{as } n \to \infty.$$

In the same way, we have

$$\operatorname{dist}_{X_t}(\mathcal{U}(t,\tau_n,x_n), B(t)\backslash N_\delta(A(t) \cap B(t)) \to 0, \quad \text{as } n \to \infty,$$

where we arrived at a contradiction, since $A(t)\backslash N_\delta(A(t) \cap B(t)$ and $B(t)\backslash N_\delta(A(t) \cap B(t)$ are disjoint closed sets and cannot be approached by a common sequence. □

Lemma 3.7 indicates that if $\mathcal{A}$ is the minimal among pullback D_X-attracting *compact* sets, then it is the minimal among pullback D_X-attracting *closed* sets. Indeed, for any a pullback D_X-attracting closed set $\widehat{C}$, $\mathcal{A} \cap \widehat{C}$ is compact and, by Lemma 3.7, is pullback D_X-attracting, so by the minimality of $\mathcal{A}$ among pullback $\mathcal{D}_X$-attracting compact sets we have $\mathcal{A} \subset \mathcal{A} \cap \widehat{C}$ and thereby $\mathcal{A} = \mathcal{A} \cap \widehat{C} \subset \widehat{C}$, as desired.

3.1.1. *Existence of pullback attractors.* The minimal components (in the sense of attraction) that we aim to collect are the omega-limit families. Namely, given a family $\widehat{D} = \{D(t) \subset X_t : t \in \mathbb{R}\}$, the pullback omega-limit set of $\widehat{D}$ by $\mathcal{U}(\cdot,\cdot)$ at time t (when it makes sense) is defined by

$$\Lambda_{X_t}\big(\widehat{D}, t\big) = \Big\{\xi \in X_t : \xi = \lim_{\tau_n \to -\infty} y_n, \text{ where } y_n \in \mathcal{U}(t,\tau_n)D(\tau_n)\Big\}, \tag{3.2}$$

which can be equivalently described (when it make sense) by set theory as

$$\Lambda_{X_t}\big(\widehat{D}, t\big) = \bigcap_{s \leqslant t} \overline{\bigcup_{\tau \leqslant s} \mathcal{U}(t,\tau)D(\tau)}^{X_t}.$$

Proposition 3.8. *(cf.* [1, Proposition 1.4]*) If the multi-valued process $\mathcal{U}(\cdot,\cdot)$ is pullback $\widehat{D}_0$-asymptotically compact, then, for any $t \in \mathbb{R}$, the set $\Lambda_{X_t}(\widehat{D}_0, t)$ given in (3.2) is a nonempty compact subset of X_t, and*

$$\lim_{\tau \to -\infty} \operatorname{dist}_{X_t}(\mathcal{U}(t,\tau)D_0(\tau), \Lambda_{X_t}(\widehat{D}_0, t)) = 0. \tag{3.3}$$

Moreover, the family $\{\Lambda_X(\widehat{D}_0, t) \subset X_t : t \in \mathbb{R}\}$ is minimal in the sense that if $\widehat{C} = \{C(t) \subset X_t : t \in \mathbb{R}\}$ is a family of compact sets such that

$$\lim_{\tau \to -\infty} \operatorname{dist}_{X_t}(\mathcal{U}(t,\tau)D_0(\tau), C(t)) = 0,$$

then $\Lambda_X(\widehat{D}_0, t) \subset C(t)$ for all $t \in \mathbb{R}$.

Proof. Let $t \in \mathbb{R}$ and consider two sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\}$ such that $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n. Now, let $\{y_n\}$ be any sequence with $y_n \in \mathcal{U}(t,\tau_n)x_n$ for all $n \in \mathbb{N}$, thus, since the multi-valued process $\mathcal{U}(\cdot,\cdot)$ is pullback $\widehat{D}_0$-asymptotically compact, then, there are a subsequence of $\{y_n\}$ (relabeled the same) and $z \in X_t$ such that y_n converge to z in X_t. Consequently, we have $z \in \Lambda_{X_t}(\widehat{D}_0, t)$, and therefore $\Lambda_{X_t}(\widehat{D}_0, t)$ is nonempty for any $t \in \mathbb{R}$.

Now, let us prove that $\Lambda_{X_t}(\widehat{D}_0, t)$ is a compact subset of X_t for all $t \in \mathbb{R}$. Let $\{y_n\} \subset \Lambda_{X_t}(\widehat{D}_0, t)$. It follows from the definition of $\Lambda_{X_t}(\widehat{D}_0, t)$ that there exist $\tau_n \leqslant t - n$ and $x_n \in D_0(\tau_n)$ such that $\operatorname{dist}_{X_t}(y_n, \mathcal{U}(t,\tau_n)x_n) \leqslant 1/n$. Since the multi-valued process $\mathcal{U}(\cdot,\cdot)$ is pullback $\widehat{D}_0$-asymptotically compact, any sequence $\{z_n\}$, with $z_n \in \mathcal{U}(t,\tau_n)x_n$ for any n, possesses a convergent subsequence in X_t. Therefore, the corresponding subsequence of $\{y_n\}$ converges in X_t to the same point.

On the other hand, let us suppose that (3.3) is not true for some $t \in \mathbb{R}$. Thus, there exist $\varepsilon > 0$, $\{\tau_n\}$, $\{x_n\}$ with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all $n \in \mathbb{R}$ such that

$$\operatorname{dist}_{X_t}(\mathcal{U}(t,\tau_n)x_n, \Lambda_{X_t}(\widehat{D}_0, t)) > \varepsilon \quad \text{for all } n \geqslant 1,$$

therefore, the contradiction arrives since the multi-valued process $\mathcal{U}(\cdot,\cdot)$ is pullback $\widehat{D}_0$-asymptotically compact.

Finally, let us consider the family of compact sets $\{C(t) \subset X_t : t \in \mathbb{R}\}$ such that

$$\lim_{\tau \to -\infty} \operatorname{dist}_{X_t}(\mathcal{U}(t,\tau)D_0(\tau), C(t)) = 0. \tag{3.4}$$

Consider $y \in \Lambda_{X_t}(\widehat{D}_0, t)$, then there exist sequences $\{\tau_n\}$, $\{x_n\}$ and $\{y_n\}$, with $\tau_n \to -\infty$, $x_n \in D_0(\tau_n)$ and $y_n \in \mathcal{U}(y,\tau_n)x_n$ for all $n \in \mathbb{N}$, such that $y_n \to y$ in X_t. Then it follows from (3.4) that $y \in \overline{C(t)}^{X_t} = C(t)$. Therefore, we conclude that $\Lambda_{X_t}(\widehat{D}_0, t) \subset C(t)$ for all $t \in \mathbb{R}$. □

Lemma 3.9. *(Negative invariance) If the multi-valued process $\mathcal{U}(\cdot,\cdot)$ is upper-semicontinuous with closed values for all $(t,\tau) \in \mathbb{R}^2_d$ and is pullback $\widehat{D}_0$-asymptotically compact, then*

$$\Lambda_{X_t}(\widehat{D}_0, t) \subset \mathcal{U}(t,\tau)\Lambda_{X_\tau}(\widehat{D}_0, \tau), \quad \textit{for all } (t,\tau) \in \mathbb{R}^2_d.$$

Proof. Let us fix $(t,\tau) \in \mathbb{R}^2_d$. Then for any $y \in \Lambda_{X_t}(\widehat{D}_0, t)$ there are sequences $y_n \in \mathcal{U}(t, \tau_n + \tau)x_n$ and $x_n \in D_0(\tau_n + \tau)$ with $\tau_n \to -\infty$, such that $y_n \to y$ in X_t.

We know that

$$\mathcal{U}(t, \tau_n + \tau)x_n \subset \mathcal{U}(t,\tau)\big[\mathcal{U}(\tau, \tau_n + \tau)x_n\big].$$

Thus, $y_n \in \mathcal{U}(t,\tau)z_n$, where $z_n \in \mathcal{U}(\tau,\tau_n+\tau)x_n$. Since $\mathcal{U}(\cdot,\cdot)$ is pullback $\widehat{D}_0$-asymptotically compact, then, there exists a subsequence of $\{z_n\}$ (relabeled the same) such that $z_n \to z \in \Lambda_{X_\tau}(\widehat{D}_0,\tau)$.

Moreover, since $\mathcal{U}(\cdot,\cdot)$ is an upper-semicontinuous process with closed values, the graph of the map $x \mapsto \mathcal{U}(t,\tau)x$ is closed, and then we have

$$y \in \mathcal{U}(t,\tau)z \subset \mathcal{U}(t,\tau)\Lambda_{X_\tau}(\widehat{D}_0,\tau).$$

□

Proposition 3.10. *If the family $\widehat{D}_0 = \{D_0(t) \subset X_t : t \in \mathbb{R}\}$ is pullback $\mathcal{D}_X$-absorbing, then*

$$\Lambda_{X_t}(\widehat{D},t) \subset \Lambda_{X_t}(\widehat{D}_0,t) \quad \text{for all} \quad \widehat{D} \in \mathcal{D}_X,\ t \in \mathbb{R}.$$

Moreover, if $\widehat{D}_0 \in \mathcal{D}_X$, then $\Lambda_{X_t}(\widehat{D}_0,t) \subset \overline{D_0(t)}^{X_t}$ for all $t \in \mathbb{R}$.

Proof. Fix $\widehat{D} \in \mathcal{D}_X$ and $t \in \mathbb{R}$. For any $y \in \Lambda_{X_t}(\widehat{D},t)$, there exist two sequences $\{\tau_n\} \subset (-\infty,t]$ and $\{y_n\} \subset X_t$, with $\tau_n \to -\infty$ and $y_n \in \mathcal{U}(t,\tau_n)D(\tau_n)$ for all $n \in \mathbb{N}$, such that $y_n \to y$ in X_t.

Since $\widehat{D}_0$ is pullback $\mathcal{D}_X$-absorbing for $\mathcal{U}(\cdot,\cdot)$, there exist $\{\tau_{n_k}\} \subset \{\tau_n\}$ with $\tau_{n_k} \leqslant t-k$, and $\mathcal{U}(t-k,\tau_{n_k})D(\tau_{n_k}) \subset \widehat{D}_0(t-k)$, for all $k \in \mathbb{N}$. Then,

$$y_{n_k} \in \mathcal{U}(t,\tau_{n_k})D(\tau_{n_k}) \subset \mathcal{U}(t,t-k)\big[\mathcal{U}(t-k,\tau_{n_k})D(\tau_{n_k})\big] \subset \mathcal{U}(t,t-k)\widehat{D}_0(t-k),$$

with $y_{n_k} \to y$ in X_t, and therefore, $y \in \Lambda_{X_t}(\widehat{D}_0,t)$.

Finally, we consider $t \in \mathbb{R}$ and we suppose that $\widehat{D}_0 \in \mathcal{D}_X$. We observe that for any $y \in \Lambda_{X_t}(\widehat{D}_0,t)$, there exist $\{\tau_n\} \subset (-\infty,t]$ with $\tau_n \to -\infty$ and $\{y_n\} \subset X_t$ with $y_n \in \mathcal{U}(t,\tau_n)\widehat{D}_0(\tau_n)$ for all $n \in \mathbb{N}$, such that $y_n \to y$ in X_t. Since $\widehat{D}_0$ is pullback $\mathcal{D}_X$-absorbing for the process $\mathcal{U}(\cdot,\cdot)$, then from certain $n \in \mathbb{N}$, $y_n \in \widehat{D}_0(t)$. Thus, $y \in \overline{D_0(t)}^{X_t}$. □

Combining the above ingredients leads to the following result.

Theorem 3.11. *Consider an upper-semicontinuous multi-valued process $\mathcal{U}(\cdot,\cdot)$ on the family of metric spaces $\{X_t\}_{t\in\mathbb{R}}$ with closed values, a pullback $\mathcal{D}_X$-absorbing family $\widehat{B}_{0,X} = \{B_{0,X}(t) \subset X_t : t \in \mathbb{R}\}$ and assume that $\mathcal{U}(\cdot,\cdot)$ is pullback $\widehat{B}_{0,X}$-asymptotically compact. Then the family $\mathcal{A} = \{\mathcal{A}(t) \subset X_t : t \in \mathbb{R}\}$ given by*

$$\mathcal{A}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}_X} \Lambda_{X_t}(\widehat{D},t)}^{X_t} \qquad \forall t \in \mathbb{R}$$

is the minimal pullback $\mathcal{D}_X$-attractor. Moreover,

(1) *If $\widehat{B}_{0,X} \in \mathcal{D}_X$, then $\mathcal{A}(t) = \Lambda_{X_t}(\widehat{B}_{0,X},t)$ for all $t \in \mathbb{R}$.*
(2) *It holds that $\mathcal{A}(t) \subset \overline{B_{0,X}(t)}^{X_t}$ for any $t \in \mathbb{R}$.*
(3) *If $\widehat{B}_{0,X} \in \mathcal{D}_X$ has closed sections and $\mathcal{D}_X$ is inclusion-closed then $\mathcal{A} \in \mathcal{D}_X$.*
(4) *If $\mathcal{A} \in \mathcal{D}_X$ and $\mathcal{U}(\cdot,\cdot)$ is a strict process, then $\mathcal{A}$ is invariant under $\mathcal{U}(\cdot,\cdot)$, i.e.,*

$$\mathcal{A}(t) = \mathcal{U}(t,\tau)\mathcal{A}(\tau) \quad \text{for any} \quad t \geqslant \tau.$$

Proof. The proof is a generalization of the fixed domain case, see, e.g., [5, Theorem 3]. First note that by Proposition 3.10 we have $\mathcal{A}(t) \subset \Lambda_{X_t}(\widehat{B}_{0,X},t)$ for all $t \in \mathbb{R}$, thus, it follows from Proposition 3.8 that $\mathcal{A}(t)$ is a compact subset of X_t for all $t \in \mathbb{R}$. Also, the pullback attractivity, negative invariance, and minimality follow immediately from Proposition 3.8, Lemma 3.9 and Proposition 3.10. The items (1)-(3) are a consequence of Propositions 3.8 and 3.10.

Now let us focus on item (4). Since $\mathcal{U}(\cdot,\cdot)$ is a strict process, for any $t \geqslant r$, we have

$$\mathcal{U}(t,r)\mathcal{A}(r) \subset \mathcal{U}(t,r)\circ\mathcal{U}(r,r+\tau)\mathcal{A}(r+\tau) = \mathcal{U}(t,r+\tau)\mathcal{A}(r+\tau), \quad \text{for all } \tau \leqslant 0. \tag{3.5}$$

Thus, since $\mathcal{A}$ pullback attracts itself, we have that

$$\lim_{\tau\to-\infty} \operatorname{dist}_{X_t}(\mathcal{U}(t,r+\tau)\mathcal{A}(r+\tau),\mathcal{A}(t)) = 0.$$

Then, given $\varepsilon > 0$ there exists $\tau(t,r,\varepsilon,\mathcal{A}) < 0$ such that

$$\operatorname{dist}_{X_t}(\mathcal{U}(t,r+\tau)\mathcal{A}(r+\tau),\mathcal{A}(t)) < \varepsilon,$$

for all $\tau \leqslant \tau(t,r,\varepsilon,\mathcal{A})$. Thus, by (3.5) we deduce that

$$\text{dist}_{X_t}(\mathcal{U}(t,r)\mathcal{A}(r),\mathcal{A}(t)) < \varepsilon, \quad \text{for all } \varepsilon > 0.$$

Therefore, $\mathcal{U}(t,r)\mathcal{A}(r) \subset \overline{\mathcal{A}(t)}^{X_t} = \mathcal{A}(t)$ for all $(t,r) \in \mathbb{R}^2_d$. ☐

The next result allows us to compare different attractors, for instance for regularity purposes. It is inspired, for instance, [16, Theorem 3.15] for the single-valued case and [5, Theorem 4] for the multi-valued but fixed domain case.

