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Small normalised solutions for a Schrödinger-Poisson system in expanding domains: Multiplicity and asymptotic behaviour

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Abstract

Given a smooth bounded domain $\Omega \subset \mathbb{R}^3$, we consider the following nonlinear Schrödinger-Poisson type system

$$\begin{cases} -\Delta u + \phi u - |u|^{p-2}u = \omega u & \text{in } \lambda\Omega, \\ -\Delta \phi = u^2 & \text{in } \lambda\Omega, \\ u > 0 & \text{in } \lambda\Omega, \\ u = \phi = 0 & \text{on } \partial(\lambda\Omega), \\ \int_{\lambda\Omega} u^2 \, dx = \rho^2 \end{cases}$$

in the expanding domain $\lambda\Omega \subset \mathbb{R}^3, \lambda > 1$ and $p \in (2, 3)$, in the unknowns (u, ϕ, ω). We show that, for arbitrary large values of the expanding parameter λ and arbitrary small values of the mass $\rho > 0$, the number of solutions is at least the Ljusternick-Schnirelmann category of $\lambda\Omega$. Moreover we show that as $\lambda \to +\infty$ the solutions found converge to a ground state of the problem in the whole space $\mathbb{R}^3$.

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Contents

1. Introduction

Elliptic systems involving the Schrödinger and the Maxwell equations have attracted a lot of interest in mathematical physics in the last decades. Many authors have studied this intriguing problem, which takes into account the interaction of a nonrelativistic particle with its own electromagnetic field and it is almost impossible to give a complete list of references on the topic. The huge literature existing deals mainly with the equations settled in the whole space $\mathbb{R}^N$, while only few works study the case of bounded domains where boundary conditions may even prevent the existence of solutions (see the work cited below). We cite here the pioneering paper of Benci and Fortunato [8] for two reasons: first, it is the inspiration of our paper, and second because it seems they first gave a deduction of the equations which describe the interaction of the matter field with the electromagnetic field in the framework of Abelian Gauge Theories, in place of the usual and classical Hartree and Thomas-Fermi-von Weizsäcker Theory of atoms and molecules (see e.g. [9,10]).

Without entering in details in the physical and mathematical derivation of the equations (beside [8], the interested reader is also referred to [3,11,12]) the search of stationary solutions

$$\psi(x,t) = u(x)e^{i\omega t} \in \mathbb{C}\,, \qquad u(x),\ \omega \in \mathbb{R},$$

of the so called Schrödinger-Maxwell system in the purely electrostatic case, i.e.

$$\phi(x,t) = \phi(x), \qquad \mathbf{A}(x,t) = \mathbf{0},$$

(where $(\phi, \mathbf{A})$ is the gauge potential of the electromagnetic field) leads to the following model problem, known as the *Schrödinger-Poisson system*:

$$\begin{cases} -\Delta u + \phi u - |u|^{p-2}u = \omega u & \text{in } \Omega, \\ -\Delta \phi = u^2 & \text{in } \Omega, \\ u, \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

It is assumed that the particle is going to be "observed" in a region Ω in $\mathbb{R}^3$ where it is confined, and p is a suitable exponent. Here u and ϕ are unknowns of the problem.

For what concern the number ω, the frequency of the wave function, there are two different points of view, depending on the problem one wants to study:

I. ω is a given datum of the problem; this means that one is interested in finding wave functions with a given frequency. Sometimes this produces restrictions on the values of ω in order to obtain solutions. For the problem in a bounded domain the reader is referred, for example, to [2,18,23,24].

II. ω is not given; in this case the wave function is completely unknown, no a priori value of the frequency is specified so the unknowns of the problem are u, ϕ and ω. In contrast to the previous case, this arises when the L^2-norm of the solutions is a priori fixed. Beside the paper [8], we cite [19,20] where a Neumann condition on ϕ is considered. The problem in the whole $\mathbb{R}^N$ is studied in [4,5] (see also the references therein) where the main difficult was to recover some compactness *à la Lions*.

We mention also [21] where both cases I. and II. are treated.

1.1. The problem addressed in this paper

In this paper we are interested in the second point of view described above since we believe it is more natural than the first. Indeed the wave function is usually unknown, so also ω has to be treated as an unknown. This has the following consequences: since the system is variational, we will find the solutions as critical points of the associated *energy functional* and the fact that ω is unknown led us to find critical points restricted to the constraint of functions with fixed L^2-norm. Hence ω will appear naturally as a Lagrange multiplier. The restriction to functions u with fixed L^2-norm has also a physical consistency since $|\psi(x,t)|^2 = u^2(x)$ and it is known that the L^2-norm of the solutions of the Schrödinger equation is constant in time.

More specifically, our aim in this paper is to show the existence of solutions (u, ϕ, ω) for the following system in an expanding domain $\lambda\Omega$, $\lambda > 1$,

$$\begin{cases} -\Delta u + \phi u - |u|^{p-2}u = \omega u & \text{in } \lambda\Omega, \\ -\Delta \phi = u^2 & \text{in } \lambda\Omega, \\ u > 0 & \text{in } \lambda\Omega, \\ u, \phi = 0 & \text{on } \partial(\lambda\Omega), \\ \displaystyle\int_{\lambda\Omega} u^2 \mathrm{d}x = \rho^2. \end{cases} \tag{P_λ}$$

The set $\Omega \subset \mathbb{R}^3$ is a smooth and bounded *domain*, i.e., open and connected. The requirement $u > 0$ is very natural, being u the modulus of the wave function (observe *en passant* that the positivity of ϕ in $\lambda\Omega$ is granted for free).

It is classical by now (see the approach in [8]) that (P_λ) can be reduced to a single equation involving a nonlocal term; indeed calling $\phi_u \in H_0^1(\Omega)$ the unique (and positive) solution of

$$-\Delta\phi = u^2 \quad \text{in } \lambda\Omega, \qquad \phi = 0 \quad \text{on } \partial(\lambda\Omega)$$

for fixed $u \in H_0^1(\lambda\Omega)$, the problem ($P_\lambda$) can be equivalently written as

$$\begin{cases} -\Delta u + \phi_u u - |u|^{p-2}u = \omega u & \text{in } \lambda\Omega, \\ u > 0 & \text{in } \lambda\Omega, \\ u = 0 & \text{on } \partial(\lambda\Omega), \\ \displaystyle\int_{\lambda\Omega} u^2 \mathrm{d}x = \rho^2, \end{cases} \tag{1.1}$$

to which we will refer from now on. In this way by a (weak) solution of (1.1) we simply mean a pair $(u, \omega) \in H_0^1(\lambda\Omega) \times \mathbb{R}$ with u having (squared) L^2-norm equals to ρ^2 for $\rho > 0$, such that

$$\forall v \in H_0^1(\lambda\Omega) : \int_{\lambda\Omega} \nabla u \nabla v \, \mathrm{d}x + \int_{\lambda\Omega} \phi_u u v \, \mathrm{d}x - \int_{\lambda\Omega} |u|^{p-2} u v \, \mathrm{d}x = \omega \int_{\lambda\Omega} u v \, \mathrm{d}x.$$

Of course, the solutions will depend on ρ and λ. By standard variational principles we know that the solutions can be found as critical points of the C^1 energy functional (it is useful to have the explicit dependence on the domain considered)

$$I(u; \lambda\Omega) := \frac{1}{2} \int_{\lambda\Omega} |\nabla u|^2 \mathrm{d}x + \frac{1}{4} \int_{\lambda\Omega} \phi_u u^2 \mathrm{d}x - \frac{1}{p} \int_{\lambda\Omega} |u|^p \mathrm{d}x$$

on the constraint given by the L^2-sphere

$$\mathcal{M}_\rho(\lambda\Omega) := \left\{ u \in H_0^1(\lambda\Omega) : \int_{\lambda\Omega} u^2 \mathrm{d}x = \rho^2 \right\}$$

and ω appears as the Lagrange multiplier. The term *ground state* is used to refer to the solution with minimal energy.

To state our result, let us recall that in [4] it has been proved that there exists $\rho_1 = \rho_1(p) > 0$ such that for any $\rho \in (0, \rho_1)$ the problem on the whole space $\mathbb{R}^3$ has a ground state solution $\mathfrak{w}_\infty$ and

$$c_\infty := \min_{u \in \mathcal{M}_\rho(\mathbb{R}^3)} I(\mathfrak{w}_\infty; \mathbb{R}^3) < 0.$$

Moreover, by the results in [14], the associated Lagrange multiplier ω_∞ is negative and the ground state $\mathfrak{w}_\infty$ is positive and radially symmetric.

Roughly speaking, our result given below states that for small value of the L^2-norm, the number of solutions is influenced by the topology of the domain, at least when the domain is very large.

Hereafter for a topological pair $Y \subset X$, $\mathrm{cat}_X(Y) \in \mathbb{N}$ denotes the Ljusternick-Schnirelmann category; if $X = Y$ we simply write $\mathrm{cat}\, X$.

The result is the following

Theorem 1.1. *Let $p \in (2,3)$ and let $N = \mathrm{cat}\,\Omega$. There exists $\rho_1 = \rho_1(p) > 0$ such that, for every $\rho \in (0, \rho_1)$ there is $\Lambda > 1$ such that for any $\lambda \in (\Lambda, +\infty)$, the problem (1.1) has at least N solutions $(u^i_{\rho,\lambda}, \omega^i_{\rho,\lambda}) \in \mathcal{M}_\rho(\lambda\Omega) \times \mathbb{R}$, $i = 1, \ldots, N$. Moreover, for every i, as $\lambda \to +\infty$,*

$$I(u^i_{\rho,\lambda}; \lambda\Omega) \to c_\infty < 0, \quad \omega^i_{\rho,\lambda} \to \omega_\infty < 0,$$

and, up to translations,

$$u^i_{\rho,\lambda} \to \mathfrak{w}_\infty \text{ in } H^1(\mathbb{R}^3).$$

Furthermore, if Ω is not contractible in itself, besides the solutions just found, there is another one $(\widetilde{u}_{\rho,\lambda}, \widetilde{\omega}_{\rho,\lambda})$ with $\widetilde{u}_{\rho,\lambda}$ nonnegative and at a higher energy level.

We remark here, once for all, that the solutions u we will find are indeed solutions in the classical sense.

1.2. Comparison with known results

Now an explicit comparison with the paper of Benci and Fortunato [8] is in order. In the paper [8] infinitely many solutions $(u_k, \omega_k)_{k\in\mathbb{N}}$ are found for the problem

$$\begin{cases} -\Delta u + \phi_u u = \omega u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega u^2 = 1 \end{cases} \tag{1.2}$$

in a fixed domain Ω. The sequence $\{u_k\}$ consists of critical points at minimax levels (over the class of sets having arbitrary large *Krasnoselskii genus*) of the energy functional restricted to the unit sphere in $L^2(\Omega)$, on which it is bounded from below. Actually, as it is observed in [18, Appendix], the same result holds by adding a nonlinearity $-|u|^{p-2}u$ in the left hand side of the equation in (1.2), for $p \in (2, 10/3)$.

