(In)consistency Operators on Quasi-Nelson Algebras

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Abstract. We propose a preliminary study of (in)consistency operators on quasi-Nelson algebras, a variety that generalizes both Nelson and Heyting algebras; our aim is to pave the way for introducing logics of formal inconsistency (LFIs) in a non-necessarily involutive setting. We show how several results that were obtained for LFIs based on distributive involutive residuated lattices can be extended to quasi-Nelson algebras and their logic. We prove that the classes of algebras thus obtained are equationally axiomatizable, and provide a twist representation for them. Having obtained some insight on filters and congruences, we characterize the directly indecomposable members of these varieties, showing in particular that two of them are semisimple. Further logical developments and extensions of the present approach are also discussed.

Keywords: Quasi-Nelson · Logics of formal inconsistency · Twist-structures · Residuated lattices

1 Introduction

Logics of formal inconsistency (LFIs) are among the most well-known and time-honoured among the inconsistency-tolerant, or paraconsistent, logical systems. Formally, an LFI is usually presented as a standard (propositional) consequence relation ($\vdash$) over a language which includes a conjunction ($\wedge$), a disjunction ($\vee$), an implication ($\Rightarrow$), truth constants ($\bot, \top$) and a negation ($\sim$) that crucially fails to satisfy the *principle of explosion*: $\varphi \wedge {\sim}\varphi \vdash \bot$. To this language one usually adds a unary *consistency* connective $\circ$ that allows one to recover explosion in a more controlled way. The intended meaning of $\circ\varphi$ is "φ is consistent", and the following weaker principle is postulated:

$$\varphi \wedge {\sim}\varphi \wedge \circ\varphi \vdash \bot \tag{1}$$

(the *finite gentle principle of explosion* of [4, p. 50]), which can be informally read as follows: "if φ is consistent and contradictory, then φ explodes".

In addition (or alternatively) to $\circ$, a dual *inconsistency* connective $\bullet$ may be employed, and $\bullet\varphi$ is interpreted as "φ is inconsistent". One may require $\bullet$ to be

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the negation-dual of $\circ$ (perhaps even taking $\bullet\varphi := \sim\circ\varphi$ as a definition) or may impose independent postulates on both. For instance, dualizing (1), one obtains:

$$\vdash \varphi \vee \sim\varphi \vee \bullet\varphi \tag{2}$$

which is indeed a valid principle of some LFIs. Under certain assumptions on the negation, it is also easily verified that (1) and (2) are equivalent via the definition $\bullet\varphi := \sim\circ\varphi$ (or its dual $\circ\varphi := \sim\bullet\varphi$; more on this below).

The starting point of the present work is the paper [5], which investigates a family of LFIs that result from adding a consistency connective to certain extensions of the *Full Lambek Calculus with Exchange and Weakening* (FL_{ew}), the substructural logic determined by the class of *commutative integral bounded residuated lattices*. The approach of [5] is fairly general and modular, but it only applies to involutive logics (i.e. those that satisfy the double negation law, $\sim\sim\varphi \Rightarrow \varphi$): the present work is a first attempt at extending this research project beyond the involutive setting.

To any given class K of residuated lattices one can associate two logical consequences in a standard way, the *truth-preserving* logic $\vdash^{\top}_{\mathsf{K}}$ and the *order-preserving* $\vdash^{\leq}_{\mathsf{K}}$. Both share the same set of valid formulas, but in general they do not coincide: $\vdash^{\top}_{\mathsf{K}}$ is the stronger one and satisfies the principle of explosion, while $\vdash^{\leq}_{\mathsf{K}}$ typically does not. The latter thus provides a natural candidate for an LFI based on residuated lattices. Assuming an axiomatization for $\vdash^{\leq}_{\mathsf{K}}$ is given (see [1]), the paper [5] describes a method for axiomatizing the order-preserving logic $\vdash^{\leq}_{\mathsf{K}^{\circ}}$ obtained by endowing each algebra in K with a consistency operator $\circ$ satisfying (the algebraic counterpart of) the gentle principle of explosion (1).

The procedure sketched above may be applied to several well-known order-preserving companions of substructural logics, including Łukasiewicz logic, Nelson's constructive logic with strong negation [11] and nilpotent minimum logic [7]; in specific cases, further insight on the resulting logics is also gained thanks to the peculiar structure of prelinear algebras (for Łukasiewicz and nilpotent minimum) and the twist representation of Nelson algebras. The method of [5] does not apply, however, to many well-known logics based on residuated lattices—e.g. Hájek's basic logic, product logic and FL_{ew} itself—because the algebras in K are required to be distributive and involutive (see below for the relevant definitions). These limitations are due to technical reasons, and the main aim of the present paper is indeed to explore the possibility of relaxing them.

We are going to show how the results of [5] can be extended if we take K to be the variety of *quasi-Nelson algebras*, a recently-introduced generalization of Nelson (and Heyting) algebras. Such a setting seems to be particularly promising for potential future research: for, on the one hand, quasi-Nelson algebras are distributive but not necessarily involutive residuated lattices; on the other, they can be represented through a Nelson-type twist construction that affords powerful insight into their structure. We therefore propose the present study as a preliminary investigation on the possibilities of extending the approach described above to more general classes of algebras and logics. We shall focus on laying the algebraic foundations, and in particular on the question of how to define and represent (in)consistency operators on quasi-Nelson algebras; we stress, however, that all the results we will establish have a straightforward logical interpretation in the setting of logics extending FL_{ew} (see Sect. 4).

The paper is organized as follows. In Sect. 2 we recall preliminary definitions and results on quasi-Nelson logic and its algebraic counterpart, the variety QN of quasi-Nelson algebras. In Sect. 3 we study inconsistency operators on quasi-Nelson algebras. The choice of focusing on inconsistency (rather than consistency) operators is motivated by the technical observation that, in the absence of involutivity, it is easier to work with the quasi-equational defining properties of inconsistency operators (Definitions 2 and 3) than with those of consistency operators (see Sect. 4.1). The difference, as we shall see, cannot be appreciated in an involutive setting such as that of [5], where consistency and inconsistency operators are perfect duals of one another. Following [5], we consider three possible definitions for an inconsistency operator (Definitions 2 and 3), which give rise to three classes of expanded quasi-Nelson algebras. We prove that all three are equationally definable (Theorems 1 and 3), thus also settling an issue left open in [5]. We extend the twist representation to quasi-Nelson algebras endowed with inconsistency operators (Theorem 4) and use it to obtain information on the congruences and filters of these new classes of algebras; we also see how the insight thus gained can improve our understanding of subvarieties. In the concluding Sect. 4 we sketch a plan for future work, focusing in particular on three directions: how to introduce consistency operators on quasi-Nelson algebras (Sect. 4.1); the logical translation of the algebraic results established so far (Sect. 4.2); and the future study of (in)consistency operators in wider settings (Sect. 4.3). To improve readability and respect space limitations, all proofs are included in the Appendix.

2 Quasi-Nelson Logic and Algebras

The class of quasi-Nelson algebras (QN) was introduced in [12] and further investigated in a number of subsequent publications [8,10,13–15]. Formally, QN can be viewed either as a subvariety of residuated lattices or as a generalization of both Nelson algebras (the algebraic counterpart of Nelson's logic) and Heyting algebras. Taking the former approach, we may define a quasi-Nelson algebra as a *commutative, integral and bounded residuated lattice*[1] $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ that further satisfies the *Nelson equation*:

$$(x \Rightarrow (x \Rightarrow y)) \wedge ({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x)) \leq x \Rightarrow y \qquad \text{(Nelson)}$$

where ${\sim} x := x \Rightarrow 0$. While (Nelson) entails that quasi-Nelson algebras satisfy distributivity ($x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$) and 3-potency ($x^3 = x^2$), they are not involutive, i.e. they need not satisfy the double negation equation ${\sim}{\sim} x = x$. In fact, the involutive members of QN are precisely the Nelson algebras, and the idempotent ones (those satisfying $x^2 = x$) are precisely the Heyting algebras.

[1] See [9] for all the unexplained terminology of universal algebra, substructural logics and (residuated) lattice theory.

Quasi-Nelson logic, the logical counterpart of QN, may be obtained by adding the *Nelson axiom* to FL_{ew} (see [9]):

$$((\varphi \Rightarrow (\varphi \Rightarrow \psi)) \wedge ({\sim}\psi \Rightarrow ({\sim}\psi \Rightarrow {\sim}\varphi))) \Rightarrow (\varphi \Rightarrow \psi).$$

The interest in quasi-Nelson algebras/logic is manyfold, and can be motivated from a number of perspectives (e.g. constructive logics, order theory, and universal algebra; see the above-mentioned papers for further details). In the present context we are mainly interested in QN as a first step in the extension of the approach of [5] to LFIs beyond the involutive setting.

A prominent feature of quasi-Nelson algebras is the twist representation, which allows one to construct each algebra $\mathbf{A} \in \mathsf{QN}$ as a special binary product of a *nuclear Heyting algebra*, i.e. a Heyting algebra $\langle H; \wedge, \vee, \to, \neg, 0, 1\rangle$ endowed with a unary *nucleus* operator $\Box$ satisfying $\Box 0 = 0$ and $x \to \Box y = \Box x \to \Box y$ (see e.g. [16] for further background on nuclei). The details of the construction are given in Definition 1 below. Recall that the set $D(\mathbf{H})$ of *dense elements* on a (nuclear) Heyting algebra $\mathbf{H}$ is given by $D(\mathbf{H}) := \{a \in H : \neg a = 0\}$.