Theorem 3.12. *Consider two families of metric spaces $\{(X_t,d_{X_t})\}$ and $\{(Y_t,d_{Y_t})\}$ with continuous embedding $X_t \subset Y_t$, for all $t \in \mathbb{R}$, respective universes $\mathcal{D}_X$ and $\mathcal{D}_Y$, and $\mathcal{D}_X \subset \mathcal{D}_Y$. Assume that $\mathcal{U}(\cdot,\cdot)$ is a time-dependent multi-valued process in both family of spaces, i.e., $\mathcal{U} : \{(t,\tau)\times X_\tau : (t,\tau) \in \mathbb{R}^2_d\} \to \{\mathcal{P}(X_t) : t \in \mathbb{R}\}$ and $\mathcal{U} : \{(t,\tau) \times Y_\tau : (t,\tau) \in \mathbb{R}^2_d\} \to \{\mathcal{P}(Y_t) : t \in \mathbb{R}\}$. For each $t \in \mathbb{R}$ denote*

$$\mathcal{A}_X(t) = \overline{\bigcup_{\widehat{D}_1 \in \mathcal{D}_X} \Lambda_{X_t}(\widehat{D}_1,t)}^{X_t}, \qquad \text{and} \qquad \mathcal{A}_Y(t) = \overline{\bigcup_{\widehat{D}_2 \in \mathcal{D}_Y} \Lambda_{Y_t}(\widehat{D}_2,t)}^{Y_t}.$$

Then $\mathcal{A}_X(t) \subset \mathcal{A}_Y(t)$ for all $t \in \mathbb{R}$.

Moreover, $\mathcal{A}_X(t) = \mathcal{A}_Y(t)$ for all $t \in \mathbb{R}$ if the following two conditions hold:

(i) $\mathcal{A}_X(t)$ is a compact subset of X_t for all $t \in \mathbb{R}$;

(ii) for any $\widehat{D}_2 \in \mathcal{D}_Y$ and $t \in \mathbb{R}$ there exist a family $\widehat{D}_1 \in \mathcal{D}_X$ and a $t^*_{\widehat{D}_1}$ such that $\mathcal{U}(\cdot,\cdot)$ is pullback $\widehat{D}_1$-asymptotically compact, and for any $s \leqslant t^*_{\widehat{D}_1}$ there exists a $\tau_s < s$ such that $\mathcal{U}(s,\tau)D_2(\tau) \subset D_1(s)$ for all $\tau \leqslant \tau_s$.

Let us suppose that the family of phase spaces does not vary, i.e. $X_t = X$ for all $t \in \mathbb{R}$. Now, let $\mathcal{D}^X_F$ be the universe of fixed bounded sets of X, i.e., $\widehat{D} \in \mathcal{D}^X_F$ if $\widehat{D} = \{D(t) = B : t \in \mathbb{R}\}$ with B a nonempty bounded subset of X.

Corollary 3.13. *Under the assumptions of Theorem 3.11 when $X_t = X$ for all $t \in \mathbb{R}$, if $\mathcal{D}^X_F \subset \mathcal{D}_X$ then $\mathcal{A}_{\mathcal{D}^X_F}(t) \subset \mathcal{A}_{\mathcal{D}_X}(t)$ for all $t \in \mathbb{R}$. Moreover if there exists $T \in \mathbb{R}$ such that $\bigcup_{t\leqslant T} B_{0,X}(t)$ is bounded in X, then $\mathcal{A}_{\mathcal{D}^X_F}(t) = \mathcal{A}_{\mathcal{D}_X}(t)$ for all $t \leqslant T$.*

3.1.2. *Equivalent pullback attractors.* In this section we will introduce the concept of equivalence of pullback attractors for time-varying spaces. Suppose we have two multi-valued time-dependent dynamical systems $(\mathcal{U}_X(\cdot,\cdot),\{X_t\}_{t\in\mathbb{R}})$, and $(\mathcal{U}_Y(\cdot,\cdot),\{Y_t\}_{t\in\mathbb{R}})$, and let us also assume that the families of associated phase spaces are connected by a family of homeomorphisms, i.e., there exists a family of maps $\{\Theta(t)\}_{t\in\mathbb{R}}$ such that $\Theta(t) : X_t \to Y_t$ is a homeomorphism for each $t \in \mathbb{R}$. Then, if there exists a homeomorphism family $\{\Theta(t)\}_{t\in\mathbb{R}}$ we are able to show an "equivalence" relationship between dynamical systems $(\mathcal{U}_X(\cdot,\cdot),\{X_t\}_{t\in\mathbb{R}})$ and $(\mathcal{U}_Y(\cdot,\cdot),\{Y_t\}_{t\in\mathbb{R}})$. With this, we will be able to show that $(\mathcal{U}_X(\cdot,\cdot),\{X_t\}_{t\in\mathbb{R}})$ has a pullback attractor if and only if, $(\mathcal{U}_Y(\cdot,\cdot),\{Y_t\}_{t\in\mathbb{R}})$ has a pullback attractor.

Definition 3.14. *We say that the families of metric spaces $\{X_t\}_{t\in\mathbb{R}}$ and $\{Y_t\}_{t\in\mathbb{R}}$ are **equivalent** if there exists a family of maps $\{\Theta(t)\}_{t\in\mathbb{R}}$ such that $\Theta(t) : X_t \to Y_t$ is a homeomorphism for each $t \in \mathbb{R}$.*

Definition 3.15. *Let $\mathcal{D}_X$ and $\mathcal{D}_Y$ be two universes on the equivalent families of metric spaces $\{X_t\}_{t\in\mathbb{R}}$ and $\{Y_t\}_{t\in\mathbb{R}}$, respectively. We say that the universes $\mathcal{D}_X$ and $\mathcal{D}_Y$ are equivalent if for any $\widehat{D} \in \mathcal{D}_X$ the family $\{\Theta(t)D(t) \subset Y_t : t \in \mathbb{R}\} \in \mathcal{D}_Y$ and for any $\widehat{C} \in \mathcal{D}_Y$ the family $\{\Theta^{-1}(t)C(t) \subset X_t : t \in \mathbb{R}\} \in \mathcal{D}_X$.*

Theorem 3.16. *Let $\{X_t\}_{t\in\mathbb{R}}$ and $\{Y_t\}_{t\in\mathbb{R}}$ be two equivalent families of metric spaces. Let $\mathcal{U}_X(\cdot,\cdot)$ be a multi-valued process on the family $\{X_t\}_{t\in\mathbb{R}}$. Then, $\mathcal{U}_Y(\cdot,\cdot)$ defined as*

$$\mathcal{U}_Y(t,\tau)y := \Theta(t) \circ \mathcal{U}_X(t,\tau) \circ \Theta^{-1}(\tau)y \quad \text{for all } (t,\tau) \in \mathbb{R}^2_d,\ y \in Y_\tau, \tag{3.6}$$

is a multi-valued process on the family $\{Y_t\}_{t\in\mathbb{R}}$. Moreover, if the multi-valued process $\mathcal{U}_X(\cdot,\cdot)$ is upper-semicontinuous, then $\mathcal{U}_Y(\cdot,\cdot)$ is also upper-semicontinuous.

Proof. It follows immediately from Definition 3.1 and Remark 3.3. ☐

Lemma 3.17. *Suppose that $\Theta : X \to Y$ is a continuou (single-valued) mapping between metric spaces (X, d_X) and (Y, d_Y). Then Θ is upper (resp. lower) semicontinuous on compact sets, i.e., for any compact set D_0 and any family $\{D_s \subset X : s \in \mathbb{R}\}$ of sets with $\lim_{s\to 0} \operatorname{dist}_X(D_s, D_0) = 0$ (resp. $\lim_{s\to 0} \operatorname{dist}_X(D_0, D_s) = 0$) we have*

$$\lim_{s\to 0} \operatorname{dist}_Y(\Theta(D_s), \Theta(D_0)) = 0, \qquad \Big(resp.\ \lim_{s\to 0} \operatorname{dist}_Y(\Theta(D_0), \Theta(D_s)) = 0\Big).$$

Proof. We first prove the upper semicontinuity by contradiction. If it does not hold, there would exist a sequence $x_n \in D_{s_n}$ with $s_n \to 0$ such that

$$\operatorname{dist}_Y(\Theta(x_n), \Theta(D_0)) \geqslant \delta, \quad \forall n \in \mathbb{N}, \tag{3.7}$$

for some $\delta > 0$. Since $\lim_{s\to 0} \operatorname{dist}_X(D_s, D_0) = 0$, for the sequence x_n there has to be a sequence $z_n \in D_0$ such that $d_X(x_n, z_n) \to 0$ as $n \to \infty$. By the compactness of D_0, up to a subsequence z_n converges to some $z_0 \in D_0$ which indicates that $x_n \to z_0$. Therefore, by the continuity of Θ we obtain $\Theta(x_n) \to \Theta(z_0)$, which contradicts (3.7). The upper semicontinuity holds.

The lower semicontinuity is proved analogously. If it were not the case, there would exist sequences $s_n \to 0$ and $x_n \in D_0$ such that

$$\operatorname{dist}_Y(\Theta(x_n), \Theta(D_{s_n})) \geqslant \delta, \quad \forall n \in \mathbb{N}, \tag{3.8}$$

for some $\delta > 0$. Since $\lim_{n\to\infty} \operatorname{dist}_X(D_0, D_{s_n}) = 0$, for each $x_n \in D_0$ there is a $z_n \in D_{s_n}$ such that $d_X(x_n, z_n) \to 0$ as $n \to \infty$. By the compactness of D_0, up to a subsequence x_n converges to some $x_0 \in D_0$ which indicates that $z_n \to x_0$. By the continuity of Θ we obtain $\Theta(x_n) \to \Theta(x_0)$ as well as $\Theta(z_n) \to \Theta(x_0)$. Therefore, $d_Y(\Theta(x_n), \Theta(z_n)) \to 0$, contradicting (3.8). Hence, the lower semicontinuity holds. □

The upper and lower semi-continuity on compact sets together imply the full continuity in Hausdorff metric on compact sets. That is, we have the following corollary.

Corollary 3.18. *Suppose that $\Theta : X \to Y$ is a continuous mapping between metric spaces (X, d_X) and (Y, d_Y). Then Θ is continuous w.r.t. Hausdorff metric for compact sets in the following sense: for any compact set D_0 and any family $\{D_s \subset X : s \in \mathbb{R}\}$ of sets with*

$$\lim_{s\to 0} \operatorname{Dist}_X(D_s, D_0) = 0, \quad then \quad \lim_{s\to 0} \operatorname{Dist}_Y(\Theta(D_s), \Theta(D_0)) = 0,$$

where Dist_X and Dist_Y denote the Hausdorff metrics on X and Y, respectively.

Remark 3.19. *Note that the full continuity in Hausdorff metric does not imply the upper or the lower semi-continuity. Therefore, Lemma 3.17 is often more applicable since in many cases, as in this paper, one would need only the upper or the lower semi-continuity, not both. For more discussion on the continuity of set-valued maps see* [2, Chapter 1].

Theorem 3.20. *Let $\{X_t\}_{t\in\mathbb{R}}$ and $\{Y_t\}_{t\in\mathbb{R}}$ be two equivalent families of metric spaces. Consider $\mathcal{U}_X(\cdot,\cdot)$ a process and an universe $\mathcal{D}_X$ on the family $\{X_t\}_{t\in\mathbb{R}}$, and let us consider the process $\mathcal{U}_Y(\cdot,\cdot)$, given in (3.6), and the universe $\mathcal{D}_Y$. If $\mathcal{A}_{\mathcal{D}_X}$ is the minimal pullback $\mathcal{D}_X$-attractor for the multi-valued process $\mathcal{U}_X(\cdot,\cdot)$, then the family $\mathcal{A}_{\mathcal{D}_Y} = \{\mathcal{A}_{\mathcal{D}_Y}(t) : t \in \mathbb{R}\}$ defined by*

$$\mathcal{A}_{\mathcal{D}_Y}(t) := \Theta(t)\mathcal{A}_{\mathcal{D}_X}(t), \quad t \in \mathbb{R},$$

is the minimal pullback $\mathcal{D}_Y$-attractor for the multi-valued process $\mathcal{U}_Y(\cdot,\cdot)$, i.e.,

(i) *$\mathcal{A}_{\mathcal{D}_Y}(t)$ is a nonempty compact subset of Y_t for each $t \in \mathbb{R}$;*

(ii) *$\mathcal{A}_{\mathcal{D}_Y}$ is pullback $\mathcal{D}_Y$-attracting, i.e.,*

$$\lim_{\tau\to-\infty} \operatorname{dist}_{Y_t}(\mathcal{U}_Y(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_Y}(t)) = 0$$

for any $\widehat{D} \in \mathcal{D}_Y$ and all $t \in \mathbb{R}$;

(iii) *$\mathcal{A}_{\mathcal{D}_Y}$ is negatively invariant under the process $\mathcal{U}_Y(\cdot,\cdot)$, i.e.,*

$$\mathcal{A}_{\mathcal{D}_Y}(t) \subset \mathcal{U}_Y(t,\tau)\mathcal{A}_{\mathcal{D}_Y}(\tau) \quad for\ any \quad t \geqslant \tau;$$

(iv) *(Minimality) if $\widehat{C} = \{C(t) : t \in \mathbb{R}\}$ is a family of compact sets which pullback $\mathcal{D}_Y$-attracts under $\mathcal{U}_Y(\cdot,\cdot)$, then $\mathcal{A}_{\mathcal{D}_Y}(t) \subset C(t)$ for all $t \in \mathbb{R}$.*

