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Coherent states of an accelerated particle

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Abstract We construct generalized coherent states (GCS) of a massive accelerated particle. This example is an important step in studying coherent states (CS) for systems with an unbounded motion and a continuous spectrum. First, we represent quantum states of the accelerated particle both known and new ones obtained by us using the method of non-commutative integration of linear differential equations. A complete set of non-stationary states for the accelerated particle is obtained. This set is expressed via elementary functions and is characterized by a continuous real parameter η, which corresponds to the initial momentum of the particle. A connection is obtained between these solutions and stationary states, which are determined by the Airy function. We solved the problem of constructing GCS, in particular, semiclassical states describing the accelerated particle, within the framework of the consistent method of integrals of motion. We have found different representations, coordinate one and in a Fock space, analyzing in detail all the parameters entering in these representations. We prove corresponding completeness and orthogonality relations. Conditions for minimizing uncertainty relations were studied, and the set of the corresponding parameters was determined. From the GCS, a family of states is isolated, which usually is called the CS. This family of states is parameterized by a real parameter σ_q, which has the meaning of the standard deviation of the coordinate at the initial time instant. The CS minimize the Robertson–Schrödinger uncertainty relation at all the time instants and the Heisenberg uncertainty relation at the initial time. The probability density is given by a Gaussian distribution with the standard deviations $\sigma_q(\tau)$ at the time τ. Coordinate mean values are moving along classical trajectories of the accelerated particles and coincide with trajectories of the maximum of the wave packets. We prove the completeness and orthogonality relations for the obtained GCS and CS. Standard deviations for the GCS and CS are calculated. On this base, and considering the change in the shape of wave packets with time, we define general conditions of the semi-classicality and a class of the CS that can be identified with semiclassical states. As follows from these conditions, in contrast to a free particle case, where CS can be considered as semiclassical states if the Compton wavelength of the particle is much less than the coordinate standard deviation at the initial time moment, after a sufficiently long time period, the CS of the accelerated particle can be always considered as semiclassical ones. It is interesting that this conclusion is matched with the one obtained in a recent work by Sazonov, in studying the Caldirola–Kanai model. Namely, there were demonstrated that the force of resistance and viscous friction prevent the spreading of a quasi-classical wave packet. Thus, the resistance force suppresses the quantum properties of the particle, increasingly highlighting the classical features in its movement over the time.

1 Introduction

Coherent states (CS) play an important role in modern quantum theory as states that provide a natural relation between quantum mechanical and classical descriptions [1–4]. They have a number of useful properties and as a consequence a wide range of applications, e.g., in semiclassical description of quantum systems, in quantization theory, in condensed matter physics, in radiation theory, in quantum computations and so on, see, e.g., Refs. [5–14]. Despite the fact that there exist a great number of publications devoted to constructing CS of different systems, an universal definition of CS and a constructive scheme of their constructing for arbitrary physical system is not known. In this relation, it should be noted that CS were first introduced and studied in detail for systems with bounded motion and discrete spectrum like harmonic oscillator, charged particle in a magnetic field and so on. Formally the problem of constructing CS for systems with quadratic Hamiltonians of the general form was solved in works by Dodonov and Man'ko, using Malkin and Man'ko integral of motion method, see cited Refs.. However, it should be noted that sometimes to extract appropriate sets of CS from the general results is not a simple task. Even for the simplest and physically important system as a

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free particle, the problem of CS construction was, in fact, solved relatively recently, in Ref. [15] following, in fact, the integral of motion method, and its special version proposed in Ref. [16]. We believe that this situation is explained by the fact that the free particle represents an unbounded motion with the continuous energy spectrum and a generalization of the initial (Glauber) scheme in constructing CS of a harmonic oscillator was not so obvious in this case. In fact, a formal application of the integral of motion method to systems with unbounded motion results in constructing the so-called generalized coherent states (GCS). In Ref. [15] the attention was paid on the fact that among families of formally constructed GCS there may exist both semiclassical states and quantum states which do not describe any semiclassical motion at all. Fixing special parameters which arise in the construction of GCS can be distinguished from physical consideration she families of semiclassical states, in particular CS as well as squeezed states.

In this article, we, using the integral of motion method, construct GCS of a massive accelerated particle. This example is a next important step in studying CS for systems with an unbounded motion and a continuous spectrum. Besides of its physical importance there is a didactic advantage of using accelerated particle CS in teaching of quantum mechanics. On this example, we once again demonstrate the existence of GCS that describe both semiclassical and purely quantum motions. In this regard, it should be said that the problem of constructing semiclassical states describing an unbounded motion with some time-dependent Hamiltonians, was considered in relatively recent works based on various approaches; see e.g., Refs. [17–20]. This interest stresses the importance of the problem under consideration. We consider constructing GCS, in particular, semiclassical states describing an accelerated particle, within the framework of the above-mentioned consistent method of integrals of motion mentioned above. In Sect. (2), we study quantum states of the accelerated particle both known and new ones obtained by us using the method of non-commutative integration of linear differential equations. In Sect. (3), we construct GCS of the accelerated particle, in different representations analyzing in detail all the parameters entering in the constructions. We prove the corresponding completeness and orthogonality relations. Standard deviations and conditions of the semi-classicality are discussed in Sec. (4). Here we define the so-called CS and a class of CS that can be identified with semiclassical states. In the Conclusion (5), we tried to list technical and physical results obtained in this article that are important in our opinion.

2 Some exact solutions of the Schrödinger equation

One of the adequate approaches to the quantum description of the rectilinear accelerated motion of a nonrelativistic particle seems to be the consideration of the motion of the particle in an uniform external field. Let we have one-dimensional motion along the x-axis, and let F be the constant force acting on the particle. The potential energy U can be taken as $U = -Fx$, such that the corresponding Hamiltonian reads ($\hat{p}_x = -i\hbar d/dx$):

$$\hat{H}_x = \frac{\hat{p}_x^2}{2m} - Fx = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - Fx. \tag{1}$$

In particular, if the particle has an electric charge e such a force can be caused by an electric field of the intensity $\mathcal{E}$, i.e., $F = e\mathcal{E}$. For example, for the particle of mass m near the Earth's surface, where the gravitational field is almost constant, it is acted upon by a constant force $F = -mg$, where g is the gravity of Earth.