Definition 1. *Let* $\mathbf{H} = \langle H; \wedge, \vee, \to, \neg, \Box, 0, 1\rangle$ *be a nuclear Heyting algebra, and let* $\nabla \subseteq H$ *be a lattice filter of* $\mathbf{H}$ *such that* $D(\mathbf{H}) \subseteq \nabla$. *Define the* quasi-Nelson twist-algebra $Tw(\mathbf{H}, \nabla) = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *with universe:*

$$A := \{\langle a_1, a_2\rangle \in H \times H : a_2 = \Box a_2,\ a_1 \vee a_2 \in \nabla,\ a_1 \wedge a_2 = 0\}$$

and operations given, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in H \times H$, *by:*

$$\begin{aligned}
1 &:= \langle 1, 0\rangle \\
0 &:= \langle 0, 1\rangle \\
\langle a_1, a_2\rangle * \langle b_1, b_2\rangle &:= \langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2)\rangle \\
\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle &:= \langle a_1 \wedge b_1, \Box(a_2 \vee b_2)\rangle \\
\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle &:= \langle a_1 \vee b_1, a_2 \wedge b_2\rangle \\
\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle &:= \langle (a_1 \to b_1) \wedge (b_2 \to a_2), \Box a_1 \wedge b_2\rangle.
\end{aligned}$$

The negation (defined as ${\sim}x := x \Rightarrow 0$) is given by ${\sim}\langle a_1, a_2\rangle = \langle a_2, \Box a_1\rangle$. Thus, if the nucleus $\Box$ is not the identity map (as an example, take $\Box x := \neg\neg x$) then $Tw(\mathbf{H}, \nabla)$ is not involutive. Each twist-algebra $Tw(\mathbf{H}, \nabla)$ belongs to QN and, conversely, the twist representation result states that *every quasi-Nelson algebra can be constructed according to Definition* 1 (see e.g. [13,16] for further details).

Before we proceed, we need to introduce a notion that plays a prominent role in the study of LFIs based on residuated lattices. The *Boolean elements* $B(\mathbf{A})$ of a bounded integral residuated lattice $\mathbf{A}$ may be defined in the following three alternative ways: (i) $B(\mathbf{A}) := \{a \in A : \exists b \in A \text{ s.t. } a \vee b = 1, a \wedge b = 0\}$; (ii) $B(\mathbf{A}) := \{a \in A : a \vee {\sim}a = 1\}$; (iii) $B(\mathbf{A}) := \{a \in A : a \wedge {\sim}a = 0\}$. Our official definition will be (ii), which is equivalent to (i), but in practice easier to work with. On the other hand, in a non-involutive setting (iii) gives a weaker notion: this is essentially because, on every residuated lattice, $a \vee {\sim}a = 1$ entails

$a \wedge \sim a = 0$, but not the other way round. $B(\mathbf{A})$ is the universe of a sublattice of $\mathbf{A}$; note that, on a quasi-Nelson twist-algebra $Tw(\mathbf{H}, \nabla)$, the Boolean elements are precisely those of the form $\langle a, \neg a \rangle$ for some $a \in B(\mathbf{H})$.

3 Inconsistency Operators

Drawing inspiration from [5], we consider three possible definitions for an inconsistency operator. As mentioned earlier, the informal reading of $\bullet a$ is "the value a is inconsistent", and we may further allow this proposition to assume only crisp (i.e. Boolean) values or not. In the involutive case, our approach is equivalent to that of [5], for the consistency operator can then be recovered as $\circ x := \sim \bullet x$.

3.1 The min and Bmin Operators

Definition 2. *A* min-inconsistency operator *on a quasi-Nelson algebra* $\mathbf{A}$ *is a unary operator* $\bullet$ *that satisfies the following quasi-equations:*

$$x \vee \sim x \vee y = 1 \qquad \textit{if and only if} \qquad \bullet\, x \leq y.$$

A Bmin-inconsistency operator *is a* min*-inconsistency operator that further satisfies the equation* $\bullet x \vee \sim \bullet x = 1$ *(ensuring that* $\bullet a$ *is a Boolean element).*

We denote by $\mathsf{QN}^{\bullet}$ (respectively, $\mathsf{QN}^{\bullet}_{\mathsf{Bm}}$) the class of all algebras $\langle \mathbf{A}, \bullet \rangle$ such that $\mathbf{A} \in \mathsf{QN}$ and $\bullet$ is a min- (resp., a Bmin-)inconsistency operator on $\mathbf{A}$. From an order-theoretic point of view, Definition 2 is precisely saying that, for each $a \in A$, the element $\bullet a$ is the dual pseudo-complement (see Definition 4) of $a \vee \sim a$. Such an element is unique if it exists, for one has:

$$\bullet\, a = \min\{b \in A : a \vee \sim a \vee b = 1\}. \tag{3}$$

Existence can be guaranteed on a finite $\mathbf{A} \in \mathsf{QN}$ (or, more generally, any algebra having a complete and completely distributive lattice reduct), for one can let:

$$\bullet\, a = \bigwedge\{b \in A : a \vee \sim a \vee b = 1\}. \tag{4}$$

However, the element $\bullet a$ given by (4) may fail to satisfy $\bullet a \vee \sim \bullet a = 1$, so a Bmin-inconsistency operator may not be definable, even when $\mathbf{A}$ is finite (see Example 2).

Example 1. Let $\mathbf{A} \in \mathsf{QN}$ be a subdirectly irreducible quasi-Nelson algebra. Defining $\bullet 0 = \bullet 1 = 0$ and $\bullet a = 1$ for $a \notin \{0, 1\}$, one obtains a min-inconsistency (in fact, a Bmin-inconsistency) operator. To see this, recall from [9, p. 202] that $\mathbf{A}$ has a unique co-atom, say $c \in A$. Then, for every element $0 < a \leq c$, we have $\sim a \leq c$ as well. For, otherwise, from $\sim a = 1$ we would have $a \leq \sim\sim a = \sim 1 = 0$, against the assumption that $0 < a$. Thus $a \vee \sim a \leq c$, which gives us $\bullet a = 1$ by (3). By the same token, for every $\langle \mathbf{A}, \bullet \rangle \in \mathsf{QN}^{\bullet}$ s.t. $\mathbf{A}$ is a subdirectly irreducible quasi-Nelson algebra, the $\bullet$ operator must be defined as indicated above. This example will also be used to show that both classes $\mathsf{QN}^{\bullet}$ and $\mathsf{QN}^{\bullet}_{\mathsf{Bm}}$ extend QN conservatively (see Theorem 2).

Proposition 1. *Let* $\langle \mathbf{A}, \bullet \rangle \in \mathsf{QN}^{\bullet}$ *and* $a, b \in A$*.*

(i) $a \in B(\mathbf{A})$ *if and only if* $\bullet a = 0$ *(in particular,* $\bullet 1 = \bullet 0 = 0$*).*
(ii) $\bullet a = \bullet(a \vee \sim a)$*.*
(iii) $\bullet \sim a = \bullet \sim \sim a$*.*
(iv) $a \vee \sim a \vee \bullet a = 1$*.*
(v) $\bullet a = \bullet a * \bullet a$*.*
(vi) $\bullet a \leq b \vee \bullet(a \vee \sim a \vee b)$*.*

Item (i) above suggests that the behaviour of $\bullet$ on each algebra $\mathbf{A} \in \mathsf{QN}$ may be read as a measure of "how Boolean" $\mathbf{A}$ is: indeed, $\mathbf{A}$ is a Boolean algebra if and only if $\bullet a = 0$ for all $a \in A$. Note that, in contrast to the involutive case of [5], it may in general happen that $\bullet \sim a \neq \bullet a \neq \bullet(a \wedge \sim a)$. This will be clarified by the twist representation for the algebras in $\mathsf{QN}^{\bullet}$ (see Proposition 5).

Theorem 1. *The conditions in Definition 2 are equivalent to the following equations:*

(i) $\bullet x \leq y \vee \bullet(x \vee \sim x \vee y)$*.*
(ii) $x \vee \sim x \vee \bullet x = 1$*.*
(iii) $\bullet 1 = 0$*.*

Hence, $\mathsf{QN}^{\bullet}$ and $\mathsf{QN}^{\bullet}_{\mathsf{Bm}}$ are varieties. Note that Theorem 1 applies to residuated lattices in general, thereby settling an issue that was left open in [5]. The following result establishes that both $\mathsf{QN}^{\bullet}$ and $\mathsf{QN}^{\bullet}_{\mathsf{Bm}}$ extend QN conservatively.

Theorem 2. *The variety* QN *of quasi-Nelson algebras is precisely the class of* $\{\bullet\}$*-free subreducts of* $\mathsf{QN}^{\bullet}_{\mathsf{Bm}}$ *(and,* a fortiori*, also of* $\mathsf{QN}^{\bullet}$*).*

3.2 The minB Operator

Definition 3. *A unary operator* $\bullet_{\mathsf{B}}$ *on a quasi-Nelson algebra* $\mathbf{A}$ *is a* minB-inconsistency operator *if the following (quasi-)equations are satisfied:*

(i) $x \vee \sim x \vee \bullet_{\mathsf{B}} x = 1$*.*
(ii) $\bullet_{\mathsf{B}} x \vee \sim \bullet_{\mathsf{B}} x = 1$*.*
(iii) $x \vee \sim x \vee y = 1$ *and* $y \vee \sim y = 1$ *imply* $\bullet_{\mathsf{B}} x \leq y$*.*

Denote by $\mathsf{QN}^{\bullet}_{\mathsf{mB}}$ the class of algebras $\langle \mathbf{A}, \bullet_{\mathsf{B}} \rangle$ such that $\mathbf{A} \in \mathsf{QN}$ and $\bullet_{\mathsf{B}}$ is a minB-inconsistency operator on $\mathbf{A}$. Every Bmin-inconsistency operator (Definition 2) is a minB-inconsistency operator, so $\mathsf{QN}^{\bullet}_{\mathsf{Bm}} = \mathsf{QN}^{\bullet} \cap \mathsf{QN}^{\bullet}_{\mathsf{mB}}$, implying that Theorem 2 applies to $\mathsf{QN}^{\bullet}_{\mathsf{mB}}$ as well. But neither $\mathsf{QN}^{\bullet}$ nor $\mathsf{QN}^{\bullet}_{\mathsf{mB}}$ is contained in the other (see Example 2). A minB-inconsistency operator is unique, if it exists, for we have:

$$\bullet_{\mathsf{B}}\, a = \min\{b \in B(\mathbf{A}) : a \vee \sim a \vee b = 1\}. \tag{5}$$

Similarly to the case of min-operators, if the sublattice of Boolean elements $B(\mathbf{A})$ is complete, then the operator $\bullet_{\mathsf{B}}$ is definable on $\mathbf{A}$ by:

$$\bullet_{\mathsf{B}}\, a := \bigwedge \{b \in B(\mathbf{A}) : a \vee \sim a \vee b = 1\}.$$

The following example should help further clarify the relationships among the classes of algebras under consideration.