Proof. (i) Since the map $\Theta(t) : X_t \to Y_t$ is continuous for all $t \in \mathbb{R}$, $\mathcal{A}_{\mathcal{D}_Y}(t) = \Theta(t)\mathcal{A}_{\mathcal{D}_X}(t)$ is a compact set in Y_t for all $t \in \mathbb{R}$.
(ii) It follows from Lemma 3.17, that

$$\begin{aligned}\lim_{\tau\to-\infty} \mathrm{dist}_{Y_t}(\mathcal{U}_Y(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_Y}(t)) &= \lim_{\tau\to-\infty} \mathrm{dist}_{Y_t}(\Theta(t)\mathcal{U}_X(t,\tau)\Theta^{-1}(\tau)D(\tau), \Theta(t)\mathcal{A}_{\mathcal{D}_X}(t)) \\ &= \lim_{\tau\to-\infty} \mathrm{dist}_{Y_t}(\Theta(t)\mathcal{U}_X(t,\tau)B(\tau), \Theta(t)\mathcal{A}_{\mathcal{D}_X}(t)) \\ &= 0,\end{aligned}$$

where $\widehat{B} = \{B(t) = \Theta^{-1}(t)D(t) : t \in \mathbb{R}\} \in \mathcal{D}_X$ with $\widehat{D} \in \mathcal{D}_Y$.
(iii) Observe that

$$\begin{aligned}\mathcal{A}_{\mathcal{D}_Y}(t) &= \Theta(t)\mathcal{A}_{\mathcal{D}_X}(t) \subset \Theta(t) \circ \mathcal{U}_X(t,\tau)\mathcal{A}_{\mathcal{D}_X}(\tau) = \Theta(t) \circ \mathcal{U}_X(t,\tau) \circ \Theta^{-1}(\tau) \circ \Theta(\tau)\mathcal{A}_{\mathcal{D}_X}(\tau) \\ &= \Theta(t) \circ \mathcal{U}_X(t,\tau) \circ \Theta^{-1}(\tau)\mathcal{A}_{\mathcal{D}_Y}(\tau) = \mathcal{U}_Y(t,\tau)\mathcal{A}_{\mathcal{D}_Y}(\tau).\end{aligned}$$

Thus, $\mathcal{A}_{\mathcal{D}_Y}(t) \subset \mathcal{U}_Y(t,\tau)\mathcal{A}_{\mathcal{D}_Y}(\tau)$ for all $(t,\tau) \in \mathbb{R}^2_d$.
(iv) Let $\widehat{C} = \{C(t) \subset Y_t : t \in \mathbb{R}\}$ be a family of compact sets which pullback $\mathcal{D}_Y$-attracts under $\mathcal{U}_Y(\cdot,\cdot)$. Then by Lemma 3.17 the family $\widehat{C}_\Theta = \{\Theta^{-1}(t)C(t) : t \in \mathbb{R}\}$ is pullback $\mathcal{D}_X$-attracting under $\mathcal{U}_X(\cdot,\cdot)$, therefore it follows from Theorem 3.11 that $\mathcal{A}_{\mathcal{D}_X}(t) \subset \Theta^{-1}(t)C(t)$ for all $t \in \mathbb{R}$. Then we conclude that $\mathcal{A}_{\mathcal{D}_Y}(t) = \Theta(t)\mathcal{A}_{\mathcal{D}_X}(t) \subset C(t)$ for all $t \in \mathbb{R}$. □

Corollary 3.21. *Let $\{X_t\}_{t\in\mathbb{R}}$ and $\{Y_t\}_{t\in\mathbb{R}}$ be two equivalent families of metric spaces. If $\mathcal{U}_X(\cdot,\cdot)$ is an upper-semicontinuous multi-valued process with closed values, dissipative, and pullback $\mathcal{D}_X$-asymptotically compact, then the multi-valued process $\mathcal{U}_Y(\cdot,\cdot)$, defined in (3.6), has a minimal pullback $\mathcal{D}_Y$-attractor on the universe $\mathcal{D}_Y$, given in Definition 3.15.*

Proof. Since the time-dependent multi-valued dynamical system $(\mathcal{U}_X(\cdot,\cdot), \{X_t\}_{t\in\mathbb{R}})$ satisfies the conditions of the Theorem 3.11, it has a minimal pullback $\mathcal{D}_X$-attractor. Then, it follows from Theorem 3.20 that the time-dependent multi-valued dynamical system $(\mathcal{U}_Y(\cdot,\cdot), \{Y_t\}_{t\in\mathbb{R}})$ also has a minimal pullback $\mathcal{D}_Y$-attractor. □

Corollary 3.22. *If $\mathcal{A}_{\mathcal{D}_X} \in \mathcal{D}_X$ and $\mathcal{U}_X(\cdot,\cdot)$ is a strict process, then $\mathcal{A}_{\mathcal{D}_Y}$ is invariant under $\mathcal{U}_Y(\cdot,\cdot)$, i.e., $\mathcal{A}_{\mathcal{D}_Y}(t) = \mathcal{U}_Y(t,\tau)\mathcal{A}_{\mathcal{D}_Y}(\tau)$ for any $t \geqslant \tau$.*

Proof. It follows from item (4) from Theorem 3.11 that $\mathcal{A}_{\mathcal{D}_X}$ is invariant under $\mathcal{U}_X(\cdot,\cdot)$. Then $\mathcal{A}_{\mathcal{D}_Y}$ is invariant under $\mathcal{U}_Y(\cdot,\cdot)$. □

3.1.3. *An example: equivalence of pullback attractors with different universes.* An important universe (or family) in the theory of dynamical systems is the universe of bounded sets. Note that this was already introduced in Corollary 3.13. Now by Theorems 3.11, 3.12, and 3.20 we shall see what relationship the pullback attractor associated with the universe of bounded sets has when it is carried by a family of homeomorphisms to another equivalent universe.

Let us consider a fixed phase space $X_t \equiv X$ and an equivalent family of phase spaces $\{Y_t\}_{t\in\mathbb{R}}$, i.e., there exists a family of homeomorphisms $\Theta(t) : X \to Y_t$ for all $t \in \mathbb{R}$. Now, let $\mathcal{D}_F^X$ be the universe of fixed bounded subsets of X, and let $\mathcal{D}_X$ be another universe on X such that $\mathcal{D}_F^X \subset \mathcal{D}_X$. Then, under the conditions of Corollary 3.13, we have that $\mathcal{A}_{\mathcal{D}_F^X}(t) \subset \mathcal{A}_{\mathcal{D}_X}(t)$ for all $t \in \mathbb{R}$, and if there exists $T \in \mathbb{R}$ such that $\bigcup_{t\leqslant T} B_{0,X}(t)$ is bounded in X, then $\mathcal{A}_{\mathcal{D}_F^X}(t) = \mathcal{A}_{\mathcal{D}_X}(t)$ for all $t \leqslant T$.

Let $\mathcal{D}_Y$ be the universe on the family $\{Y_t\}_{t\in\mathbb{R}}$, which is equivalent to universe $\mathcal{D}_X$, see Definition 3.15, and let us denote by $\mathcal{D}^Y_{1,F} := \{\Theta(t)D : D \in \mathcal{D}_F^X,\ t \in \mathbb{R}\}$, $\mathcal{A}_{\mathcal{D}^Y_{1,F}} := \{\Theta(t)\mathcal{A}_{\mathcal{D}_F^X}(t) : t \in \mathbb{R}\}$ and $\mathcal{A}_{\mathcal{D}_Y} := \{\Theta(t)\mathcal{A}_{\mathcal{D}_X}(t) : t \in \mathbb{R}\}$. Then, it follows from Theorem 3.20 that $\mathcal{A}_{\mathcal{D}^Y_{1,F}}$ and $\mathcal{A}_{\mathcal{D}_Y}$ are pullback attractors of the time-dependent multi-valued process $\mathcal{U}_Y(\cdot,\cdot)$ w.r.t. the universes $\mathcal{D}^Y_{1,F}$ and $\mathcal{D}_Y$, respectively, and

$$\mathcal{A}_{\mathcal{D}^Y_{1,F}}(t) \subset \mathcal{A}_{\mathcal{D}_Y}(t) \quad \text{for all } t \in \mathbb{R}.$$

Moreover, if there exists a $T \in \mathbb{R}$ such that $\bigcup_{t\leqslant T} B_{0,X}(t)$ is bounded in X, then $\mathcal{A}_{\mathcal{D}^Y_{1,F}}(t) = \mathcal{A}_{\mathcal{D}_Y}(t)$ for all $t \leqslant T$.

On the other hand, let us denote by $\mathcal{D}^Y_{2,F}$ the universe of bounded sets on the family $\{Y_t\}_{t\in\mathbb{R}}$ in the following sense $\widehat{D} \in \mathcal{D}^Y_{2,F}$ if $\widehat{D} = \{D(t) \subset B_{Y_t}(0,R) : t \in \mathbb{R}\}$ for some $R > 0$, (where $B_{Y_t}(0,R)$ is the open ball centered at zero with radius R). Let us assume that $\mathcal{D}^Y_{2,F} \subset \mathcal{D}_Y$.

Let us assume that there is a pullback $\mathcal{D}_Y$-absorbing family $\widehat{B}_{0,Y} \in \mathcal{D}_Y$, and also that there exists a $T \in \mathbb{R}$ such that $B_{0,Y}(t) \subset B_{Y_t}[0,R]$ for all $t \leqslant T$, and some $R > 0$ (where $B_{Y_t}[0,R]$ is the closed ball with center at zero and radius R). Then, by Theorem 3.11 for any $t \leqslant T$, we get

$$\begin{aligned}\mathcal{A}_{\mathcal{D}_Y}(t) = \Lambda_{Y_t}(\widehat{B}_{0,Y},t) &= \bigcap_{s\leqslant t}\overline{\bigcup_{\tau\leqslant s}\mathcal{U}_Y(t,\tau)B_{0,Y}(\tau)}^{Y_t} \\ &\subset \bigcap_{s\leqslant t}\overline{\bigcup_{\tau\leqslant s}\mathcal{U}_Y(t,\tau)B_{Y_\tau}[0,R]}^{Y_t} = \Lambda_{Y_t}(\widehat{B}_{Y,R},t) \\ &\subset \overline{\bigcup_{\widehat{D}_F\in\mathcal{D}^Y_{2,F}}\Lambda_{Y_t}(\widehat{D}_F,t)}^{Y_t} = \mathcal{A}_{\mathcal{D}^Y_{2,F}}(t),\end{aligned}$$

where $\widehat{B}_{Y,R} = \{B_{Y_t}[0,R] : t \in \mathbb{R}\}$. With this and Theorem 3.12, we conclude that

$$\mathcal{A}_{\mathcal{D}_Y}(t) = \mathcal{A}_{\mathcal{D}^Y_{2,F}}(t), \quad \text{and} \quad \mathcal{A}_{\mathcal{D}_X}(t) = \Theta^{-1}(t)\mathcal{A}_{\mathcal{D}^Y_{2,F}}(t) \text{ for all } t \leqslant T.$$

In addition, if $\bigcup_{t\leqslant T} B_{0,X}(t)$ is also bounded in X, then

$$\mathcal{A}_{\mathcal{D}^X_F}(t) = \mathcal{A}_{\mathcal{D}_X}(t) = \Theta^{-1}(t)\mathcal{A}_{\mathcal{D}^Y_{2,F}}(t) \text{ and } \mathcal{A}_{\mathcal{D}^Y_{1,F}}(t) = \mathcal{A}_{\mathcal{D}_Y}(t) = \mathcal{A}_{\mathcal{D}^Y_{2,F}}(t) \text{ for all } t \leqslant T.$$

3.2. **Pullback attractor of the 3D-Navier-Stokes equation: Construction.** In this section we are interested in studying the asymptotic behavior, in the pullback sense, of the weak solutions of the problem 3D-(NS), on the families of Hilbert spaces $\{H_t\}_{t\in\mathbb{R}}$. For this, we will consider the hypotheses **(H1)-(H3)** and the hypothesis **(HF)** and $f \in L^2_{loc}(\mathbb{R}; V_t^*)$.

3.2.1. *Generation of an upper semicontinuous multi-valued process.* Given an initial time $\tau \in \mathbb{R}$ and an initial datum $u_\tau \in H_\tau$, we denote by $\Psi(\tau,u_\tau)$ a set of weak solutions to 3D-(NS) in $[\tau,+\infty)$ with initial datum $u_\tau \in H_\tau$:

$$\Psi(\tau,u_\tau) := \left\{ u(\cdot,\tau,u_\tau) \;\middle|\; \begin{array}{l} u \text{ is a weak solution to 3D-(NS) with} \\ u(\tau,\tau,u_\tau) = u_\tau \text{ satisfying (2.20)} \end{array} \right\}.$$

Then, it follows from Theorem 2.18 and Theorem 2.21 that the family $\Psi(\tau,u_\tau)$ is not empty. Now, we can define a family of multi-valued maps $\mathcal{U}(\cdot,\cdot) : \{(t,\tau)\times H_\tau : (t,\tau) \in \mathbb{R}^2_d\} \to \{\mathcal{P}(H_t) : t \in \mathbb{R}\}$ by

$$\mathcal{U}(t,\tau)u_\tau := \{u(t) : u \in \Psi(\tau,u_\tau)\} \in \mathcal{P}(H_t), \quad u_\tau \in H_\tau,\ \tau \leqslant t. \tag{3.9}$$

The following lemma will allow us to show that the evolution process associated with the weak solutions of 3D-(NS), given in (3.9), is upper-semicontinuous and has closed values, see Corollary 3.24. The energy method will play a key role, e.g. [18, 26].

Theorem 3.23. *Assume conditions of Theorem 2.18 and the hypothesis* **(HF)** *hold. Then for any strongly convergent sequence $\{u^m_\tau\}$, $\tau \in \mathbb{R}$, of initial data with $u^m_\tau \to u_\tau$ in H_τ and any sequence $\{u_m\} \subset \Psi(\tau,u^m_\tau)$, there exist a subsequence of $\{u_m\}$ (relabeled the same) and $u \in \Psi(\tau,u_\tau)$ such that*

$$u_m(s) \to u(s) \quad \textit{strongly in } H_s \textit{ for any } s \geqslant \tau. \tag{3.10}$$

Proof. Let us fix $T \in \mathbb{R}$ with $T > \tau$. In the same way as Theorem 2.21 and Corollary 2.22 and making use of the energy inequality given in (2.20), $\{u_m\}$ is bounded in $L^\infty(\tau,T;H_t)$ and $L^2(\tau,T;V_t)$, and the sequence $\{\frac{\partial u_m}{\partial t}\}$ is bounded in $L^{4/3}(\tau,T;V_t^*)$. By the Aubin-Lions compactness Lemma, there exist a subsequence of

$\{u_m\}$ (relabeled the same) and $u \in L^\infty(\tau, T; H_t) \cap L^2(\tau, T; V_t)$ with $\frac{\partial u}{\partial t} \in L^{4/3}(\tau, T; V_t^*)$ such that

$$\begin{cases} u_m \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(\tau, T; H_t), \\ u_m \rightharpoonup u & \text{weakly in } L^2(\tau, T; V_t), \\ \dfrac{\partial u_m}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} & \text{weakly in } L^{4/3}(\tau, T; V_t^*), \\ u_m \to u & \text{strongly in } L^2(\tau, T; H_t). \end{cases} \tag{3.11}$$

Again, in the same way as [32, Theorem 2.2] the convergences given in (3.11) allow us to pass to limit in the weak formulation (2.13). Moreover, it is not difficult to check that $u(\tau) = u_\tau$. Therefore, we have that $u \in \Psi(\tau, u_\tau)$.