Below, we recall known stationary solutions of the Schrödinger equation with the Hamiltonian (1) and construct new non-stationary solutions of the corresponding time-dependent Schrödinger equation using the method of non-commutative integration of linear differential equations [21–23].

2.1 Stationary states

In the coordinate representation, stationary states $\psi_E(x)$ satisfy the Schrödinger equation $\hat{H}_x\psi_E(x) = E\psi_E(x)$,

$$\frac{d^2\psi_E(x)}{dx^2} + \left(\frac{2m}{\hbar^2}\right)(E + Fx)\psi_E(x) = 0. \tag{2}$$

In the potential field under consideration the energy levels form a continuous nondegenerate spectrum, $+\infty > E > -\infty$. The corresponding motion is finite toward $x = -\infty$ and infinite toward $x = +\infty$. Introducing a dimensionless variable

$$\xi = \left(x + \frac{E}{F}\right)\left(\frac{2mF}{\hbar^2}\right)^{1/3}, \tag{3}$$

one can reduce Eq. (2) to the form $\psi''(\xi) + \xi\psi(\xi) = 0$. A solution of the latter equation, which is finite for all x, reads (see Ref. [24]):

$$\psi_E(x) = \psi(\xi) = A\mathrm{Ai}(-\xi),\ \mathrm{Ai}(\xi) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}u^3 + u\xi\right)du. \tag{4}$$

The function $\mathrm{Ai}(\xi)$ is the so-called Airy function, see Ref. [25], and $A = [2m/(\hbar^2\sqrt{F})]^{1/3}$ is a normalization factor which provides the normalization of the functions $\psi_E(x)$ to the delta function of the energy,

$$\int_{-\infty}^{\infty} \psi_E(x)\psi_{E'}(x)dx = \delta\big(E' - E\big). \tag{5}$$

2.2 A complete set of non-stationary solutions

It is convenient to introduce the dimensionless operators q and $\hat{p}_q$ and the time τ as:

$$q = xl^{-1},\ \hat{p}_q = -i\partial_q = \frac{l}{\hbar}\hat{p}_x,\ \tau = \frac{\hbar}{ml^2}t \tag{6}$$

such that

$$\hat{H}_x = \frac{\hbar^2}{ml^2}H_q,\ \ H_q = \frac{\hat{p}_q^2}{2} - F_q q,\ \ F_q = \frac{ml^3}{\hbar^2}F_x.$$

In new variables (6) the evolution is described by the Schrödinger equation of the form:

$$\begin{aligned} &i\hbar\partial_t\Psi(x,t) = \hat{H}_x\Psi(x,t) \Longrightarrow \hat{S}\chi(q,\tau) = 0,\ \hat{S} = i\partial_\tau - \hat{H}_q\ , \\ &\chi(q,\tau) = \sqrt{l}\Psi\left(lq, \frac{ml^2}{\hbar}\tau\right),\ |\Psi(x,t)|^2dx = |\chi(q,\tau)|^2dq\ . \end{aligned} \tag{7}$$

Some solutions of the Schrödinger equation (7) can be constructed using the method of non-commutative integration of linear differential equations. To this end we note that symmetry operators $\hat{Y}_a,\ [\hat{Y}_a,\ \hat{S}] = 0,\ a = 1,\ 2,\ 3,\ 4,$ of the Schrödinger equation (7),

$$\hat{Y}_1 = -i,\ \ \hat{Y}_2 = \partial_q - iF_q\tau,\ \ \hat{Y}_3 = \tau\partial_q - \frac{i}{2}\big(F_q\tau^2 + 2q\big),\ \ \hat{Y}_4 = \partial_\tau + F\hat{Y}_3$$

form a four-dimensional solvable Lie algebra with nonzero commutation relations

$$[\hat{Y}_2, \hat{Y}_3] = \hat{Y}_1, [\hat{Y}_3, \hat{Y}_4] = -\hat{Y}_2. \tag{8}$$

We define an irreducible λ-representation (see Refs. [22, 23]) of the Lie algebra (8) by operators that act on functions of a variable $\eta \in (-\infty,\ \infty)$ and are parameterized by two real parameters j_0 and $j_1 \geq 0$,

$$\ell_1 = ij_0,\ \ \ell_2 = i(-\eta j_0 + j_1),\ \ \ell_3 = \partial_\eta,\ \ \ell_4 = \frac{i}{2}\eta(\eta j_0 - 2j_1). \tag{9}$$

We will look for a complete set of solutions to the Schrödinger equation, which is parameterized by η, in the form:

$$\chi(q,\ \tau|\eta) = \int_{-\infty}^{+\infty} dj_0 \int_0^{+\infty} dj_1\ \chi(q,\ \tau|\eta,\ j_0,\ j_1)$$

where functions $\chi(q,\ \tau|\eta,\ j_0,\ j_1)$ are found as a solution to a system of first-order differential equations

$$\Big(\ell_a + \hat{Y}_a\Big)\chi(q,\tau|\eta, j_0, j_1) = 0. \tag{10}$$

Then the general solution of the Schrödinger equation (7) is given by the following integral:

$$\chi(q,\tau) = \int_{-\infty}^{+\infty} C(\eta)\chi(q,\tau|\eta)d\eta \tag{11}$$

where $C(\eta)$ is an arbitrary function such that the integral in (11) converges.