Example 2. [5, Fig. 2, p. 1236] depicts an eight-element *nilpotent minimum algebra* (let us call it $\mathbf{A}_8$) on which two different consistency operators $\circ$ and $\circ_{\mathsf{B}}$ are definable (note that there is a mistake in the table of the monoid operation: it should have $f * f = f$ instead of $f * f = a$). Nilpotent minimum algebras are a subvariety of Nelson algebras (see e.g. [8]), so $\mathbf{A}_8 \in \mathsf{QN}$. Dualizing the definitions (i.e. letting $\bullet x := \sim \circ x$ and $\bullet_{\mathsf{B}} x := \sim \circ_{\mathsf{B}} x$), we can endow $\mathbf{A}_8$ with a (necessarily unique) min-inconsistency operator $\bullet$ and an (also unique) minB-inconsistency operator $\bullet_{\mathsf{B}}$ which do not coincide. Thus $\langle \mathbf{A}_8, \bullet \rangle \in \mathsf{QN}^{\bullet} - \mathsf{QN}^{\bullet}_{\mathsf{mB}}$ and $\langle \mathbf{A}_8, \bullet_{\mathsf{B}} \rangle \in \mathsf{QN}^{\bullet}_{\mathsf{mB}} - \mathsf{QN}^{\bullet}$. Since $\langle \mathbf{A}_8, \bullet \rangle \notin \mathsf{QN}^{\bullet}_{\mathsf{Bm}}$, we also see that it is not possible to define a Bmin-inconsistency operator on $\mathbf{A}_8$.

The counterpart of Proposition 1 for minB-inconsistency operators is the following:

Proposition 2. *Let* $\langle \mathbf{A}, \bullet \rangle \in \mathsf{QN}^{\bullet}_{\mathsf{mB}}$ *and* $a, b \in A$.

(i) $a \in B(\mathbf{A})$ *if and only if* $\bullet_{\mathsf{B}} a = 0$ *(thus* $\bullet_{\mathsf{B}} 1 = \bullet_{\mathsf{B}} 0 = 0$ *and* $\bullet_{\mathsf{B}} \bullet_{\mathsf{B}} a = 0$*).*
(ii) $\bullet_{\mathsf{B}} a = \bullet_{\mathsf{B}} (a \vee \sim a) = \sim\sim \bullet_{\mathsf{B}} a$.
(iii) $\bullet_{\mathsf{B}} \sim a = \bullet_{\mathsf{B}} \sim\sim a$.
(iv) $\bullet_{\mathsf{B}} a \vee b \vee \bullet_{\mathsf{B}} b \in B(\mathbf{A})$.
(v) $\bullet_{\mathsf{B}} a \leq b \vee \bullet_{\mathsf{B}} (b \vee \sim b) \vee \bullet_{\mathsf{B}} (a \vee \sim a \vee b)$.

Also in this case the class of algebras introduced in Definition 3 is equational, thus settling the corresponding open issue from [5].

Theorem 3. *The conditions in Definition 3 are equivalent to the following equations:*

(i) $x \vee \sim x \vee \bullet_{\mathsf{B}} x = 1$.
(ii) $\bullet_{\mathsf{B}} x \vee \sim \bullet_{\mathsf{B}} x = 1$.
(iii) $\bullet_{\mathsf{B}} 1 = 0$.
(iv) $\bullet_{\mathsf{B}} x \leq y \vee \bullet_{\mathsf{B}} (y \vee \sim y) \vee \bullet_{\mathsf{B}} (x \vee \sim x \vee y)$.

Hence, $\mathsf{QN}^{\bullet}_{\mathsf{mB}}$ *is a variety.*

3.3 Twist Representation

We now proceed to extend the twist representation of quasi-Nelson algebras given in Sect. 2 to algebras in $\mathsf{QN}^{\bullet}$. This will provide us with further insight into their structure and a useful tool for establishing arithmetical properties.

Definition 4. *Given a (nuclear) Heyting algebra* $\mathbf{H}$*, we shall denote by* $\urcorner$ *the unary operation that realizes the* dual pseudo-complement (dpc) *given (whenever it exists) by* $\urcorner a = \min\{b \in H : a \vee b = 1\}$ *for all* $a \in H$.

Proposition 3. *Let* $\mathbf{A} = Tw(\mathbf{H}, \nabla)$ *be a twist-algebra over a nuclear Heyting algebra* $\mathbf{H}$*. Assume the element* $\urcorner b$ *exists for all* $b \in \nabla$*. Then, defining* $\bullet \langle a_1, a_2 \rangle := \langle \urcorner (a_1 \vee a_2), \neg \urcorner (a_1 \vee a_2) \rangle$ *for all* $\langle a_1, a_2 \rangle \in A$*, we have* $\langle \mathbf{A}, \bullet \rangle \in \mathsf{QN}^{\bullet}$.

Proposition 4. *Let* $\mathbf{A} = Tw(\mathbf{H}, \nabla) \in \mathsf{QN}$ *be endowed with an operation* $\bullet$ *such that* $\langle \mathbf{A}, \bullet \rangle \in \mathsf{QN}^{\bullet}$*. Then the element* $\ulcorner b$ *exists for all* $b \in \nabla$*, and* $\bullet \langle a_1, a_2 \rangle := \langle \ulcorner (a_1 \vee a_2), \neg \ulcorner (a_1 \vee a_2) \rangle$ *for all* $\langle a_1, a_2 \rangle \in A$*.*

Propositions 3 and 4 give us the announced representation result.

Theorem 4. *Every quasi-Nelson algebra endowed with a* min*-inconsistency operator* $\bullet$ *can be constructed as* $Tw(\mathbf{H}, \nabla)$ *in accordance with Proposition 3.*

Theorem 4 can be used to establish an equivalence between an algebraic category based on $\mathsf{QN}^{\bullet}$ and a suitably defined category having as objects tuples of type $\langle \mathbf{H}, \nabla \rangle$, as done in [6] for Nelson algebras. In turn, on such an equivalence one might build a "two-sorted" topological duality for $\mathsf{QN}^{\bullet}$ (see [17]).

The twist representation also provides us with an easy way to establish arithmetical properties of $\mathsf{QN}^{\bullet}$ and its subvarieties. For instance, it is easy to verify that the following (in)equalities are satisfied on twist-algebras – the two inequalities being, in general, strict (cf. Proposition 1): $\bullet \sim (x \wedge \sim x) = \bullet(x \wedge \sim x) \leq \bullet \sim x \leq \bullet x = \bullet x * \bullet x$. As another application, in the next proposition we consider, from the perspective of twist-algebras, a few (in)equalities (corresponding to subvarieties of $\mathsf{QN}^{\bullet}$) that play a prominent role in the study of logics of formal inconsistency.

Proposition 5. *Let* $\langle \mathbf{A} = Tw(\mathbf{H}, \nabla), \bullet \rangle \in \mathsf{QN}^{\bullet}$*, let* $a \in \nabla$ *and* $\langle b_1, b_2 \rangle \in A$*.*

(i) $\bullet x \leq \bullet(x \wedge \sim x)$ *holds iff* $\ulcorner a = \ulcorner \Box a$*.*
(ii) $\bullet \sim x \leq \bullet(x \wedge \sim x)$ *holds iff* $\Box b_1 \vee b_2 \vee \ulcorner \Box (b_1 \vee b_2) = 1$*.*
(iii) $\bullet x \leq \bullet \sim x$ *holds iff* $b_1 \vee b_2 \vee \ulcorner (\Box b_1 \vee b_2) = 1$*.*
(iv) $x \wedge \bullet x = 0$ *holds iff* $\nabla = \{1\}$ *(and* $\mathbf{H}$ *is a Boolean algebra).*
(v) $\sim x \wedge \bullet x = 0$ *holds iff* $\Box a = \Box b_1 \vee \neg b_1 = 1$*.*
(vi) $\bullet x \vee \sim \bullet x = 1$ *holds iff* $\ulcorner a \vee \neg \ulcorner a = 1$*.*

3.4 Filters and Congruences

In this section we take a look at filters and congruences on quasi-Nelson algebras endowed with inconsistency operators; here, too, we shall profit from the insight gained with the twist representation.

An *implicative filter* of a residuated lattice $\mathbf{A}$ is a lattice filter $F \subseteq A$ that is further closed under the monoid operation, i.e., $a * b \in F$ whenever $a, b \in F$. Implicative filters are in one-to-one correspondence with congruences on commutative residuated lattices [9, Thm 3.47] via the maps defined as follows. To an implicative filter $F \subseteq A$ one associates the congruence $\theta_F := \{\langle a, b \rangle \in A \times A : a \Rightarrow b, b \Rightarrow a \in F\}$, and to a congruence $\theta \in \mathrm{Con}(\mathbf{A})$ one associates the implicative filter $1/\theta$.

The above correspondence applies to quasi-Nelson algebras as well. Given $\langle \mathbf{A}, \bullet \rangle \in \mathsf{QN}^{\bullet}$ and a congruence $\theta \in \mathrm{Con}(\mathbf{A}, \bullet)$, —that is, θ is compatible with the operations of $\mathbf{A}$ and also with the $\bullet$ operator—the associated implicative filter $1/\theta$ will satisfy the following: if $a \Rightarrow b, b \Rightarrow a \in 1/\theta$, then $\bullet a \Rightarrow \bullet b \in 1/\theta$.