Even though Benci and Fortunato work on the unit sphere in $L^2(\Omega)$, their result is true for every value of radius $\rho > 0$, namely on $\mathcal{M}_\rho(\Omega)$. Summing up, the result of [8] follows for the problem

$$\begin{cases} -\Delta u + \phi_u u - |u|^{p-2}u = \omega u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega u^2 = \rho^2 \end{cases}$$

for any $p \in (2, 10/3)$, $\rho > 0$, and they also obtain that

(a) the solutions u_k are possibly sign changing, except of course the solution which minimizes the related functional $I(\cdot\,;\Omega)$ restricted to $\mathcal{M}_\rho(\Omega)$,
(b) the sequence of critical levels $\{I(u_k;\Omega)\}$ is diverging,
(c) the sequence of Lagrange multipliers $\{\omega_k\}$ is diverging.

Our paper gives a contribution in the sense that solutions different from those found in [8] are furnished here. Indeed for large (expanding) domains we find solutions that

(a') are all positive,
(b') are at a negative level of the energy functional,
(c') all have negative related Lagrange multipliers,
(d') when the domain is larger and larger the Lagrange multipliers, as well the solutions u, are converging to the solutions in the whole space.

However we can prove this just for solutions with small L^2-norm and our result just follows for $p \in (2,3)$ and not in the whole range $(2, 10/3)$.

Our approach is slightly different from that in [8] in the sense that we use another topological invariant which also adapts to non even functionals and permits to gain the positivity of the solutions. Indeed, by means of the Ljusternick-Schnirelmann category and the barycenter map introduced by Benci and Cerami in [6], we are able to find a number of solutions depending on the "topological complexity" of the domain $\lambda\Omega$. The main difficulty here is due to the presence of a nonlocal term driven by the Green function; in fact, fine estimates have to be found in order to deal with the Green function and compare suitable infima when the domain is expanding, and implement the barycenter map on suitable sublevels of the energy functional.

As it will be clear in Section 2, the restrictions on p, ρ and λ in the main theorem are due to: *(i)* the boundedness from below of the functional, *(ii)* the evaluation of the barycenters of functions in suitable sublevels of the functional $I(\cdot\,;\lambda\Omega)$, and *(iii)* the radial property of the ground state solution $\mathfrak{w}_\infty$ for the limit problem in the whole $\mathbb{R}^3$.

We observe that problems in expanding domains have attracted attention in the mathematical literature and also have been studied in other contexts; see e.g., [1,13]. However this is the first paper where, besides a multiplicity result, a convergence result is also presented.

1.3. Organization of the paper

The organization of the paper is the following: in Section 2 we introduce basic notations, we recall some well known facts and give the variational framework of the problem. In Section 3 the barycenter map is introduced; it will be a fundamental tool in order to employ the Ljusternick-Schnirelmann theory. Here is seen the role played by large values of λ. In Section 4 the proof of Theorem 1.1 is given. In a final Appendix we show the striking difference between the whole space and a bounded domain for what concerns the minimum of the functional restricted to the L^2-sphere.

2. Some preliminaries

Let us start by introducing few notations.

In all the paper $\Omega \subset \mathbb{R}^3$ is a (smooth and) bounded domain such that $0 \in \Omega$. Hereafter the open ball centred in $x_0 \in \mathbb{R}^3$ with radius $\widehat{r} > 0$ will be denoted with $B_{\widehat{r}}(x_0)$. Note that, for $\lambda > 1$

and $x_0 \neq 0$, we have $\lambda B_{\hat{r}}(x_0) = B_{\lambda\hat{r}}(\lambda x_0)$, so that $\lambda B_{\hat{r}}(x_0) \neq B_{\lambda\hat{r}}(x_0)$. We use $d(x, D)$ to denote the distance between a point $x \in \mathbb{R}^3$ and $D \subset \mathbb{R}^3$. Also let us fix, from now on, a small $r > 0$ such that $B_r := B_r(0) \subset \Omega$ and, setting

$$\Omega_r^+ := \left\{ x \in \mathbb{R}^3 : d(x, \Omega) \le r \right\}$$
$$\Omega_r^- := \left\{ x \in \Omega : d(x, \partial\Omega) \ge r \right\},$$

the sets

$$\Omega,\ \Omega_r^+,\ \text{and } \Omega_r^- \text{ are homotopically equivalent.} \tag{2.1}$$

This holds thanks to the fact that $\partial\Omega$ is a smooth compact manifold and the Tubular Neighbourhood Theorem applies (see [15, pp. 137-141]).

In particular, for any $\lambda > 1$, the same is true for

$$\lambda\Omega, \quad \lambda\Omega_r^+ \quad \text{and} \quad \lambda\Omega_r^-.$$

In all the paper r will be the fixed value above.

The symbol $o_s(1)$ denotes a quantity which goes to zero as $s \to +\infty$.

We denote with $|\cdot|_{L^q(D)}$ the usual L^q-norm, and let $H_0^1(D)$ be the usual Sobolev space endowed with equivalent (squared) norm

$$\|u\|^2_{H_0^1(D)} = \int_D |\nabla u|^2 \mathrm{d}x.$$

As noted earlier, for every fixed $u \in H_0^1(D)$, the problem

$$\begin{cases} -\Delta\phi = u^2 & \text{in } D, \\ \phi = 0 & \text{on } \partial D \end{cases} \tag{2.2}$$

possesses a unique solution $\phi_u \in H_0^1(D)$. A list of properties of ϕ_u can be found e.g., in [2, Lemma 1.1]. We just observe here that, if $\mu_1(D)$ denotes the first eigenvalue of the Laplacian in the domain D,

$$\int_D |\nabla\phi_u|^2 \mathrm{d}x = \int_D \phi_u u^2 \mathrm{d}x \le |\phi_u|_{L^2(D)} |u^2|_{L^2(D)} \le \mu_1^{-1/2}(D) \|\phi_u\|_{H_0^1(D)} \|u\|^2_{H_0^1(D)}$$

and thus,

$$\int_D \phi_u u^2 \mathrm{d}x \le \mu_1^{-1}(D) \|u\|^4_{H_0^1(D)}. \tag{2.3}$$

We use the convention that given a function in $H_0^1(D)$ we will denote with the same letter the function trivially extended to the whole $\mathbb{R}^3$, which then belongs to $H^1(\mathbb{R}^3)$. In particular, for

$u \in H_0^1(D) \subset H^1(\mathbb{R}^3)$ if $\phi_u \in H_0^1(D)$ is the unique solution of (2.2), we also have $\phi_u \in H^1(\mathbb{R}^3)$ and

$$-\Delta\phi_u = u^2 \text{ in } D, \quad \phi_u = 0 \text{ in } \mathbb{R}^3 \setminus D.$$

However, it is important to establish explicitly the difference between ϕ_u and the unique solution φ_u of the problem

$$-\Delta\varphi = u^2 \text{ in } \mathbb{R}^3.$$

We will formulate the relationship between these two functions (see Fact 4 below).

We know that the solutions (u, ω) of the problem

$$\begin{cases} -\Delta u + \phi_u u - |u|^{p-2}u = \omega u & \text{in } D, \\ u > 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \\ \displaystyle\int_D u^2 \mathrm{d}x = \rho^2, \end{cases}$$

in a smooth bounded domain D can be characterized as critical points of the C^1 functional in $H_0^1(D)$

$$I(u; D) := \frac{1}{2}\int_D |\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_D \phi_u u^2 \mathrm{d}x - \frac{1}{p}\int_D |u|^p \mathrm{d}x$$

restricted to the manifold

$$\mathcal{M}_\rho(D) := \Big\{u \in H_0^1(D) : \int_D u^2 \mathrm{d}x = \rho^2\Big\}, \qquad \rho > 0.$$

Analogously, in case of the whole space, the solutions (u, ω) of the problem

$$\begin{cases} -\Delta u + \varphi_u u - |u|^{p-2}u = \omega u & \text{in } \mathbb{R}^3, \\ u > 0 & \text{in } \mathbb{R}^3, \\ \displaystyle\int_{\mathbb{R}^3} u^2 \mathrm{d}x = \rho^2 \end{cases}$$

can be characterized as critical points of the C^1 functional in $H^1(\mathbb{R}^3)$

$$I(u; \mathbb{R}^3) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \varphi_u u^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \mathrm{d}x$$

restricted to the manifold

$$\mathcal{M}_\rho(\mathbb{R}^3) := \Big\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} u^2 \mathrm{d}x = \rho^2 \Big\}, \qquad \rho > 0.$$

In both cases, the Lagrange multiplier associated to the critical point is the parameter ω which appears in the related problem.

We recall now the following facts that will be fundamental in the whole paper. We limit ourselves to the case $p \in (2,3)$ which is our interest here, even though some facts are also true for $p \in (2, 10/3)$.

Fact 1. For every $\rho > 0$, the functional $I(\cdot\,; D)$ is bounded from below and coercive on $\mathcal{M}_\rho(D)$. This is proved in [4, Lemma 3.1] in the case $D = \mathbb{R}^3$, but it is easy to see that it holds on every domain D (see, for instance, [20, Proposition 3.3]).

Fact 2. For every $\rho > 0$, it is

$$c_\infty := \inf_{u \in \mathcal{M}_\rho(\mathbb{R}^3)} I(u; \mathbb{R}^3) \in (-\infty, 0).$$

For a proof see [5, pp. 2498-2499]. The inequality $c_\infty < 0$ is strongly based on the fact that the domain is the whole $\mathbb{R}^3$, since suitable scalings are used that are not allowed in a bounded domain. In fact, in a bounded domain the infimum is strictly positive (at least for small ρ) as we will show in the Appendix.