On the other hand, let $\widehat{u}_m$ and $\widehat{u}$ be the trivial extension, given in (2.2), of u_m and u, respectively. Then by Proposition 2.6 the sequence $\{\widehat{u}_m\}$ is equicontinuous in $H^{-1}(\mathbb{R}^3)^3$ on $[\tau, T]$, in fact

$$\begin{aligned} \|\widehat{u}_m(t) - \widehat{u}_m(s)\|_{H^{-1}(\mathbb{R}^3)^3} &\leqslant \int_s^t \left\| \frac{\partial u_m}{\partial t}(r) \right\|_{V_r^*} dr \\ &\leqslant |t-s|^{1/4} \left(\int_\tau^T \left\| \frac{\partial u_m}{\partial t}(r) \right\|_{V_r^*}^{4/3} dr \right)^{3/4}, \quad \text{for all } \tau \leqslant s \leqslant t \leqslant T. \end{aligned}$$

Also, in the same way as (2.27), we have that $u_m' \rightharpoonup u'$ weakly in $L^2(\tau + \delta, T; V_t^*)$ for any $\delta > 0$. Therefore, by [3, Theorem II.5.12], $\{\widehat{u}_m\}$ is bounded in $C([\tau + \delta, T]; L^2(\mathbb{R}^3)^3)$, for any $\delta > 0$. Therefore, by the Arzelà-Ascoli Theorem, up to a subsequence, there follows that

$$\widehat{u}_m \to \widehat{u} \quad \text{strongly in } C([\tau + \delta, T]; H^{-1}(\mathbb{R}^3)^3) \quad \text{for any } \delta > 0.$$

Hence, by the boundness of $\{\widehat{u}_m\}$ in $C([\tau + \delta, T]; L^2(\mathbb{R}^3)^3)$ for any $\delta > 0$, and $u_m(\tau) \to u_\tau$, we conclude that

$$\widehat{u}_m(s) \rightharpoonup \widehat{u}(s) \quad \text{weakly in } L^2(\mathbb{R}^3)^3 \text{ for any } \tau \leqslant s \leqslant T \tag{3.12}$$

which implies that

$$u_m(s) \rightharpoonup u(s) \quad \text{weakly in } H_s \text{ for any } \tau \leqslant s \leqslant T. \tag{3.13}$$

Thus, it follows from energy inequality (2.17) that the estimate

$$|z(r)|_r^2 \leqslant |z(s)|_s^2 + 2\int_s^r (f(\theta), z(\theta))_{-1,\theta} d\theta, \quad \tau \leqslant s \leqslant r \leqslant T,$$

holds for $z = u_m$ and $z = u$, it follows that the functions $J_m, J : [\tau, T] \to \mathbb{R}$ defined by

$$\begin{cases} J_m(r) = |u_m(r)|_r^2 - 2\displaystyle\int_\tau^r (f(\theta), u_m(\theta))_{-1,\theta} d\theta, \\ J(r) = |u(r)|_r^2 - 2\displaystyle\int_\tau^r (f(\theta), u(\theta))_{-1,\theta} d\theta, \end{cases} \tag{3.14}$$

are non-increasing and continuous, and by (3.11) satisfy

$$J_m(r) \to J(r) \quad \text{a.e. } r \in (\tau, T).$$

Now, we will prove that $J_m(r) \to J(r)$ for any $r \in [\tau, T]$. Let us consider the following extended energy functionals

$$\begin{aligned} \widehat{J}_m(r) &= |||\widehat{u}_m(r)|||^2 - 2\int_\tau^r \langle\langle \widehat{f}(\theta), \widehat{u}_m(\theta) \rangle\rangle d\theta, \\ \widehat{J}(r) &= |||\widehat{u}(r)|||^2 - 2\int_\tau^r \langle\langle \widehat{f}(\theta), \widehat{u}(\theta) \rangle\rangle d\theta. \end{aligned}$$

Thus, it follows from (3.14) that $\widehat{J}_m$ and $\widehat{J}$ are non-increasing and continuous, and satisfy

$$\widehat{J}_m(r) \to \widehat{J}(r) \quad \text{a.e. } r \in (\tau, T).$$

Now, consider a fixed $t^* \in (\tau, T]$ and an increasing sequence $\{t_k\} \subset [\tau, t^*)$ such that $t_k \to t^*$ and $\widehat{J}_m(t_k) \to \widehat{J}(t_k)$ for all $k \geqslant 1$. Thus, by the continuity of $\widehat{J}$ and $\widehat{J}_m$, for any $\epsilon > 0$ there exist $M, K > 0$ such that

$$|\widehat{J}(t_k) - \widehat{J}(t^*)| \leqslant \frac{\epsilon}{2} \quad \text{for } k \geqslant K,$$
$$|\widehat{J}_m(t_K) - \widehat{J}(t_K)| \leqslant \frac{\epsilon}{2} \quad \text{for } m \geqslant M.$$

Since $\widehat{J}_m$ is a non-increasing function, we have that

$$\begin{aligned}\widehat{J}_m(t^*) - \widehat{J}(t^*) &\leqslant \widehat{J}_m(t_K) - \widehat{J}(t^*)\\ &\leqslant |\widehat{J}_m(t_K) - \widehat{J}(t_K)| + |\widehat{J}(t_K) - \widehat{J}(t^*)| \leqslant \epsilon\end{aligned}$$

for all $m \geqslant M$, and consequently $\limsup_{m\to\infty} \widehat{J}_m(t^*) \leqslant \widehat{J}(t^*)$. Taking into account that

$$\int_\tau^{t^*} \langle\langle \widehat{f}(\theta), \widehat{u}_m(\theta)\rangle\rangle d\theta \to \int_\tau^{t^*} \langle\langle \widehat{f}(\theta), \widehat{u}(\theta)\rangle\rangle d\theta,$$

we deduce that $\limsup_{m\to\infty} |||\widehat{u}_m(t^*)||| \leqslant |||\widehat{u}(t^*)|||$. Then, from the latter and (3.12), we have

$$\begin{aligned}\limsup_{m\to\infty} |u_m(t^*) - u(t^*)|_{t^*}^2 &= \limsup_{m\to\infty} |||\widehat{u}_m(t^*) - \widehat{u}(t^*)|||^2\\ &= \limsup_{m\to\infty} |||\widehat{u}_m(t^*)|||^2 - 2\liminf_{m\to\infty}((\widehat{u}_m(t^*), \widehat{u}(t^*))) + \limsup_{m\to\infty} |||\widehat{u}(t^*)|||^2\\ &\leqslant |||\widehat{u}(t^*)|||^2 - 2|||\widehat{u}(t^*)|||^2 + |||\widehat{u}(t^*)|||^2 = 0.\end{aligned}$$

Hence, we conclude that (3.10) holds for all $s \in [\tau, T]$. □

Corollary 3.24 (Upper semicontinuity)**.** *Let $f \in L^2_{loc}(\mathbb{R}; V_t^*)$. Suppose that the conditions* **(H1)**-**(H2)** *and* **(HF)** *hold. Then the time-dependent multi-valued bi-parametric family $\mathcal{U}(\cdot,\cdot)$ defined in* (3.9) *is an upper-semicontinuous multi-valued process with closed values on the family $\{H_t\}_{t\in\mathbb{R}}$.*

Proof. It follows from Theorem 3.23. □

3.2.2. *Asymptotic compactness and the pullback attractor.* In this subsection, we will prove that the multi-valued process defined in (3.9) has a pullback attractor w.r.t. some tempered universe. For this, we will show that there is a family of pullback absorbing bounded sets, and that the process is pullback asymptotically compact. The asymptotic compactness will be proved with the energy method, which was used also in the proof of Theorem 3.23.

The following lemma is fundamental because it guarantees the dissipativity of the system.

Lemma 3.25. *Let us consider $f \in L^2_{loc}(\mathbb{R}; V_t^*)$. Suppose that conditions* **(H1)**-**(H3)** *and* **(HF)** *hold. Then, for any $u_\tau \in H_\tau$, any weak solution $u(\cdot) = u(\cdot, \tau; u_\tau) \in \Psi(\tau, u_\tau)$ satisfies the following estimates, for all $t \geqslant \tau$,*

$$|u(t)|_t^2 \leqslant e^{-\lambda_{1,t}(t-\tau)}|u_\tau|_\tau^2 + \int_\tau^t e^{-\lambda_{1,t}(t-s)}|f(s)|_{-1,s}^2 ds, \tag{3.15}$$

and

$$\int_\tau^t |\nabla u(s)|_s^2 ds \leqslant |u_\tau|_\tau^2 + \int_\tau^t |f(s)|_{-1,s}^2 ds. \tag{3.16}$$

Proof. Given $\tau \in \mathbb{R}$ and $u_\tau \in H_\tau$, let $u \in \Psi(\tau, u_\tau)$ for all $t \geqslant \tau$ a weak solution to 3D-(NS). Then u satisfies

$$\frac{1}{2}\frac{d}{dt}|u|_t^2 + |\nabla u|_t^2 \leqslant (f(t), u(t))_{-1,t} \quad \text{a.e. } t \geqslant \tau, \tag{3.17}$$

since $b_t(u, u, u) = 0$. Thus, we obtain

$$\frac{1}{2}\frac{d}{dt}|u|_t^2 + \frac{1}{2}|\nabla u|_t^2 \leqslant \frac{1}{2}|f(t)|_{-1,t}^2 \quad \text{a.e. } t \geqslant \tau. \tag{3.18}$$

Now, multiplying by $e^{\lambda_{1,t}s}$, we have

$$\frac{d}{ds}\left(e^{\lambda_{1,t}s}|u|_s^2\right) \leqslant e^{\lambda_{1,t}s}|f(s)|_{-1,s}^2.$$

Therefore, integrating in s from τ to t, we deduce that

$$|u(t)|_t^2 \leqslant e^{-\lambda_{1,t}(t-\tau)}|u_\tau|_\tau^2 + e^{-\lambda_{1,t}t}\int_\tau^t e^{\lambda_{1,t}s}|f(s)|_{-1,s}^2 ds,$$

for all $t \geqslant \tau$. Lastly, note that from (3.18), we also deduce that

$$\int_\tau^t |\nabla u(s)|_s^2 ds \leqslant |u_\tau|_\tau^2 + \int_\tau^t |f(s)|_{-1,s}^2 ds,$$

for all $t \geqslant \tau$. □

Definition 3.26 (Tempered universe)**.** *Let $\mathcal{D}_\lambda$ be the universe of all families of subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}, D(t) \subset H_t$ and $D(t) \neq \varnothing\}$ such that*

$$\lim_{\tau \to -\infty} e^{\lambda_{1,\tau}\tau} \sup_{u_\tau \in D(\tau)} |u_\tau|_\tau^2 = 0.$$

Now, let us denote by

$$\mathcal{I}_*^{2,\lambda} := \left\{ f \in L^2_{loc}(\mathbb{R}; V_t^*) : \int_{-\infty}^t e^{\lambda_{1,t}s}|f(s)|_{-1,s}^2 ds < +\infty \quad \text{for all } t \in \mathbb{R} \right\}.$$

Corollary 3.27. *Let $f \in \mathcal{I}_*^{2,\lambda}$ and suppose that the conditions* **(H1)-(H3)** *and* **(HF)** *hold. Then, the family $\widehat{B}_0 = \{B_{H_t}[0, \mathcal{R}(t)] : t \in \mathbb{R}\}$ is pullback $\mathcal{D}_\lambda$-absorbing for the multi-valued process $\mathcal{U}(\cdot,\cdot)$, where*

$$\mathcal{R}^2(t) = 1 + \int_{-\infty}^t e^{-\lambda_{1,t}(t-s)}|f(s)|_{-1,s}^2 ds. \tag{3.19}$$

Moreover, $\widehat{B}_0 \in \mathcal{D}_\lambda$.

Proof. The existence of a pullback absorbing set follows immediately from Lemmas 3.25. Following the same reasoning as [21, Lemma 7.1], we obtain

$$\lim_{\tau \to -\infty} e^{\lambda_{1,\tau}\tau}\mathcal{R}^2(\tau) = 0.$$

Consequently, the family $\widehat{B}_0 \in \mathcal{D}_\lambda$. □

Lemma 3.28. *Let $f \in \mathcal{I}_*^{2,\lambda}$ and suppose that the conditions* **(H1)-(H3)** *and* **(HF)** *hold. Then, given $h > 0$, for any $t \in \mathbb{R}$ and for each $\widehat{D} \in \mathcal{D}_\lambda$, there exists $\tau_1(t,h,\widehat{D}) < t - 3h - 3$ such that for all $\tau \leqslant \tau_1(t,h,\widehat{D})$, $u_\tau \in D(\tau)$, and $u \in \Psi(\tau, u_\tau)$ it holds*

$$\begin{cases} |u(r)|_r \leqslant \varrho_1(t), & \forall r \in [t-h-1, t], \\ \displaystyle\int_{r-1}^r |\nabla u(s)|_s^2 ds \leqslant \varrho_2(t), & \forall r \in [t-h, t], \end{cases}$$

where

$$\varrho_1^2(t) = 1 + e^{\lambda_{1,t}(1+h-t)}\int_{-\infty}^t e^{\lambda_{1,t}s}|f(s)|_{-1,s}^2 ds,$$

$$\varrho_2(t) = \varrho_1^2(t) + \int_{t-h-1}^t |f(s)|_{-1,s}^2 ds.$$

Proof. Let $t \in \mathbb{R}$. Then, it follows from the Lemma 3.25 that for any $h > 0$ and for each $r \in [t-h-1, t]$, one has

$$\begin{aligned} |u(r)|_r^2 &\leqslant e^{-\lambda_{1,r}(r-\tau)}|u_\tau|_\tau^2 + \int_\tau^r e^{-\lambda_{1,r}(r-s)}|f(s)|_{-1,s}^2 ds \\ &\leqslant e^{\lambda_{1,t}(1+h-t)}e^{\lambda_{1,t}\tau}|u_\tau|_\tau^2 + e^{\lambda_{1,t}(1+h-t)}\int_{-\infty}^t e^{\lambda_{1,t}s}|f(s)|_{-1,s}^2 ds, \end{aligned}$$

for all $\tau \leqslant t-h-1$. Thus, by Definition 3.26, for $t \in \mathbb{R}$ and $h > 0$, there exists $\tau_1(t,h,\widehat{D}) \in \mathbb{R}$ such that

$$|u(r)|_r^2 \leqslant 1 + e^{\lambda_{1,t}(1+h-t)} \int_{-\infty}^t e^{\lambda_{1,t}s} |f(s)|_{-1,s}^2 ds, \tag{3.20}$$

for all $u_\tau \in D(\tau)$, $\tau \leqslant \tau_1(t,h,\widehat{D})$, and $r \in [t-h-1,t]$. Thus, denoting by

$$\varrho_1^2(t) := 1 + e^{\lambda_{1,t}(1+h-t)} \int_{-\infty}^t e^{\lambda_{1,t}s} |f(s)|_{-1,s}^2 ds,$$

we have

$$|u(r)|_r \leqslant \varrho_1(t),$$

for all $u_\tau \in D(\tau)$, $\tau \leqslant \tau_1(t,h,\widehat{D})$, and $r \in [t-h-1,t]$.