We say that an implicative filter F closed under this rule is a $\bullet$-*filter* (cf. the definition of $\circ$-filter in [5, Fig. 2, p. 1236], and note that we do not need to impose condition (F2), because we know to be dealing with varieties). It is easy to verify that each $\bullet$-filter F is also closed under the following rule: if $a \in F$, then $\sim \bullet a \in F$ (in fact, Corollary 1 entails that this rule is not only necessary but also sufficient for defining $\bullet$-filters). Indeed, from $a \in F$ we have $1 \Rightarrow a = a \in F$ and $a \Rightarrow 1 = 1 \in F$, so we immediately obtain $\bullet a \Rightarrow \bullet 1 = \bullet a \Rightarrow 0 = \sim \bullet a \in F$.

The isomorphism between implicative filters and congruences of each $\mathbf{A} \in \mathsf{QN}$ is preserved when we consider an algebra $\langle \mathbf{A}, \bullet \rangle \in \mathsf{QN}^{\bullet}$ and its $\bullet$-filters: indeed, it is easy to verify that θ_F is compatible with $\bullet$ if (and only if) F is a $\bullet$-filter. This entails, in particular, that any algebra $\langle \mathbf{A}, \bullet \rangle$ constructed as in Example 1 is simple (i.e. has exactly two congruences). Indeed, if $\langle \mathbf{A}, \bullet \rangle$ had a non-trivial congruence $Id_A \neq \theta \neq A \times A$, then $1/\theta$ would be a non-trivial $\bullet$-filter. But there are no $\bullet$-filters on $\mathbf{A}$ except $\{1\}$ and A itself, for $a \in 1/\theta$ implies $\sim \bullet a = \sim 1 = 0 \in 1/\theta$ whenever $0 \neq a \neq 1$. A similar reasoning shows that the algebra defined in Example 2 (endowed either with $\bullet$ or with $\bullet_{\mathsf{B}}$) is also simple. Indeed, any algebra $\langle \mathbf{A}, \bullet \rangle$ will be simple as long as $\mathbf{A}$ is subdirectly irreducible, and the algebra in Example 2 witnesses that the converse is not true: $\langle \mathbf{A}_8, \bullet \rangle$ is simple even though $\mathbf{A}_8$ is not a subdirectly irreducible residuated lattice.

$\bullet$-filters can be characterized via the twist construction, building on a description of implicative filters on quasi-Nelson twist-algebras [16, Prop. 4.1].

Proposition 6. *A subset $G \subseteq A$ of $\mathbf{A} = Tw(\mathbf{H}, \nabla) \in \mathsf{QN}$ is an implicative filter if and only if $G = (F \times H) \cap A$, where F is a lattice filter of $\mathbf{H}$.*

Consider an algebra $\langle \mathbf{A} = Tw(\mathbf{H}, \nabla), \bullet \rangle \in \mathsf{QN}^{\bullet}$. By the preceding result, every $\bullet$-filter $G \subseteq A$ has the shape $G = (F \times H) \cap A$, with F a lattice filter of $\mathbf{H}$. In such a case, moreover, one will have $\neg \ulcorner a \in F$ whenever $a \in \nabla \cap F$. This property appears to be a relativized version of the *normality* considered in [18]: the latter, indeed, corresponds precisely to the special case where $F \subseteq \nabla$. This consideration suggests the following definition.

Definition 5. *Let $\mathbf{H}$ be a (nuclear) Heyting algebra and le $\nabla \subseteq H$ be a filter such that $\ulcorner a$ exists for all $a \in \nabla$. Given a lattice filter $F \subseteq H$, we say that F is* ∇-normal *if $\neg \ulcorner a \in F$ whenever $a \in \nabla \cap F$.*

The family of all ∇-normal filters (for a fixed ∇) is closed under arbitrary intersections, and so forms a complete lattice. Two alternative characterizations of ∇-normality are given in the following lemma.

Lemma 1. *Let $\mathbf{H}$ be a Heyting algebra and let $\nabla, F \subseteq H$ be lattice filters. Assuming $\ulcorner a$ exists for all $a \in \nabla$, the following are equivalent:*

(i) F is ∇-normal.
(ii) For all $a, b \in \nabla$, if $a \to b \in F$, then $\ulcorner b \to \ulcorner a \in F$.
(iii) For all $a, b \in \nabla$, if $a \to b, b \to a \in F$, then $\ulcorner a \to \ulcorner b \in F$.

Having singled out the notion of ∇-normality allows us to smoothly extend Proposition 6 as follows.

Proposition 7. *A subset $G \subseteq A$ of an algebra $\langle \mathbf{A} = Tw(\mathbf{H}, \nabla), \bullet\rangle \in \mathsf{QN}^{\bullet}$ is a $\bullet$-filter if and only if $G = (F \times H) \cap A$, where F is a ∇-normal filter of $\mathbf{H}$.*

The preceding result implicitly contains a characterization of the congruences on each algebra $\langle \mathbf{A} = Tw(\mathbf{H}, \nabla\rangle, \bullet) \in \mathsf{QN}^{\bullet}$ in terms of $\mathrm{Con}(\mathbf{H})$, and suggests that further insight on $\mathsf{QN}^{\bullet}$ might be obtained by importing results on normal filters (e.g. Thm. 4.3, Cor. 4.5) from [18]. Before explaining this, let us note that Proposition 7 entails (Lemmas 4.7, 4.8 and) Theorem 4.9 from [5]. Indeed, phrased in our notation, the latter states that, given $\langle \mathbf{A} = Tw(\mathbf{H}, \nabla), \bullet\rangle \in \mathsf{QN}^{\bullet}$, we have that:

(i) $G := (\nabla \times H) \cap A$ is a proper $\bullet$-filter $\mathbf{A}$ iff ∇ is proper and normal;
(ii) if ∇ is the minimal filter of $\mathbf{H}$, then G is the minimal filter of $\mathbf{A}$.

The only mismatch is that the normality mentioned in (i) is the standard notion from [18]; but normality and ∇-normality here coincide, because $F = \nabla$. The following characterization of $\bullet$-filters also follows from Proposition 7.

Corollary 1. *Let $\langle \mathbf{A} = Tw(\mathbf{H}, \nabla), \bullet\rangle \in \mathsf{QN}^{\bullet}$ and let $G \subseteq A$ be an implicative filter of $\mathbf{A}$. The following are equivalent:*

(i) G is a $\bullet$-filter.
(ii) $\sim \bullet a \in G$ whenever $a \in G$.

Unlike some of the previous propositions, the proof of Corollary 1 appears to rely in an essential way on the twist representation; we therefore do not know whether the result applies to more general classes of residuated lattices, or even to the involutive ones considered in [5]. While the twist representation does not seem essential to the following result, it enables us to provide a straightforward proof (cf. [5, Thm. 3.8]).

Theorem 5. *An algebra $\langle \mathbf{A}, \bullet\rangle \in \mathsf{QN}^{\bullet}$ is directly indecomposable if and only if $B(\mathbf{A}) = \{0, 1\}$. In consequence, we also have that $B(\mathbf{A}) = \{0, 1\}$ whenever $\langle \mathbf{A}, \bullet\rangle$ is subdirectly irreducible.*

In the case of a quasi-Nelson algebra $\mathbf{A} = Tw(\mathbf{H}, \nabla)$, we know that the lattice of congruences $\mathrm{Con}(\mathbf{A})$ is isomorphic to the lattice $\mathrm{Con}\langle H; \wedge, \vee, \to, \neg, 0, 1\rangle$ of congruences of $\mathbf{H}$ viewed simply as a Heyting algebra (the nucleus $\Box$ does not alter the congruences of the underlying Heyting algebra reduct). This result, proved in [13, Prop. 8], may also be obtained as a corollary of Proposition 6, and we may employ Proposition 7 to obtain a similar result about $\mathrm{Con}\langle \mathbf{A}, \bullet\rangle$.

As observed earlier, for each algebra $\langle \mathbf{A} = Tw(\mathbf{H}, \nabla), \bullet\rangle \in \mathsf{QN}^{\bullet}$, we have that $\mathrm{Con}\langle \mathbf{A} = Tw(\mathbf{H}, \nabla), \bullet\rangle$ is isomorphic to the lattice of $\bullet$-filters on $\langle \mathbf{A}, \bullet\rangle$, which is isomorphic (by Proposition 7) to the lattice of ∇-normal filters of $\mathbf{H}$. These, in turn, are easily seen to be in one-to-one correspondence with the Heyting algebra congruences of $\mathbf{H}$ that satisfy the following property: for all $a, b \in \nabla$ such that $\langle a, b\rangle \in \theta$, we have $\langle \ulcorner a, \ulcorner b\rangle \in \theta$. This result may not appear very informative, but one needs to keep in mind that the role of $\mathbf{H}$ in the representation given in Theorem 4 is not exactly that of a standard algebra, for the dual pseudo-complement operation is only required to be defined on the elements of ∇. Obviously, when $\nabla = H$, we recover the well-known correspondence between the congruences of a Heyting algebra endowed with a dual pseudo-complement and its normal filters [18, Thm. 3.3].

We state below a stronger version of Theorem 5 for quasi-Nelson algebras endowed with a minB-inconsistency operator (cf. [5, Thm. 3.16]).

Theorem 6. *For each* $\langle \mathbf{A}, \bullet_{\mathsf{B}} \rangle \in \mathsf{QN}^{\bullet}_{\mathsf{mB}}$*, the following are equivalent:*

(i) $\langle \mathbf{A}, \bullet_{\mathsf{B}} \rangle$ *is simple.*
(ii) $\langle \mathbf{A}, \bullet_{\mathsf{B}} \rangle$ *is subdirectly irreducible.*
(iii) $\langle \mathbf{A}, \bullet_{\mathsf{B}} \rangle$ *directly indecomposable.*
(iv) $B(\mathbf{A}) = \{0, 1\}$.

