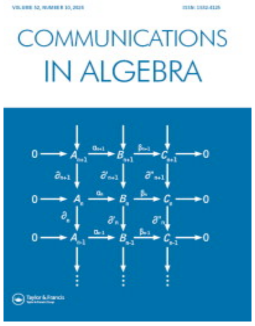

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Complete group of central units in some group rings over $\mathbb{Z}$ and $\mathbb{Z}[\theta_p]$

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ABSTRACT

In the present work, we contribute with new results that enabled us to give an explicit description of the groups of central units of integral group rings and group rings with coefficients in $\mathbb{Z}[\theta]$, for certain groups G, where θ is a $p-$primitive root of unity and p is a prime number. In particular, we find a set of generators for the case where G is the Heisenberg Group.

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1. Introduction

The problem of finding an explicit description of the group of units of group rings, or even the group of central units, is a very hard problem.

Some work has been done in this direction. For example, in [11] Li and Parmenter computed explicitly the group of central units of $\mathbb{Z}A_5$. Also, in [2], Ferraz found a way of computing the rank of the central units in $\mathbb{Z}G$, and by doing this, as an application, he was able to prove that, in the cases which $G = A_n$, for $n = 1, 2, 3, 4, 7, 8, 9, 12$, the rank is zero and the group of central units is trivial. More work on the group of central units can be found in [1].

Also, in [3] Ferraz used the so called "Hoechsmann's units" to present an explicit set of independent generators for the abelian free part of the group of units of $\mathbb{Z}C_p$, for all $p < 67$. Following this, in [4], Ferraz and Kitani found a set of independent generators of the abelian free part of the group of units of $\mathbb{Z}C_{p^n}$, given certain conditions on p^n, and in [5], Ferraz and Marcuz were able to find similar results for the cases where $G = C_2 \times C_p$ or $G = C_2 \times C_2 \times C_p$. Again, aiming to describe the group of central units, in [6] we found a way of expliciting a set of independent generators of the abelian free part of the group of central units of $\mathbb{Z}G$, for some cases where G is a metacyclic or metabelian group.

Also, recently in [7] we could provide a way of calculating a set of independent generators of the abelian free part of the group of units of $\mathbb{Z}[\theta]G$, where θ is a p-primitive root of unit (p an odd prime) and G an elementary abelian finite group, for some values of p - here p needs to be a regular prime number, so that a result that depends of Kummer's lemma can be used, and also it is important that $p < 67$, so that we could use the units given in [2].

The works cited above give explicit descriptions of the groups of units or central units, but there is also a lot of work regarding theoretical and structural results, and also many ways of finding subgroups

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of finite index—not necessarily the full groups of units or central units. A good reference for such results is the book [10].

But in most cases it is an open problem to describe a full set of generators for the group of units or central units of a group ring. In the present paper, we mainly continue the work that was done in [6], giving a description of the group of central units of new group rings over non-abelian groups—with coefficients in $\mathbb{Z}$ and $\mathbb{Z}[\theta]$, with θ a p-primitive root of unity, for certain primes p.

We fix some notations: here, A and B will denote finite groups. For any group G, G' will denote its derived subgroup. For any set X, $\widehat{X}$ will denote the sum of all the elements in X. For any ring R, $U(R)$ will denote its group of units. Finally, if $S \subset \mathbb{Z}(A \times B)$ we denote by $\tilde{U}(S)$ the group of normalized units in S that commute with all elements of A and B (i.e., $\tilde{U}(S)$ contains only normalized central units of $\mathbb{Z}(A \times B)$).

The ideas presented in the second section are inspired by [12] and [6], and we give an internal description of the group of central units of new integral group rings $\mathbb{Z}G$, i.e., we find explicitly a set of generators of this group.

The ideas presented in the third section are inspired by [7], and we use the results presented in the second section to find a description of units in certain group rings $\mathbb{Z}[\theta]G$ with coefficients in $\mathbb{Z}[\theta]$.

Finally, in the fourth section we apply the results in an explicit example—the Heisenberg group.

2. Integral group rings—groups of type $A \times B$

We will assume that A is abelian. First, we define the following morphisms:

$$\varepsilon_A : \mathbb{Z}(A \times B) \to \mathbb{Z}B,$$

which maps every element $a \in A$ to the identity element 1, and analogously we define

$$\varepsilon_B : \mathbb{Z}(A \times B) \to \mathbb{Z}A,$$

which maps every element $b \in B$ to the identity element 1.
We denote $K = \ker(\varepsilon_A) \cap \ker(\varepsilon_B)$.

Then we have the following commutative diagram, where $\Delta(A)$ and $\Delta(B)$ are the kernels of the augmentation maps of $\mathbb{Z}A$ and $\mathbb{Z}B$, respectively, and ι is always the inclusion map:

$$\begin{array}{ccccccccc}
 & & \{0\} & & \{0\} & & \{0\} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\{0\} & \longrightarrow & K & \xhookrightarrow{\iota} & \ker(\varepsilon_A) & \xrightarrow{\varepsilon_B} & \Delta(A) & \longrightarrow & \{0\} \\
 & & \iota \downarrow & & \iota \downarrow & & \iota \downarrow & & \\
\{0\} & \longrightarrow & \ker(\varepsilon_B) & \xhookrightarrow{\iota} & \mathbb{Z}(A \times B) & \xrightarrow{\varepsilon_B} & \mathbb{Z}A & \longrightarrow & \{0\} \\
 & & \varepsilon_A \downarrow & & \varepsilon_A \downarrow & & \varepsilon_A \downarrow & & \\
\{0\} & \longrightarrow & \Delta(B) & \xhookrightarrow{\iota} & \mathbb{Z}B & \xrightarrow{\varepsilon_B} & \mathbb{Z} & \longrightarrow & \{0\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \{0\} & & \{0\} & & \{0\} & &
\end{array}$$

We note that the rows and columns in the diagram above are splitting short exact sequences of abelian groups under addition (just consider the maps $(\iota - \varepsilon_B) : \ker(\varepsilon_A) \to K$, $(\iota - \varepsilon_A) : \ker(\varepsilon_B) \to K$), $(\iota - \varepsilon_B) : \mathbb{Z}(A \times B) \to \ker(\varepsilon_B)$ and $(\iota - \varepsilon_B) : \mathbb{Z}(A \times B) \to \ker(\varepsilon_B)$).