In [4] it has been proved that there exists $\rho_1 = \rho_1(p) > 0$ such that for any $\rho \in (0, \rho_1)$, all the minimizing sequences for c_∞ are precompact in $H^1(\mathbb{R}^3)$ up to translations, and converging to a positive ground state $\mathfrak{w}_\infty$ in $H^1(\mathbb{R}^3)$, which is the one appearing in the statement of Theorem 1.1, so that

$$c_\infty = \min_{u \in \mathcal{M}_\rho(\mathbb{R}^3)} I(u; \mathbb{R}^3) = I(\mathfrak{w}_\infty; \mathbb{R}^3) \in (-\infty, 0).$$

More explicitly, if $\{u_n\} \subset \mathcal{M}_\rho(\mathbb{R}^3)$ is a minimizing sequence for c_∞ and vanishing occurs, then $u_n \rightharpoonup 0$ and $\|u_n\| \nrightarrow 0$ in $H^1(\mathbb{R}^3)$, since $c_\infty < 0$. Then the well known Lions' lemma [17] implies that

$$\sup_{y \in \mathbb{R}^3} \int_{B_1(y)} u_n^2 \mathrm{d}x \geq \delta > 0, \quad \text{for some } \delta > 0.$$

Hence there exist $\{y_n\} \subset \mathbb{R}^3$ and $\mathfrak{w}_\infty \in \mathcal{M}_\rho(\mathbb{R}^3)$ such that $v_n := u_n(\cdot + y_n) \in \mathcal{M}_\rho(\mathbb{R}^3)$ and

$$v_n \to \mathfrak{w}_\infty \quad \text{in } H^1(\mathbb{R}^3), \qquad I(\mathfrak{w}_\infty; \mathbb{R}^3) = c_\infty.$$

Furthermore, $|y_n| \to +\infty$. Indeed, since

$$\int_{B_1} u_n^2(x + y_n) \mathrm{d}x = \int_{B_1(y_n)} u_n^2 \mathrm{d}x \geq \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} u_n^2 \mathrm{d}x - o_n(1) \geq \frac{\delta}{2} > 0,$$

if the sequence $\{y_n\}$ were bounded, for a large $R > 0$,

$$\int_{B_R} u_n^2 \mathrm{d}x \geq \int_{B_1(y_n)} u_n^2 \mathrm{d}x \geq \frac{\delta}{2} > 0, \quad \text{for every } n \in \mathbb{N}$$

which is a contradiction since $u_n \to 0$ in $L^2(B_R)$.

Finally, as proved in [14, Theorem 0.1], the ground state $\mathfrak{w}_\infty$ is radially symmetric for $\rho \in (0, \rho_1)$, if necessary by reducing ρ_1.

Fact 3. There is a Lagrange multiplier ω_∞ associated to $\mathfrak{w}_\infty$, that satisfies

$$\rho^2 \omega_\infty = \int_{\mathbb{R}^3} |\nabla \mathfrak{w}_\infty|^2 \mathrm{d}x + \int_{\mathbb{R}^3} \varphi_\infty \mathfrak{w}_\infty^2 \, \mathrm{d}x - \int_{\mathbb{R}^3} \mathfrak{w}_\infty^p \mathrm{d}x, \tag{2.4}$$

where we have written for simplicity $\varphi_\infty := \varphi_{\mathfrak{w}_\infty}$. Let us recall the argument which shows that ω_∞ is negative. By [14] we know that, setting

$$\mathfrak{w}_\infty = \rho^{\frac{4}{4-3(p-2)}} \mathfrak{v}(\rho^{\frac{2(p-2)}{4-3(p-2)}} \cdot),$$

$\mathfrak{v}$ is a radial constrained (to the L^2-sphere in $H^1(\mathbb{R}^3)$) minimizer for the functional defined on $H^1(\mathbb{R}^3)$ by

$$J(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{\rho^{\alpha(p)}}{4} \int_{\mathbb{R}^3} \varphi_u u^2 \mathrm{d}x - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \mathrm{d}x$$

where

$$\alpha(p) := \frac{8(3-p)}{10-3p} > 0 \quad \text{if} \quad 2 < p < 3.$$

Then

$$J(\mathfrak{v}) = \min \left\{ J(u) : u \in H^1(\mathbb{R}^3) \text{ and } |u|_{L^2(\mathbb{R}^3)} = 1 \right\}$$

and $\mathfrak{v}$ is a solution of the problem

$$\begin{cases} -\Delta v + \rho^{\alpha(p)} \varphi_v v - |v|^{p-2} v = \omega v & \text{in } \mathbb{R}^3, \\ v > 0 & \text{in } \mathbb{R}^3, \\ \displaystyle\int_{\mathbb{R}^3} v^2 = 1, & \end{cases}$$

with ω as its Lagrange multiplier, which evidently satisfies

$$\omega = \int_{\mathbb{R}^3} |\nabla \mathfrak{v}|^2 \mathrm{d}x + \rho^{\alpha(p)} \int_{\mathbb{R}^3} \varphi_{\mathfrak{v}} \mathfrak{v}^2 \, \mathrm{d}x - \int_{\mathbb{R}^3} \mathfrak{v}^p \mathrm{d}x. \tag{2.5}$$

Possibly by reducing ρ_1, for $\rho \in (0, \rho_1)$ it is $\omega < 0$; see [14, Proposition 1.3]. Note that

$$\int_{\mathbb{R}^3} |\nabla \mathfrak{w}_\infty|^2 \mathrm{d}x = \rho^{\gamma(p)} \int_{\mathbb{R}^3} |\nabla \mathfrak{v}|^2 \mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}^3} \mathfrak{w}_\infty^p \mathrm{d}x = \rho^{\gamma(p)} \int_{\mathbb{R}^3} \mathfrak{v}^p \mathrm{d}x,$$

where

$$\gamma(p) := \frac{8-2(p-2)}{10-3p} > 0 \quad \text{if} \quad 2 < p < 3,$$

and

$$\int_{\mathbb{R}^3} \varphi_\infty \mathfrak{w}_\infty^2 \, \mathrm{d}x = \rho^{\alpha(p)+\gamma(p)} \int_{\mathbb{R}^3} \varphi_{\mathfrak{v}} \mathfrak{v}^2 \, \mathrm{d}x.$$

Thus, by using (2.4) and (2.5), we infer that

$$\begin{aligned} \rho^2 \omega_\infty &= \rho^{\gamma(p)} \Big[\int_{\mathbb{R}^3} |\nabla \mathfrak{v}|^2 \mathrm{d}x + \rho^{\alpha(p)} \int_{\mathbb{R}^3} \varphi_{\mathfrak{v}} \mathfrak{v}^2 \, \mathrm{d}x - \int_{\mathbb{R}^3} \mathfrak{v}^p \mathrm{d}x \Big] \\ &= \rho^{\gamma(p)} \omega. \end{aligned}$$

Therefore, for $\rho \in (0, \rho_1)$, it is also $\omega_\infty < 0$.

Fact 4. Let us consider a smooth bounded domain $D \subset \mathbb{R}^3$ and $u \in H_0^1(D)$. The unique solution ϕ_u to the problem

$$-\Delta \phi = u^2 \text{ in } D, \quad \phi = 0 \text{ on } \partial D$$

and the unique solution φ_u to the problem

$$-\Delta \varphi = u^2 \text{ in } \mathbb{R}^3$$

are such that

$$\int_D \phi_u u^2 \mathrm{d}x = \int_D \varphi_u u^2 \mathrm{d}x - \int_{D\times D} H(x, y; D) u^2(x) u^2(y) \mathrm{d}x \mathrm{d}y, \tag{2.6}$$

where $H(\cdot, \cdot; D)$ denotes the *smooth part* of the Green's function for $-\Delta$ in D (see [16]). This function has the following relevant properties useful for our purpose.

(1) The function $H(\cdot, \cdot; D)$ is nonnegative.
(2) We have

$$M_D := |H(\cdot, \cdot; D)|_{L^\infty(D\times D)} < +\infty. \tag{2.7}$$

(3) If $\lambda > 1$, then

$$H(x, y; \lambda D) = \frac{1}{\lambda} H\left(\frac{x}{\lambda}, \frac{y}{\lambda}; D\right), \quad \forall (x, y) \in (\lambda D) \times (\lambda \overline{D}).$$

(4) If $D_1 \subset D_2 \subset \mathbb{R}^3$ are smooth bounded domains, then

$$H(x, y; D_2) < H(x, y; D_1), \quad \forall\, (x, y) \in D_1 \times \overline{D}_1. \tag{2.8}$$

As two important consequences, we highlight the following:

i) Given a family $\{u_\lambda\}_{\lambda>1}$ of functions such that $u_\lambda \in \mathcal{M}_\rho(\lambda D)$, using the property (3) of the function H, we have the estimate

$$0 \leq \int_{\lambda D\times\lambda D} H(x, y; \lambda D) u_\lambda^2(x) u_\lambda^2(y)\mathrm{d}x\mathrm{d}y \leq \frac{M_D}{\lambda}\left(\int_{\lambda D} u_\lambda^2(x)\mathrm{d}x\right)^2 = \frac{M_D}{\lambda}\rho^4. \tag{2.9}$$

ii) If $D_1 \subset D_2 \subset \mathbb{R}^3$ are smooth bounded domains, using the trivial extension of functions, if $u \in H_0^1(D_1)$, then $u \in H_0^1(D_2)$, and

$$\int_{D_1} \phi_1 u^2 < \int_{D_2} \phi_2 u^2,$$

where ϕ_i for $i \in \{1, 2\}$ denotes the unique solution of the problem

$$-\Delta\phi = u^2 \text{ in } D_i, \quad \phi = 0 \text{ on } \partial D_i.$$

Therefore $I(u; D_1) < I(u; D_2)$.

3. The role of large λ

In this section we consider some interesting and fundamental results which are true specifically for expanding domains. All the results involve a limit in λ and the role of taking a large λ is then evident.

Let us start with few notations. For $u \in H^1(\mathbb{R}^3)$ with compact support we define the barycenter map by

$$\beta(u) := \frac{\displaystyle\int_{\mathbb{R}^3} x|\nabla u|^2\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\mathrm{d}x} \in \mathbb{R}^3.$$

It is a continuous map.
For $x \in \mathbb{R}^3$ and $0 < \widehat{r} < \widehat{R}$, $A_{\widehat{R},\widehat{r},x} := B_{\widehat{R}}(x) \setminus \overline{B_{\widehat{r}}(x)}$ is the annulus centred in x and radii $\widehat{r}$, $\widehat{R}$; for $\lambda > 1$ and $x \neq 0$, it holds that $\lambda A_{\widehat{R},\widehat{r},x} = A_{\lambda\widehat{R},\lambda\widehat{r},\lambda x}$, from which $\lambda A_{\widehat{R},\widehat{r},x} \neq A_{\lambda\widehat{R},\lambda\widehat{r},x}$. If $x = 0$, we simply write $A_{\widehat{R},\widehat{r}}$. For $\lambda > 1$ and $R \in (r, +\infty)$, where r is the fixed number in the beginning of Section 2, the set

$$\left\{u \in \mathcal{M}_\rho(A_{\lambda R,\lambda r,x}) :\ \beta(u) = x\right\}$$

is not empty and

$$\begin{aligned} a(R, r, \lambda, x) &:= \inf \Big\{ I(u; A_{\lambda R, \lambda r, x}) : u \in \mathcal{M}_\rho(A_{\lambda R, \lambda r, x}),\ \beta(u) = x \Big\} \\ &\geq \inf \Big\{ I(u; A_{\lambda R, \lambda r, x}) : u \in \mathcal{M}_\rho(A_{\lambda R, \lambda r, x}) \Big\} \\ &> -\infty, \end{aligned}$$

where the last inequality is due to **Fact 1** with $D = A_{\lambda R, \lambda r, x}$.