Hence, $\mathsf{QN}^{\bullet}_{\mathsf{mB}}$ *(and* $\mathsf{QN}^{\bullet}_{\mathsf{Bm}}$*) are semisimple varieties.*

4 Future Work

4.1 Consistency Operators

On every quasi-Nelson algebra $\mathbf{A}$, a max-consistency operator $\circ$ may be introduced, as in [5, Def. 3.3], by the prescription:

$$x \wedge {\sim} x \wedge y = 0 \qquad \text{if and only if} \qquad y \leq \circ x$$

giving us $\circ a = \max\{b \in A : a \wedge {\sim} a \wedge b = 0\}$ for all $a \in A$. Many of the results established in the previous sections, including the twist representation (see below), can be obtained for the consistency operator $\circ$ as well. There are, however, certain additional technical difficulties. For instance, it is not clear at this point whether an analogue of Theorem 3 (entailing that the resulting class of algebras is equational) can be established: this is essentially due to the fact that the term $y \vee {\sim} y$ appearing in item (iv) of Theorem 3 is the same that defines the Boolean elements, but its negation-dual $y \wedge {\sim} y$, as we have observed, does have the same effect in a non-involutive setting. Strategies for overcoming such difficulties will have to be explored in future research.

On the other hand, a twist representation can be established, and the analogue of Theorem 4 would state that the consistency operator is given, on every twist-algebra, by $\circ\langle a_1, a_2 \rangle := \langle \neg - (a_1 \vee a_2), -(a_1 \vee a_2) \rangle$, where the unary operation $-$ is defined as follows:

$$- a = \min\{b \in \Box[H] : \Box(a \vee b) = 1\}. \tag{6}$$

(6) is precisely saying that $-a$ is the dual pseudo-complement of $a \in \Box[H]$ computed in the Heyting algebra $\Box[H] := \{a \in H : \Box a = a\}$ of fixpoints of the nucleus operator. Using this representation it is not hard to verify, for instance, that when both a min-inconsistency operator $\bullet$ and a max-consistency operator $\circ$ are defined on a quasi-Nelson algebra $\mathbf{A} = Tw(\mathbf{H}, \nabla)$, the inequality

$\sim \bullet x \leq \circ x$ is always satisfied. On the other hand, the converse inequality ($\circ x \leq \sim \bullet x$) holds if and only if $-a = \Box \neg a$ for all $a \in \nabla$, in which case the term $\sim \bullet x$ may be taken as a definition of $\circ x$. A perfect duality between the two operators, however, is only reached if we impose the stronger requirement $\neg a = -a$ for all $a \in \nabla$ (in which case we may also define $\bullet x := \sim \circ x$, as in the involutive setting). These considerations suggest that the exploration of non-involutive algebras simultaneously endowed with consistency and inconsistency operators may also prove to be an interesting direction for future research.

4.2 LFIs Based on Quasi-Nelson Algebras

As mentioned earlier, the logical interpretation of the results presented in the previous sections is straightforward. Following [5], we can proceed by first defining truth-preserving logics $\vdash^{\top}_{\mathsf{K}}$ for $\mathsf{K} \in \{\mathsf{QN}^{\bullet}, \mathsf{QN}^{\bullet}_{\mathsf{Bm}}, \mathsf{QN}^{\bullet}_{\mathsf{mB}}\}$. For $\mathsf{K} = \mathsf{QN}^{\bullet}$, the counterparts of the rules (A1), (Max) and (CNG) of [5, Definition 5.1] would be the following:

$$\text{(A1')} \quad \frac{}{\varphi \vee \sim \varphi \vee \bullet \varphi} \qquad \text{(Min)} \quad \frac{\varphi \vee \sim \varphi \vee \psi}{\bullet \varphi \Rightarrow \psi} \qquad \text{(CNG)} \quad \frac{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi}{\bullet \varphi \Rightarrow \bullet \psi}$$

(Note that these formulations of (A1') and (Min) appear closer to their algebraic counterparts than (A1) and (Max) in [5].) In a similar way one may define rules for the truth-preserving logic $\vdash^{\top}_{\mathsf{K}}$ with $\mathsf{K} \in \{\mathsf{QN}^{\bullet}_{\mathsf{Bm}}, \mathsf{QN}^{\bullet}_{\mathsf{mB}}\}$. By restricting the application of these rules as indicated in [5, Def. 5.5], we can then obtain order-preserving logics $\vdash^{\leq}_{\mathsf{K}}$ for $\mathsf{K} \in \{\mathsf{QN}^{\bullet}, \mathsf{QN}^{\bullet}_{\mathsf{Bm}}, \mathsf{QN}^{\bullet}_{\mathsf{mB}}\}$, thus recovering the subsequent logical results of [5, Section 5] in the setting of quasi-Nelson algebras endowed with inconsistency operators. We intend to pursue this in future publications, alongside the study of logics based on algebras endowed with consistency operators (defined as in the preceding subsection) and logics that include both consistency and inconsistency connectives.

4.3 (In)consistency Operators Beyond the Quasi-Nelson Setting

As mentioned in the Introduction, the present paper aims at establishing an algebraic background for extending the approach of [5] to LFIs beyond the setting of distributive involutive residuated lattices. Quasi-Nelson algebras, while non-necessarily involutive, are still a quite special subclass of residuated lattices (members of QN are, in particular, distributive and 3-potent), but some of the results presented here can be proved in a more general setting. As a next step in this direction, we speculate that the classes of residuated lattices introduced in the recent papers [2,3] might be a promising starting point. These are much more general residuated lattices that, while not necessarily satisfying any of the above-mentioned requirements (involutivity, distributivity, n-potency, integrality or commutativity), are still representable as twist-algebras. One may thus hope to obtain suitable generalizations of the results presented in the present paper, including those that appear to rely more heavily on the twist construction (e.g. Proposition 7 and Corollary 1). We leave this as a last suggestion for future investigations.

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Appendix: Proofs

Proof (Proposition 1). Items (i) to (iv) are straightforward consequences of Definition 2. We shall prove (v) later on, as it follows directly from the twist representation. Let us prove (vi). Using Definition 2, it suffices to show that $a \vee \sim a \vee b \vee \bullet(a \vee \sim a \vee b) = 1$ for all $a, b \in A$. Since $a \leq a \vee \sim a \vee b$, we have $a \vee \sim a \vee b \geq \sim a \geq \sim(a \vee \sim a \vee b)$. Hence, we can use item (iv) to obtain the result: $a \vee \sim a \vee b \vee \bullet(a \vee \sim a \vee b) = (a \vee \sim a \vee b) \vee \sim(a \vee \sim a \vee b) \vee \bullet(a \vee \sim a \vee b) = 1$.

Proof (Theorem 1). We have seen in Proposition 1 that the three conditions in the statement are consequences of Definition 2. Conversely, assume conditions (i)–(iii) hold on a quasi-Nelson algebra $\mathbf{A}$, and let a, b be such that $a \vee \sim a \vee b = 1$. By (i) and (iii), we have $\bullet a \leq b \vee \bullet(a \vee \sim a \vee b) = b \vee \bullet 1 = b \vee 0 = b$, as required. Conversely, assume $\bullet a \leq b$. Then, by (ii), we have $1 = a \vee \sim a \vee \bullet a \leq a \vee \sim a \vee b$, as required.

Proof (Theorem 2). Recall that all the classes under consideration are varieties. Thus, assuming an equation $\varphi = \psi$ in the $\{\bullet\}$-free language does not hold in QN, let $\mathbf{A} \in \mathsf{QN}$ be a subdirectly irreducible algebra witnessing this. We can then apply Example 1 to obtain an algebra $\langle \mathbf{A}, \bullet \rangle \in \mathsf{QN}^{\bullet}_{\mathsf{Bm}}$ that does not satisfy $\varphi = \psi$.

Proof (Proposition 2). Items (i)–(iii) are immediate consequences of Definition 3. Regarding (iv), using items (ii) and (i) of Definition 3, we have:

$$\begin{aligned}
&(\bullet_{\mathsf{B}} a \vee b \vee \bullet_{\mathsf{B}} b) \vee \sim(\bullet_{\mathsf{B}} a \vee b \vee \bullet_{\mathsf{B}} b) \\
&= (\bullet_{\mathsf{B}} a \vee b \vee \bullet_{\mathsf{B}} b) \vee (\sim \bullet_{\mathsf{B}} a \wedge \sim b \wedge \sim \bullet_{\mathsf{B}} b) \\
&= (\bullet_{\mathsf{B}} a \vee b \vee \bullet_{\mathsf{B}} b \vee \sim \bullet_{\mathsf{B}} a) \wedge (\bullet_{\mathsf{B}} a \vee b \vee \bullet_{\mathsf{B}} b \vee \sim b) \wedge (\bullet_{\mathsf{B}} a \vee b \vee \bullet_{\mathsf{B}} b \vee \sim b \sim \bullet_{\mathsf{B}} b) \\
&\geq (\bullet_{\mathsf{B}} a \vee \sim \bullet_{\mathsf{B}} a) \wedge (b \vee \bullet_{\mathsf{B}} b \vee \sim b) \wedge (\bullet_{\mathsf{B}} b \vee \sim \bullet_{\mathsf{B}} b) = 1 \wedge 1 \wedge 1 = 1.
\end{aligned}$$

Let us now prove (v). Note that, by the previous items (ii) and (iv), we have $b \vee \bullet_{\mathsf{B}}(b \vee \sim b) \vee \bullet_{\mathsf{B}}(a \vee \sim a \vee b) = \bullet_{\mathsf{B}}(a \vee \sim a \vee b) \vee b \vee \bullet_{\mathsf{B}} b \in B(\mathbf{A})$. Then, by Definition 3 (iii), it will be sufficient to show that $a \vee \sim a \vee b \vee \bullet_{\mathsf{B}}(b \vee \sim b) \vee \bullet_{\mathsf{B}}(a \vee \sim a \vee b) = 1$. Indeed, following the proof of Proposition 1 (vi), we can show that $a \vee \sim a \vee b \vee \bullet_{\mathsf{B}}(a \vee \sim a \vee b) = 1$. From $a \leq a \vee \sim a \vee b$, we have $a \vee \sim a \vee b \geq \sim a \geq \sim(a \vee \sim a \vee b)$. Hence, Definition 3 (i) gives us the required result: $a \vee \sim a \vee b \vee \bullet(a \vee \sim a \vee b) = (a \vee \sim a \vee b) \vee \sim(a \vee \sim a \vee b) \vee \bullet(a \vee \sim a \vee b) = 1$.