A solution of Eq. (10), we seek in the form:

$$\begin{aligned} \chi(q,\tau|\eta, j_0, j_1) =& (2\pi)^{-1/4}w(j_1)\delta(j_0 - 1) \\ &\times \exp\left\{-\frac{i}{2}\left[\tau\eta^2 - 2\eta(\tau F_q - j_1) - 2\eta(q + \tau j_1) + F_q\tau^2(\eta - j_1) + \frac{F_q^2}{3}\tau^3\right]\right\}. \end{aligned} \tag{12}$$

Substituting representation (12) into Eq. (7), we obtain $j_1^2 w(j_1) = 0$, which implies $w(j_1) = \delta(j_1)$. Taking this into account and integrating $\chi(q,\ \tau|\eta,\ j_0,\ j_1)$ over the parameters j_0 and j_1, we finally obtain:

$$\chi(q,\tau|\eta) = (2\pi)^{-1/4}\exp\left\{-\frac{i}{2}\left[(\eta\tau - 2q)(\eta + \tau F_q) + \frac{F_q^2}{3}\tau^3\right]\right\}. \tag{13}$$

These constructed solutions don't have a finite norm and are parameterized by the real parameter η. However, the solutions satisfy completeness and orthogonality relations:

$$\int_{-\infty}^{+\infty} \chi^*(q,\tau|\eta)\chi(q,\tau|\eta')dq = \delta(\eta-\eta')$$
$$\int_{-\infty}^{+\infty} \chi^*(q,\tau|\eta)\chi(q',\tau|\eta)d\eta = \delta(q-q'). \tag{14}$$

One can find a connection between solutions (13) and the stationary states (4) in dimensionless variables (6). Stationary states $\chi_\varepsilon(q, \tau)$ in the dimensionless variables, satisfy the equation

$$\hat{S}\chi_\varepsilon(q,\tau)=0, \quad H_q\chi_\varepsilon(q,\tau)=\varepsilon\chi_\varepsilon(q,\tau). \tag{15}$$

We will search these solutions in the form:

$$\chi_\varepsilon(q,\tau)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} Q_\varepsilon^*(\eta)\chi(q,\tau|\eta)d\eta. \tag{16}$$

Then it follows from Eq. (15) that functions $Q_\varepsilon(\eta)$ must satisfy the equation

$$i\big(F_q\ell_3-\ell_4\big)Q_\varepsilon(\eta)=\varepsilon Q_\varepsilon(\eta) \tag{17}$$

which has the following solutions:

$$Q_\varepsilon(\eta)=(2\pi F_q^2)^{-1/4}\exp\left[\frac{i}{2F}\left(\frac{\eta^3}{3}-2\varepsilon\eta\right)\right]$$
$$\int_{-\infty}^{+\infty} Q_\varepsilon^*(\eta)Q_{\varepsilon'}(\eta)d\eta=\delta(\varepsilon-\varepsilon'). \tag{18}$$

Finally, taking into account Eq. (18), we obtain the explicit form for the stationary states $\chi_\varepsilon(q, \tau)$:

$$\chi_\varepsilon(q,\tau)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} Q_\varepsilon^*(\eta)\chi(q,\tau|\eta)d\eta=\chi_\varepsilon(q)\exp(-i\varepsilon\tau)$$
$$\chi_\varepsilon(q)=\frac{2^{1/3}}{F_q^{1/6}}\mathrm{Ai}(-\xi),\quad \xi=\left(q+\frac{\varepsilon}{F_q}\right)\big(2F_q\big)^{1/3},$$
$$\hat{H}_q\chi_\varepsilon(q)=\varepsilon\chi_\varepsilon(q) \tag{19}$$

where $\mathrm{Ai}(\xi)$ is the Airy function; see Eq. (4). Equation (19) represents the relationship between the new non-stationary solutions (13) and the stationary states $\chi_\varepsilon(q, \tau)$.

One can calculate the Wigner function $W(p_q, q, \tau)$ that corresponds to solutions (13),

$$W(p_q,q,\tau)=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\chi^*\left(q-\frac{q'}{2},\tau\middle|\eta\right)\chi\left(q+\frac{q'}{2},\tau\middle|\eta\right)e^{-ip_qq'}dq'$$
$$=\frac{1}{2\pi\hbar}\delta\big(\eta+F_q\tau-p_q\big). \tag{20}$$

The obtained representation reveals the physical meaning of the parameter η. It is the particle momentum at the initial time moment. Note that the Wigner functions for a particle in a variable uniform field were obtained in Ref. [26].

Note that the constructed solutions (13) form a complete and orthogonal set and are parameterized by a continuous real parameter η. Moreover, these solutions are expressed via elementary functions, which can be useful in various applications.

3 GCS of an accelerated particle

3.1 Integrals of motion

First, we pass to creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$,

$$\hat{a}=\frac{q+i\hat{p}_q}{\sqrt{2}},\ \hat{a}^\dagger=\frac{q-i\hat{p}_q}{\sqrt{2}},\ \big[\hat{a},\hat{a}^\dagger\big]=1. \tag{21}$$

In terms of these operators, the Hamiltonian $\hat{H}_q$ reads:

$$\hat{H}_q=\frac{1}{4}\Big[\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger-\big(\hat{a}^\dagger\big)^2-\hat{a}^2\Big]-\frac{F_q}{\sqrt{2}}\big(\hat{a}+\hat{a}^\dagger\big). \tag{22}$$

It can't be reduced to the first canonical form for a quadratic combination of creation and annihilation operators, which is the oscillator-like form, by any canonical transformation; this indicates that the spectrum of $\hat{H}$ is continuous, as shown in Ref. [13].

Let us construct an integral of motion $\hat{A}(\tau)$ linear in the operators $\hat{q}$ and $\hat{p}_q$,

$$\hat{A}(\tau) = f(\tau)\hat{q} + ig(\tau)\hat{p}_q + \varphi(\tau). \tag{23}$$

Here $f(\tau)$, $g(\tau)$ and $\varphi(\tau)$ are some complex functions on the time τ. For the operator $\hat{A}(\tau)$ to be an integral of motion, it has to commute with the equation operator $\hat{S} = i\partial_\tau - \hat{H}_q$,

$$\left[\hat{S}, \hat{A}(\tau)\right] = 0. \tag{24}$$

In case if the Hamiltonian $\hat{H}_q$ is self-adjoint, the adjoint operator $\hat{A}^\dagger(\tau)$ is an integral of motion as well. We also demand

$$\left[\hat{A}(\tau), \hat{A}^\dagger(\tau)\right] = 1 \tag{25}$$

for $\hat{A}(\tau)$ and $\hat{A}^\dagger(\tau)$ to be annihilation and creation operators.