In the same way as (3.20) and making uses of (3.16), we obtain

$$\int_{r-1}^r |\nabla u(s)|_s^2 ds \leqslant \varrho_1^2(t) + \int_{t-h-1}^t |f(s)|_{-1,s}^2 ds =: \varrho_2(t),$$

for all $u_\tau \in D(\tau)$, $\tau \leqslant \tau_1(t,h,\widehat{D})$, and $r \in [t-h,t]$. ☐

Lemma 3.29 (Asymptotic compactness). *Let $f \in \mathcal{I}_*^{2,\lambda}$ and suppose that conditions* **(H1)-(H3)** *and* **(HF)** *hold. Then, the time-dependent multi-valued process $\mathcal{U}(\cdot,\cdot)$ given in (3.9) is pullback $\mathcal{D}_\lambda$-asymptotically compact.*

Proof. Similarly to Theorem 3.23, we shall demonstrate the asymptotic compactness of the multi-valued process using the energy method, e.g. [25, Proposition 22] or [26, Lemma 5.11]. Given $t_0 \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\lambda$, let $\{u_{\tau_m}\}$ be any sequence, where $u_{\tau_m} \in D(\tau_m)$ for each $m \in \mathbb{N}$ and $\tau_m \to -\infty$ as $m \to \infty$. Now, let us consider the sequence $\{w_m\}$, such that $w_m \in \mathcal{U}(t_0,\tau_m)u_{\tau_m}$ for each $m \in \mathbb{N}$. Then we will prove that the sequence $\{w_m\}$ is relatively compact in H_{t_0}. In fact: for each $m \in \mathbb{N}$ let us denote by $u_m(\cdot) \in \Psi(\tau_m, u_{\tau_m})$ a weak solution of 3D-(NS) associated to the initial condition $u_{\tau_m} \in D(\tau_m)$, such that $w_m = u_m(t_0)$ for all $m \in \mathbb{N}$. Then, since $\tau_m \to -\infty$, it follows from Lemma 3.28 that, for t_0 and some $h > 0$, there exists $m_0(t_0,h) \in \mathbb{N}$ such that

$$\begin{aligned} &|u_m(r)|_r \leqslant \varrho_1(t_0), && \forall r \in [t_0-h-1,t_0], \\ &\int_{r-1}^r |\nabla u_m(s)|_s^2 ds \leqslant \varrho_2(t_0), && \forall r \in [t_0-h,t_0], \end{aligned} \tag{3.21}$$

for all $m \geqslant m_0(t_0,h)$.

Since for each $m \geqslant m_0(t_0,h)$, the function u_m is a weak solution of 3D-(NS) on $[t_0-h,t_0]$. Thus, with the same reasoning as Theorem 3.23, there exists a subsequence (relabeled the same) and a limit function u that converge in the same sense as in (3.11). Moreover, $u \in C([t_0-h,t_0];H_t)$ and u is a weak solution of 3D-(NS) on (t_0-h,t_0), and as in (3.13)), we have

$$u_m(s) \rightharpoonup u(s) \quad \text{weakly in } H_s \text{ for any } t_0-h \leqslant s \leqslant t_0. \tag{3.22}$$

Moreover, putting $z = u$ and u_m, z satisfies the following energy inequality

$$|z(t)|_t^2 \leqslant |z(s)|_s^2 + 2\int_s^t (f(r),z(r))_{-1,r} dr,$$

for all $t_0-h \leqslant s \leqslant t \leqslant t_0$. Then, we can define the functions $J_m, J : [t_0-h,t_0] \to \mathbb{R}$ as

$$J_m(t) = |u_m(t)|_t^2 - 2\int_{t_0-h}^t (f(r),u_m(r))_{-1,r} dr$$

$$J(t) = |u(t)|_t^2 - 2\int_{t_0-h}^t (f(r),u(r))_{-1,r} dr,$$

which are non-increasing and continuous, and satisfy

$$J_m(t) \to J(t) \quad \text{a.e. } t \in (t_0-h,t_0).$$

Therefore, with the properties of functionals $J(\cdot)$ and $J_m(\cdot)$, the convergence given in (3.22), and the same reasoning as in Theorem 3.23, we obtain that

$$u_m \to u \quad \text{strongly in } C([t_0-h,t_0],H_t).$$

In particular $u_m(t_0) \to u(t_0)$ in H_{t_0}, so the proof is completed. $\square$

Theorem 3.30. *Let $f \in \mathcal{I}_*^{2,\lambda}$ and suppose that conditions* **(H1)-(H3)** *and* **(HF)** *hold. Then there exists the minimal pullback $\mathcal{D}_\lambda$-attractor $\mathcal{A}_\lambda \in \mathcal{D}_\lambda$ for the multi-valued processes $\mathcal{U}(\cdot,\cdot)$.*

Proof. First, note that the existence of a family pullback $\mathcal{D}_\lambda$-absorbing, $\widehat{B}_0 = \{B_{H_t}[0,\mathcal{R}(t)] : t \in \mathbb{R}\}$, was given by the Corollary 3.27, where the radius function $\mathcal{R} : \mathbb{R} \to (0,+\infty)$ was set in (3.19). On the other hand, the asymptotic compactness of the multi-valued process $\mathcal{U}(\cdot,\cdot)$ is given by Lemma 3.29. Then, It follows from Theorem 3.11 that there exists a minimal pullback $\mathcal{D}_\lambda$-attractor associated with the time-dependent multi-valued process $\mathcal{U}(\cdot,\cdot)$. $\square$

3.2.3. *Pullback attractor of the equivalent system (CC-NS).* Now, we are interested in showing the existence of a pullback attractor associated with the problem (CC-NS) on the Hilbert space H. For this, we will consider the hypotheses **(H1)-(H3)** and **(HF)** and $\widehat{f} \in L^2_{loc}(\mathbb{R}; H)$. On the other hand, let $\{\Theta(t) : H \to H_t : t \in \mathbb{R}\}$ be the family of homeomorphisms defined as

$$\Theta(t)(v_1, v_2, v_3) = (u_1, u_2, u_3), \tag{3.23}$$

where

$$u_k(x) = \sum_{i=1}^{3} \frac{\partial r_k}{\partial y_i}(\overline{r}(x,t),t) \cdot v_i(\overline{r}(x,t)) \quad \text{ for all } x \in \mathcal{O}_t,\ k = 1,2,3.$$

Then, given an initial time $\tau \in \mathbb{R}$ and an initial datum $v_\tau \in H$, we denote by $\widehat{\Psi}(\tau, v_\tau)$ the set of all $v(\cdot;\tau,v_\tau) : [\tau,+\infty) \to \mathbb{R}^3$ that are weak solutions to (CC-NS) with initial datum $v_\tau \in H$, such that

$$u(t;\tau,u_\tau) = \Theta(t)v(t;\tau,v_\tau) \in \{u(t) : u \in \Psi(\tau,u_\tau)\} \tag{3.24}$$

for all $t \geqslant \tau$, where $u_\tau = \Theta(\tau)v_\tau \in H_\tau$. Define

$$\widehat{\Psi}(\tau,v_\tau) = \left\{ v(\cdot\,;\tau,v_\tau) \,\middle|\, \begin{array}{l} v \text{ is weak solution to (CC-NS) with } v(\tau) = v_\tau \text{ such} \\ \text{that } u(\cdot;\tau;u_\tau) \text{ given in (3.24) belongs to } \Psi(\tau,u_\tau) \end{array} \right\}.$$

Then, it follows from Theorem 2.18 that the family $\widehat{\Psi}(\tau,v_\tau)$ is not empty. Now, we can define a family of multi-valued maps $\widehat{\mathcal{U}}(\cdot,\cdot) : \{(t,\tau)\times H : (t,\tau) \in \mathbb{R}^2_d\} \to \mathcal{P}(H)$ given by

$$\widehat{\mathcal{U}}(t,\tau)v_\tau = \left\{ v(t) : v \in \widehat{\Psi}(\tau,v_\tau) \right\} \in \mathcal{P}(H), \quad v_\tau \in H,\ \tau \leqslant t. \tag{3.25}$$

Lemma 3.31. *The time-dependent dynamical systems $(\mathcal{U}(\cdot,\cdot),\{H_t\}_{t\in\mathbb{R}})$ and $(\widehat{\mathcal{U}}(\cdot,\cdot),H)$, given in (3.9) and (3.25), respectively, are equivalents.*

Proof. By Remark 2.17 and Theorem 2.18 the time-dependent multi-valued families of maps $\{\widehat{\mathcal{U}}(t,\tau) : H \to \mathcal{P}(H) : (t,\tau) \in \mathbb{R}^2_d\}$ and $\{\mathcal{U}(t,\tau) : H_\tau \to \mathcal{P}(H_t) : (t,\tau) \in \mathbb{R}^2_d\}$ satisfy the following relationship

$$\widehat{\mathcal{U}}(t,\tau)v := \Theta^{-1}(t) \circ \mathcal{U}(t,\tau) \circ \Theta(\tau)v \in \mathcal{P}(H) \quad \text{ for all } (t,\tau) \in \mathbb{R}^2_d,\ v \in H.$$

$\square$

Definition 3.32. *We define $\widehat{\mathcal{D}}_\lambda$ the class of all families of subsets $\widehat{D} = \{D(t) : t \in \mathbb{R},\ D(t) \subset H$ and $D(t) \neq \varnothing\}$ such that*

$$\lim_{\tau\to-\infty} e^{\lambda_1 \tau} \sup_{v_\tau \in D(\tau)} |u_\tau|^2 = 0,$$

where λ_1 is the first eigenvalue of $-\Delta$ on $H^1_0(\mathcal{O})^n$,i.e.,

$$\lambda_1 := \min_{w \in H^1_0(\mathcal{O})^n \setminus\{0\}} \frac{\|\nabla w\|^2_{L^2(\mathcal{O})^n}}{\|w\|^2_{L^2(\mathcal{O})^n}}.$$

Lemma 3.33. *The universes $\mathcal{D}_\lambda$ and $\widehat{\mathcal{D}}_\lambda$, given in the Definitions 3.26 and 3.32, respectively, are equivalents.*

Proof. It follows immediately from the definitions of universes 3.26 and 3.32, and the family of homeomorphisms $\{\Theta(t)\}_{t\in\mathbb{R}}$, given in (3.23). $\square$

Theorem 3.34. *The time-dependent multi-valued bi-parametric family* $\widehat{\mathcal{U}}(\cdot,\cdot)$ *(defined in (3.25)) is an upper-semicontinuous multi-valued process with closed values on* H*. Moreover, the family* $\widehat{\mathcal{A}}_\lambda = \left\{\widehat{\mathcal{A}}_\lambda(t) : t \in \mathbb{R}\right\}$*, defined as*

$$\widehat{\mathcal{A}}_\lambda(t) := \Theta^{-1}(t)\mathcal{A}_\lambda(t), \quad \text{for all } t \in \mathbb{R},$$

is a pullback $\widehat{\mathcal{D}}_\lambda$*-attractor for the multi-valued process* $\widehat{\mathcal{U}}(\cdot,\cdot)$*.*

Proof. It follows from Theorem 3.20. □

4. Dynamics of 2D-Navier-Stokes equations on a non-cylindrical domain

In this section we study the 2D case of the Navier-Stokes equation (NS) on a non-cylindrical domain. Since in this case the weak solutions are unique and generate a single-valued process, we are able to show, under certain conditions, the finite fractal dimension of the pullback attractor. The fractal dimension problem for the multi-valued 3D case remains open.

4.1. **Fractal dimension of sets in time-varying spaces: theoretical results.** We first establish a theoretical result on the fractal dimension of pullback attractors that applies to time-varying phase spaces.

Definition 4.1. *Let* X *be a metric space and* K *a compact subset of* X*. The fractal dimension of* K *is defined by*

$$d_f^X(K) = \limsup_{r\to 0} \frac{\log N_r(K,X)}{-\log r},$$

where $N_r(K,X)$ *is the minimum number of balls of radius* r*, centered at some point of* K*, that cover* K*.*

The fractal dimension is also called *the upper box-counting dimension* in the literature (e.g., [7, 15, 8, 34]).

Theorem 4.2. *(cf. [27, Theorem 4.11]) Let* $t_0 \in \mathbb{R}$ *be fixed and consider* $\{X_t\}_{t\in(-\infty,t_0]}$ *and* $\{W_t\}_{t\in(-\infty,t_0]}$ *two families of normed vector spaces such that* W_t *is compactly embedded in* X_t *for every* $t \leqslant t_0$*. Assume that* $\{\mathscr{C}_t\}_{t\in(-\infty,t_0]}$ *is a family of bounded subsets of* $\{X_t\}_{t\in(-\infty,t_0]}$ *and there exists a positive constant* $\varrho_0 = \varrho_0(t_0) > 0$ *such that for each* $t \in \mathbb{R}$ *there is an* $u_t \in \mathscr{C}_t$ *satisfying*

$$\mathscr{C}_t \subset B_{X_t}(u_t, \varrho_0) \quad \text{for all } t \leqslant t_0. \tag{4.1}$$

Also suppose that there exists a family of operators $\{L_t : X_{t-1} \to W_t\}_{t\in(-\infty,t_0]}$ *such that*

1. $\mathscr{C}_t \subset L_t\mathscr{C}_{t-1}$ *for all* $t \leqslant t_0$*;*
2. *there exists a function* $\kappa : (-\infty, t_0] \to (0, +\infty)$ *such that*
$$\|L_t x - L_t y\|_{W_t} \leqslant \kappa(t)\|x-y\|_{X_{t-1}}, \ \text{for all } \ x, y \in \mathscr{C}_{t-1}, \ t \leqslant t_0;$$
3. *there is an* $\mathcal{N}_{t_0} \in \mathbb{N}$ *such that*
$$\sup_{s\leqslant t_0} N^s_{1/4\kappa(s)} \leqslant \mathcal{N}_{t_0}, \tag{4.2}$$
where $N^t_\epsilon := N_\epsilon(B_{W_t}(0,1), X_t)$ *for any* $\epsilon > 0$ *and* $t \leqslant t_0$*.*

Then,

$$\sup_{s\leqslant t_0} d_f^{X_s}(\mathscr{C}_s) \leqslant \frac{\log \mathcal{N}_{t_0}}{\log 2}.$$