Proof (Theorem 3). We have seen in Proposition 2 that an operator $\bullet_{\mathsf{B}}$ given as per Definition 3 satisfies all the equations in the statement. Conversely, supposing these equations hold, let us prove that Definition 3 (iii) is satisfied. Let then $a, b \in A$ be elements of $\mathbf{A} \in \mathsf{QN}$ such that $a \vee \sim a \vee b = b \vee \sim b = 1$. Then, using (iv) and (iii), we have $\bullet_{\mathsf{B}} a \leq b \vee \bullet_{\mathsf{B}}(b \vee \sim b) \vee \bullet_{\mathsf{B}}(a \vee \sim a \vee b) = b \vee \bullet_{\mathsf{B}} 1 \vee \bullet_{\mathsf{B}} 1 = b \vee 0 \vee 0 = b$, as required.

Proof (Proposition 3). Let us preliminarily note that $Tw(\mathbf{H}, \nabla)$ is closed under the new operation. Indeed, on the one hand, if $\langle a_1, a_2 \rangle \in A$, then $a_1 \vee a_2 \in \nabla$, so $\neg(a_1 \vee a_2)$ exists in $\mathbf{H}$. Furthermore, we have $\urcorner(a_1 \vee a_2) \wedge \neg \urcorner(a_1 \vee a_2) = 0$ and, since $\Box \neg x = \neg x$, also $\Box \neg \urcorner(a_1 \vee a_2) = \neg \urcorner(a_1 \vee a_2)$.

To prove the main statement, for the "if" part, let us check that $\langle a_1, a_2 \rangle \vee \sim\langle a_1, a_2 \rangle \vee \bullet\langle a_1, a_2 \rangle = \langle 1, 0 \rangle$. Recalling that $\Box a_1 \wedge a_2 \leq \Box a_1 \wedge \Box a_2 = \Box(a_1 \wedge a_2) = \Box 0 = 0$, we easily obtain: $\langle a_1, a_2 \rangle \vee \sim\langle a_1, a_2 \rangle \vee \bullet\langle a_1, a_2 \rangle = \langle a_1 \vee a_2 \vee \urcorner(a_1 \vee a_2), a_2 \wedge \Box a_1 \wedge \neg \urcorner(a_1 \vee a_2) \rangle = \langle 1, 0 \wedge \neg \urcorner(a_1 \vee a_2) \rangle = \langle 1, 0 \rangle$.

For the "only if" part, assume $\langle a_1, a_2 \rangle \vee \sim\langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle = \langle 1, 0 \rangle$. In the light of the preceding computations, we note that only the first component gives us some information; namely that $a_1 \vee a_2 \vee b_1 = 1$. Then, by the property of the dpc, we have $-(a_1 \vee a_2) \leq b_1$. As to the second component, we need to show that $b_2 \leq \neg \urcorner(a_1 \vee a_2)$. By the property of the pseudo-complement, this is equivalent to $\urcorner(a_1 \vee a_2) \wedge b_2 = 0$. The latter, in turn, follows from $\urcorner(a_1 \vee a_2) \leq b_1$ and the assumption that $b_1 \wedge b_2 = 0$, for we have $\urcorner(a_1 \vee a_2) \wedge b_2 \leq b_1 \wedge b_2 = 0$.

Proof (Proposition 4). Let $b \in \nabla$. Then $\langle b, 0 \rangle \in A$. Let $\bullet\langle b, 0 \rangle = \langle a_1, a_2 \rangle$. We claim that $a_1 = \urcorner b$. We therefore need to show that, for all $c \in H$, we have $b \vee c = 1 \quad \text{iff} \quad a_1 \leq c$. Assume $a_1 \leq c$. From Definition 2 we have: $\langle 1, 0 \rangle = \langle b, 0 \rangle \vee \sim\langle b, 0 \rangle \vee \bullet\langle b, 0 \rangle = \langle b \vee 0 \vee a_1, 0 \wedge \Box b \wedge a_2 \rangle = \langle b \vee a_1, 0 \rangle$, which, in particular, gives us $1 = b \vee a_1 \leq b \vee c$, as required.

Conversely, assume $b \vee c = 1$. Then, considering for instance the element $\langle c, \neg c \rangle \in A$, we have: $\langle b, 0 \rangle \vee \sim\langle b, 0 \rangle \vee \langle c, \neg c \rangle = \langle b \vee 0 \vee c, 0 \wedge \Box b \wedge \neg c \rangle = \langle b \vee c, 0 \rangle = \langle 1, 0 \rangle$. Thus, we may apply Definition 2 to obtain $\bullet\langle b, 0 \rangle \leq \langle c, \neg c \rangle$, giving us in particular $a_1 \leq c$, as required.

For the second claim in the statement, given $\langle a_1, a_2 \rangle \in A$, let $\bullet\langle a_1, a_2 \rangle = \langle b_1, b_2 \rangle$. By Proposition 1 (i), it suffices to compute $\bullet(\langle a_1, a_2 \rangle \vee \sim\langle a_1, a_2 \rangle) = \bullet\langle a_1 \vee a_2, 0 \rangle$. Since $a_1 \vee a_2 \in \nabla$, we can apply the above reasoning to obtain $b_1 = \urcorner(a_1 \vee a_2)$. Hence $\bullet\langle a_1, a_2 \rangle = \langle \urcorner(a_1 \vee a_2), b_2 \rangle$, and we know that $\urcorner(a_1 \vee a_2) \wedge b_2 = 0$, i.e. (by the property of the pseudo-complement) $b_2 \leq \neg \urcorner(a_1 \vee a_2)$. It remains to show that $\neg \urcorner(a_1 \vee a_2) \leq b_2$. Since $\langle a_1, a_2 \rangle \vee \sim\langle a_1, a_2 \rangle \vee \langle \urcorner(a_1 \vee a_2), \neg \urcorner(a_1 \vee a_2) \rangle = \langle 1, 0 \rangle$, by Definition 4 we have $\langle b_1, b_2 \rangle \leq \langle \urcorner(a_1 \vee a_2), \neg \urcorner(a_1 \vee a_2) \rangle$, giving us, in particular, $\neg \urcorner(a_1 \vee a_2) \leq b_2$.

Proof (Proposition 5). Let us preliminarily verify that the following (in)equalities hold on every twist-algebra: $\bullet \sim(x \wedge \sim x) = \bullet(x \wedge \sim x) \leq \bullet \sim x \leq \bullet x = \bullet x * \bullet x$. Given $\langle b_1, b_2 \rangle \in A$, to establish $\bullet \sim(x \wedge \sim x) = \bullet(x \wedge \sim x)$, let us compute: $\bullet(\langle b_1, b_2 \rangle \wedge \sim\langle b_1, b_2 \rangle) = \bullet\langle 0, \Box(b_2 \vee \Box b_1) \rangle = \bullet\langle 0, \Box(b_1 \vee b_2) \rangle = \langle \urcorner \Box(b_1 \vee b_2), \neg \urcorner \Box(b_1 \vee b_2) \rangle = \bullet\langle \Box(b_1 \vee b_2), 0 \rangle = \bullet \sim\langle 0, \Box(b_1 \vee b_2) \rangle = \bullet \sim(\langle b_1, b_2 \rangle \wedge \sim\langle b_1, b_2 \rangle)$. Now observe that from $\Box b_1 \vee b_2 \leq \Box(b_1 \vee b_2) = \Box(\Box b_1 \vee b_2)$ we have $\urcorner \Box(b_1 \vee b_2) \leq \urcorner$

$(\Box b_1 \vee b_2)$, which justifies the inequality $\bullet(x \wedge {\sim} x) \leq \bullet{\sim} x$. To justify the inequality $\bullet{\sim} x \leq \bullet x$, suffice it to observe that $\neg(\Box b_1 \vee b_2) \leq \neg(b_1 \vee b_2)$ because $b_1 \vee b_2 \leq \Box b_1 \vee b_2$. Finally, regarding $\bullet = \bullet x * \bullet x$, recall that the idempotent elements on a twist-algebra are precisely those of the form $\langle b_1, \neg b_1\rangle$ for some $b_1 \in H$.

(i). By the preceding observations, we have $\bullet\langle b_1, b_2\rangle \leq \bullet(\langle b_1, b_2\rangle \wedge {\sim}\langle b_1, b_2\rangle)$ if and only if $\neg(b_1 \vee b_2) \leq \neg\Box(b_1 \vee b_2)$. Note that the inequality $\neg\Box(b_1 \vee b_2) \leq \neg(b_1 \vee b_2)$ holds in general. Since $b_1 \vee b_2 \in \nabla$, the result easily follows.

(ii). Recalling the preliminary observations, we easily see that $\bullet{\sim}\langle b_1, b_2\rangle \leq \bullet(\langle b_1, b_2\rangle \wedge {\sim}\langle b_1, b_2\rangle)$ if and only if $\neg(\Box b_1 \vee b_2) \leq \neg\Box(b_1 \vee b_2)$ if and only if $\neg(\Box b_1 \vee b_2) = \neg\Box(b_1 \vee b_2)$. By the property of the dpc, the latter is in turn equivalent to $\Box b_1 \vee b_2 \vee \neg\Box(b_1 \vee b_2) = 1$.