To satisfy Eq. (24), the functions $f(\tau)$, $g(\tau)$, and $\varphi(\tau)$ have to obey the following equations:

$$\dot{f}(\tau) = 0,\ \dot{g}(\tau) - if(\tau) = 0,\ \ \dot{\varphi}(\tau) + iF_q g(\tau) = 0 \tag{26}$$

where derivatives in τ are denoted by dots above. The general solution of Eq. (26) reads:

$$f(\tau) = c_1,\ g(\tau) = c_2 + ic_1\tau,\ \ \varphi(\tau) = F_q c_1 \frac{\tau^2}{2} - iF_q c_2 \tau + c_3 \tag{27}$$

where c_1,c_2 and c_3 are arbitrary constants. Note that the constant c_3 in Eq. (23) is reduced as a result of the substitution $z \to z + c_3$, therefore, without loss of generality, we further set $c_3 = 0$. It follows from Eq. (25) that

$$2\mathrm{Re}\big(g^*(\tau)f(\tau)\big) = 2\mathrm{Re}\big(c_1^* c_2\big) = 1 \Longrightarrow |c_2||c_1|\cos(\mu_2 - \mu_1) = \frac{1}{2} \tag{28}$$

where $c_1 = |c_1|e^{i\mu_1}$ and $c_2 = |c_2|e^{i\mu_2}$, $\mu_1 \in [0; 2\pi)$, $\mu_2 \in [0; 2\pi)$. Taking all that into account, we obtain:

$$\begin{aligned} q &= g^*(\tau)\hat{A}(\tau) + g(\tau)\hat{A}^\dagger(\tau) - 2\mathrm{Re}\big(g^*(\tau)\varphi(\tau)\big)\,, \\ i\hat{p}_q &= c_1^*\hat{A}(\tau) - c_1\hat{A}^\dagger(\tau) - 2i\mathrm{Im}\big(c_1^*\varphi(\tau)\big)\,. \end{aligned} \tag{29}$$

3.2 GCS

Consider the eigenvalue problem $\hat{A}(\tau)|z,\ \tau\rangle = z(\tau)|z,\ \tau\rangle$ for the annihilation operator $\hat{A}(\tau)$. In the general case eigenvalues $z(\tau)$ that correspond to eigenvectors $|z,\ \tau\rangle$ depend on the time τ. However, if $\hat{A}(\tau)$ is the integral of motion and, at the same time $|z,\ \tau\rangle$ are normalized solutions of the corresponding Schrödinger equation $\hat{S}|z,\ \tau\rangle = 0$, these eigenvalues do not depend on time. A simple proof of this statement follows from the fact that if $\hat{A}(\tau)$ is the integral of motion its mean value in the state $|z,\ \tau\rangle$ does not depend on time. Thus,

$$\left\langle z,\ \tau\middle|\hat{A}(\tau)\middle|z,\ \tau\right\rangle = z(\tau) = \mathrm{const}\ = z.$$

A more formal proof, based on the equations $\hat{S}|z,\ \tau\rangle = 0$ and $|z,\ \tau\rangle \neq 0$ is given by a chain of relations:

$$\begin{aligned} \left[\hat{S}, \hat{A}(\tau)\right]|z,\tau\rangle &= \hat{S}(z(\tau)|z,\tau\rangle) \\ &= i\dot{z}(\tau)|z,\tau\rangle + z(\tau)\hat{S}|z,\tau\rangle = i\dot{z}(\tau)|z,\tau\rangle = 0 \Longrightarrow \\ z(\tau) &= \mathrm{const}\ = z\,. \end{aligned} \tag{30}$$

Thus, in what follows, the above-mentioned eigenvalue problem looks as follows:

$$\hat{A}(\tau)|z,\tau\rangle = z|z,\tau\rangle, \quad \langle z,\tau \mid z,\tau\rangle = 1 \tag{31}$$

where in the general case z is a complex number.

It follows from Eqs. (29) and (31) that

$$\begin{aligned} q(\tau) &\equiv \langle z,\tau|q|z,\tau\rangle = q_0 + p_0\tau + F_q\frac{\tau^2}{2},\ \ q_0 = 2\mathrm{Re}\big(c_2^* z\big)\,, \\ p(\tau) &\equiv \left\langle z,\tau\middle|\hat{p}\middle|z,\tau\right\rangle = p_0 + F_q\tau,\ \ p_0 = 2\mathrm{Im}\big(c_1^* z\big)\,, \\ z &= \left\langle z,\tau\middle|\hat{A}(\tau)\middle|z,\tau\right\rangle = c_1 q(\tau) + ig(\tau)p(\tau) + \varphi(\tau) = c_1 q_0 + ic_2 p_0\,. \end{aligned} \tag{32}$$

The mean values $q(\tau)$ and $p(\tau)$ correspond to the classical trajectory of accelerated by a constant force F_q particle. They satisfy the classical Hamilton equations with the Hamiltonian H_q.

Being written in the coordinate representation, Eq. (31) reads:

$$\left[c_1 q + \varphi(\tau) + g(\tau)\partial_q\right]\Phi_z(q,\tau) = z\Phi_z(q,\tau), \quad \Phi_z(q,\tau) \equiv \langle q \mid z,\tau\rangle. \tag{33}$$

The general solution of this equation has the form

$$\Phi_z(q,\tau) = N\exp\left[-\frac{c_1}{g(\tau)}\frac{q^2}{2} + \frac{z-\varphi(\tau)}{g(\tau)}q + \chi(\tau,z)\right] \tag{34}$$

where $\chi(\tau,\, z)$ is an arbitrary function on τ and z and N is a normalization constant.

One can see that the functions $\Phi_z(q,\, \tau)$ can be written in terms of the mean values $q(\tau)$ and $p(\tau)$,

$$\Phi_z(q,\tau) = N\exp\left\{ip(\tau)q - \frac{c_1}{2g(\tau)}[q-q(\tau)]^2 + \phi(\tau,z)\right\} \tag{35}$$

where $\phi(\tau,\, z)$ is an arbitrary function on τ and z.