Note that the above infimum does not depend on the choice of the point x; indeed, for any $x \in \mathbb{R}^3$ and $0 < \widehat{r} < \widehat{R}$, we have (see Fact 4)

$$H(y, z; A_{R,r,x}) = H(y - x, z - x; A_{R,r}), \quad \forall\ (y, z) \in A_{R,r,x} \times \overline{A}_{R,r,x}$$

and then every term in the functional is invariant under translations. Let us set $a(R, r, \lambda) := a(R, r, \lambda, 0)$. We also use the notations

$$b_\lambda := \inf_{u \in \mathcal{M}_\rho(B_{\lambda r})} I(u; B_{\lambda r}) \qquad \text{and} \qquad c_\lambda := \inf_{u \in \mathcal{M}_\rho(\lambda\Omega)} I(u; \lambda\Omega).$$

Of course, the numbers $a(R, r, \lambda)$, b_λ and c_λ (besides c_∞) also depend on $\rho > 0$. However, we do not make explicit this dependence in the notation.

Now we use the fact that the families of infima with fixed L^2-norm are bounded from below. Let us choose $\lambda > 1$ and consider a smooth bounded domain $D \subset \mathbb{R}^3$; if $u \in \mathcal{M}_\rho(\lambda D)$ (so that $u \in \mathcal{M}_\rho(\mathbb{R}^3)$), then there exists $m < 0$ such that

$$m \leq \int_{\mathbb{R}^3} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p} |u|^p \right) \mathrm{d}x = \int_{\lambda D} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p} |u|^p \right) \mathrm{d}x < I(u; \lambda D);$$

see [4, Lemma 3.1]. Therefore, for the different kinds of domains in which we are interested, i.e., $D = B_r$, $D = \Omega$ and $D = A_{R,r}$, it is $m \leq b_\lambda$, $m \leq c_\lambda$ and $m \leq a(R, r, \lambda)$ for all $\lambda > 1$. As a consequence,

$$m \leq \liminf_{\lambda \to +\infty} b_\lambda, \quad m \leq \liminf_{\lambda \to +\infty} c_\lambda \quad \text{and} \quad m \leq \liminf_{\lambda \to +\infty} a(R, r, \lambda).$$

Remark 3.1. For a smooth bounded domain $D \subset \mathbb{R}^3$ and $u \in \mathcal{M}_\rho(D)$, we have by (2.6)

$$\begin{aligned} I(u; D) &= \int_D \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p} |u|^p \right) dx + \frac{1}{4} \int_D \phi_u u^2 dx \\ &= \int_D \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{4} \varphi_u u^2 - \frac{1}{p} |u|^p \right) dx - \frac{1}{4} \int_{D \times D} H(x, y; D) u^2(x) u^2(y) dx dy, \end{aligned}$$

from which

$$I(u; D) = I(u; \mathbb{R}^3) - \frac{1}{4} \int\limits_{D \times D} H(x, y; D) u^2(x) u^2(y) dx dy. \tag{3.1}$$

Using the previous remark, we obtain the following inequalities involving the infima:

$$c_\infty \leq \liminf_{\lambda \to +\infty} a(R, r, \lambda), \quad c_\infty \leq \liminf_{\lambda \to +\infty} b_\lambda \quad \text{and} \quad c_\infty \leq \liminf_{\lambda \to +\infty} c_\lambda. \tag{3.2}$$

Indeed, by definition there are families $\{u_\lambda\}_{\lambda>1}$, $\{v_\lambda\}_{\lambda>1}$ and $\{w_\lambda\}_{\lambda>1}$ such that, for every $\lambda > 1$, $u_\lambda \in \mathcal{M}_\rho(A_{\lambda R, \lambda r})$, $v_\lambda \in \mathcal{M}_\rho(B_{\lambda r})$, $w_\lambda \in \mathcal{M}_\rho(\lambda\Omega)$ (hence $u_\lambda, v_\lambda, w_\lambda \in \mathcal{M}_\rho(\mathbb{R}^3)$) and for which the inequalities

$$I(u_\lambda; A_{\lambda R, \lambda r}) < a(R, r, \lambda) + \frac{1}{\lambda}, \quad I(v_\lambda; B_{\lambda r}) < b_\lambda + \frac{1}{\lambda}, \quad I(w_\lambda; \lambda\Omega) < c_\lambda + \frac{1}{\lambda}$$

hold. But then, using (3.1) with $D = A_{\lambda R, \lambda r}$, $D = B_{\lambda r}$ and $D = \lambda\Omega$, respectively,

$$c_\infty \leq I(u_\lambda; \mathbb{R}^3) < a(R, r, \lambda) + \frac{1}{\lambda} + \frac{1}{4} \int\limits_{A_{\lambda R, \lambda r} \times A_{\lambda R, \lambda r}} H(x, y; A_{\lambda R, \lambda r}) u_\lambda^2(x) u_\lambda^2(y) dx dy,$$

$$c_\infty \leq I(v_\lambda; \mathbb{R}^3) < b_\lambda + \frac{1}{\lambda} + \frac{1}{4} \int\limits_{B_{\lambda r} \times B_{\lambda r}} H(x, y; B_{\lambda r}) v_\lambda^2(x) v_\lambda^2(y) dx dy,$$

$$c_\infty \leq I(w_\lambda; \mathbb{R}^3) < c_\lambda + \frac{1}{\lambda} + \frac{1}{4} \int\limits_{\lambda\Omega \times \lambda\Omega} H(x, y; \lambda\Omega) w_\lambda^2(x) w_\lambda^2(y) dx dy,$$

and the conclusion easily follows by using (2.9).

The next two propositions will serve as a preparation for the main result of this section, Proposition 3.4. Recall that $\rho_1 > 0$ was previously introduced in Section 2.

Proposition 3.2. *Let $2 < p < 3$. For any $\rho \in (0, \rho_1)$ it is*

$$\liminf_{\lambda \to +\infty} a(R, r, \lambda) > c_\infty.$$

Proof. Assume by contradiction that, along a subsequence $\lambda_n \to +\infty$ it holds

$$\lim_{\lambda \to +\infty} a(R, r, \lambda_n) = c_\infty.$$

Besides, consider a sequence $\{u_n\}$ with $u_n \in \mathcal{M}_\rho(A_{\lambda_n R, \lambda_n r})$, such that $\beta(u_n) = 0$ and it also satisfies

$$a(R, r, \lambda_n) \leq I(u_n; A_{\lambda_n R, \lambda_n r}) < a(R, r, \lambda_n) + o_n(1).$$

Hence, using (3.1) with $D = A_{\lambda_n R, \lambda_n r}$, we have

$$a(R, r, \lambda_n) + o_n(1) \leq I(u_n; \mathbb{R}^3) < a(R, r, \lambda_n) + o_n(1)$$

so that, by our assumption,

$$I(u_n;\mathbb{R}^3)\to c_\infty.$$

Thus, we may suppose that

$$\{u_n\}\subset\mathcal{M}_\rho(\mathbb{R}^3),\quad \beta(u_n)=0 \text{ and } I(u_n;\mathbb{R}^3)\to c_\infty. \tag{3.3}$$

Due to the coercivity of $I(\cdot,\mathbb{R}^3)$, the sequence $\{u_n\}$ has to be bounded. Moreover, $\|u_n\|\nrightarrow 0$; otherwise, by using (2.3),

$$\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x,\ \int_{\mathbb{R}^3}\varphi_{u_n}u_n^2\mathrm{d}x,\ \int_{\mathbb{R}^3}|u_n|^p\mathrm{d}x\to 0$$

and by (3.3)

$$c_\infty+o_n(1)=I(u_n;\mathbb{R}^3)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}^3}\varphi_{u_n}u_n^2\mathrm{d}x-\frac{1}{p}\int_{\mathbb{R}^3}|u_n|^p\mathrm{d}x=o_n(1),$$

we obtain $c_\infty=0$, in contradiction with $c_\infty<0$. Hence, as in Fact 2 of Section 2, there exists a sequence $\{y_n\}\subset\mathbb{R}^3$ with $|y_n|\to+\infty$ such that $v_n:=u_n(\cdot+y_n)\in\mathcal{M}_\rho(\mathbb{R}^3)$ and

$$v_n\to\mathfrak{w}_\infty\quad\text{in } H^1(\mathbb{R}^3),\qquad I(\mathfrak{w}_\infty;\mathbb{R}^3)=c_\infty. \tag{3.4}$$

The convergence in (3.4) can equivalently be written as $w_n:=u_n(\cdot+y_n)-\mathfrak{w}_\infty\to 0$; i.e.,

$$u_n(x)=w_n(x-y_n)+\mathfrak{w}_\infty(x-y_n)\quad\text{with } w_n\to 0 \text{ in } H^1(\mathbb{R}^3).$$

Since $I(\cdot;A_{\lambda_nR,\lambda_nr})$ is rotationally invariant, we can assume that

$$y_n=(y_n^1,0,0)\quad\text{with}\quad y_n^1<0.$$

Claim 1: $\displaystyle\int_{B_{\lambda_nr/2}(y_n)}|\nabla u_n|^2\mathrm{d}x\to\int_{\mathbb{R}^3}|\nabla\mathfrak{w}_\infty|^2\mathrm{d}x.$

In fact, denoting with

$$\sigma_{1,n}:=\int_{B_{\lambda_nr/2}(y_n)}|\nabla u_n|^2\mathrm{d}x=\int_{B_{\lambda_nr/2}(y_n)}|\nabla(w_n(x-y_n)+\mathfrak{w}_\infty(x-y_n))|^2\mathrm{d}x\,,$$

$$\sigma_{2,n}:=\int_{B_{\lambda_nr/2}(y_n)}2|\nabla w_n(x-y_n)||\nabla\mathfrak{w}_\infty(x-y_n)|\mathrm{d}x\,,$$

$$\sigma_{3,n}:=\int_{B_{\lambda_nr/2}(y_n)}|\nabla w_n(x-y_n)|^2\mathrm{d}x\,,$$

$$\sigma_{4,n} := \int_{B_{\lambda_n r/2}(y_n)} |\nabla \mathfrak{w}_\infty(x-y_n)|^2 \mathrm{d}x\,,$$

we have

$$\sigma_{3,n} - \sigma_{2,n} + \sigma_{4,n} \le \sigma_{1,n} \le \sigma_{3,n} + \sigma_{2,n} + \sigma_{4,n}. \tag{3.5}$$