Proof. Let $t \leqslant t_0$. Then, it follows from (1), (2) and (4.1) that

$$\mathscr{C}_t = \left[L_t\mathscr{C}_{t-1}\right] \cap \mathscr{C}_t \subset B_{W_t}(L_t u_{t-1}, \kappa(t)\varrho_0) \subset \bigcup_{i=1}^{N^t_{1/4\kappa(t)}} B_{X_t}\left(\tilde{u}_{t,i}, \frac{\varrho_0}{4}\right),$$

where $\tilde{u}_{t,i} \in X_t$ for all $i \in \{1, \cdots, N^t_{1/4\kappa(t)}\}$. Assuming without lost of generality that $\mathscr{C}_t \cap B_{X_t}\left(\tilde{u}_{t,i}, \frac{\varrho_0}{4}\right) \neq \varnothing$ for any $i \in \left\{1, \ldots, N^t_{1/4\kappa(t)}\right\}$ we can take $u_{t,i} \in \mathscr{C}_t$ such that

$$\mathscr{C}_t \subset \bigcup_{i=1}^{N^t_{1/4\kappa(t)}} B_{X_t}\left(u_{t,i}, \frac{\varrho_0}{2}\right), \quad \text{for all } t \leqslant t_0.$$

Then we have that $N_{X_t}(\mathscr{C}_t, \varrho_0/2^k) \leqslant \mathcal{N}_{t_0}$. Then, proceeding by induction, it is proven that

$$N_{X_t}(\mathscr{C}_t, \varrho_0/2^k) \leqslant \mathcal{N}_{t_0}^k, \quad \text{for all } k \geqslant 1,\ t \leqslant t_0.$$

Consequently, in a standard way, we arrive at

$$d_f^{X_t}(\mathscr{C}_t) \leqslant \frac{\log \mathcal{N}_{t_0}}{\log 2} \quad \text{for all } t \leqslant t_0.$$

□

Lemma 4.3. *(cf.* [7, Lemma 4.2]*) Let X, Y be two normed spaces. Consider $\mathscr{C} \subset X$, and $f : \mathscr{C} \to Y$ a Hölder continuous function with exponent θ, $\theta \in (0,1]$, i.e., there exists an $L > 0$ such that*

$$\|f(x) - f(y)\|_Y \leqslant L\|x - y\|_X^\theta,$$

for all $x, y \in \mathscr{C}$. Then,

$$d_f^Y(f(\mathscr{C})) \leqslant \frac{1}{\theta} d_f^X(\mathscr{C}).$$

Now we apply the previous results to pullback attractors of single-valued processes. Before that, for the readers' convenience let us briefly recall the existence of a pullback attractor. The readers are referred to [19, 9, 13, 5, 21, 26], see also Theorem 3.11 for the corresponding multi-valued version.

Definition 4.4. *A bi-parametric family of operators $\{\mathcal{U}(t,\tau) : X_\tau \to X_t : (\tau, t) \in \mathbb{R}_d^2\}$ is called a single-valued evolution process, or simply called a process, on a family of complete metric spaces $\{(X_t, d_{X_t})\}_{t\in\mathbb{R}}$ if*

(i) *$\mathcal{U}(\tau,\tau) = \mathrm{Id}_{X_\tau}$ (identity on X_τ) for any $\tau \in \mathbb{R}$,*
(ii) *$\mathcal{U}(t,s) \circ \mathcal{U}(s,\tau) = \mathcal{U}(t,\tau)$ for any $\tau \leqslant s \leqslant t$.*

In addition, a process $\{\mathcal{U}(t,\tau)\}$ is called continuous, if each $\mathcal{U}(t,\tau)$ is a continuous operator.

Lemma 4.5. *(cf.* [19, Theorem 23]*) Consider a continuous process $\mathcal{U}(\cdot,\cdot)$ on the family of metric spaces $\{X_t\}_{t\in\mathbb{R}}$, a pullback $\mathcal{D}_X$-absorbing family $\widehat{B}_{0,X} = \{B_{0,X}(t) \subset X_t : t \in \mathbb{R}\}$ and assume that $\mathcal{U}(\cdot,\cdot)$ is pullback $\widehat{B}_{0,X}$-asymptotically compact. Then, the family $\mathcal{A} = \{\mathcal{A}(t) \subset X_t : t \in \mathbb{R}\}$ given by*

$$\mathcal{A}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}_X} \Lambda_{X_t}(\widehat{D}, t)}^{X_t}, \quad \forall t \in \mathbb{R},$$

is the minimal pullback $\mathcal{D}_X$-attractor.

Theorem 4.6. *Let $\{X_t\}_{t\in\mathbb{R}}$ and $\{W_t\}_{t\in\mathbb{R}}$ be two families of normed vector spaces such that W_t is compactly embedded in X_t for every $t \in \mathbb{R}$. Consider that the single-valued process $\mathcal{U}_X(\cdot,\cdot)$ has a pullback $\mathcal{D}_X$-attractor $\mathcal{A}_{\mathcal{D}_X}$. Assume that for any $(t,s) \in \mathbb{R}_d^2$ there exists a $L_{t,s} > 0$ such that*

$$\|\mathcal{U}_X(t,s)x_1 - \mathcal{U}_X(t,s)x_2\|_{X_t} \leqslant L_{t,s}\|x_1 - x_2\|_{X_s}, \text{ for all } x_1, x_2 \in \mathcal{A}_{\mathcal{D}_X}(s). \tag{4.3}$$

Moreover, we assume that there exists $t_0 \in \mathbb{R}$ such that

(1) *there is a positive constant $\varrho_0 = \varrho(t_0)$ such that for each $t \leqslant t_0$ there is an $u_t \in \mathcal{A}_{\mathcal{D}_X}(t)$ satisfying*

$$\mathcal{A}_{\mathcal{D}_X}(t) \subset B_{X_t}(u_t, \varrho_0) \quad \text{for all } t \leqslant t_0; \tag{4.4}$$

(2) *there exists a function $\kappa : \mathbb{R} \to (0, +\infty)$ such that*

$$\|\mathcal{U}_X(t,t-1)x - \mathcal{U}_X(t,t-1)y\|_{W_t} \leqslant \kappa(t)\|x - y\|_{X_{t-1}},$$

for all $x, y \in \mathcal{A}_{\mathcal{D}_X}(t-1)$, $t \leqslant t_0$;

(3) *there is an $\mathcal{N}_{t_0} \in \mathbb{N}$ such that*

$$\sup_{s\leqslant t_0} N_{1/4\kappa(s)}^s \leqslant \mathcal{N}_{t_0}, \tag{4.5}$$

where $N_\epsilon^t := N_\epsilon(B_{W_t}(0,1), X_t)$ for any $\epsilon > 0$ and $t \leqslant t_0$.

Then

$$\sup_{s\in\mathbb{R}} d_f^{X_s}(\mathcal{A}_{\mathcal{D}_X}(s)) \leqslant \frac{\log \mathcal{N}_{t_0}}{\log 2}.$$

Proof. Applying the Theorem 4.2 with $L_t := \mathcal{U}_X(t, t-1)$ and $\mathscr{C}_t = A_{\mathcal{D}_X}(t)$ for each $t \leqslant t_0$, we have

$$d_f^{X_t}(\mathcal{A}_{\mathcal{D}_X}(t)) \leqslant \frac{\log \mathcal{N}_{t_0}}{\log 2} \text{ for all } t \leqslant t_0. \tag{4.6}$$

On the other hand, it follows from the invariance of the pullback attractor $\mathcal{A}_{\mathcal{D}_X}$ that

$$\mathcal{A}_{\mathcal{D}_X}(t) = \mathcal{U}_X(t, t_0)\mathcal{A}_{\mathcal{D}_X}(t_0) \text{ for all } t > t_0.$$

Therefore, it follows from estimate (4.6), the Lipschitz condition (4.3), and Lemma 4.3 that

$$d_f^{X_t}(\mathcal{A}_{\mathcal{D}_X}(t)) \leqslant \frac{\log \mathcal{N}_{t_0}}{\log 2} \text{ for all } t > t_0.$$

□

Remark 4.7. *The result to estimate the fractal dimension of pullback attractors with non-cylindrical domains was developed in* [12, Lemma 6.1]*,* [13, Theorem 4.4] *and* [27, Theorem 4.11]*. Here our result, Theorem 4.6, is new in the following sense: The proof presented in* [27] *is given for a family of time-dependent maps, acting on a family of time-dependent compact sets, satisfying the hypotheses of negative invariance, Lipschitzianity, etc., for all time $t \in \mathbb{R}$. On the other hand, our hypotheses need not hold for all time $t \in \mathbb{R}$ (as required by* [27, Theorem 4.11]*), but only need hold for $t \leqslant t_0$. It is essentially the invariance of the pullback attractor which contributed to the improvement.*

The following result establishes the relationship of the fractal dimension of pullback attractors associated with equivalent dynamical systems.

Theorem 4.8. *Let $\{X_t\}_{t\in\mathbb{R}}$ and $\{Y_t\}_{t\in\mathbb{R}}$ be two equivalent families of normed spaces. Consider the time-dependent single-valued evolution processes $\mathcal{U}_X(\cdot,\cdot)$, and $\mathcal{U}_Y(\cdot,\cdot)$ satisfying the relation given in Theorem 3.16. Let $\mathcal{A}_{\mathcal{D}_X}$ and $\mathcal{A}_{\mathcal{D}_Y}$ be the pullback $\mathcal{D}_X$-attractor and the pullback $\mathcal{D}_Y$-attractor, associated with $\mathcal{U}_X(\cdot,\cdot)$ and $\mathcal{U}_Y(\cdot,\cdot)$, respectively. Then,*

i) if the homeomorphism family $\{\Theta(t) : X_t \to Y_t : t \in \mathbb{R}\}$ is a family of Lipschitz transformations, i.e., for each $t \in \mathbb{R}$ there exists a $\ell_t > 0$ such that

$$\|\Theta(t)x - \Theta(t)y\|_{Y_t} \leqslant \ell_t \|x - y\|_{X_t} \text{ for all } x, y \in X_t.$$

Then $d_f^{Y_t}(\mathcal{A}_{\mathcal{D}_Y}(t)) \leqslant d_f^{X_t}(\mathcal{A}_{\mathcal{D}_X}(t))$ for all $t \in \mathbb{R}$;

ii) if the homeomorphism family $\{\Theta(t) : X_t \to Y_t : t \in \mathbb{R}\}$ is a family of bi-Lipschitz transformations, i.e., for each $t \in \mathbb{R}$ there are positive numbers $\ell_{1,t}, \ell_{2,t}$ such that

$$\ell_{1,t}\|x - y\|_{X_t} \leqslant \|\Theta(t)x - \Theta(t)y\|_{Y_t} \leqslant \ell_{2,t}\|x - y\|_{X_t} \text{ for all } x, y \in X_t.$$

Then $d_f^{Y_t}(\mathcal{A}_{\mathcal{D}_Y}(t)) = d_f^{X_t}(\mathcal{A}_{\mathcal{D}_X}(t))$ for all $t \in \mathbb{R}$.

Proof. It follows directly from Lemma 4.3. □

4.2. **Pullback attractor of the 2D-Navier-Stokes equation: Fractal dimension.** In this section, we are interested in estimating the fractal dimension of the pullback attractor of the 2D-Navier-Stokes equation (NS) on the time-dependent family of Hilbert spaces $\{H_t\}_{t\in\mathbb{R}}$.

Note that under the hypotheses **(H1)-(H2)** and $f \in L^2_{loc}(\mathbb{R}; V_t^*)$, given $\tau \in \mathbb{R}$ and $u_\tau \in H_\tau$, Theorem 2.18 guarantees the existence of a unique weak solution $u(\cdot) = u(\cdot, \tau, u_\tau)$ of the system 2D-(NS) that passes through $u_\tau \in H_\tau$ in the instant of time $\tau \in \mathbb{R}$, i.e., $u(\tau) = u_\tau$. This allows us to define the bi-parametric family of single-valued maps $\{\mathcal{U}(t, \tau) : H_\tau \to H_t : (t, \tau) \in \mathbb{R}^2_d\}$ as

$$\mathcal{U}(t, \tau) : H_\tau \to H_t \quad \text{by} \quad \mathcal{U}(t, \tau)u_\tau = u(t, \tau, u_\tau), \tag{4.7}$$

where $u(\cdot) = u(\cdot, \tau, u_\tau)$ is the unique weak solution of 2D-(NS) associated to the initial condition $u_\tau \in H_\tau$ for any $(t, \tau) \in \mathbb{R}^2_d$.

Corollary 4.9. *Let $f \in L^2_{loc}(\mathbb{R}; V_t^*)$. Then the bi-parametric family $\mathcal{U}(\cdot,\cdot)$ defined in (4.7) is a continuous process.*

Proof. In the same way as [32, Theorem 2.8] or [38, Theorem 4.8], let $u(t)$, $v(t)$ be two weak solutions of 2D-(NS) associated with the initial conditions u_τ and v_τ, respectively. Now, denote by $w := u - v$ and take w as a test function in the weak formulation, thus we have

$$\frac{1}{2}\frac{d}{dt}|w|_t^2 + |\nabla w|_t^2 + b_t(u,u,w) - b_t(v,v,w) = 0.$$

From (2.9) we have that

$$\begin{aligned} b_t(u,u,w) - b_t(v,v,w) &= b_t(u,u,w) - b_t(u,v,w) + b_t(u,v,w) - b_t(v,v,w) \\ &= b_t(u,w,w) + b_t(w,v,w) = b_t(w,v,w) \\ &\leqslant c_t\|w\|^2_{L^4(\mathcal{O}_t)^2}|\nabla v|_t \leqslant c_t d_t^2 |w|_t |\nabla w|_t |\nabla v|_t, \end{aligned}$$

where we have used the Ladyzhenskaya inequality, i.e., $\|w\|_{L^4(\mathcal{O}_t)^2} \leqslant d_t |w|_t^{1/2} |\nabla w|_t^{1/2}$. Now, applying Young inequality we obtain

$$\frac{d}{dt}|w|_t^2 + |\nabla w|_t^2 \leqslant c_{1,t}|\nabla v|_t^2 |w|_t^2, \tag{4.8}$$

where $c_{1,t} = c_t^2 d_t^4$. Hence, by integrating and applying the Gronwall inequality we have

$$|\mathcal{U}(t,\tau)u_\tau - \mathcal{U}(t,\tau)v_\tau|_t^2 \leqslant |u_\tau - v_\tau|_\tau^2 \exp\left\{c_{1,t}\int_\tau^t |\nabla v(r)|_r^2 dr\right\},$$

for any $u_\tau, v_\tau \in H_\tau$ and $(t,\tau) \in \mathbb{R}_d^2$. Therefore, the time-dependent evolution process $\mathcal{U}(\cdot,\cdot)$ is continuous. □

Remark 4.10. *Let $(t,\tau) \in \mathbb{R}_d^2$ and $u_\tau, v_\tau \in H_\tau$. Making $u(\cdot) = \mathcal{U}(\cdot,\tau)u_\tau$ and $v(\cdot) = \mathcal{U}(\cdot,\tau)v_\tau$, and using (4.8), we have*

$$\int_s^t |\nabla u(r) - \nabla v(r)|_r^2 dr \leqslant |u(s) - v(s)|_s^2 + c_{1,t}\int_s^t |\nabla v(r)|_r^2 |u(r) - v(r)|_r^2 dr, \tag{4.9}$$

for all $\tau \leqslant s \leqslant t$.