Remark 2.1. In the paragraph above, we are using the fact that a short exact sequence of abelian groups $\{e\} \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to \{e\}$ splits if, and only if, there is a morphism $\alpha' : G \to H$ such that $\alpha'(\alpha(h)) = h$, for all $h \in H$—see Theorem 2.1.4 of [10].

This leads us to the following commutative diagram, where for any $S \subset \mathbb{Z}(A \times B)$, $\tilde{U}(S)$ denotes the group of normalized units (i.e., units with augmentation 1) in S that commute with $A \times B$ (it is equivalent to say that they commute with B, since A is abelian and every element in A commutes with every element in B):

$$
\begin{array}{ccccccccc}
 & & \{1\} & & \{1\} & & \{1\} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\{1\} & \longrightarrow & \tilde{U}(1+K) & \xhookrightarrow{\iota} & \tilde{U}(1+\ker(\varepsilon_A)) & \xrightarrow{\varepsilon_B} & \tilde{U}(1+\Delta(A)) & \longrightarrow & \{1\} \\
 & & \iota \downarrow & & \iota \downarrow & & \iota \downarrow & & \\
\{1\} & \longrightarrow & \tilde{U}(1+\ker(\varepsilon_B)) & \xhookrightarrow{\iota} & \tilde{U}(\mathbb{Z}(A \times B)) & \xrightarrow{\varepsilon_B} & \tilde{U}(\mathbb{Z}A) & \longrightarrow & \{1\} \\
 & & \varepsilon_A \downarrow & & \varepsilon_A \downarrow & & \varepsilon_A \downarrow & & \\
\{1\} & \longrightarrow & \tilde{U}(1+\Delta(B)) & \xhookrightarrow{\iota} & \tilde{U}(\mathbb{Z}B) & \xrightarrow{\varepsilon_B} & \tilde{U}(\mathbb{Z}) = 1 & \longrightarrow & \{1\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \{1\} & & \{1\} & & \{1\} & &
\end{array}
$$

Proposition 2.2. *The rows and columns of the diagram above are all splitting short exact sequences of abelian groups under multiplication.*

Proof. It is easy to see that the rows and columns are all short exact sequences, and that the third row and column split. We have to prove that the first and second rows and columns split as well. We again make use of the Remark 2.1.

For the first row, we define the map $c_1 : \tilde{U}(1+\ker(\varepsilon_A)) \to \tilde{U}(1+K)$ as $c_1(x) = x(\varepsilon_B(x))^{-1}$. It is clear that this map is well defined - it is useful to remember that all elements of $\tilde{U}(1+\ker(\varepsilon_A))$ are central, by definition, therefore x is central here. We easily verify that c_1 is indeed a morphism:

$$c_1(xy) = xy(\varepsilon_B(xy))^{-1} = xy(\varepsilon_B(x)\varepsilon_B(y))^{-1} =$$

$$x(\varepsilon_B(x))^{-1}y(\varepsilon_B(y)^{-1}) = c_1(x)c_1(y),$$

the last row of equations above follows from the fact that all these elements are central.

Now we verify that $c_1(\iota(u)) = u$, for all $u \in \tilde{U}(1+K)$:

$$c_1(\iota(u)) = c_1(u) = u(\varepsilon_B(u))^{-1} = u1^{-1} = u.$$

So the first row splits. For the first column we can do exactly the same thing, defining the map $c_2 : \tilde{U}(1+\ker(\varepsilon_B)) \to \tilde{U}(1+K)$ as $c_2(x) = x(\varepsilon_A(x))^{-1}$.

Therefore, we are just left to prove that the second row and column split. In order to do so, we define the maps $c_3(x) : \tilde{U}(\mathbb{Z}(A \times B)) \to \tilde{U}(1+\ker(\varepsilon_B))$ as $c_3(x) = x(\varepsilon_A(x))^{-1}$ and $c_4(x) : \tilde{U}(\mathbb{Z}(A \times B)) \to \tilde{U}(1+\ker(\varepsilon_A))$ as $c_3(x) = x(\varepsilon_B(x))^{-1}$. For the same argument above, those maps are morphisms that witness the fact that the second row and column are splitting sequences.

□

Remark 2.3. Let $\{e\} \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to \{e\}$ be a splitting short exact sequence of abelian groups with $\alpha' : G \to H$ such that $\alpha'(\alpha(h)) = h$, for all $h \in H$. Then it is well known that there is an isomorphism $\theta : G \to H \times K$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle id} & & \\ 1 & \longrightarrow & H & \xrightarrow{\iota} & H \times K & \xrightarrow{\pi} & K & \longrightarrow & 1 \end{array}$$

where ι and π are the natural inclusion and projection, respectively.