With the change of variable $z := x - y_n$ we easily deduce that

$$\sigma_{2,n} = \int_{B_{\lambda_n r/2}} 2|\nabla w_n||\nabla \mathfrak{w}_\infty|\mathrm{d}z \le 2\|w_n\|_{H^1(\mathbb{R}^3)}\Big(\int_{\mathbb{R}^3} |\nabla \mathfrak{w}_\infty|^2\mathrm{d}x\Big)^{1/2} \to 0,$$

$$\sigma_{3,n} = \int_{B_{\lambda_n r/2}} |\nabla w_n|^2\mathrm{d}z \le \|w_n\|^2_{H^1(\mathbb{R}^3)} \to 0,$$

$$\sigma_{4,n} = \int_{B_{\lambda_n r/2}} |\nabla \mathfrak{w}_\infty|^2\mathrm{d}z \to \int_{\mathbb{R}^3} |\nabla \mathfrak{w}_\infty|^2\mathrm{d}x.$$

Thus, by (3.5), we get the claim. On the other hand, setting $\Theta_n := A_{\lambda_n R,\lambda_n r} \cap B_{\lambda_n r/2}(y_n)$, we have

$$\int_{B_{\lambda_n r/2}(y_n)} |\nabla u_n|^2\mathrm{d}x = \int_{\Theta_n} |\nabla u_n|^2\mathrm{d}x + \int_{B_{\lambda_n r/2}(y_n)\setminus\Theta_n} |\nabla u_n|^2\mathrm{d}x.$$

But $B_{\lambda_n r/2}(y_n) \setminus \Theta_n = B_{\lambda_n r/2}(y_n) \setminus A_{\lambda_n R,\lambda_n r}$ and, being $\operatorname{supp} u_n \subset A_{\lambda_n R,\lambda_n r}$, from Claim 1

$$\int_{B_{\lambda_n r/2}(y_n)\setminus\Theta_n} |\nabla u_n|^2\mathrm{d}x = 0$$

holds. As a consequence, $\int_{\Theta_n} |\nabla u_n|^2\mathrm{d}x \to \int_{\mathbb{R}^3} |\nabla \mathfrak{w}_\infty|^2\mathrm{d}x$. In virtue of this we have

Claim 2: $\int_{\Upsilon_n} |\nabla u_n|^2\mathrm{d}x \to 0$, where $\Upsilon_n := A_{\lambda_n R,\lambda_n r} \setminus B_{\lambda_n r/2}(y_n)$.

Indeed, the inequality

$$\int_{A_{\lambda_n R,\lambda_n r}} |\nabla u_n|^2\mathrm{d}x \le \int_{A_{\lambda_n R,\lambda_n r}} |\nabla w_n(x-y_n)|^2\mathrm{d}x + \int_{A_{\lambda_n R,\lambda_n r}} 2|\nabla w_n(x-y_n)||\nabla \mathfrak{w}_\infty(x-y_n)|\mathrm{d}x$$
$$+ \int_{A_{\lambda_n R,\lambda_n r}} |\nabla \mathfrak{w}_\infty(x-y_n)|^2\mathrm{d}x$$

can be rewritten as

$$\int_{A_{\lambda_n R,\lambda_n r}} |\nabla u_n|^2 \mathrm{d}x \leq \int_{\mathbb{R}^3} |\nabla w_n(x-y_n)|^2 \mathrm{d}x + \int_{\mathbb{R}^3} 2|\nabla w_n(x-y_n)||\nabla \mathfrak{w}_\infty(x-y_n)| \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla \mathfrak{w}_\infty(x-y_n)|^2 \mathrm{d}x.$$

Hence, due to the invariance under translations of the integrals,

$$\int_{A_{\lambda_n R,\lambda_n r}} |\nabla u_n|^2 \mathrm{d}x \leq \|w_n\|^2_{H^1(\mathbb{R}^3)} + 2\|w_n\|_{H^1(\mathbb{R}^3)} \Big(\int_{\mathbb{R}^3} |\nabla \mathfrak{w}_\infty|^2 \mathrm{d}x\Big)^{1/2} + \int_{\mathbb{R}^3} |\nabla \mathfrak{w}_\infty|^2 \mathrm{d}x$$
$$= o_n(1) + \int_{\mathbb{R}^3} |\nabla \mathfrak{w}_\infty|^2 \mathrm{d}x,$$

from which

$$\limsup_{n\to\infty} \int_{A_{\lambda_n R,\lambda_n r}} |\nabla u_n|^2 \mathrm{d}x \leq \int_{\mathbb{R}^3} |\nabla \mathfrak{w}_\infty|^2 \mathrm{d}x.$$

On the other hand,

$$\int_{A_{\lambda_n R,\lambda_n r}} |\nabla u_n|^2 \mathrm{d}x = \int_{\Upsilon_n} |\nabla u_n|^2 \mathrm{d}x + \int_{\Theta_n} |\nabla u_n|^2 \mathrm{d}x \geq \int_{\Theta_n} |\nabla u_n|^2 \mathrm{d}x,$$

so that

$$\liminf_{n\to\infty} \int_{A_{\lambda_n R,\lambda_n r}} |\nabla u_n|^2 \mathrm{d}x \geq \int_{\mathbb{R}^3} |\nabla \mathfrak{w}_\infty|^2 \mathrm{d}x.$$

As a consequence,

$$\lim_{n\to\infty} \int_{A_{\lambda_n R,\lambda_n r}} |\nabla u_n|^2 \mathrm{d}x = \int_{\mathbb{R}^3} |\nabla \mathfrak{w}_\infty|^2 \mathrm{d}x$$

and from

$$\int_{\Upsilon_n} |\nabla u_n|^2 \mathrm{d}x = \int_{A_{\lambda_n R,\lambda_n r}} |\nabla u_n|^2 \mathrm{d}x - \int_{\Theta_n} |\nabla u_n|^2 \mathrm{d}x$$

we have

$$\int_{\Upsilon_n} |\nabla u_n|^2 \mathrm{d}x \to 0,$$

proving the claim.

Finally, since

$$0=\beta(u_n)=\int_{A_{\lambda_n R,\lambda_n r}} x^1|\nabla u_n|^2 dx=\int_{\Theta_n} x^1|\nabla u_n|^2 dx+\int_{\Upsilon_n} x^1|\nabla u_n|^2 dx$$
$$<-\frac{\lambda_n r}{2}\left(\int_{\mathbb{R}^3}|\nabla \mathfrak{w}_\infty|^2 dx+o_n(1)\right)+\lambda_n R\int_{\Upsilon_n}|\nabla u_n|^2 dx$$

we deduce that

$$\frac{r}{2R}\int_{\mathbb{R}^3}|\nabla \mathfrak{w}_\infty|^2 dx+o_n(1)<\int_{\Upsilon_n}|\nabla u_n|^2 dx,$$

in contradiction with Claim 2. The proof is thereby completed. □

To prove Theorem 1.1, we will need to deal with radial functions. Since it is not clear if b_λ is achieved on a radial minimizer (see e.g., [22, Theorem 1.7]) we introduce the radial setting. For any $\rho>0$, $\lambda>1$, let

$$\mathcal{M}^*_\rho(B_{\lambda r}):=\Big\{u\in\mathcal{M}_\rho(B_{\lambda r}): u \text{ is radial}\Big\}$$
$$=\Big\{u\in H^1_0(B_{\lambda r}): |u|_{L^2(B_{\lambda r})}=\rho,\ u \text{ is radial}\Big\}$$

and

$$b^*_\lambda:=\inf_{u\in\mathcal{M}^*_\rho(B_{\lambda r})} I(u;B_{\lambda r}). \tag{3.6}$$

By using the arguments of [8,21], it is easy to see that b^*_λ is achieved on a function that we denoted by $\mathfrak{w}^*_{B_{\lambda r}}$.

Proposition 3.3. *Let* $2<p<3$. *For any* $\rho\in(0,\rho_1)$, *it is*

$$\lim_{\lambda\to\infty} b_\lambda=\lim_{\lambda\to\infty} b^*_\lambda=\lim_{\lambda\to\infty} c_\lambda=c_\infty.$$

Proof. Let $\mathfrak{w}_\infty$ be a positive radial ground state solution of the problem in the whole space $\mathbb{R}^3$, and $h:[0,+\infty)\to[0,1]$ a decreasing, C^∞ function such that

$$h(t):=\begin{cases}1, & t\le 1,\\ 0, & t\ge 2.\end{cases}$$

For $T\ge r/2$ consider the function $h_T\in C^\infty_0(\mathbb{R}^3)$ given by $h_T(x):=h(|x|/T)$, and define $w_T:=\mathfrak{w}_\infty h_T$. Note that $w_T\to\mathfrak{w}_\infty$ in $H^1(\mathbb{R}^3)$, when $T\to+\infty$. Also let $t_T>0$ be such that $t_T w_T\in\mathcal{M}_\rho(B_{2T})$.

After fixing $T \geq r/2$, the number $\widehat{\lambda} = \widehat{\lambda}(T) := 2T/r \geq 1$ is such that for every $\lambda \geq \widehat{\lambda}$, $B_{2T} \subset B_{\lambda r} \subset \lambda\Omega$. Then for every $\lambda \geq \widehat{\lambda}$, denoting $\phi_T := \phi_{t_T w_T}$, by using (3.1) with $D = B_{\lambda r}$ and $D = \lambda\Omega$, we have

$$b_\lambda \leq b_\lambda^* \leq I(t_T w_T; B_{\lambda r}) = I(t_T w_T; \mathbb{R}^3) - \frac{t_T^4}{4\lambda} \int_{B_{\lambda r} \times B_{\lambda r}} H\left(\frac{x}{\lambda}, \frac{y}{\lambda}; B_r\right) w_T^2(x) w_T^2(y) \mathrm{d}x\mathrm{d}y$$

and

$$c_\lambda \leq I(t_T w_T; \lambda\Omega) = I(t_T w_T; \mathbb{R}^3) - \frac{t_T^4}{4\lambda} \int_{\lambda\Omega \times \lambda\Omega} H\left(\frac{x}{\lambda}, \frac{y}{\lambda}; \Omega\right) w_T^2(x) w_T^2(y) \mathrm{d}x\mathrm{d}y.$$

Therefore, using (2.9) with $D = B_r$ and $D = \Omega$ respectively,

$$\limsup_{\lambda \to +\infty} b_\lambda \leq \limsup_{\lambda \to +\infty} b_\lambda^* \leq I(t_T w_T; \mathbb{R}^3) \text{ and } \limsup_{\lambda \to +\infty} c_\lambda \leq I(t_T w_T; \mathbb{R}^3), \quad \forall T \geq r/2. \tag{3.7}$$

Since $w_T \to \mathfrak{w}_\infty$ in $L^2(\mathbb{R}^3)$, by the definition of t_T we have

$$t_T^2 = \frac{\rho^2}{\int_{\mathbb{R}^3} w_T^2 \, \mathrm{d}x} \to 1.$$

It follows that $t_T \to 1$. In particular, $I(t_T w_T; \mathbb{R}^3) \to I(\mathfrak{w}_\infty; \mathbb{R}^3) = c_\infty$ as $T \to +\infty$, and then, by (3.7), we infer that

$$\limsup_{\lambda \to +\infty} b_\lambda \leq \limsup_{\lambda \to +\infty} b_\lambda^* \leq c_\infty \quad \text{and} \quad \limsup_{\lambda \to +\infty} c_\lambda \leq c_\infty.$$

We can conclude by (3.2). □

With the above propositions in hand, we can provide a proof of the main result in this section. Given a number $l \in \mathbb{R}$, we define the sublevel set

$$[I(\cdot\,; \lambda\Omega)]^l := \left\{u \in \mathcal{M}_\rho(\lambda\Omega) : I(u; \lambda\Omega) \leq l\right\}.$$

For the next proposition, note that $M_{B_r} > M_\Omega$ since $B_r \subset \Omega$, in virtue of (2.7) and (2.8).