(iii). As with the preceding item, we have $\bullet\langle b_1, b_2\rangle \leq \bullet{\sim}\langle b_1, b_2\rangle$ iff $\neg(b_1 \vee b_2) \leq \neg(\Box b_1 \vee b_2)$ iff $\neg(b_1 \vee b_2) = \neg(\Box b_1 \vee b_2)$. By the property of the dpc, the latter is equivalent to $b_1 \vee b_2 \vee \neg(\Box b_1 \vee b_2) = 1$.

(iv). Assume $\mathbf{A} \models x \wedge \bullet x = 0$, and let $a \in \nabla$. Then, considering $\langle a, 0\rangle \in A$, compute $\langle a, 0\rangle \wedge \bullet\langle a, 0\rangle = \langle a \wedge \neg a, \Box(0 \vee \neg\neg a)\rangle$. The assumption then gives us, in particular, $\Box(0 \vee \neg\neg a) = \Box(\neg\neg a) = 1$. From the latter, recalling that $\neg\Box x = \neg x$, we obtain $\neg\Box\neg\neg a = \neg\neg\neg a = 0 = \neg 1$. Since $\neg a \leq \neg\neg\neg a$, we conclude that $\neg a = 0$. The latter gives us $a = a \vee 0 = a \vee \neg a = 1$. So $\nabla = \{1\}$. Notice that, by construction, $D(\mathbf{H}) \subseteq \nabla$. Since $D(\mathbf{H}) = \{b \vee \neg b : b \in H\}$, we see that $\nabla = \{1\}$ entails that $\mathbf{H}$ is a Boolean algebra (and, in consequence, the nucleus is the identity map).

Conversely, assume $\nabla = \{1\}$. Then $\bullet\langle b_1, b_2\rangle = \langle\neg 1, \neg\neg 1\rangle = \langle 0, 1\rangle$ for all $\langle b_1, b_2\rangle \in A$.

(v). Assume $\mathbf{A} \models {\sim} x \wedge \bullet x = 0$, and let $\langle b_1, b_2\rangle \in A$. Let us compute ${\sim}\langle b_1, b_2\rangle \wedge \bullet\langle b_1, b_2\rangle = \langle b_2 \wedge \neg(b_1 \vee b_2), \Box(\Box b_1 \vee \neg\neg(b_1 \vee b_2))\rangle = \langle b_2 \wedge \neg(b_1 \vee b_2), \Box(b_1 \vee \neg\neg(b_1 \vee b_2))\rangle$. We thus see that the assumption implies, in particular, $b_2 \wedge \neg(b_1 \vee b_2) = 0$, which (by the property of the pseudo-complement $\neg$) is equivalent to $\neg(b_1 \vee b_2) \leq \neg b_2$. By the property of the dpc, the latter is in turn equivalent to $b_1 \vee b_2 \vee \neg b_2 = 1$. But $b_1 \leq \neg b_2$ (because $b_1 \wedge b_2 = 0$), so $b_1 \vee b_2 \vee \neg b_2 = b_2 \vee \neg b_2$. This means that, for every $b_2 \in \Box[H]$, we have $b_2 \vee \neg b_2 = 1$. Thus, in particular, the Heyting algebra $\mathbf{H}_\Box$ is Boolean; notice also that $b_2 \vee \neg b_2 = 1$ is equivalent to the statement $\Box c \vee \neg c = 1$ for all $c \in H$, because $\Box b_2 = b_2$ and $\neg\Box c = \neg c$.

Now let $a \in \nabla$, so $\langle a, 0\rangle, \langle 0, \Box a\rangle \in A$. Taking then $b_1 = 0$ and $b_2 = \Box a$, ${\sim}\langle 0, \Box a\rangle \wedge \bullet\langle 0, \Box a\rangle = \langle\Box a \wedge \neg\Box a, \Box(0 \vee \neg\neg\Box a)\rangle = \langle 0, 1\rangle$. From the second component, we have $\Box\neg\neg\Box a = \neg\neg\Box a = 1$, which negating both sides gives us $\neg\neg\neg\Box a = 0 = \neg 1$. Since $x \leq \neg\neg x$, we thus have $\neg\Box a = 0$. Then $1 = \Box a \vee \neg\Box a = \Box a \vee 0 = \Box a$, as claimed.

Conversely, let $\langle b_1, b_2\rangle \in A$. We need to show that $b_2 \wedge \neg(b_1 \vee b_2) = 0$ and $\Box(b_1 \vee \neg\neg(b_1 \vee b_2)) = 1$. Since $b_2 \in \Box[H]$, the assumptions give us $b_2 \vee \neg b_2 = 1$. The latter gives us $b_1 \vee b_2 \vee \neg b_2 = 1$, which is equivalent (by the property of the dpc) to $\neg(b_1 \vee b_2) \leq \neg b_2$. By the property of the pseudo-complement, the latter is in turn equivalent to $b_2 \wedge \neg(b_1 \vee b_2) = 0$, which is the first required equality. To obtain the second, from $b_2 \wedge \neg(b_1 \vee b_2) = 0$ we have, again by the property of the pseudo-complement, $b_2 \leq \neg \ulcorner (b_1 \vee b_2)$. From $b_2 \leq \neg \ulcorner (b_1 \vee b_2)$ we obtain $b_1 \vee b_2 \leq b_1 \vee \neg \ulcorner (b_1 \vee b_2)$. Since $b_1 \vee b_2 \in \nabla$, we have that $b_1 \vee \neg \ulcorner (b_1 \vee b_2) \in \nabla$ as well. So we may use the assumptions to conclude that $\square(b_1 \vee \neg \ulcorner (b_1 \vee b_2)) = 1$.

(vi). Observe that $\bullet\langle a_1, a_2\rangle \vee \sim\bullet\langle a_1, a_2\rangle = \langle \ulcorner (a_1 \vee a_2) \vee \neg \ulcorner (a_1 \vee a_2), \neg \ulcorner (a_1 \vee a_2) \wedge \square \ulcorner (a_1 \vee a_2)\rangle = \langle \ulcorner (a_1 \vee a_2) \vee \neg \ulcorner (a_1 \vee a_2), 0\rangle$. The latter equality follows from the nucleus properties: we have $\neg \ulcorner (a_1 \vee a_2) \wedge \square \ulcorner (a_1 \vee a_2) \leq \square\neg \ulcorner (a_1 \vee a_2) \wedge \square \ulcorner (a_1 \vee a_2) = \square(\neg \ulcorner (a_1 \vee a_2) \wedge \ulcorner (a_1 \vee a_2)) = \square 0 = 0$. The only non-trivial condition imposed by the identity $\bullet x \vee \sim\bullet x = 1$ is thus $\ulcorner (a_1 \vee a_2) \vee \neg \ulcorner (a_1 \vee a_2) = 1$.

Proof (Lemma 1). It will be useful to preliminary state the following lemma (we write $\ulcorner a, \ulcorner b$ etc. meaning that the relevant properties hold whenever the dual pseudo-complements of the elements a, b etc. exist in the Heyting algebra $\mathbf{H}$).

Lemma 2. *For every Heyting algebra* $\mathbf{H}$ *and for every* $a, b \in H$*, we have* $\neg \ulcorner (a \to b) \leq \ulcorner b \to \ulcorner a$.

Proof. Since $\ulcorner$ is order-reversing and $\ulcorner (x \wedge y) = \ulcorner x \vee \ulcorner y$, from $a \wedge (a \to b) \leq b$ we have $\ulcorner b \leq \ulcorner (a \wedge (a \to b)) = \ulcorner a \vee \ulcorner (a \to b)$. From the latter inequality, using distributivity, we obtain $\ulcorner b \wedge \neg \ulcorner (a \to b) \leq (\ulcorner a \vee \ulcorner (a \to b)) \wedge \neg \ulcorner (a \to b) = (\ulcorner a \wedge \neg \ulcorner (a \to b)) \vee (\ulcorner (a \to b) \wedge \neg \ulcorner (a \to b)) = (\ulcorner a \wedge \neg \ulcorner (a \to b)) \vee 0 = \ulcorner a \wedge \neg \ulcorner (a \to b)$. Thus, in particular, we have $\ulcorner b \wedge \neg \ulcorner (a \to b) \leq \ulcorner a$, which by residuation gives us $\neg \ulcorner (a \to b) \leq \ulcorner b \to \ulcorner a$, as required.

We can now easily prove the equivalence among the three items in the statement of Lemma 1. To show that (i) entails (ii), assume F is ∇-normal and $a \to b \in F$ for some $a, b \in \nabla$. Then $\neg \ulcorner (a \to b) \in F$, so we can apply Lemma 2 (iii) to obtain $\ulcorner b \to \ulcorner a$, as required. It is clear that (ii) entails (iii). To conclude the proof, assuming F satisfies (iii), let us prove (i). Let $a \in \nabla \cap F$. Then $a \to 1 = 1 \in F$ and $1 \to a = a \in F$. Hence we can apply the hypothesis to obtain $\ulcorner a \to \ulcorner 1 \in F$. But $\ulcorner a \to \ulcorner 1 = \ulcorner a \to 0 = \neg \ulcorner a$, so we are done.

Proof (Proposition 7). We know that every implicative filter $G \subseteq \mathbf{A}$ has the shape $G = (F \times H) \cap A$, where F is a lattice filter of $\mathbf{H}$ (Proposition 6). It remains to show that G is a $\bullet$-filter if and only if F is ∇-normal. Let us first assume that G is a $\bullet$-filter, and let $a \in \nabla \cap F$. Then $\langle a, 0\rangle \in G$ and, as observed earlier, $\bullet\langle a, 0\rangle \Rightarrow \langle 0, 1\rangle = \sim\bullet\langle a, 0\rangle = \langle \neg \ulcorner a, \square \ulcorner a\rangle \in G$. This means that $\neg \ulcorner a \in F$, as required.