This isomorphism θ is defined for $g \in G$ as $\theta(g) = (\alpha'(g), \beta(g))$. Let $(h, k) \in H \times K$, then θ^{-1} (the inverse of θ) is defined as $\theta^{-1}(h, k) = g\alpha((\alpha'(g))^{-1}h)$.

Remark 2.4. Remark 2.3 and Proposition 2.2 lead us to the conclusion that, if $\tilde{U}(1 + K)$, $\tilde{U}(\mathbb{Z}B)$ and $\tilde{U}(\mathbb{Z}A)$ are known, then $\tilde{U}(\mathbb{Z}(A \times B))$ is also known.

In fact, Proposition 2.2 tells us that $\tilde{U}(\mathbb{Z}(A \times B)) \cong \tilde{U}(1 + \ker(\varepsilon_A)) \times \tilde{U}(\mathbb{Z}B)$, and that $\tilde{U}(1 + \ker(\varepsilon_A)) \cong \tilde{U}(1 + K) \times \tilde{U}(1 + \Delta(A)) = \tilde{U}(1 + K) \times \tilde{U}(\mathbb{Z}A)$, therefore:

$$\tilde{U}(\mathbb{Z}(A \times B)) \cong \tilde{U}(1 + K) \times \tilde{U}(\mathbb{Z}B) \times \tilde{U}(\mathbb{Z}A)$$

We recall that $\tilde{U}(\mathbb{Z}G)$ is known for some groups G—see, for example [6]. Therefore, for such cases, in order to know completely the full group of normalized central units in $\mathbb{Z}(A \times B)$, we just need to evaluate, in these cases, the group $\tilde{U}(1 + K)$. So this is what we will do next.

2.1. The group $\tilde{U}(1 + K)$

From now on, B will denote a group satisfying the following property: if $b \in B$ but $b \notin B'$, then the conjugacy class of b is $B'b$, or equivalently, $\gamma_b = \widehat{B'}b$, where γ_b is the sum of all the elements in the conjugacy class of b (i.e., the sum of the elements of type xbx^{-1}, for $x \in B$), and we recall that A will denote an abelian group.

First of all, we will prove that $K = \ker(\varepsilon_A) \cap \ker(\varepsilon_B)$ is generated, as a $\mathbb{Z}$-module, by elements of the form $(1 - b)(1 - a)$, with $b \in B$ and $a \in A$:

In fact, we know (see [14]) that $\ker(\varepsilon_A) = \langle(1 - a)\rangle_{a\in A}$ and $\ker(\varepsilon_B) = \langle(1 - b)\rangle_{b\in B}$ - as ideals in $\mathbb{Z}(A \times B)$.

In particular, an element $x \in K$ can be written as $x = \sum\limits_{a\in A} \beta_a(1 - a)$, where $\beta_a \in \mathbb{Z}(A \times B)$. Then, we have:

$$\varepsilon_B(x) = \sum_{a\in A} \varepsilon_B(\beta_a(1 - a)) = \sum_{a\in A} \varepsilon_B(\beta_a)\varepsilon_B(1 - a) = \sum_{a\in A} \varepsilon_B(\beta_a)(1 - a) = 0$$

as $\tilde{a}(1 - a) = \tilde{a} - \tilde{a}a = -(1 - \tilde{a}) + (1 - \tilde{a}a)$ for all $\tilde{a}, a \in A$ and all the elements of A commute with all elements of B, we can assume that $supp(\beta_a) \subset B$, for all a.

We also have that $\{(1-a)\}_{a\in A}$ is linearly independent over $\mathbb{Z}B$, so the equation above implies $\varepsilon_B(\beta_a) = 0$, for all a, and this proves that $\beta_a \in \langle 1 - b\rangle_{b\in B}$, and $x \in \langle(1 - b)(1 - a)\rangle_{b\in B, a\in A}$ - as a $\mathbb{Z}(A \times B)$-ideal.

But again, $\tilde{a}(1 - a) = -(1 - \tilde{a}) + (1 - \tilde{a}a)$ and $\tilde{b}(1 - b) = -(1 - \tilde{b}) + (1 - \tilde{b}b)$, for all $a, \tilde{a} \in A$, $b, \tilde{b} \in B$. Since a and b always commute, we can assume $\alpha_{a,b} \in \mathbb{Z}$ in the expression below:

$$x = \sum_{a\in A, b\in B} \alpha_{a,b}(1 - a)(1 - b),$$

This completes the proof that K is generated by elements of the form $(1 - b)(1 - a)$ as a $\mathbb{Z}$-module.

Now let us consider an element $\alpha \in 1 + K$ central in $\mathbb{Z}(A \times B)$. By the above, we know that there are $\alpha_{a,b} \in \mathbb{Z}$ such that:

$$\alpha = 1 + \sum_{a \in A, b \in B} \alpha_{a,b}(1 - b)(1 - a).$$

Since α is central in $\mathbb{Z}(A \times B)$, we have that $b\alpha b^{-1} = \alpha$. Let $\mathcal{C}_B$ be the set of all conjugacy classes of B. We have:

$$\alpha = 1 + \sum_{c \in \mathcal{C}_B, a \in A} \alpha_{a,c}(1 - a) \sum_{b \in c} (1 - b),$$

where $\alpha_{a,c}$ are integers. Also, since B has the property that, if $b \notin B'$ then $\gamma_b = b\widehat{B'}$:

$$\alpha = \sum_{\substack{a \in A \\ c \in \mathcal{C}_B \\ \text{with } c \cap B' = \varnothing}} \alpha_{a,c} \sum_{b \in B'} (1 - b_c b)(1 - a) + \sum_{b \in B', a \in A} \alpha_{a,b}(1 - b)(1 - a), \tag{2.1}$$

where b_c is a representative of the class c. Now we define the following morphism:

$$\pi : \tilde{U}(\mathbb{Z}(A \times B)) \to \tilde{U}\left(\mathbb{Z}\left(\frac{A \times B}{(A \times B)'}\right)\right) = \tilde{U}\left(\mathbb{Z}\left(A \times \frac{B}{B'}\right)\right),$$

which extends linearly the map $(a, b) \mapsto (a, \overline{b})$, where $\overline{b} = bB'$.