Proposition 3.4. *Let $2 < p < 3$. For any $\rho \in (0, \rho_1)$, there exists $\Lambda = \Lambda(\rho) > 1$ such that, for every $\lambda \geq \Lambda$ it holds that*

$$u \in [I(\cdot\,; \lambda\Omega)]^{l(\lambda)} \implies \beta(u) \in \lambda\Omega_r^+,$$

where

$$l(\lambda) := b_\lambda^* + \frac{1}{\lambda}\left(1 + \frac{M_{B_r}\rho^4}{4}\right).$$

Proof. For any $\lambda > 1$ there is a function $u_\lambda \in \mathcal{M}_\rho(B_{\lambda r})$ for which

$$I(u_\lambda; B_{\lambda r}) < b_\lambda + \frac{1}{\lambda}.$$

Then, using the fact that the smooth part of the Green's function on $\lambda\Omega$ is positive and (3.1) with $D = B_{\lambda r}$, we get

$$\begin{aligned} I(u_\lambda; \mathbb{R}^3) - \frac{1}{4}\int\limits_{\lambda\Omega\times\lambda\Omega} H(x, y; \lambda\Omega)u_\lambda^2(x)u_\lambda^2(y)\mathrm{d}x\mathrm{d}y &< I(u_\lambda; \mathbb{R}^3) \\ &< b_\lambda + \frac{1}{\lambda} + \frac{1}{4}\int\limits_{B_{\lambda r}\times B_{\lambda r}} H(x, y; B_{\lambda r})u_\lambda^2(x)u_\lambda^2(y)\mathrm{d}x\mathrm{d}y, \end{aligned}$$

from which, using (3.1) with $D = \lambda\Omega$ and the estimate (2.9) with $D = B_r$, the inequalities

$$c_\lambda \le I(u_\lambda; \lambda\Omega) < b_\lambda + \frac{1}{\lambda} + \frac{M_{B_r}\rho^4}{4\lambda} \le l(\lambda), \tag{3.8}$$

hold. Hence, the sublevel is not empty.

We argue now by contradiction. Assume that there exists a sequence of numbers $\{\lambda_n\}$ with $\lambda_n \to +\infty$ and $u_n \in \mathcal{M}_\rho(\Omega_n)$ is such that $I(u_n; \Omega_n) \le l(\lambda_n)$ but $x_n := \beta(u_n) \notin \lambda_n\Omega_r^+$. From here and until the end of the proof, let

$$\Omega_n := \lambda_n\Omega, \quad R > \operatorname{diam}\Omega, \quad A_n := A_{\lambda_n R, \lambda_n r, x_n}, \quad c_n := c_{\lambda_n}, \quad b_n^* := b_{\lambda_n}^*.$$

Claim: The inclusion $\Omega_n \subset A_n$ holds.

Of course, from the chain of implications

$$\begin{aligned} x_n \notin \lambda_n\Omega_r^+ &\implies d(x_n/\lambda_n, \Omega) > r \implies d(x_n, \Omega_n) > \lambda_n r \implies B_{\lambda_n r}(x_n) \cap \Omega_n = \emptyset \\ &\implies \overline{B_{\lambda_n r}(x_n)} \cap \Omega_n = \emptyset \end{aligned}$$

we obtain that

$$\Omega_n \subset \mathbb{R}^3 \setminus \overline{B_{\lambda_n r}(x_n)}. \tag{3.9}$$

Now let $y_n \in \Omega_n$ be an arbitrary point. Note that

$$z_n \in \Omega_n \implies |z_n - y_n| < \operatorname{diam}\Omega_n = \lambda_n \operatorname{diam}\Omega < \lambda_n R \implies z_n \in B_{\lambda_n R}(y_n).$$

Since supp $u_n \subset \Omega_n \subset B_{\lambda_n R}(y_n)$, this implies that $x_n = \beta(u_n) \in B_{\lambda_n R}(y_n)$. Therefore, $y_n \in B_{\lambda_n R}(x_n)$ and, by the arbitrariness of y_n, it follows that

$$\Omega_n \subset B_{\lambda_n R}(x_n). \tag{3.10}$$

From (3.9) and (3.10) we get the Claim.

We assert now that

$$a(R, r, \lambda_n, x_n) < c_n + \frac{1}{\lambda_n} + \frac{M_{B_r}\rho^4}{4\lambda_n} < b_n^* + \frac{2}{\lambda_n} + \frac{M_{B_r}\rho^4}{2\lambda_n}. \tag{3.11}$$

Indeed, we can take a sequence $\{v_n\}$ with $v_n \in \mathcal{M}_\rho(\Omega_n)$ such that

$$I(v_n; \Omega_n) < c_n + \frac{1}{\lambda_n}.$$

Then by the Claim, the fact that the smooth part of the Green's function on A_n is positive, and (3.1) with $D = \Omega_n$, we deduce that

$$I(v_n; \mathbb{R}^3) - \frac{1}{4}\int\limits_{A_n \times A_n} H(x, y; A_n) v_n^2(x) v_n^2(y) \mathrm{d}x\mathrm{d}y < I(v_n; \mathbb{R}^3)$$
$$< c_n + \frac{1}{\lambda_n} + \frac{1}{4}\int\limits_{\Omega_n \times \Omega_n} H(x, y; \Omega_n) v_n^2(x) v_n^2(y) \mathrm{d}x\mathrm{d}y.$$

From these inequalities, using (3.1) with $D = A_n$, the estimate (2.9) with $D = \Omega$, and (2.8) with $D_1 = B_r$, $D_2 = \Omega$, we get

$$a(R, r, \lambda_n, x_n) \leq I(v_n; A_n) < c_n + \frac{1}{\lambda_n} + \frac{M_{B_r}\rho^4}{4\lambda_n}.$$

Therefore, in virtue of (3.8) we infer (3.11). But then, since $a(R, r, \lambda_n, x_n) = a(R, r, \lambda_n)$, we can write

$$a(R, r, \lambda_n) < b_n^* + \frac{2}{\lambda_n} + \frac{M_{B_r}\rho^4}{2\lambda_n}$$

which, together with Proposition 3.3, implies that

$$\liminf_{n\to\infty} a(R, r, \lambda_n) \leq \liminf_{n\to\infty} b_n^* = c_\infty.$$

This contradicts Proposition 3.2, so that the proof is ended. □

We recall that $\mathfrak{w}^*_{B_{\lambda r}} \in \mathcal{M}^*_\rho(B_{\lambda r})$ denotes a positive, radial ground state; i.e., $\mathfrak{w}^*_{B_{\lambda r}}$ is such that

$$I(\mathfrak{w}^*_{B_{\lambda r}}; B_{\lambda r}) = b^*_\lambda;$$

see (3.6). We now consider the continuous map $\Psi_{\lambda,r} : \lambda\Omega_r^- \to H^1_0(\lambda\Omega)$, given by

$$[\Psi_{\lambda,r}(y)](x) := \begin{cases} \mathfrak{w}^*_{B_{\lambda r}}(|x-y|), & \text{if } x \in B_{\lambda r}(y), \\ 0, & \text{if } x \in \lambda\Omega \setminus B_{\lambda r}(y), \end{cases} \qquad \text{for every } y \in \lambda\Omega_r^-.$$

Let us fix $y \in \lambda\Omega_r^-$; keeping in mind the terms of the functional, we make explicit the relation between integrals involving $\Psi_{\lambda,r}(y)$ and the corresponding integrals involving $\mathfrak{w}^*_{B_{\lambda r}}$. For the gradient terms we have

$$\int_{\lambda\Omega} \left|\nabla[\Psi_{\lambda,r}(y)](x)\right|^2 \mathrm{d}x = \int_{B_{\lambda r}(y)} \left|\nabla \mathfrak{w}^*_{B_{\lambda r}}(|x-y|)\right|^2 \mathrm{d}x = \int_{B_{\lambda r}} |\nabla \mathfrak{w}^*_{B_{\lambda r}}(\xi)|^2 \mathrm{d}\xi.$$

The nonlinear terms are such that

$$\int_{\lambda\Omega} \left|[\Psi_{\lambda,r}(y)](x)\right|^p \mathrm{d}x = \int_{B_{\lambda r}(y)} \left|\mathfrak{w}^*_{B_{\lambda r}}(|x-y|)\right|^p \mathrm{d}x = \int_{B_{\lambda r}} |\mathfrak{w}^*_{B_{\lambda r}}(\xi)|^p \mathrm{d}\xi.$$

With respect to the nonlocal nonlinearities, first note that

$$\int_{\lambda\Omega\times\lambda\Omega} \frac{[\Psi_{\lambda,r}(y)]^2(x)[\Psi_{\lambda,r}(y)]^2(z)}{|x-z|} \mathrm{d}x\mathrm{d}z =$$
$$\int_{B_{\lambda r}(y)\times B_{\lambda r}(y)} \frac{[\mathfrak{w}^*_{B_{\lambda r}}(|x-y|)]^2[\mathfrak{w}^*_{B_{\lambda r}}(|z-y|)]^2}{|x-z|} \mathrm{d}x\mathrm{d}z = \int_{B_{\lambda r}\times B_{\lambda r}} \frac{[\mathfrak{w}^*_{B_{\lambda r}}(\xi)]^2[\mathfrak{w}^*_{B_{\lambda r}}(\zeta)]^2}{|\xi-\zeta|} \mathrm{d}\xi\mathrm{d}\zeta.$$

Then, since

$$(x,z) = (\xi+y, \zeta+y) \in B_{\lambda r}(y) \times B_{\lambda r}(y) \quad \Longleftrightarrow \quad (\xi,\zeta) \in B_{\lambda r} \times B_{\lambda r},$$

it follows that the terms involving the smooth parts of the Green functions satisfy

$$\begin{aligned} \sigma_{\lambda,r}(y) &:= \int_{\lambda\Omega\times\lambda\Omega} H(x,z;\lambda\Omega)[\Psi_{\lambda,r}(y)]^2(x)[\Psi_{\lambda,r}(y)]^2(z)\mathrm{d}x\mathrm{d}z \\ &= \int_{B_{\lambda r}\times B_{\lambda r}} H(\xi+y,\zeta+y;\lambda\Omega)[\mathfrak{w}^*_{B_{\lambda r}}(\xi)]^2[\mathfrak{w}^*_{B_{\lambda r}}(\zeta)]^2 \mathrm{d}\xi\mathrm{d}\zeta. \end{aligned}$$