Inspired by [23], we need an additional hypothesis on the external force f to obtain high regularity of the weak solutions of 2D-(NS), and with this being able to show the smoothing property of the time-dependent dynamical system.

(H4) Let us consider the external force $f \in L^2_{loc}(\mathbb{R}; H_t)$ satisfying the piecewise bounded integrability condition on the family $\{H_t\}_{t\in\mathbb{R}}$, i.e.,

$$M_f(t) := \sup_{s\leqslant t}\int_{s-1}^s |f(s)|_s^2 ds < \infty, \quad \text{for all } t \in \mathbb{R}.$$

Note that **(H4)** is a condition between the translation-bounded and the tempered conditions, see [10, Proposition 4.1]. In fact, **(H4)** is equivalent to $\sup_{s\leqslant 0}\int_{s-1}^s |f(s)|_s^2 ds < \infty$.

Lemma 4.11. *Suppose that hypotheses* **(H1)**-**(H4)** *hold. Then, any weak solution to 2D-(NS) satisfies the following estimates*

$$|\mathcal{U}(t,s)u_s|_t^2 \leqslant e^{-\lambda_{1,t}(t-s)}|u_s|_s^2 + \lambda_{1,t}^{-1}(1+\lambda_{1,t}^{-1})M_f(t),$$

for all $u_s \in H_s$, $s \leqslant t$, and

$$\int_s^t |\nabla\mathcal{U}(r,s)u_s|_r^2 dr \leqslant |u_s|_s^2 + \lambda_{1,t}^{-1}M_f(t), \quad \textit{if } s \in [t-1,t],$$

$$\int_{t-1}^t |\nabla\mathcal{U}(r,s)u_s|_r^2 dr \leqslant e^{-\lambda_{1,t}(t-1-s)}|u_s|_s^2 + 2\lambda_{1,t}^{-1}(1+\lambda_{1,t}^{-1})M_f(t), \quad \textit{if } s < t-1.$$

Proof. Let $u_s \in H_s$ and denote by $u(\cdot) = \mathcal{U}(\cdot,s)u_s$. Then, from (3.17) and Poincaré inequality, we have u satisfying

$$\frac{d}{dt}|u|_t^2 + \lambda_{1,t}|u|_t^2 \leqslant \frac{1}{\lambda_{1,t}}|f(t)|_t^2, \tag{4.10}$$

Thus, multiplying (4.10) by $e^{\lambda_{1,t}t}$ and integrating from s to t, we get

$$|u(t)|_t^2 \leqslant e^{-\lambda_{1,t}(t-s)}|u(s)|_s^2 + \lambda_{1,t}^{-1}e^{-\lambda_{1,t}t}\int_s^t e^{\lambda_{1,t}\theta}|f(\theta)|_\theta^2 d\theta. \tag{4.11}$$

By the hypothesis (H4), can estimate the last term on the left side of inequality (4.11) as

$$\begin{aligned}
\lambda_{1,t}^{-1}e^{-\lambda_{1,t}t}\int_s^t e^{\lambda_{1,t}\theta}|f(\theta)|_\theta^2 d\theta &\leqslant \lambda_{1,t}^{-1}e^{-\lambda_{1,t}t}\int_{-\infty}^t e^{\lambda_{1,t}\theta}|f(\theta)|_\theta^2 d\theta \\
&= \lambda_{1,t}^{-1}e^{-\lambda_{1,t}t}\sum_{n=0}^{\infty}\int_{t-(n+1)}^{t-n} e^{\lambda_{1,t}\theta}|f(\theta)|_\theta^2 d\theta \\
&\leqslant \lambda_{1,t}^{-1}e^{-\lambda_{1,t}t}\sum_{n=0}^{\infty} e^{\lambda_{1,t}(t-n)}\int_{t-(n+1)}^{t-n}|f(\theta)|_\theta^2 d\theta \\
&\leqslant \lambda_{1,t}^{-1}(1-e^{-\lambda_{1,t}})^{-1}M_f(t) \\
&\leqslant \lambda_{1,t}^{-1}(1+\lambda_{1,t}^{-1})M_f(t).
\end{aligned}$$

Therefore, by (4.11) we have

$$|u(t)|_t^2 \leqslant e^{-\lambda_{1,t}(t-s)}|u_s|_s^2 + \lambda_{1,t}^{-1}(1+\lambda_{1,t}^{-1})M_f(t), \tag{4.12}$$

for all $u_s \in H_s$, $t \geqslant s$.

On the other hand, again from (3.17), u satisfies

$$\frac{d}{dt}|u|_t^2 + |\nabla u|_t^2 \leqslant \frac{1}{\lambda_{1,t}}|f(t)|_t^2. \tag{4.13}$$

Then, integrating from s to t, we obtain

$$\int_s^t |\nabla u(r)|_r^2 dr \leqslant |u_s|_s^2 + \frac{1}{\lambda_{1,t}}\int_s^t |f(r)|_r^2 dr \leqslant |u_s|_s^2 + \lambda_{1,t}^{-1}M_f(t),$$

for all $u_s \in H_s$, $t \geqslant s$, in particular for $s \in [t-1,t]$.

Now, let us assume that $s < t-1$. Integrating in (4.13) from $t-1$ to t and using the inequality (4.12), we obtain

$$\begin{aligned}
\int_{t-1}^t |\nabla u(r)|_r^2 dr &\leqslant |u(t-1)|_{t-1}^2 + \frac{1}{\lambda_{1,t}}\int_{t-1}^t |f(r)|_r^2 dr \\
&\leqslant |u(t-1)|_{t-1}^2 + \lambda_{1,t}^{-1}M_f(t) \\
&\leqslant e^{-\lambda_{1,t-1}(t-1-s)}|u_s|_s^2 + \lambda_{1,t-1}^{-1}(1+\lambda_{1,t-1}^{-1})M_f(t-1) + \lambda_{1,t}^{-1}M_f(t) \\
&\leqslant e^{-\lambda_{1,t}(t-1-s)}|u_s|_s^2 + 2\lambda_{1,t}^{-1}(1+\lambda_{1,t}^{-1})M_f(t).
\end{aligned}$$

for all $u_s \in H_s$, and $s < t-1$. □

Corollary 4.12. *Suppose that conditions* **(H1)-(H4)** *hold. Let $\widehat{D} \in \mathcal{D}_\lambda$, $t_0 \in \mathbb{R}$, and $h > 0$. Then, there exists a $\tau(t_0,h,\widehat{D}) \in \mathbb{R}$ such that for all $u_s \in D(s)$ with $s \leqslant \tau(t_0,h,\widehat{D})$,*

$$|\mathcal{U}(t,s)u_s|_t \leqslant \mathcal{R}_1(t_0) \quad \textit{for all } t \in [t_0-h-2,t_0],$$

and

$$\int_{t-1}^t |\nabla\mathcal{U}(r,s)u_s|_r^2 dr \leqslant \mathcal{R}_2(t_0) \quad \textit{for all } t \in [t_0-h-1,t_0],$$

where $\mathcal{R}_1^2(t_0) := 1 + \lambda_{1,t_0}^{-1}(1+\lambda_{1,t_0}^{-1})M_f(t_0)$ and $\mathcal{R}_2(t_0) := 1 + 2\lambda_{1,t_0}^{-1}(1+\lambda_{1,t_0}^{-1})M_f(t_0)$.

Proof. By Lemma 4.11, we know that

$$|\mathcal{U}(t,s)u_s|_t^2 \leqslant e^{-\lambda_{1,t}(t-s)}|u_s|_s^2 + \lambda_{1,t}^{-1}(1+\lambda_{1,t}^{-1})M_f(t), \quad \text{for any} \quad u_s \in H_s,\ s \leqslant t.$$

Therefore, for $\widehat{D} \in \mathcal{D}_\lambda$, $t_0 \in \mathbb{R}$ and $h > 0$, there exists a $\tau(t_0,h,\widehat{D}) \in \mathbb{R}$ such that

$$|\mathcal{U}(t,s)u_s|_t^2 \leqslant 1 + \lambda_{1,t_0}^{-1}(1+\lambda_{1,t_0}^{-1})M_f(t_0), \tag{4.14}$$

for all $s \leqslant \tau(t_0, h, \widehat{D})$ and $t \in [t_0 - h - 2, t_0]$.

On the other hand, using (4.13), we deduce that

$$\int_{t-1}^{t} |\nabla u(r)|_r^2 dr \leqslant |u(t-1)|_{t-1}^2 + \lambda_{1,t}^{-1} M_f(t).$$

Then, from (4.14), we get

$$\int_{t-1}^{t} |\nabla \mathcal{U}(r,s)u_s|_r^2 dr \leqslant 1 + 2\lambda_{1,t_0}^{-1}(1 + \lambda_{1,t_0}^{-1}) M_f(t_0),$$

for all $s \leqslant \tau(t_0, h, \widehat{D})$ and $t \in [t_0 - h - 1, t_0]$. □

Lemma 4.13. *Suppose that conditions* **(H1)-(H4)** *hold. Then, there exist positive non-decreasing functions* $C_f(\cdot), \widehat{C}_f(\cdot) : \mathbb{R} \to (0, +\infty)$, *such that given* $t_0 \in \mathbb{R}$, $\widehat{D} \in \mathcal{D}_\lambda$ *and* $h > 0$, *there exists a* $\tau(t_0, h, \widehat{D}) \in \mathbb{R}$ *such that*

$$|\nabla \mathcal{U}(t,s)u_s|_t \leqslant C_f(t_0), \quad \text{and} \quad \int_{t-1}^{t} |A\mathcal{U}(r,s)u_s|_r^2 dr \leqslant \widehat{C}_f(t_0), \tag{4.15}$$

for all $u_s \in D(s)$, *with* $s \leqslant \tau(t_0, h, \widehat{D})$, *and* $t \in [t_0 - h, t_0]$, *where*

$$C_f^2(t_0) := \left[1 + \lambda_{1,t_0}^{-1}(1 + \lambda_{1,t_0}^{-1}) M_f(t_0) + \lambda_{1,t_0}^{-1} M_f(t_0)\right] \cdot e^{a(t_0)}$$

and

$$\widehat{C}_f(t_0) := C_f^2(t_0) + \tilde{c}_{t_0} C_f^4(t_0) \mathcal{R}^2(t_0) + 2M_f(t_0).$$

Proof. Let $t_0 \in \mathbb{R}$ and $\widehat{D} = \{D(t) \subset H_t : t \in \mathbb{R}\} \in \mathcal{D}_\lambda^H$, and denote by $u(t) = \mathcal{U}(t,s)u_s$ with $s \leqslant t \leqslant t_0$ and $u_s \in D(s)$. Then, taking Au as a test function, we get

$$\frac{1}{2}\frac{d}{dt}|\nabla u|_t^2 + |Au|_t^2 + b_t(u, u, Au) = (f(t), Au)_t. \tag{4.16}$$

Note that by (2.10), we have

$$|b_t(u,u,Au)| \leqslant c_t |u|_t^{1/2} |\nabla u|_t |Au|_t^{3/2} \leqslant \frac{1}{4}|Au|_t^2 + \frac{\tilde{c}_t}{2}|u|_t^2 |\nabla u|_t^4.$$

Now, by using (4.16), we obtain

$$\frac{d}{dt}|\nabla u|_t^2 + |Au|_t^2 \leqslant 2|f(t)|_t^2 + \tilde{c}_t |u|_t^2 |\nabla u|_t^4 = 2|f(t)|_t^2 + \tilde{c}_t |u|_t^2 |\nabla u|_t^2 |\nabla u|_t^2. \tag{4.17}$$

Considering $\mathbf{z}(t) := |\nabla u|_t^2$, $\mathbf{a}(t) := \tilde{c}_t |u|_t^2 |\nabla u|_t^2$, and $\mathbf{b}(t) := 2|f(t)|_t^2$. Then, the inequality (4.17) has the following form

$$\frac{d\mathbf{z}}{dt}(t) \leqslant \mathbf{b}(t) + \mathbf{a}(t)\mathbf{z}(t). \tag{4.18}$$

Then, it follows from Corollary 4.12 that there exists a $\tau(t, h, \widehat{D}) \in \mathbb{R}$ such that

$$\begin{cases} \displaystyle\int_{t-1}^{t} \mathbf{a}(\theta) d\theta \leqslant a(t_0) := \tilde{c}_{t_0} \mathcal{R}_1^2(t_0) \mathcal{R}_2(t_0), \\ \displaystyle\int_{t-1}^{t} \mathbf{b}(\theta) d\theta \leqslant b(t_0) := 2M_f(t_0), \end{cases} \tag{4.19}$$

for all $u_s \in D(s)$, $s \leqslant \tau(t_0, h, \widehat{D})$ and $t \in [t_0 - h - 1, t_0]$. Furthermore, from (4.18) we deduce that

$$\frac{d}{d\ell}\left(e^{-\int_r^\ell \mathbf{a}(\theta)d\theta} \mathbf{z}(\ell)\right) \leqslant \mathbf{b}(\ell) e^{-\int_r^\ell \mathbf{a}(\theta)d\theta}.$$

Now, integrating from r to t, with $r \in [t-1, t]$, we have that

$$e^{-\int_r^t \mathbf{a}(\theta)d\theta} \mathbf{z}(t) \leqslant \mathbf{z}(r) + \int_r^t \mathbf{b}(\ell) e^{-\int_r^\ell \mathbf{a}(\theta)d\theta} d\ell.$$

Then, by using (4.19), we get

$$\mathbf{z}(t) \leqslant \mathbf{z}(r)e^{a(t_0)} + \int_r^t \mathbf{b}(\ell)e^{\int_\ell^t \mathbf{a}(\theta)d\theta}d\ell \leqslant \mathbf{z}(r)e^{a(t_0)} + b(t_0)e^{a(t_0)},$$

for all $r \in [t-1, t]$. Thus, integrating with respect to r, from $t-1$ to t, we have that

$$\mathbf{z}(t) \leqslant e^{a(t_0)} \left(\int_{t-1}^t \mathbf{z}(r)dr + b(t_0) \right).$$

Thus

$$|\nabla u(t)|_t^2 \leqslant e^{a(t_0)} \left(\int_{t-1}^t |\nabla u(r)|_r^2 dr + b(t_0) \right), \text{ for all } t \in [t_0 - h - 1, t_0].$$

Then, again from Corollary 4.12, we have

$$|\nabla \mathcal{U}(t,s)u_s|_t \leqslant C_f(t_0), \text{ for all } s \leqslant \tau(t, \widehat{D}) \text{ and } t \in [t_0 - h - 1, t_0], \tag{4.20}$$

where $C_f^2(t_0) = \left[\mathcal{R}_2(t_0) + \lambda_{1,t_0}^{-1} M_f(t_0)\right] \cdot e^{a(t_0)}$. Now, taking into account (4.17), we have

$$\int_{t-1}^t |Au(r)|_r^2 dr \leqslant |\nabla u(t-1)|_{t-1}^2 + \tilde{c}_t \int_{t-1}^t |u(r)|_r^2 |\nabla u(r)|_r^4 dr + 2\int_{t-1}^t |f(r)|_r^2 dr.$$

It follows from Corollary 4.12 and (4.20) that

$$\int_{t-1}^t |A\mathcal{U}(r,s)u_s|_r^2 dr \leqslant C_f^2(t_0) + \tilde{c}_{t_0} C_f^4(t_0)\mathcal{R}_1^2(t_0) + 2M_f(t_0),$$

for all $u_s \in D(s)$, $s \leqslant \tau(t_0, h, \widehat{D})$ and $t \in [t_0 - h, t_0]$. □

Corollary 4.14. *Under the conditions of Lemma 4.13. The family $\widehat{B}_V = \left\{B_{V_t}[0, C_f(t)] : t \in \mathbb{R}\right\}$ is pullback $\mathcal{D}_\lambda$-absorbing, for the process $\mathcal{U}(\cdot,\cdot)$ defined in (4.7). Moreover the family $\widehat{B}_V \in \mathcal{D}_\lambda$.*

Proof. It follows directly from Lemma 4.13. □

4.2.1. *Pullback attractor of 2D-(NS).* The existence of the pullback $\mathcal{D}_\lambda$-attractor for process $\left(\mathcal{U}(\cdot,\cdot), \{H_t\}_{t\in\mathbb{R}}\right)$ was given in [38, Theorem 6.4]. Now, with the regularity of the weak solutions, obtained in the previous results, we are able to give a simpler demonstration of the existence of a minimal pullback $\mathcal{D}_\lambda$-attractor.