We have, by equation (2.1), for $\alpha \in \tilde{U}(\mathbb{Z}(A \times B))$ (in particular, α is a unit):

$$\pi(\alpha) = 1 + \sum_{\substack{a \in A \\ c \in \mathcal{C}_B \\ \text{with } c \cap B' = \varnothing}} \alpha_{a,c}(1 - b_c)(1 - a)|B'| \tag{2.2}$$

Now, similarly to what was done in Theorems 1.2 and 1.4 of [6], we define:

$$w_1 := 1 + \sum_{\substack{a \in A \\ c \in \mathcal{C}_B \\ \text{with } c \cap B' = \varnothing}} \alpha_{a,c}(1 - b_c)(1 - a)\widehat{B'},$$

and we want to define $w_2 \in \ker(\pi)$ such that $w_1 w_2 = \alpha$, and this is what we are going to do now. Since the product $w_1 w_2$ results in a unit, both w_1 and w_2 would be units.

We have that, if $w_2 \in \ker(\pi)$, then:

$$w_1 w_2 = w_2 + \sum_{\substack{a \in A \\ c \in \mathcal{C}_B \\ \text{with } c \cap B' = \varnothing}} \alpha_{a,c}(1 - b_c)(1 - a)\widehat{B'} w_2$$

$$= w_2 + \sum_{\substack{a \in A \\ c \in \mathcal{C}_B \\ \text{with } c \cap B' = \varnothing}} \alpha_{a,c}(1 - b_c)(1 - a)\widehat{B'},$$

and if we want the expression above to be equal to our unit α, we must have:

$$w_2 = \alpha - \sum_{\substack{a \in A \\ c \in \mathcal{C}_B \\ \text{with } c \cap B' = \varnothing}} \alpha_{a,c}(1 - b_c)(1 - a)\widehat{B'}, \tag{2.3}$$

using the expression in (2.1), we get that, defining w_2 like above, in fact $w_2 \in \ker(\pi)$. Also, by construction, we would have that $w_1 w_2 = \alpha$. Hence, we define the unit w_2 as in equation (2.6).

This leads us to the conclusion that, in fact, w_1 and w_2 are (normalized central) units in $\mathbb{Z}(A \times B)$ - since their product is itself a unit.

We write another expression for w_2 that does not depend on α, and is a consequence of (2.1) and (2.3). To do so, we first substitute the term α in equation (2.3) by the expression in equation (2.1). Doing this and putting the $\alpha_{a,c}$'s in evidence, we get that:

$$w_2 = \sum_{\substack{a\in A\\ c\in\mathcal{C}_B\\ \text{with } c\cap B'=\varnothing}} \alpha_{a,c}\left[\sum_{b\in B'}(1-b_cb)(1-a)-(1-b_c)(1-a)\widehat{B'}\right]+$$

$$+\sum_{\substack{b\in B'\\ a\in A}}\alpha_{a,b}(1-b)(1-a)=$$

$$= \sum_{\substack{a\in A\\ c\in\mathcal{C}_B\\ \text{with } c\cap B'=\varnothing}} \alpha_{a,c}\sum_{b\in B'}((1-b_cb)(1-a)-(b-b_cb)(1-a))+$$

$$+\sum_{\substack{b\in B'\\ a\in A}}\alpha_{a,b}(1-b)(1-a)=$$

$$= \sum_{\substack{a\in A\\ c\in\mathcal{C}_B\\ \text{with } c\cap B'=\varnothing}} \alpha_{a,c}\sum_{b\in B'}(1-b)(1-a)+\sum_{\substack{b\in B'\\ a\in A}}\alpha_{a,b}(1-b)(1-a)=$$

and the above one gives us, by swapping the terms, the following:

$$w_2 = 1+\sum_{\substack{b\in B'\\ a\in A}}\alpha_{a,b}(1-b)(1-a)+\sum_{\substack{a\in A\\ c\in\mathcal{C}_B\\ \text{with } c\cap B'=\varnothing}}\alpha_{a,c}(|B'|-\widehat{B'})(1-a) \tag{2.4}$$

the expression above makes it clear that $supp(W_2)\subset A\times B'$.

We denote by $\mathbb{Z}_n$ the ring of integers modulo n. Now we define the morphism:

$$\pi_{|B'|}:\tilde{U}\left(\mathbb{Z}\left(A\times\frac{B}{B'}\right)\right)\to\tilde{U}\left(\mathbb{Z}_{|B'|}\left(A\times\frac{B}{B'}\right)\right),$$

which takes the (integer) coefficients to their classes modulo $|B'|$.

Now we define the following groups of normalized central units:

$$W_1 := \left\{u=1+\frac{w-1}{|B'|}\widehat{B'} : w\in\pi^{-1}(\ker(\pi_{|B'|})), \varepsilon_A(w)=\varepsilon_B(w)=1\right\} \tag{2.5}$$

$$W_2 := \left\{u\in\tilde{U}(\mathbb{Z}(A\times B'))\cap Z(\mathbb{Z}(A\times B)) : \varepsilon_A(u)=\varepsilon_B(u)=1\right\}, \tag{2.6}$$

where $Z(\mathbb{Z}(A\times B))$ denotes the center of the group ring $\mathbb{Z}(A\times B)$.