We now demand the functions $\Phi_z(q,\, \tau)$ satisfy the Schrödinger equation

$$\hat{S}\Phi_z(q,\tau) = 0 \tag{36}$$

where the operator $\hat{S}$ is defined in Eq. (7). Thus, we fix the function $\phi(\tau,\, z)$ to be:

$$\phi(\tau,z) = -\frac{i}{2}\int_0^\tau\left[p^2(\tau') + \frac{f(\tau')}{g(\tau')}\right]d\tau'. \tag{37}$$

The density probability $\rho(q,\, \tau)$ generated by the functions $\Phi_z(q,\, \tau)$ has the form:

$$\rho(q,\tau) = |\Phi_z(q,\tau)|^2 = N^2\exp\left\{-\frac{[q-q(\tau)]^2}{2|g(\tau)|^2} + 2\mathrm{Re}\phi(\tau,z)\right\}. \tag{38}$$

Considering the normalization integral, we find the constant N,

$$\int_{-\infty}^{\infty}\rho(q,\tau)dq = 1 \Rightarrow N = \left(2\pi|g(\tau)|^2\right)^{-1/4}\exp(-\mathrm{Re}\phi(\tau,z)). \tag{39}$$

Thus, normalized solutions of the Schrödinger equation that are eigenfunctions of the annihilation operator $\hat{A}(\tau)$ have the form:

$$\Phi_z(q,\tau) = \frac{1}{\sqrt{\sqrt{2\pi}|g(\tau)|}}\exp\left\{ip(\tau)q - \frac{f(\tau)}{g(\tau)}\frac{[q-q(\tau)]^2}{2} + i\mathrm{Im}\phi(\tau,z)\right\}. \tag{40}$$

whereas the corresponding probability density reads:

$$\rho_z(q,\tau) = \frac{1}{\sqrt{2\pi}|g(\tau)|}\exp\left\{-\frac{[q-q(\tau)]^2}{2|g(\tau)|^2}\right\}. \tag{41}$$

In what follows we call solutions (40) the time-dependent generalized CS. It should be noted that, in fact, we have a family of states parametrized by two complex constants c_1 and c_2 that satisfy restriction (28). Additional restrictions on the constants c_1 and c_2 transform these states into CS of the accelerated particle, see below.

Substituting the explicit form of trajectories (32) into Eq. (37), obtain the function $\phi(\tau,\, z)$ in the following form:

$$\phi(\tau,z) = -\frac{i}{2}\left(F_q^2\frac{\tau^3}{3} + F_q p_0\tau^2 + p_0^2\tau\right) - \frac{1}{2}\ln\frac{g(\tau)}{c_2}. \tag{42}$$

Thus we obtain a general formula for GCS of an accelerated particle,

$$\begin{aligned}\Phi_z(q,\tau) = &\frac{1}{\sqrt{\sqrt{2\pi}\frac{|c_2|}{c_2}g(\tau)}}\\ &\times\exp\left\{i\left[p(\tau)q - \frac{1}{2}p_0^2\tau\right] - \frac{c_1}{g(\tau)}\frac{[q-q(\tau)]^2}{2} - \frac{i}{2}F_q\left(\frac{F_q}{3}\tau + p_0\right)\tau^2\right\}.\end{aligned} \tag{43}$$

Setting $F_q = 0$ in Eq. (43), we obtain the time-dependent generalized CS of a free particle, see Ref. [15].

Next we will demonstrate that GCS satisfy the completeness condition. To this end we first consider the action of the displacement operator $\mathcal{D}(z,\,\tau) = \exp\Big[z\hat{A}^{\dagger}(\tau) - z^{*}\hat{A}(\tau)\Big]$ on the vacuum vector $|0,\,\tau\rangle$ in the coordinate representation:

$$\begin{aligned}\tilde{\Phi}_z(q,\tau) =& \mathcal{D}(z,\tau)\Phi_0(q,\tau) = \exp\left[-\frac{|z|^2}{2} + z\hat{A}^{\dagger}(\tau)\right]\Phi_0(q,\tau),\\ \Phi_0(q,\tau) =& \langle q \mid 0,\tau\rangle = \frac{1}{\sqrt{\sqrt{2\pi}\frac{|c_2|}{c_2}g(\tau)}}\\ &\times \exp\left[-\frac{c_1}{g(\tau)}\frac{\left(q - F_q\frac{\tau^2}{2}\right)^2}{2} + iF_q\left(q - F_q\frac{\tau^2}{6}\right)\tau\right].\end{aligned} \tag{44}$$

Thus, taking the explicit forms of the mean values (32) into account, we obtain:

$$\begin{aligned}\tilde{\Phi}_z(q,\tau) &= \exp\left\{-\frac{|z|^2}{2} + \left[c_1^{*}\left(q - g^{*}(\tau)\frac{z}{2}\right) + \varphi(\tau)\right]z\right\}\Phi_0\big(q - g^{*}(\tau)z,\tau\big)\\ &= \exp\left(-\frac{i}{2}q_0p_0\right)\Phi_z(q,\tau)\,.\end{aligned} \tag{45}$$

The states $\Phi_z(q,\,\tau)$ and $\tilde{\Phi}_z(q,\,\tau)$ differ by a phase factor only.

Now we will show that the states $\tilde{\Phi}_z(q,\,\tau)$ satisfy the completeness condition, which will give us the completeness condition for GCS. To this end, It is useful to introduce the vacuum vector $|0,\,\tau\rangle$ at a given time instant, $\hat{A}(\tau)|0,\,\tau\rangle = 0$, and the corresponding Fock space,

$$\begin{aligned}&|n,\tau\rangle = \frac{\left[\hat{A}^{\dagger}(\tau)\right]^n}{\sqrt{n!}}|0,\tau\rangle,\quad \langle n,\tau|n,\tau\rangle = 1\quad n = 0,1,2,\ldots\\ &\hat{A}(\tau)|n,\tau\rangle = \sqrt{n}|n-1,\tau\rangle,\quad \hat{A}^{\dagger}(\tau)|n,\tau\rangle = \sqrt{n+1}|n+1,\tau\rangle.\end{aligned} \tag{46}$$

Using representation (44) and definitions (46), one derives the following form for the states $\tilde{\Phi}_z(q,\,\tau)$:

$$\tilde{\Phi}_z(q,\tau) = \exp\left[-\frac{|z|^2}{2}\right]\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\langle q|n,\tau\rangle. \tag{47}$$