Summing up, we can write

$$I(\Psi_{\lambda,r}(y);\lambda\Omega) = I(\mathfrak{w}^*_{B_{\lambda r}};\mathbb{R}^3) - \frac{\sigma_{\lambda,r}(y)}{4}.$$

But then, by using (3.1) (first with $D = \lambda\Omega$, $u = \Psi_{\lambda,r}(y)$, then with $D = B_{\lambda r}$, $u = \mathfrak{w}^*_{B_{\lambda r}}$), (2.9) with $D = B_r$, we find that

$$\begin{aligned}
I(\Psi_{\lambda,r}(y);\lambda\Omega) &< I(\mathfrak{w}^*_{B_{\lambda r}};\mathbb{R}^3) \\
&= I(\mathfrak{w}^*_{B_{\lambda r}};B_{\lambda r}) + \frac{1}{4}\int_{B_{\lambda r}\times B_{\lambda r}} H(x,z;B_{\lambda r})[\mathfrak{w}^*_{B_{\lambda r}}(x)]^2[\mathfrak{w}^*_{B_{\lambda r}}(z)]^2 \mathrm{d}x\mathrm{d}z \\
&= b^*_\lambda + \frac{1}{4}\int_{B_{\lambda r}\times B_{\lambda r}} H(x,z;B_{\lambda r})[\mathfrak{w}^*_{B_{\lambda r}}(x)]^2[\mathfrak{w}^*_{B_{\lambda r}}(z)]^2 \mathrm{d}x\mathrm{d}z \\
&< b^*_\lambda + \frac{M_{B_r}\rho^4}{4\lambda} \\
&< l(\lambda).
\end{aligned}$$

Taking $\lambda \geq \Lambda$, by Proposition 3.4, it follows that $\beta(\Psi_{\lambda,r}(y)) = y$.

Therefore, for $\lambda \geq \Lambda$, we have the following diagram of continuous maps

$$\lambda\Omega^-_r \xrightarrow{\Psi_{\lambda,r}} [I(\cdot\,;\lambda\Omega)]^{l(\lambda)} \xrightarrow{\beta} \lambda\Omega^+_r \simeq \lambda\Omega^-_r$$

and the composition is homotopic to the identity map of $\lambda\Omega^-_r$.

With the above ingredients in hands the next result is standard, but we present the proof for the reader's convenience.

Proposition 3.5. *Let $2 < p < 3$ and $\rho \in (0,\rho_1)$. Then, for any $\lambda \geq \Lambda$ (the one given in Proposition 3.4) it holds*

$$\operatorname{cat}[I(\cdot\,;\lambda\Omega)]^{l(\lambda)} \geq \operatorname{cat}\lambda\Omega = \operatorname{cat}\Omega.$$

Proof. Assume that $\operatorname{cat}[I(\cdot\,;\lambda\Omega)]^{l(\lambda)} = n$; thus

$$[I(\cdot\,;\lambda\Omega)]^{l(\lambda)} = F_1 \cup \ldots \cup F_n,$$

where each F_i is closed and contractible in $[I(\cdot\,;\lambda\Omega)]^{l(\lambda)}$. Hence, for each $i \in \{1,\ldots,n\}$, there exists

$$h_i \in C\Big([0,1]\times F_i, [I(\cdot\,;\lambda\Omega)]^{l(\lambda)}\Big)$$

with

$$h_i(0,u) = u \quad \text{and} \quad h_i(1,u) = w_i \in [I(\cdot\,;\lambda\Omega)]^{l(\lambda)}, \quad \forall u \in F_i\,.$$

Note that the sets $K_i := \Psi^{-1}_{\lambda,r}(F_i)$ are closed and satisfy

$$\lambda\Omega_r^- = K_1 \cup \ldots \cup K_n.$$

Now we claim that each K_i is contractible in $\lambda\Omega_r^+$. Indeed, for any $i \in \{1, \ldots, n\}$ fixed, the map

$$g_i : [0,1] \times K_i \to \lambda\Omega_r^+ \quad \text{defined by} \quad g_i(t, y) := \beta\big(h_i\left(t, \Psi_{\lambda,r}(y)\right)\big)$$

is continuous and such that, for all $y \in K_i$,

$$\begin{aligned} g_i(0, y) &= \beta\left(\Psi_{\lambda,r}(y)\right) = y, \\ g_i(1, y) &= \beta(w_i) \in \lambda\Omega_r^+ \qquad \text{(by Proposition 3.4).} \end{aligned}$$

Then, by (2.1),

$$\operatorname{cat}\Omega = \operatorname{cat}_{\lambda\Omega_r^+}(\lambda\Omega_r^-) \le n = \operatorname{cat}[I(\cdot\,;\lambda\Omega)]^{l(\lambda)}.$$

This concludes the proof. □

4. Proof of Theorem 1.1

Let us recall here a compactness condition useful to implement variational methods. In general, if H is a Hilbert space, $\mathcal{M} \subset H$ a submanifold and $I : H \to \mathbb{R}$ a C^1 functional, we say that I satisfies the Palais-Smale condition on $\mathcal{M}$ if any sequence $\{u_n\} \subset \mathcal{M}$ such that

$$\{I(u_n)\} \text{ is convergent and } (I_{|\mathcal{M}})'(u_n) \to 0$$

possesses a subsequence converging to some $u \in \mathcal{M}$. We will also say that I constrained to $\mathcal{M}$ satisfies the (PS) condition.

It is known that our functional $I(\cdot\,;\lambda\Omega)$ constrained to $\mathcal{M}_\rho(\lambda\Omega)$ satisfies the (PS) condition (see [18, Appendix]); hence by the Ljusternick-Schnirelmann theory and Proposition 3.5, for a fixed $\rho \in (0, \rho_1)$ and for every $\lambda \ge \Lambda = \Lambda(\rho)$, the functional $I(\cdot\,;\lambda\Omega)$ constrained to $\mathcal{M}_\rho(\lambda\Omega)$ has at least $N := \operatorname{cat}(\lambda\Omega) = \operatorname{cat}\Omega$ distinct critical points $\{u^i_{\rho,\lambda}\}_{i=1,\ldots,N}$ with energy satisfying

$$c_\lambda \le I(u^i_{\rho,\lambda};\lambda\Omega) \le b^*_\lambda + \frac{1}{\lambda}\left(1 + \frac{M_{B_r}\rho^4}{4}\right), \quad i = 1, \ldots, N, \tag{4.1}$$

and the right hand side can be made less than $c_\infty/2$ by Proposition 3.3, up to taking a greater value of Λ. This means that $\sup_{\lambda\ge\Lambda} I(u^i_{\rho,\lambda};\lambda\Omega) < 0$. The Lagrange multipliers $\{\omega^i_{\rho,\lambda}\}_{i=1,\ldots,N} \subset \mathbb{R}$ are associated to the critical points.

Moreover by (4.1), Proposition 3.3 and (2.9) (with $D = \Omega$) it is, for each $i = 1, \ldots, N$,

$$I(u^i_{\rho,\lambda};\lambda\Omega) = I(u^i_{\rho,\lambda};\mathbb{R}^3) - \frac{1}{4}\int\limits_{\lambda\Omega\times\lambda\Omega} H(x,y;\lambda\Omega)[u^i_{\rho,\lambda}]^2(x)[u^i_{\rho,\lambda}]^2(y)\mathrm{d}x\mathrm{d}y \to c_\infty$$

as $\lambda \to +\infty$ so that, in particular, each family $\{u^i_{\rho,\lambda}\}_{\lambda\ge\Lambda}$ (extended by zero outside of $\lambda\Omega$) provides a minimizing sequence for c_∞ as $\lambda \to +\infty$. Therefore, up to translations (recall Fact 2 in Section 2), for each $i \in \{1, \ldots, N\}$,

$$u^i_{\rho,\lambda} \to \mathfrak{w}_\infty > 0 \text{ in } H^1(\mathbb{R}^3) \quad \text{as } \lambda \to +\infty.$$

Using (2.4), the solutions also satisfy

$$\begin{aligned}\rho^2\omega^i_{\rho,\lambda} &= \int_{\lambda\Omega} |\nabla u^i_{\rho,\lambda}|^2 dx + \int_{\lambda\Omega} \phi_{u^i_{\rho,\lambda}} [u^i_{\rho,\lambda}]^2\, dx - \int_{\lambda\Omega} |u^i_{\rho,\lambda}|^p dx \\ &= \int_{\mathbb{R}^3} |\nabla \mathfrak{w}_\infty|^2 dx + \int_{\mathbb{R}^3} \varphi_\infty \mathfrak{w}^2_\infty\, dx - \int_{\mathbb{R}^3} \mathfrak{w}^p_\infty dx + o_\lambda(1) \\ &= \rho^2\omega_\infty + o_\lambda(1),\end{aligned}$$

from which, for every $i \in \{1, \dots, N\}$, $\lim_{\lambda\to+\infty} \omega^i_{\rho,\lambda} = \omega_\infty < 0$.

Defining, as usual, $u^+ := \max\{u, 0\}$ we see that the whole previous analysis is valid for the functional

$$I^+(u; \lambda\Omega) := \frac{1}{2}\int_{\lambda\Omega} |\nabla u|^2 dx + \frac{1}{4}\int_{\lambda\Omega} \phi_u u^2 dx - \frac{1}{p}\int_{\lambda\Omega} (u^+)^p dx$$

restricted to $\mathcal{M}_\rho(\lambda\Omega)$. Then we have, for any fixed $\rho \in (0, \rho_1)$ and $\lambda \geq \Lambda$, at least $\operatorname{cat}(\lambda\Omega) = \operatorname{cat}\Omega$ solutions $(u_{\rho,\lambda}, \omega_{\rho,\lambda})$ of

$$\begin{cases} -\Delta u + \phi_u u - (u^+)^{p-1} = \omega u & \text{in } \lambda\Omega, \\ u = 0 & \text{on } \partial(\lambda\Omega), \end{cases}$$

with $u_{\rho,\lambda}$ nonnegative, $\int_{\lambda\Omega} u^2_{\rho,\lambda} dx = \rho^2$ and $\omega_{\rho,\lambda} < 0$. The maximum principle allows to conclude that $u_{\rho,\lambda} > 0$ in Ω.