First of all, we note that $W_1, W_2\subset\tilde{U}(1+K)$, $w_1\in W_1$—since $\pi(w_1)\in\ker\pi_{|B'|}$ and $w_1=1+\frac{w_1-1}{|B'|}\widehat{B'}$.

We also note that, in the definition of W_1, w and $\tilde{w}$, would define the same unit (i.e., $1+\frac{w-1}{|B'|}\widehat{B'}=1+\frac{\tilde{w}-1}{|B'|}\widehat{B'}$) if, and only if $\pi(w)=\pi(\tilde{w})$, i.e., $1+\frac{w-1}{|B'|}\widehat{B'}=1+\frac{\tilde{w}-1}{|B'|}\widehat{B'}$ if, and only if $\pi(w)=\pi(\tilde{w})$.

We also note that $\left(1+\frac{w-1}{|B'|}\widehat{B'}\right)\left(1+\frac{\tilde{w}-1}{|B'|}\widehat{B'}\right)=\left(1+\frac{w\tilde{w}-1}{|B'|}\widehat{B'}\right)$.

Therefore, we have that:

$$W_1 \cong \left\{x \in \ker(\pi_{|B'|}) : \varepsilon_A(x) = \varepsilon_B(x) = 1\right\} \leq \tilde{U}\left(\mathbb{Z}\left(A \times \frac{B}{B'}\right)\right), \tag{2.7}$$

where ε_A and ε_B are considered as functions from $\mathbb{Z}\left(A \times \frac{B}{B'}\right)$.

So far, we have that $\tilde{U}(1+K) = W_1 W_2$. To prove that this product is direct, we just need to show that $W_1 \cap W_2 = \{1\}$. In fact, suppose that the element $u = \left(1 + \frac{w-1}{|B'|}\widehat{B'}\right)$ of W_1 is also in W_2. As $W_2 \subset \mathbb{Z}(A \times B')$, we can assume $supp(w) \subset A$. Since $u \in W_2$, we have $1 = \varepsilon_B(u) = 1 + \varepsilon_B(w) - 1 = w$, giving us that $w = 1$, which implies $u = 1$.

Then $\tilde{U}(1+K) = W_1 \times W_2$.

We just proved the following theorem:

Theorem 2.5. *Suppose that A is a finite abelian group and B is a finite group such that, for all $b \in B$ with $b \notin B'$, $\gamma_b = b\widehat{B'}$. Then:*

$$\tilde{U}(\mathbb{Z}(A \times B)) \cong \tilde{U}(\mathbb{Z}A) \times \tilde{U}(\mathbb{Z}B) \times W_1 \times W_2.$$

The congruence given in the theorem above is described in Remark 2.3.

3. Central units of $\mathbb{Z}[\theta]G$

In this section, we adapt the results of [7] to the context of central units of group rings of type $\mathbb{Z}[\theta]G$, where θ is a p-primitive root of unity. From now on, p will denote an odd prime number, and G will denote a finite group.

As usual, $Z(U_1(\mathbb{Z}H))$ will denote the group of central units of $\mathbb{Z}H$ with augmentation 1 and $Z(U_1(\mathbb{Z}[\theta]H))$ will denote the group of central units of $\mathbb{Z}[\theta]H$ with augmentation congruent to 1 mod $(\theta - 1)$.

We also define the following morphisms, for $a_{i,h}, a_h \in \mathbb{Z}$, $b_h \in \mathbb{Z}[\theta]$, $C_p = \langle x \rangle$:

$\sigma_0 : \mathbb{Z}(C_p \times G) \to \mathbb{Z}[\theta]G$, given by

$$\sum_{\substack{i=0,\dots,p-1 \\ h \in G}} a_{i,h} x^i h \mapsto \sum_{\substack{i=0,\dots,p-1 \\ h \in G}} a_{i,h} \theta^i h.$$

$\pi_0 : \mathbb{Z}(C_p \times G) \to \mathbb{Z}G$, given by

$$\sum_{\substack{i=0,\dots,p-1 \\ h \in G}} a_{i,h} x^i h \mapsto \sum_{\substack{i=0,\dots,p-1 \\ h \in G}} a_{i,h} h.$$

$\alpha_0 : \mathbb{Z}[\theta]G \to \mathbb{F}_p G$, which takes the coefficients to their classes modulo $\theta - 1$:

$$\sum_{h \in G} b_h h \mapsto \sum_{h \in G} \overline{b_h} h.$$

$\rho_0 : \mathbb{Z}G \to \mathbb{F}_p G$, which takes the coefficients to their classes modulo p:

$$\sum_{h \in G} a_h h \mapsto \sum_{h \in G} \overline{a_h} h.$$

Then, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}(C_p \times G) & \xrightarrow{\pi_0} & \mathbb{Z}G \\ \downarrow{\sigma_0} & & \downarrow{\rho_0} \\ \mathbb{Z}[\theta]G & \xrightarrow{\alpha_0} & \mathbb{F}_p G \end{array}$$

In [13, Lemmas 2.1–2.7] Low proved that $\ker(\pi_0)$ and $\ker(\alpha_0)$ are isomorphic rings under the morphism σ_0. Using an essentially identical argument, we will prove the same in the case where σ, π, ρ and α are restrictions of the original morphisms to the centers of the rings. We have the following commutative diagram:

$$\begin{array}{ccc} Z(\mathbb{Z}(C_p \times G)) & \xrightarrow{\pi} & Z(\mathbb{Z}G) \\ \sigma \downarrow & & \downarrow \rho \\ Z(\mathbb{Z}[\theta]G) & \xrightarrow{\alpha} & Z(\mathbb{F}_pG) \end{array}$$

From now on, let us denote $K^* = \ker(\pi)$ and $M^* = \ker(\alpha)$—following the same notation as in [13].