Conversely, assume F is ∇-normal and $\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle, \langle b_1, b_2\rangle \Rightarrow \langle a_1, a_2\rangle \in G$. This means, in particular, that $a_1 \to b_1, b_1 \to a_1, a_2 \to b_2, b_2 \to a_2 \in F$. Since $x \to y \leq x \to (y \vee z)$, from $a_1 \to b_1, a_2 \to b_2 \in F$ we obtain $a_1 \to (b_1 \vee b_2), a_2 \to (b_1 \vee b_2) \in F$, hence $(a_1 \to (b_1 \vee b_2)) \wedge (a_2 \to (b_1 \vee b_2)) = (a_1 \vee a_2) \to (b_1 \vee b_2) \in F$ as well. Symmetrically, from $b_1 \to a_1, b_2 \to a_2 \in F$ we obtain $(b_1 \vee b_2) \to (a_1 \vee a_2) \in F$. Since $a_1 \vee a_2, b_1 \vee b_2 \in \nabla$, we can use Lemma 1 to obtain $\ulcorner (a_1 \vee a_2) \to \ulcorner (b_1 \vee b_2) \in F$. Since $\ulcorner (a_1 \vee a_2) \to \ulcorner (b_1 \vee b_2)$ is the first component of $\bullet\langle a_1, a_2\rangle \Rightarrow \bullet\langle b_1, b_2\rangle$, this allows us to conclude that $\bullet\langle a_1, a_2\rangle \Rightarrow \bullet\langle b_1, b_2\rangle \in G$, as required.

Proof (Corollary 1). We have already observed that every $\bullet$-filter satisfies (ii). For the converse, let $\mathbf{A} = Tw(\mathbf{H}, \nabla)$, so we can use Proposition 7. Hence, $G = (F \times H) \cap A$, where F is a ∇-normal lattice filter of $\mathbf{H}$. Let $a = \langle a_1, a_2 \rangle$ and $b = \langle b_1, b_2 \rangle$ be such that $\langle a_1, a_2 \rangle \Rightarrow \langle b_1, b_2 \rangle, \langle b_1, b_2 \rangle \Rightarrow \langle a_1, a_2 \rangle \in G$. This means, in particular, that $a_1 \to b_1, b_1 \to a_1, a_2 \to b_2, b_2 \to a_2 \in F$. Since $x \to y \leq x \to (y \vee z)$ holds on Heyting algebras, from $a_1 \to b_1, a_2 \to b_2 \in F$ we obtain $a_1 \to (b_1 \vee b_2), a_2 \to (b_1 \vee b_2) \in F$, hence $(a_1 \to (b_1 \vee b_2)) \wedge (a_2 \to (b_1 \vee b_2)) = (a_1 \vee a_2) \to (b_1 \vee b_2) \in F$ as well. Symmetrically, from $b_1 \to a_1, b_2 \to a_2 \in F$ we obtain $(b_1 \vee b_2) \to (a_1 \vee a_2) \in F$. Since $a_1 \vee a_2, b_1 \vee b_2 \in \nabla$, we can use Lemma 1 to obtain $\ulcorner(a_1 \vee a_2) \to \ulcorner(b_1 \vee b_2) \in F$. Since $\ulcorner(a_1 \vee a_2) \to \ulcorner(b_1 \vee b_2)$ is the first component of $\bullet\langle a_1, a_2 \rangle \Rightarrow \bullet\langle b_1, b_2 \rangle$, this allows us to conclude that $\bullet\langle a_1, a_2 \rangle \Rightarrow \bullet\langle b_1, b_2 \rangle = \bullet a \Rightarrow \bullet b \in G$, as required.

Proof (Theorem 5). Supposing $\mathbf{A}$ is not directly indecomposable, let $\mathbf{A} = \mathbf{B} \times \mathbf{C}$. Then the element $\langle 1^{\mathbf{B}}, 0^{\mathbf{C}} \rangle$ is Boolean and $B(\mathbf{A}) \neq \{0, 1\}$. Conversely, let $\mathbf{A} = Tw(\mathbf{H}, \nabla)$. As observed earlier, every Boolean element is involutive and an idempotent, that is, every element in $B(\mathbf{A})$ is of type $a = \langle a_1, \neg a_1 \rangle$ for some $a_1 \in H$ with $a_1 = \Box a_1$ and $a_1 \vee \neg a_1 = 1$ (hence, a_1 is a Boolean element of $\mathbf{H}$). Now, assuming $a \in B(\mathbf{A}) - \{0, 1\}$, let us consider the up-set $G := [a)$. We claim that G is a $\bullet$-filter. It is easy to see that G is closed under $*$ (hence, it is an implicative filter). It follows that $G = (F \times H) \cap A$, where $F = [a_1)$. Since a_1 is Boolean, we have $a_1 = \neg \ulcorner a_1$ (Lemma 2). It follows that F is ∇-normal. Indeed, letting $b_1 \in \nabla \cap F$ (i.e. $a_1 \leq b_1$), we have $\ulcorner b_1 \leq \ulcorner a_1$ and $a_1 = \neg \ulcorner a_1 \leq \neg \ulcorner b_1$. Hence, F is ∇-normal and, by Proposition 7, G is a $\bullet$-filter. A similar reasoning shows that $J := [\sim a)$ is a $\bullet$-filter. Note that in the lattice of $\bullet$-filters of $\mathbf{A}$, we have $G \wedge J = \{1\}$ and $G \vee J = A$. The former claim follows from the observation that $c \geq a, \sim a$ entails $c \geq a \vee \sim a = 1$. As to the latter, we have that $a, \sim a \in G \vee J$ entails $a \wedge \sim a = 0 \in G \vee J$. Thus, considering the associated congruences θ_G and θ_J, we have $\theta_G \wedge \theta_J = Id_A$ and $\theta_G \vee \theta_J = A \times A$. Hence, θ_G and θ_J are (non-trivial) factor congruences of $\mathbf{A}$ (it is obvious that they permute, for quasi-Nelson algebras (as a subclass of residuated lattices) are congruence-permutable). We would then conclude that $\mathbf{A} = \mathbf{A}/\theta_G \times \mathbf{A}/\theta_J$, contradicting our hypothesis that $\mathbf{A}$ was directly indecomposable.

Proof (Theorem 6). Clearly (i) implies (ii), which implies (iii). The only non-trivial implications are from (iii) to (iv) and from (iv) to (i). For the former, we can reason as in the proof of Theorem 5. As to the latter, assuming $B(\mathbf{A}) = \{0, 1\}$, we have $\bullet_{\mathsf{B}} a = 1$ for all $a \in A - B(\mathbf{A})$ (item (iii) of Proposition 2). It follows that the only $\bullet_{\mathsf{B}}$-filters of $\mathbf{A}$ are $\{1\}$ and A itself. Hence, $\mathbf{A}$ has only two congruences, and is simple.

References

1. Bou, F., et al.: Logics preserving degrees of truth from varieties of residuated lattices. J. Log. Comput. **19**(6), 1031–1069 (2009)
2. Busaniche, M., Galatos, N., Marcos, M.A.: Twist structures and Nelson Conuclei. Stud. Logica **110**(4), 949–987 (2022)

3. Busaniche, M., Rivieccio, U.: Nelson Conuclei and nuclei: the twist construction beyond involutivity. Studia Logica (To appear)
4. Carnielli, W.A., Marcos, J.: A taxonomy of C-systems. In: Paraconsistency, pp. 1–94. CRC Press (2002)
5. Esteva, F., Figallo-Orellano, A., Flaminio, T., Godo, L.: Logics of formal inconsistency based on distributive involutive residuated lattices. J. Log. Comput. **31**(5), 1226–1265 (2021)
6. Esteva, F., Figallo-Orellano, A., Flaminio, T., Godo, L.: Some categorical equivalences for Nelson algebras with consistency operators. In: 19th World Congress of the International Fuzzy Systems Association (IFSA), 12th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT)
7. Esteva, F., Godo, L.: Monoidal t-norm based logic: towards a logic for left-continuous t-norms. Fuzzy Sets Syst. **124**(3), 271–288 (2001)
8. Flaminio, T., Rivieccio, U.: Prelinearity in (quasi-)Nelson logic. Fuzzy Sets Syst. **445**, 66–89 (2022)
9. Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: an algebraic glimpse at substructural logics. In: Studies in Logic and the Foundations of Mathematics, vol. 151. Elsevier, Amsterdam (2007)
10. Nascimento, T., Rivieccio, U.: Negation and implication in quasi-Nelson logic. Logic. Investig. **27**(1), 107–123 (2021)
11. Nelson, D.: Constructible falsity. J. Symb. Log. **14**, 16–26 (1949)
12. Rivieccio, U., Spinks, M.: quasi-Nelson algebras. Electron. Notes Theor. Comput. Sci. **344**, 169–188 (2019)
13. Rivieccio, U., Spinks, M.: Quasi-Nelson; or, non-involutive Nelson algebras. In: Fazio, D., Ledda, A., Paoli, F. (eds.) Algebraic Perspectives on Substructural Logics. LNCS, vol. 57, pp. 133–168. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-52163-9_8
14. Rivieccio, U.: Fragments of quasi-Nelson: two negations. J. Appl. Log. **7**, 499–559 (2020)
15. Rivieccio, U.: Fragments of Quasi-Nelson: the algebraizable core. Logic J. IGPL **30**(5), 807–839 (2021). https://doi.org/10.1093/jigpal/jzab023
16. Rivieccio, U.: Quasi-N4-lattices. Soft. Comput. **26**(6), 2671–2688 (2022)
17. Rivieccio, U., Jung, A.: A duality for two-sorted lattices. Soft. Comput. **25**(2), 851–868 (2021)
18. Sankappanavar, H.: Heyting algebras with dual pseudocomplementation. Pacific J. Math. **117**(2), 405–415 (1985)