The final part of Theorem 1.1 is proved by using the same ideas of [7] if Ω (and hence $\lambda\Omega$) is not contractible in itself. Indeed, in this case the compact set $K := \overline{\Psi_{\lambda,r}(\lambda\Omega^-_r)} \subset \mathcal{M}_\rho(\lambda\Omega)$ can not be contractible in the sublevel $[I(\cdot\,; \lambda\Omega)]^{l(\lambda)}$. Take now $\widehat{u} \in \mathcal{M}_\rho(\lambda\Omega) \setminus K$ such that $\widehat{u} \geq 0$; thus $\widehat{u} \not\equiv 0$. Then $I(\widehat{u}; \lambda\Omega) > l(\lambda)$, and the cone

$$\mathfrak{C} := \left\{ t\widehat{u} + (1-t)u : t \in [0, 1], u \in K \right\}$$

does not contain the zero function (observe that the functions in K are nonnegative). Then, the projection of the cone on $\mathcal{M}_\rho(\lambda\Omega)$

$$\mathrm{P}(\mathfrak{C}) := \left\{ \rho\frac{w}{|w|_{L^2(\lambda\Omega)}} : w \in \mathfrak{C} \right\} \subset \mathcal{M}_\rho(\lambda\Omega),$$

is well defined. Let

$$m(\lambda) := \max_{\mathrm{P}(\mathfrak{C})} I(\cdot\,; \lambda\Omega) > l(\lambda).$$

Since $K \subset \mathrm{P}(\mathfrak{C}) \subset \mathcal{M}_\rho(\lambda\Omega)$ and $\mathrm{P}(\mathfrak{C})$ is contractible in $[I(\cdot\,;\lambda\Omega)]^{m(\lambda)}$, we infer that K is also contractible in $[I(\cdot\,;\lambda\Omega)]^{m(\lambda)}$. On the other hand, recalling that K is not contractible in $[I(\cdot\,;\lambda\Omega)]^{l(\lambda)}$ and the Palais-Smale condition is satisfied, we conclude that there is another nonnegative critical point $\widetilde{u}_{\rho,\lambda}$ for the functional with energy level between $l(\lambda)$ and $m(\lambda)$; hence another Lagrange multiplier $\widetilde{\omega}_{\rho,\lambda}$, which is associated to $\widetilde{u}_{\rho,\lambda}$. In other words, there is another solution $(\widetilde{u}_{\rho,\lambda}, \widetilde{\omega}_{\rho,\lambda})$ to the problem with $\widetilde{u}_{\rho,\lambda}$ nonnegative. However, we cannot guarantee as before the positivity of this additional solution $\widetilde{u}_{\rho,\lambda}$, since we do not know if it is at a nonpositive level of the functional and if the associated Lagrange multiplier $\widetilde{\omega}_{\rho,\lambda}$ is negative.

Author contributions

The authors contributed equally to the writing of this article. All authors read and approved the final manuscript.

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Appendix A

The aim of this section is twofold.

i) On one hand, we show the strike difference between the case of the whole space $\mathbb{R}^3$ and the situation for a bounded domain D, from the point of view of the minimum of the functional with fixed L^2-norm equal to ρ. In fact, while in $\mathbb{R}^3$ the minimum of the constrained functional is negative (see Fact 2) for any $\rho > 0$, in the bounded domain the minimum is positive for $\rho \le \rho_D$, a constant depending on the same domain.
ii) On the other hand, we show that if the domain is expanding (namely we consider λD with $\lambda \to +\infty$), then $\rho_{\lambda D} \to 0$. As a consequence, when ρ is fixed, if the domain is expanding we will have $\rho_{\lambda D} < \rho$ and then the computations which led to get the positivity of the minimum are not allowed.

To address the first issue, let D be a smooth bonded domain. Let us start by recalling that

Lemma 5.1. *(see [20, Lemma 3.1 and Remark 3.2]) If $D \subset \mathbb{R}^3$ is a smooth bounded domain, $p \in (2,6)$ and $r \in (0,p)$, then for every $u \in H^1(D)$ it is*

$$|u|^p_{L^p(D)} \le C_D^{p-r}\,\|u\|^{p-r}_{H^1(D)}|u|^r_{L^2(D)}$$

where

$$C_D = \sup_{u\in H^1(D)\setminus\{0\}} \frac{|u|_{L^{\frac{2(p-r)}{2-r}}(D)}}{\|u\|_{H^1(D)}} > 0.$$

In particular, for every $u \in H_0^1(D)$ it holds

$$|u|^p_{L^p(D)} \le C_D^{p-r}\left(1+\frac{1}{\mu_1(D)}\right)^{\frac{p-r}{2}} |\nabla u|^{p-r}_{L^2(D)}|u|^r_{L^2(D)},$$

where $\mu_1(D)$ is the first eigenvalue of the Laplacian in $H_0^1(D)$.

By choosing $r = p-2$ the Lemma furnishes, for $u \in \mathcal{M}_\rho(D)$,

$$|u|^p_{L^p(D)} \le C_D^2\left(1+\frac{1}{\mu_1(D)}\right)|\nabla u|^2_{L^2(D)}|u|^{p-2}_{L^2(D)} =: \widetilde{C}_D|\nabla u|^2_{L^2(D)}\rho^{p-2}. \tag{A.1}$$

From (A.1) it follows that for a suitable small ρ, for example

$$\rho \le \left(\frac{p}{4\widetilde{C}_D}\right)^{1/(p-2)} =: \rho_D, \tag{A.2}$$

it is $|u|^p_{L^p(D)} \le \frac{p}{4}|\nabla u|^2_{L^2(D)}$ and therefore

$$I(u;D) > \frac{1}{2}|\nabla u|^2_{L^2(D)} - \frac{1}{p}|u|^p_{L^p(D)} \ge \frac{1}{2}|\nabla u|^2_{L^2(D)} - \frac{1}{4}|\nabla u|^2_{L^2(D)} \ge \frac{\rho^2}{4}\mu_1(D) > 0. \tag{A.3}$$

The constant ρ_D is not optimal; however, the above arguments show the following result that we were not able to find in the literature.

Proposition 5.2. *Let $D \subset \mathbb{R}^3$ be a bounded and smooth domain. There is a constant $\rho_D > 0$ such that, for any $\rho \in (0, \rho_D]$, it is*

$$\min_{u\in\mathcal{M}_\rho(D)} I(u;D) > 0.$$

The fact that the infimum is achieved is standard and follows since we have compactness being in a bounded domain.

We stress that in the above result the value ρ_D depends on the domain. Furthermore, the statement of Proposition 5.2 does not hold when we fix the L^2-norm a priori and consider the minimum of the constrained functional on a family of expanding domains.

To see this, let D be a smooth bounded domain, and fix $\rho \in (0, \rho_D]$. We have the following

Claim: with the above notations

$$\{\widetilde{C}_{\lambda D}\}_{\lambda>1} \quad \text{is diverging as} \quad \lambda \to +\infty.$$

Assuming the Claim for a moment, since ρ is fixed and λ is increasing, we get that $\rho_{\lambda D} < \rho$, for a suitable large λ. Thus (A.2) is violated and we cannot guarantee anymore neither the second inequality in (A.3) nor the subsequent positivity of the infimum on an expanding domain.

In fact, we have seen in Proposition 3.3 that with ρ fixed, considering the expanding domains $B_{\lambda r}$ and $\lambda\Omega$, the infimum of the functional is negative for large values of λ. It is not surprising then, since we can see $\mathbb{R}^3$ in some sense as the limit of $B_{\lambda r}$ as $\lambda \to +\infty$, that $c_\infty < 0$ (see Fact 2).

Let us show the Claim; in first place, taking any $u \in H_0^1(D) \setminus \{0\}$ and $\lambda > 1$, the function defined by

$$x \in \lambda D \quad \longmapsto \quad v(x) := \frac{1}{\lambda^{3/2}} u(x/\lambda) \in \mathbb{R}$$

is such that $v \in H_0^1(\lambda D)$. Since

$$\int_{\lambda D} v^2 \mathrm{d}x = \int_D u^2 \mathrm{d}x \quad \text{and} \quad \int_{\lambda D} |\nabla v|^2 \mathrm{d}x = \frac{1}{\lambda^2} \int_D |\nabla u|^2 \mathrm{d}x,$$

we have

$$\mu_1(\lambda D) \leq \frac{\|v\|^2_{H_0^1(\lambda D)}}{|v|^2_{L^2(\lambda D)}} = \frac{1}{\lambda^2} \frac{\|u\|^2_{H_0^1(D)}}{|u|^2_{L^2(D)}}.$$

By the arbitrariness of u, this implies that

$$\mu_1(\lambda D) \leq \frac{1}{\lambda^2} \mu_1(D). \tag{A.4}$$

On the other hand, let us choose a point x_0 in D and $\delta > 0$ such that $B_\delta(x_0) \subset D$. Also we fix a function $w \in C_0^\infty(B_\delta(x_0))$ such that $w > 0$ in $B_\delta(x_0)$. For any $\lambda > 1$, $B_\delta(\lambda x_0) \subset B_{\lambda\delta}(\lambda x_0) \subset \lambda D$, and the function

$$x \in B_\delta(\lambda x_0) \quad \longmapsto \quad w_\lambda(x) := w\,(x - (\lambda - 1)x_0) \in \mathbb{R}$$

is such that

$$\int_{B_\delta(\lambda x_0)} w_\lambda^p \mathrm{d}x = \int_{B_\delta(x_0)} w^p \mathrm{d}x, \qquad \int_{B_\delta(\lambda x_0)} w_\lambda^2 \mathrm{d}x = \int_{B_\delta(x_0)} w^2 \mathrm{d}x,$$

and

$$\int_{B_\delta(\lambda x_0)} |\nabla w_\lambda|^2 \mathrm{d}x = \int_{B_\delta(x_0)} |\nabla w|^2 \mathrm{d}x.$$

Hence $w_\lambda \in H^1(\lambda D)$, and (being $2(p-r)/(2-r) = 4/(4-p) \in (2, 6)$)

$$0<\frac{|w|_{L^{4/(4-p)}(B_\delta(x_0))}}{\|w\|_{H^1(B_\delta(x_0))}}=\frac{|w_\lambda|_{L^{4/(4-p)}(B_\delta(\lambda x_0))}}{\|w_\lambda\|_{H^1(B_\delta(\lambda x_0))}}\leq\sup_{u\in H^1(\lambda D)\setminus\{0\}}\frac{|u|_{L^{4/(4-p)}(\lambda D)}}{\|u\|_{H^1(\lambda D)}}=C_{\lambda D},$$

where the expression in the right hand side of the last inequality is the optimal constant in the embedding $H^1(\lambda D)\hookrightarrow L^{4/(4-p)}(\lambda D)$. But then, recalling the definition of $\widetilde{C}_{\lambda D}$ and (A.4) we find that

$$\lim_{\lambda\to+\infty}\widetilde{C}_{\lambda D}=\lim_{\lambda\to+\infty}C^2_{\lambda D}\left(1+\frac{1}{\mu_1(\lambda D)}\right)\geq\lim_{\lambda\to+\infty}C^2_{\lambda D}\left(1+\frac{\lambda^2}{\mu_1(D)}\right)=+\infty$$

which proves the Claim.

Data availability

No data was used for the research described in the article.

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