With the help the completeness property of the states $|n,\,\tau\rangle$,

$$\sum_{n=0}^{\infty}|n,\tau\rangle\langle n,\tau| = 1,\quad \forall\tau \tag{48}$$

one can find the overlapping of the CS and prove the corresponding completeness relations:

$$\begin{aligned}&\int_{-\infty}^{+\infty}\Big(\tilde{\Phi}_{z'}(q,\tau)\Big)^{*}\tilde{\Phi}_z(q,\tau)dq = \exp\left(z'^{*}z - \frac{\left|z'\right|^2 + |z|^2}{2}\right),\quad \forall\tau\,;\\ &\int\int\Big(\tilde{\Phi}_z(q,\tau)\Big)^{*}\tilde{\Phi}_z\big(q',\tau\big)d^2z = \pi\delta\big(q - q'\big)d^2z = d\mathrm{Re}z d\mathrm{Im}z,\quad \forall\tau\,.\end{aligned} \tag{49}$$

Equation (49) implies already the completeness relation for the GCS,

$$\int\int\big(\Phi_z(q,\,\tau)\big)^{*}\Phi_z\big(q',\,\tau\big)d^2z = \pi\delta\big(q - q'\big),\quad \forall\tau.$$

4 Standard deviations and conditions of semi-classicality

Calculating standard deviations $\sigma_q(\tau)$, $\sigma_p(\tau)$, and the characteristic quantity $\sigma_{qp}(\tau)$, with respect to the GCS, we obtain:

$$\begin{aligned}\sigma_q(\tau) &= \sqrt{\langle\big(\hat{q} - \langle q\rangle\big)^2\rangle} = \sqrt{\langle q^2\rangle - \langle q\rangle^2} = |g(\tau)|\,,\\ \sigma_p(\tau) &= \sqrt{\langle\big(\hat{p} - \langle p\rangle\big)^2\rangle} = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = |f(\tau)| = |c_1|\,,\\ \sigma_{qp}(\tau) &= \frac{1}{2}\big\langle\big(\hat{q} - \langle q\rangle\big)\big(\hat{p} - \langle p\rangle\big) + \big(\hat{p} - \langle p\rangle\big)\big(\hat{q} - \langle q\rangle\big)\big\rangle\end{aligned}$$

$$= i\left[1/2 - g(\tau) f^*(\tau)\right]. \tag{50}$$

One can easily see that the GCS, for any set of the parameters c_1 and c_2, minimize the Robertson-Schrödinger uncertainty relation [27, 28],

$$\sigma_q^2(\tau)\sigma_p^2(\tau) - \sigma_{qp}^2(\tau) = 1/4. \tag{51}$$

Let us consider the Heisenberg uncertainty relation for the GCS. Taking into account relations (28) for constants c_1 and c_2, we obtain

$$\sigma_q(\tau)\sigma_p(\tau) = \sqrt{\frac{1}{4} + \left[|c_2||c_1|\sin(\mu_2 - \mu_1) + |c_1|^2\tau\right]^2} \geq \frac{1}{2}. \tag{52}$$

Then using Eq. (50), we find $\sigma_q(0) = \sigma_q = |c_2|$ and $\sigma_p(0) = \sigma_p = |c_1|$, such that at $\tau = 0$ Eq. (52) implies:

$$\sigma_q\sigma_p = |c_2||c_1| = \sqrt{\frac{1}{4} + [|c_2||c_1|\sin(\mu_2 - \mu_1)]^2} \geq \frac{1}{2}. \tag{53}$$

It follows from (28) that $|c_i| \neq 0, i = 1, 2$ say that the left-hand side of Eq. (53) is minimized when $\mu_1 = \mu_2 = \mu$, which provides the minimization of the Heisenberg uncertainty relation for the CS at the initial time instant,

$$\sigma_q(\tau)\sigma_p(\tau)\big|_{\tau=0} = \frac{1}{2}. \tag{54}$$

In what follows, we consider the GCS with the above restriction $\mu_1 = \mu_2 = \mu$. Namely, such states we call simply CS of a accelerated particle. In this case relation (28), $2\mathrm{Re}\left(c_1^* c_2\right) = 1$, takes the form:

$$|c_2||c_1| = 1/2 \Longrightarrow c_2^* = c_1^{-1}/2. \tag{55}$$

One can see that the constant μ does not enter the CS. Thus, in what follows we set $\mu = 0$. Then

$$\begin{aligned} &c_2 = |c_2| = \sigma_q,\ c_1 = |c_1| = \sigma_p = 1/(2\sigma_q)\,, \\ &g(\tau) = \left(\sigma_q + \frac{i\tau}{2\sigma_q}\right),\ \ \sigma_q(\tau) = |g(\tau)| = \sqrt{\sigma_q^2 + \frac{\tau^2}{4\sigma_q^2}}\,. \end{aligned} \tag{56}$$

It follows from Eq. (56), that for any time instant τ the Heisenberg uncertainty relation for the CS takes the form:

$$\sigma_q(\tau)\sigma_p(\tau) = \frac{1}{2}\sqrt{1 + \frac{\tau^2}{4\sigma_q^4}} \geq \frac{1}{2}. \tag{57}$$

Finally, taking into account Eq. (32), we obtain the following coordinate representation for the CS of an accelerated particle:

$$\Phi_z^{\sigma_q}(q,\tau) = \frac{\exp\left\{i\left[p(\tau)q - \frac{p_0^2}{2}\tau\right] - \frac{[q-q(\tau)]^2}{4\left(\sigma_q^2 + i\tau/2\right)} - \frac{i}{2}F_q\left(F_q\frac{\tau}{3} + p_0\right)\tau^2\right\}}{\sqrt{\left(\sigma_q + \frac{i\tau}{2\sigma_q}\right)\sqrt{2\pi}}}. \tag{58}$$

We stress that, in fact, we have constructed a family of the CS parametrized by one real parameter σ_q. Each set of the CS in the family has its specific initial standard deviations $\sigma_q > 0$. The CS from a family with a given σ_q can be also labeled by the quantum number z,

$$z = \frac{q_0}{2\sigma_q} + i\sigma_q p_0 \tag{59}$$

which is in one to one correspondence with the corresponding classical trajectory initial data, $q_0 = 2\sigma_q \mathrm{Re}z,\ \ p_0 = (\mathrm{Im}z)/\sigma_q$. Thus, we will take σ_q and z as independent parameters of the constructed CS.