Lemma 3.1. *As in the commutative diagram above, we have that $K^* \cong M^*$, an isomorphism being the restriction $\sigma_{|\ker(\pi)}$.*

Proof. It is easy to see that $\sigma(\ker(\pi)) \subset \ker(\alpha)$. As a consequence of Low's result, we only need to prove that $\sigma_{|\ker(\pi)} : \ker(\pi) \to \ker(\alpha)$ is surjective.

Let $U \in M^*$. Setting T a set of representatives of all conjugacy classes of G, we have that:

$$U = \sum_{g \in T} a_g \gamma_g,$$

where $a_g = \sum_{j=0}^{p-2} b_{g,j}\theta^j \in \mathbb{Z}[\theta]$, for certain integers $b_{g,j}$.

Now we define:

$$\tilde{V} = \sum_{g \in T} \left(\sum_{j=0}^{p-2} b_{g,j} x^j \right) \gamma_g.$$

Since $\alpha(U) = 0$, we have that $\sum_{j=0}^{p-2} b_{g,j}\theta^j \equiv 0 \ (mod\ (\theta - 1))$, for all g, and consequently we have that $\sum_{j=0}^{p-2} b_{g,j} \equiv 0 \ (mod\ p)$. Then $\pi(\sum_{j=0}^{p-2} b_{g,j}x^j) = pA_g$, for a certain $A_g \in \mathbb{Z}$.

Now, define $V = \tilde{V} - \widehat{C_p} \sum_{g \in T} A_g \gamma_g$. Then we have $V \in \ker(\pi)$ with $\sigma(V) = U$, as we wanted.

□

Now we redefine our maps to get the following commutative diagram:

$$\begin{array}{ccc} Z(U_1(\mathbb{Z}(C_p \times G))) & \xrightarrow{\pi} & Z(U_1(\mathbb{Z}G)) \\ \sigma \downarrow & & \downarrow \rho \\ Z(U_1(\mathbb{Z}[\theta]G)) & \xrightarrow{\alpha} & Z(U_1(\mathbb{F}_pG)) \end{array} \tag{3.1}$$

In this diagram, we call $K = \ker(\pi)$ and $M = \ker(\alpha)$. It is useful to recall that $U_1(\mathbb{Z}[\theta]G)$ stands for the units which have augmentation congruent to 1 mod $(\theta - 1)$, and not just those with augmentation 1.

We have that $(1+K^*)$ and $(1+M^*)$ are isomorphic as monoids (under multiplication) by Lemma 3.1. Also, K and M are, respectively, the groups of units of these monoids. Therefore, we have the following:

Lemma 3.2. *$\sigma : K \to M$ is an isomorphism of groups.*

In particular, the lemma above tells us that, if $\alpha(u) = 1$, then $u \in Im(\sigma)$.

Now, our aim is to prove a result analogous to the main theorem in [7]. If $Z(U_1(\mathbb{Z}(C_p \times G)))$ is a known group, i.e., if we know a complete set of generators, we would like σ to be surjective, so that $Z(U_1(\mathbb{Z}[\theta]G))$ is also known. And we will prove now that this is true under certain conditions on G.

Remark 3.3. Knowing the central units of augmentation congruent to 1 modulo $(\theta - 1)$ is enough to know every other central unit—see the introduction of [7].

Theorem 3.4. *Let G be a finite group of exponent p with the properties below,*

$$\begin{aligned} &(i) \quad Z(G) = G'; \\ &(ii) \quad \text{if } g \in G \text{ but } g \notin G', \text{ then } \gamma_g = g\widehat{G'} = g\widehat{Z(G)}, \end{aligned} \tag{3.2}$$

where $Z(G)$ denotes the center of G and γ_g denotes the sum of the elements in the conjugacy class of g.

Then σ is surjective.

Remark 3.5. All groups considered here are special p-groups, and all extraspecial p-groups of exponent p satisfy the hypotheses of the theorem.

Proof. The proof will be an adaptation of what was done in [7].

Let $u \in Z(U_1(\mathbb{Z}[\theta]G))$, we want to prove that $u \in Im(\sigma)$.

We can write:

$$u = \sum_{g \in T} a_g \gamma_g + \sum_{g \in Z(G)} a_g g,$$

for certain values $a_g \in \mathbb{Z}[\theta]$, where T is a full set of representatives of the conjugacy classes of $G \backslash Z(G)$.