Theorem 4.15 (Existence and regularity of a pullback $\mathcal{D}_\lambda$-attractor))**.** *Let the conditions* **(H1)**-**(H4)** *hold. Then the family $\mathcal{A}_\lambda = \{\mathcal{A}_\lambda(t) \subset V_t : t \in \mathbb{R}\}$ defined by*

$$\mathcal{A}_\lambda(t) = \Lambda_{H_t}(\widehat{B}_V, t) = \bigcap_{s\leqslant t} \overline{\bigcup_{\tau \leqslant s} \mathcal{U}(t,\tau)B_{V_\tau}[0, C_f(\tau)]}^{H_t} \subset B_{V_t}[0, C_f(t)], \quad \forall t \in \mathbb{R},$$

is the unique pullback $\mathcal{D}_\lambda$-attractor for the solution process $\mathcal{U}(\cdot,\cdot)$ belonging to $\mathcal{D}_\lambda$, where $\widehat{B}_V$ is given in the Corollary 4.14.

Proof. The evolution process $\mathcal{U}(\cdot,\cdot)$ is dissipative due to Corollary 4.14 with the family $\widehat{B}_V = \{B_{V_t}[0, C_f(t)] : t \in \mathbb{R}\}$ that is pullback $\mathcal{D}_\lambda$-absorbing. Regarding asymptotic compactness, let us consider the sequence $\{u_{\tau_n}\}$ such that $u_{\tau_n} \in B_{V_{\tau_n}}[0, C_f(\tau_n)]$ for all $n \in \mathbb{N}$ and $\tau_n \to -\infty$. Now, let $t \geqslant \tau_n$ for all $n \in \mathbb{N}$, and consider the sequence $\{\mathcal{U}(t,\tau_n)u_{\tau_n}\}$. We affirm that sequence $\{\mathcal{U}(t,\tau_n)u_{\tau_n}\}$ has a convergent subsequence in H_t. Indeed, it follows from Corollary 4.14 that the family $\widehat{B}_V \in \mathcal{D}_\lambda$, so that there exists an $m_0(t) \in \mathbb{N}$ such that $\mathcal{U}(t,\tau_n)u_{\tau_n} \in B_{V_t}[0, C_f(t)]$ for all $n \geqslant m_0(t)$. Since $B_{V_t}[0, C_f(t)]$ is a compact set of H_t (for any $t \in \mathbb{R}$), the sequence $\{\mathcal{U}(t,\tau_n)u_{\tau_n}\}_{n \geqslant m_0(t)}$ has a convergent subsequence in H_t, therefore the asymptotic compactness is proven. Then, applying Theorem 3.11, there exists a minimal pullback $\mathcal{D}_\lambda$-attractor $\mathcal{A}_\lambda$ that belongs to the universe $\mathcal{D}_\lambda$. □

4.2.2. *Fractal dimension of the pullback attractor.* A consequence of the following result is that the evolution process $\mathcal{U}(\cdot,\cdot)$, defined in (4.7), satisfies the smoothing property on the pullback $\mathcal{D}_\lambda$-attractor $\mathcal{A}_\lambda$ constructed in Theorem 4.15.

Proposition 4.16. *Given $t \in \mathbb{R}$ and $\widehat{D} = \{D(t) \subset H_t : t \in \mathbb{R}\} \in \mathcal{D}_\lambda$, then there exist $\kappa_t > 0$ (that is a positive non-decreasing function of t) and $\tau(t,\widehat{D}) \in \mathbb{R}$ such that*

$$|\nabla u(t;s,u_s) - \nabla v(t;s,v_s)|_t \leqslant \kappa_t |u(t-1;s,u_s) - v(t-1;s,v_s)|_{t-1}$$

for all $u_s, v_s \in D(s)$ and $s \leqslant \tau(t,\widehat{D})$, where κ_t is a non-decreasing function of t given by $\kappa_t := C_{4,t}\big[C_f^4(t) + C_f(t)\widehat{C}_f(t)\big]\big[1 + c_{1,t}C_f^2(t)\exp\{c_{1,t}C_f^2(t)\}\big]$.

Proof. Let us denote by $u(t) = \mathcal{U}(t,s)u_0$, $v(t) = \mathcal{U}(t,s)v_0$ and by $w := u - v$. Then, by taking Aw as a test function, we have

$$\frac{1}{2}\frac{d}{dt}|\nabla w|_t^2 + |Aw|_t^2 + b_t(u,u,Aw) - b_t(v,v,Aw) = 0.$$

Let us focus on term $b_t(u,u,Aw) - b_t(v,v,Aw)$. By using (2.10), we have

$$\begin{aligned}
b_t(u,u,Aw) - b_t(v,v,Aw) &= b_t(u,u,Aw) - b_t(u,v,Aw) + b_t(u,v,Aw) - b_t(v,v,Aw)\\
&= b_t(u,w,Aw) + b_t(w,v,Aw)\\
&\leqslant c_t|u|_t^{1/2}|\nabla u|_t^{1/2}|\nabla w|_t^{1/2}|Aw|_t^{3/2} + c_t|w|_t^{1/2}|\nabla w|_t^{1/2}|\nabla v|_t^{1/2}|Av|_t^{1/2}|Aw|_t\\
&\leqslant \frac{C_{4,t}}{2}\Big[|\nabla u|_t^4|\nabla w|_t^2 + |\nabla w|_t^2|\nabla v|_t|Av|_t\Big] + \frac{1}{2}|Aw|_t^2.
\end{aligned}$$

Consequently, we obtain

$$\frac{d}{dt}|\nabla w|_t^2 + |Aw|_t^2 \leqslant C_{4,t}\big(|\nabla u|_t^4 + |\nabla v|_t|Av|_t\big)|\nabla w|_t^2.$$

Integrating in time, we have

$$|\nabla w(t)|_t^2 + \int_r^t |Aw(\theta)|_t^2 d\theta \leqslant |\nabla w(r)|_r^2 + C_{4,t}\int_r^t \big(|\nabla u(\theta)|_\theta^4 + |\nabla v(\theta)|_\theta|Av(\theta)|_\theta\big)|\nabla w(\theta)|_\theta^2 d\theta.$$

Applying the Gronwall inequality, we obtain

$$|\nabla w(t)|_t^2 \leqslant |\nabla w(r)|_r^2 \exp\left\{C_{4,t}\int_r^t \big(|\nabla u(\theta)|_\theta^4 + |\nabla v(\theta)|_\theta|Av(\theta)|_\theta\big)d\theta\right\}$$

Now, integrating in r between $t-1$ to t, we obtain

$$|\nabla w(t)|_t^2 \leqslant \exp\left\{C_{4,t}\int_{t-1}^t \big(|\nabla u(\theta)|_\theta^4 + |\nabla v(\theta)|_\theta|Av(\theta)|_\theta\big)d\theta\right\}\int_{t-1}^t |\nabla w(r)|_r^2 dr. \tag{4.21}$$

Thus, by the inequality (4.9) and Lemma 4.13, there exist $\kappa_t > 0$ and $\tau(t,\widehat{D}) \in \mathbb{R}$ such that

$$|\nabla w(t)|_t^2 \leqslant \kappa_t |w(t-1)|_{t-1}^2, \quad \text{for all } s \leqslant \tau(t,\widehat{D}).$$

□

Lemma 4.17. *Given $t \in \mathbb{R}$, there exists $\kappa_t > 0$, which is a non-decreasing function of t, such that*

$$|\nabla\mathcal{U}(t,t-1)u - \nabla\mathcal{U}(t,t-1)v|_t \leqslant \kappa_t|u - v|_{t-1}$$

for all $u, v \in \mathcal{A}_\lambda(t-1)$ and $t \in \mathbb{R}$, where κ_t was given in Proposition 4.16.

Proof. This is proved in the same way as Proposition 4.16, and taking into consideration, by Theorem 4.15, that $\mathcal{A}_\lambda(t) \subset B_{V_t}[0, C_f(t)]$ for all $t \in \mathbb{R}$. □

The following results are introduced in order to help prove the condition (4.2) in Theorem 4.2 when we apply the abstract results to our Navier-Stokes problem.

Proposition 4.18. *(cf.* [27, Corollary 4.14] *) For each $t \in \mathbb{R}$, the set $\Omega_t := \bigcup_{s\leqslant t} \mathcal{O}_s$ is bounded in $\mathbb{R}^2$. Then, for any $\epsilon > 0$, we have*

$$\sup_{s\leqslant t} N_\epsilon\big(B_{H^1(\mathcal{O}_s)^2}(0,1), L^2(\mathcal{O}_s)^2\big) \leqslant N_\epsilon\big(B_{H^1(\Omega_t)^2}(0,1), L^2(\Omega_t)^2\big). \tag{4.22}$$

Remark 4.19. *It follows from Theorem 4.15 that there exists a non-decreasing function $\varrho : \mathbb{R} \to (0,+\infty)$ (given as $\varrho(t) := 2\lambda_{1,t}^{-1} C_f(t)$ for all $t \in \mathbb{R}$) such that for each $t \in \mathbb{R}$ there exists a $u_t \in \mathcal{A}_\lambda(t)$ satisfying*

$$\mathcal{A}_\lambda(t) \subset B_{H_t}\big(u_t, \varrho(t)\big) \ \textit{for all } t \in \mathbb{R}.$$

On the other hand, if we denote by $N_\epsilon^t := N_\epsilon(B_{V_t}(0,1), H_t)$ for any $\epsilon > 0$ and $t \in \mathbb{R}$, then, by the inequality (4.22), $\mathcal{N}_t := \sup_{s\leqslant t} N_\epsilon^s < \infty$ for any $t \in \mathbb{R}$.

Theorem 4.20. *Assume that all conditions from Theorem 4.15 hold. Then the pullback attractor $\mathcal{A}_\lambda$ associated with the time-dependent dynamical system $(\mathcal{U}(\cdot,\cdot), \{H_t\}_{t\in\mathbb{R}})$ has finite fractal dimension, that is,*

$$\sup_{s\leqslant t} d_f^{H_s}(\mathcal{A}_\lambda(s)) \leqslant \frac{\log \mathcal{N}_t}{\log 2} < +\infty, \ \textit{for all } t \in \mathbb{R}.$$

Proof. Denote by $\mathcal{U}_t := \mathcal{U}(t, t-1)$ for all $t \in \mathbb{R}$. It follows from Lemma 4.17 that the evolution process $\mathcal{U}(\cdot,\cdot)$ satisfies the smoothing property on pullback $\mathcal{D}_\lambda$-attractor $\mathcal{A}_\lambda$, given in the Theorem 4.15, i.e.,

$$|\nabla \mathcal{U}_t u - \nabla \mathcal{U}_t v|_t \leqslant \kappa_t |u - v|_{t-1}, \ \text{ for all } u, v \in \mathcal{A}_\lambda(t-1), \ t \in \mathbb{R}.$$

Therefore, it follows from Theorem 4.6 and Remark 4.19 that

$$\sup_{s\leqslant t} d_f^{H_s}(\mathcal{A}_\lambda(s)) \leqslant \frac{\log \mathcal{N}_t}{\log 2}, \ \text{ for all } t \in \mathbb{R}.$$

□

Corollary 4.21. *Suposse that all conditions from Theorem 4.15 hold. Then the pullback attractor $\mathcal{A}_\lambda$ has finite fractal dimension on the family of Banach spaces $\{V_t\}_{t\in\mathbb{R}}$, that is,*

$$\sup_{s\leqslant t} d_f^{V_s}(\mathcal{A}_\lambda(s)) \leqslant \sup_{s\leqslant t} d_f^{H_s}(\mathcal{A}_\lambda(s)) \leqslant \frac{\log \mathcal{N}_t}{\log 2}, \ \textit{for all } t \in \mathbb{R}.$$

Proof. It follows immediately from Theorem 4.20 and Lemma 4.3. □

Finally, we note that the pullback attractor of the transformed system 2D-(CC-NS) has also finite fractal dimension on the Hilbert space H. To see this, let us define the bi-parametric family of single-valued maps $\{\widehat{\mathcal{U}}(t,\tau) : H \to H : (t,\tau) \in \mathbb{R}_d^2\}$ as

$$\widehat{\mathcal{U}}(t,\tau) : H \to H \quad \text{by} \quad \widehat{\mathcal{U}}(t,\tau) v_\tau = v(t;\tau, v_\tau), \tag{4.23}$$

where $v(\cdot) = v(\cdot;\tau, v_\tau)$ is the unique weak solution of 2D-(CC-NS) associated to the initial condition $v_\tau \in H$ for any $(t,\tau) \in \mathbb{R}_d^2$, and in the same way as Definition 3.32 and Theorem 3.34 we define the pullback $\widehat{\mathcal{D}}_\lambda$-attractor $\widehat{\mathcal{A}}_\lambda$ of $(\widehat{\mathcal{U}}(\cdot,\cdot), H)$. Then, it follows from Lemma 2.3 and Theorem 4.8 that

$$\sup_{s\leqslant t} d_f^{H}\big(\widehat{\mathcal{A}}_\lambda(s)\big) = \sup_{s\leqslant t} d_f^{H_s}\big(\mathcal{A}_\lambda(s)\big) \leqslant \frac{\log \mathcal{N}_t}{\log 2}, \ \text{ for all } t \in \mathbb{R}.$$

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

DATA AVAILABILITY

We do not analyse or generate any datasets, because our work proceeds within a theoretical and mathematical approach.

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