If $\sigma_q < 1/2$ or $\sigma_p < 1/2$ the accelerated particle CS are, at the initial time instant, the so-called squeezed states; see Ref. [12].

The probability densities that corresponds to the CS (58) are:

$$\rho_z^{\sigma_q}(q,\tau) = \frac{1}{\sqrt{2\pi\sigma_q^2(\tau)}}\exp\left\{-\frac{[q-q(\tau)]^2}{2\sigma_q^2(\tau)}\right\}. \tag{60}$$

One can see that at any time instant τ the probability densities (60) are given by Gaussian distributions with standard deviations $\sigma_q(\tau)$.

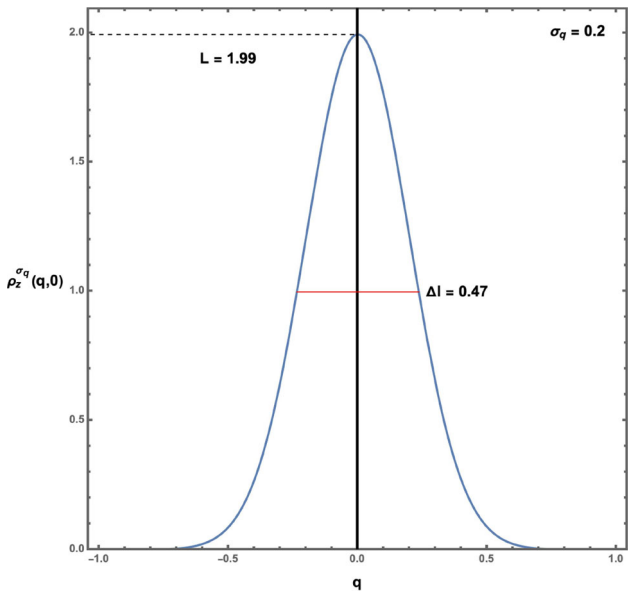

Fig. 1 The shape of the wave packet at the initial moment of time for $\sigma_q = 0.2$ and $q_0 = 0$

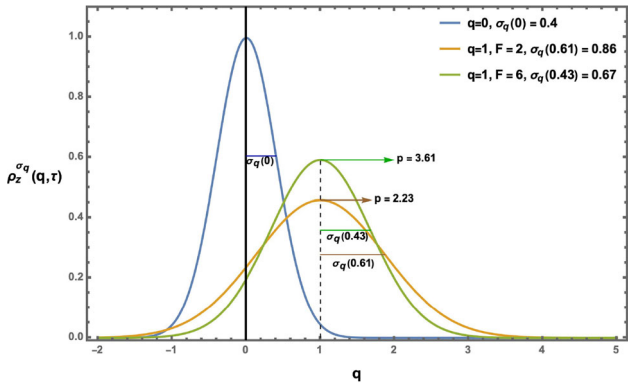

Fig. 2 Evolution of the probability density for $q_0 = 0$, $p_0 = 1$, $\sigma_q = 0.4$. Blue color shows the particle distribution density at the initial moment of time with initial conditions $q_0 = 0$, $p_0 = 1$. The standard deviation at the initial time is chosen to be $\sigma_q = 0.4$. Yellow color shows the evolution of the distribution density of an accelerated particle, which is acted upon by a force $F = 2$ and which is located at point $q = 1$ and has momentum $p = 2.23$. Green color shows the evolution of the distribution density of an accelerated particle at the same point $q = 1$ with momentum $p = 3.61$, which is acted upon by a force $F = 6$

Let us consider the shape of the particle wave packet (the shape of the probability density) at the initial time instant. Equation (60) implies that this packet has the height $L = 1/(\sqrt{2\pi}\sigma_q)$ and the half-width $\Delta l = \sqrt{8\ln 2}\sigma_q$,

$$\Delta l = \frac{1}{L}\sqrt{\frac{4\ln 2}{\pi}} \approx 0.939\frac{1}{L}. \tag{61}$$

The same relation holds true for the all the GCS.

Figure 1 shows the wave packet corresponding to the CS (58) at the initial time for $\sigma_q = 0.2$ and $q_0 = 0$.

Now consider the change in the shape of the wave packet over the time. The coordinate mean values $\langle q\rangle = q(\tau) = q_0 + p_0\tau + F_q\tau^2/2$ are moving along the classical trajectory with the particle momentum $\langle p\rangle = p(\tau) = p_0 + F_q\tau$ and the constant acceleration F_q. With the same momentum and the acceleration are moving the maxima of the probability densities (60). The half-width $\Delta l(\tau) = \sqrt{8\ln 2}\sigma_q(\tau)$ of a given Gaussian wave packet does not depend on the force F acting on the particle. This force affects the magnitude of the shift of the wave packet as a whole along the coordinate axis q per time unit.

The maximum of the probability density (60) is located at the point $q > q_0$ at the time

$$\tau = \tau_q = \begin{cases} \left[\sqrt{\left(\frac{p_0}{F_q}\right)^2 + \frac{2(q-q_0)}{F_q}} - \frac{p_0}{F_q}\right], & F_q > 0 \\ (q-q_0)/p_0, & F_q = 0 \end{cases} \tag{62}$$

and is characterized by the standard deviation Ω_q,

$$\Omega_q = \sigma_q(\tau_q) = \sqrt{\sigma_q^2 + \frac{1}{4\sigma_q^2}\left[\sqrt{\left(\frac{p_0}{F_q}\right)^2 + \frac{2(q-q_0)}{F_q}} - \frac{p_0}{F_q}\right]^2}$$
$$< \Omega_q\big|_{F_q=0} = \sqrt{\sigma_q^2 + \left(\frac{q-q_0}{2p_0\sigma_q}\right)^2}. \tag{63}$$

The spreading of the wave packet of an accelerated particle at a point q is less than the spreading of the wave packet of a free particle arriving at the same point. This blurring decreases the larger the F:

$$\Omega_q = \sigma_q + \frac{(q-q_0)}{4F_q\sigma_q^3} + O\left(\frac{1}{F_q}\right)^{3/2}. \tag{64}$$

Let us illustrate what has been said with the graph, Fig. 2. We see that the greater the force F, the less the spreading of the wave packet corresponding to the particle at point $q = 1$.