First, we prove that $u^p \in Im(\sigma)$. To do so, we evaluate $\alpha(u^p)$:

$$\alpha(u^p) = \alpha(u)^p = \left(\sum_{g \in T} \overline{a_g} \gamma_g + \sum_{g \in Z(G)} \overline{a_g} g \right)^p,$$

since γ_g is central and $\mathbb{F}_p$ is a ring of characteristic p, we can write the expression above as:

$$\sum_{g \in T} \overline{a_g}^p \gamma_g^p + \sum_{g \in Z(G)} \overline{a_g}^p g^p,$$

since the exponent of G is p, G has the properties (3.2), and $\overline{a_g} \in \mathbb{F}_p$, we can write it as

$$\sum_{g \in T} \overline{a_g} g^p \widehat{Z(G)}^p + \sum_{g \in Z(G)} \overline{a_g},$$

and since $p | \widehat{Z(G)}^p$ and the augmentation of u is congruent to 1 module $(\theta - 1)$, the above expression reduces to $\overline{1}$. Therefore, $u^p \in \ker(\alpha)$ and, by Lemma 3.2:

$$u^p \in Im(\sigma) \tag{3.3}$$

Now, for every integer $1 \leq s \leq p - 1$, we define the $\mathbb{Z}$-linear morphisms:

$$\psi_s : \mathbb{Z}[\theta]G \to \mathbb{Z}[\theta]G$$

$$\theta \mapsto \theta^s,$$

i.e., the morphisms ψ_s are induced by the action of $Gal(\mathbb{Q}(\theta)/\mathbb{Q})$ on $\mathbb{Z}[\theta]G$.

We note that if $w \in Z(U_1(\mathbb{Z}[\theta]G))$, then $\psi_s(w) \in Z(U_1(\mathbb{Z}[\theta]G))$ too, for all s.

We have that $\alpha(u) = \alpha(\psi_s(u))$, so $(u\psi_s(u)^{-1}) \in \ker(\alpha)$, and by Lemma 3.2:

$$u\psi_s(u)^{-1} \in Im(\sigma), \text{ for all } 1 \leq s \leq p - 1. \tag{3.4}$$

We also define the following unit:

$$v = \prod_{s=1}^{p-1} \psi_s(u) \in Z(U_1(\mathbb{Z}[\theta]G))$$

We have that $\psi_s(v) = v$, for all s. We can write v uniquely as:

$$v = \sum_{i=1}^{p-2} z_i \theta^i,$$

for certain $z_i \in \mathbb{Z}G$. And we have:

$$v = \psi_s(v) = \sum_{i=0}^{p-2} z_i \theta^{si}.$$

Now we write each z_i as $z_i = \sum_{h \in G} a_{i,h} h$, for certain $a_{i,h} \in \mathbb{Z}$, and we conclude that the coefficient of h in v is:

$$\sum_{i=0}^{p-2} a_{i,h}\theta^i = \sum_{i=0}^{p-2} a_{i,h}\theta^{si} = \psi_s(\sum_{i=0}^{p-2} a_{i,h}\theta^i),$$

i.e., the coefficient of h is fixed by $Gal(\mathbb{Q}(\theta)/\mathbb{Q})$. Thus, by the correspondence given in the Fundamental Theorem of Galois Theory, we have that this coefficient is rational, and therefore integer.

So we conclude that $v \in \mathbb{Z}G \subset \mathbb{Z}(C_p \times G)$, and we can write $v = \sigma(v)$. In particular, we have:

$$v \in Im(\sigma) \tag{3.5}$$

By (3.4), we have that:

$$\prod_{s=1}^{p-1} u\psi_s(u)^{-1} = u^{p-1} v \in Im(\sigma),$$

then, by (3.5), we have that $u^{p-1} \in Im(\sigma)$ and, finally, by (3.3), we conclude that $u \in Im(\sigma)$, as we wanted to prove. □

4. Concrete example: the Heisenberg group

We now illustrate how to apply the previous results by giving a concrete example.

First of all, let us show that there are examples of groups with properties (3.2) that can be treated with our results. To do so, we will find a set of generators of $Z(U_1(\mathbb{Z}(C_p \times H)))$ and $Z(U_1(\mathbb{Z}[\theta] \times H))$, where H is the famous Heisenberg group modulo p, and p is an odd prime number - to make things simple, we will consider the example $p = 5$, but we could choose any regular prime number less than 67 and would still be able to compute with more work (see [6, 7]).

Remark 4.1. The regularity comes out to be necessary because, in order to be able to compute the complete group of units of $\mathbb{Z}G$ when G is a finite elementary abelian group, one needs Kummer's Lemma (see [7, 9]), and the necessity that $p < 67$ comes from the fact that the units found in [2] only generate the full group of units of $\mathbb{Z}C_p$ when $p < 67$ (as the author points out), since for such primes we have that the real class number of the number field $\mathbb{Q}(\theta + \theta^{-1})$ is 1 (see [15], Lemma 8.1, Theorem 8.2 and result (a) in p. 421—in the "Tables" section).

In fact, for $n \geq 2$, we could define:

$$H_n = \langle a_1, \ldots, a_n, b | a_1^p = \cdots = a_n^p = b^p = 1; a_i a_j = a_j a_i, \forall i, j \geq 1;$$
$$a_1 b = b a_1; b a_k b^{-1} a_k^{-1} = a_1, \forall k \geq 2 \rangle$$

Analogously to what happens in the case $n = 2$ - which is the Heisenberg group modulo p - we have that $Z(H_n) = \langle a_1 \rangle = H_n'$, and the conjugacy class of an element of type $g = a_2^{i_2} \ldots a_n^{i_n} b^j$ is the set $\{g, a_1 g, a_1^2 g, \ldots, a_1^{p-1} g\}$. In fact, any such groups would fit our hypotheses.

From now on, we will consider only the case $H := H_2$, with $p = 5$, to make things more simple.

In Section 6 of [6], we find a basis for the free abelian part of $Z(U_1(\mathbb{Z}H))$. More specifically, we have: If we take $u(g) = g^4 + g - 1$ for all g, then:

$$\widetilde{W_2} = \langle a_1 \rangle \times \langle u(a_1) \rangle ,$$

and setting $V = \{u(a_1)^p, u(a_1 b)^p, u(a_1 b^2)^p, \ldots, u(a_1 b^{p-1})^p, u(b)^p\}$, we have:

$$\widetilde{W_1} = \left\langle 1 + \frac{(w-1)\widehat{a_1^p}}{p} \, | w \in V \right\rangle ,$$

and $Z(U_1(\mathbb{Z}H)) = \widetilde{W_1} \times \widetilde{W_2}$.

Then, by Remark 2.4, we have that:

$$\tilde{U}(\mathbb{Z}C_5 \times H) \cong \tilde{U}(1 + K) \times \tilde{U}(\mathbb{Z}C_5) \times \tilde{U}(\mathbb{Z}H),$$

and we know that (see, for example, [2]), if $C_5 = \langle x \rangle$, then $\tilde{U}(\mathbb{Z}C_5) = \langle x, x^4 + x - 1 \rangle$ and $\tilde{U}(\mathbb{Z}H) = \widetilde{W_1} \times \widetilde{W_2}$, as written above.

Finally, we know, by Theorem 2.5, that

$$\tilde{U}(1 + K) = W_1 \times W_2,$$

Where:

$$W_1 := \left\{ u = 1 + \frac{w-1}{|H'|} \widehat{H'} : w \in \pi^{-1}(\ker(\pi_{|H'|})), \varepsilon_{C_5}(w) = \varepsilon_H(w) = 1 \right\}$$

$$W_2 := \left\{ u \in \tilde{U}(\mathbb{Z}(C_5 \times H')) \cap Z(\mathbb{Z}(C_5 \times H)) : \varepsilon_{C_5}(u) = \varepsilon_H(u) = 1 \right\}$$

Now, we compute explicitly W_1 and W_2.

For W_2, we note that, since $H' = Z(H) = \langle a_1 \rangle \cong C_5$, then by [9], we have that W_2 is generated by units of type $g^4 + g - 1$, where $g = x^i a_1^j$ $(0 \leq i, j \leq 4)$.

The hard part is to compute W_1:

First, we note that $\pi_{|H'|}$ can be viewed as a map of type:

$$U(\mathbb{Z}(C_5 \times (C_5 \times C_5))) \to U(\mathbb{Z}_5(C_5 \times (C_5 \times C_5)))$$

Here, $C_5 \times (C_5 \times C_5) = \langle x \rangle \times (\langle \overline{a_2} \rangle \times \langle \overline{b} \rangle)$.

We know that $U(\mathbb{Z}(C_5)^3)$ has an abelian free part of rank 31 (since the rank of $U(\mathbb{Z}(C_p)^n)$ is $\frac{1}{2}(p - 3)(p^n - 1)/(p-1)$). Then, by [9] we have that the elements $u(xy), u(\overline{a_2} b^j), u(\overline{b})$ generate $U(\mathbb{Z}(C_5 \times (C_5 \times C_5)))$ and are all independent, for $y \in \langle \overline{a_2} \rangle \times \langle \overline{b} \rangle$, $0 \leq j \leq 4$ (again we are letting $u(g) := g^4 + g - 1$, for all g).

Therefore, every unit $w \in U(\mathbb{Z}(C_5 \times (C_5 \times C_5)))$ is of the form:

$$u(\overline{b})^{\alpha_{\overline{b}}} \prod_y u(xy)^{\beta_y} \prod_i u(\overline{a_2} \overline{b}^i)^{\gamma_i},$$

therefore, if $\varepsilon_H(w) = \varepsilon_{C_p}(w) = 1$, then:

$$\begin{aligned} \alpha_{\overline{b}} &= -\beta_{\overline{b}} \\ \beta_{\overline{a_2}\overline{b^i}} &= -\gamma_i \\ \sum_y \beta_y &= 0 \end{aligned} \tag{4.1}$$

so that the factors would cancel out after applying ε_{C_p} or ε_H.

We also know, by [8] that:

$$\ker(\pi_{|H'|}) = \{u^p : u \in U(\mathbb{Z}(\langle x\rangle \times (\langle \overline{a_2}\rangle \times \left\langle \overline{b}\right\rangle))\} \tag{4.2}$$

Therefore, combining equations (4.2), (4.1) and the expression for W_1, we know completely an explicit set of generators for W_1.

Thus, we were able to show a complete set of generators for the abelian free part of $\tilde{U}(\mathbb{Z}C_5 \times H)$ by just applying Remark 2.3. Just to make things clearer, if

$$\theta_1 : \tilde{U}(\mathbb{Z}(A \times B)) \to \tilde{U}(1 + \ker(\varepsilon_A)) \times \tilde{U}(\mathbb{Z}B) \text{ and}$$

$$\theta_2 : \tilde{U}(\mathbb{Z}(1 + \ker(\varepsilon_A))) \to \tilde{U}(1 + K) \times \tilde{U}(\mathbb{Z}A)$$

are the isomorphisms given in Remark 2.3, then we have a complete set of generators of $\tilde{U}(\mathbb{Z}(A \times B))$ by simply suitably applying θ_1^{-1} and θ_2^{-1} (also given in Remark 2.3) to the units we found above.

Also, applying the map σ and Theorem 3.2, we also have an explicit set of generators for the abelian free part of $Z(U_1(\mathbb{Z}[\theta]H))$, where θ is a 5-primitive root of unity.

The example we just showed could be easily computed for other regular primes $p \leq 67$, using basically the same argument (see [6, 7]).

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