To consider the question which CS can be treated as representing a semiclassical particle motion, we have to return to the initial dimensional variables x and t (6) and to the initial wave function $\Psi_z^{\sigma_q}(x,\,t)$ written in these variables,

$$\Phi_z^{\sigma_q}(q,\,\tau) = \sqrt{l}\Psi_z^{\sigma_q}\left(lq,\,\frac{ml^2}{\hbar}\tau\right).$$

Taking into account that

$$x(t) = lq(\tau) = x_0 + \frac{p_0^x}{m}t + \frac{F_x}{m}\frac{t^2}{2},\;\; p_0 = \frac{l}{\hbar}p_0^x\,,$$
$$p_x(t) = \frac{\hbar}{l}p(\tau) = p_0^x + F_x t\,,$$
$$\sigma_x(0) = l\sigma_q(0) = l\sigma = \sigma_x,\; \sigma_x^2(t) = \sigma_x^2 + \frac{\hbar^2}{4m^2\sigma_x^2}t^2\,, \tag{65}$$

we obtain

$$\Psi_z^{\sigma_q}(x,t) = \frac{1}{\sqrt{\left(\sigma_x + \frac{i\hbar}{2m\sigma_x}t\right)\sqrt{2\pi}}}$$
$$\times \exp\left\{\frac{i}{\hbar}\left[\left(p_x(\tau)x - \frac{p_0^{x2}}{2m}t\right) - \frac{F_x}{m}\left(\frac{F_x}{3}t + p_0^x\right)\frac{t^2}{2}\right] - \frac{[x-x(t)]^2}{4\left(\sigma_x^2 + \frac{\hbar}{2m}it\right)}\right\},$$
$$\rho_z^{\sigma_q}(x,t) = \left|\Psi_z^{\sigma_q}(x,t)\right|^2 = \frac{1}{\sqrt{\left(\sigma_x^2 + \frac{\hbar^2}{4m^2\sigma_x^2}t^2\right)2\pi}}\exp\left\{-\frac{1}{2}\frac{[x-x(t)]^2}{\sigma_x^2 + \frac{\hbar^2}{4m^2\sigma_x^2}t^2}\right\}. \tag{66}$$

The shape of distribution (66) that corresponds to the semiclassical motion must change with the time slowly in a certain sense. This change is associated with a change of the quantity $\sigma_x^2(t)$, see Eq. (65). We suppose that the semiclassical motion implies the quantity $\sigma_x^2(t)$ (or equivalently $\frac{\hbar^2}{4\,m^2\sigma_x^2}t^2$) is much less than the distance square that the particle passes for the same time t. Then, the semi-classicality condition reads:

$$\frac{\hbar^2 t^2}{4\sigma_x^2} \ll \left(p_0^x t + \frac{F_x t^2}{2}\right)^2. \tag{67}$$

It can be rewritten in a different form:

$$\frac{\lambda}{\left|1 + \frac{\lambda}{2\pi\hbar}F_x\frac{t}{2}\right|} \ll 4\pi\sigma_x,\;\; \lambda = \frac{2\pi\hbar}{p_x} \tag{68}$$

where λ is the Compton wavelength of the particle.

Thus, CS of a free particle ($F_x = 0$) can be considered as semiclassical states if the Compton wavelength of the particle is much less than the coordinate standard deviation σ_x at the initial time moment, see Ref. [15]. However, if $F_x \neq 0$ and after a sufficiently long time period, CS of an accelerated particle can be always considered as semiclassical ones.

5 Concluding remarks

We study quantum states of the accelerated particle both known and new ones obtained by us using the method of non-commutative integration of linear differential equations. A complete set of non-stationary states (13) for the accelerated particle is obtained. This set is expressed via elementary functions and is characterized by a continuous real parameter η, which corresponds to the initial momentum of the particle. A connection is obtained between these solutions and stationary states, which are determined by the Airy function (19).

We solved the problem of constructing GCS, in particular, semiclassical states describing the accelerated particle, within the framework of the consistent method of integrals of motion. We have found different representations, coordinate one and in a Fock space, analyzing in detail all the parameters entering in these representations. We prove the corresponding completeness and orthogonality relations. Conditions for minimizing uncertainty relations, were studied and the set of the corresponding parameters was determined. From the GCS a family of states is isolated, which usually is called the CS. This family of states is parameterized by the real parameter σ_q, which has the meaning of the standard deviation of the coordinate at the initial time instant. The CS minimize the uncertainty relation (51) at all the time instants and the Heisenberg uncertainty relation (54) at the initial time. The probability density (60) is given by a Gaussian distribution with the standard deviations $\sigma_q(\tau)$ and the constructed CS are wave packets that are solutions to the Schrödinger equation for the accelerated particle. Coordinate mean values are moving along classical trajectories of the accelerated particles and coincide with trajectories of the maximum of the wave packets. We prove the completeness and orthogonality relations for the obtained GCS and CS.

Standard deviations for the GCS and CS are calculated. On this base, and considering the change in the shape of wave packets with time, we define general conditions of the semi-classicality and a class of CS that can be identified with semiclassical states. As follows from this conditions, in contrast to a free particle case, where CS can be considered as semiclassical states if the Compton wavelength of the particle is much less than the coordinate standard deviation σ_x at the initial time moment, see Ref. [15], after a sufficiently long time period, the CS of the accelerated particle can be always considered as semiclassical ones. It is interesting that this conclusion is matched with the one obtained in Ref. [18] in studying the Caldirola–Kanai model. Namely, there were demonstrated that the force of resistance and viscous friction prevent the spreading of a quasi-classical wave packet. Thus, the resistance force suppresses the quantum properties of the particle, increasingly highlighting the classical features in its movement over time.

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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