Ground State for a Schrödinger–Born–Infeld System via an Approximating Procedure

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Abstract

In this paper we discuss some results on the existence of solutions for an elliptic system appearing in physical sciences. In particular the system appears when we look at standing wave solutions in the electrostatic situation for the Schrödinger equation coupled, with the minimal coupling rule, with the electromagnetic equations of Born–Infeld theory. Many difficulties appear, especially due to the fact we are in an unbounded domain (the whole space $\mathbb{R}^3$) and to the intrinsic nonlinear nature of the equations. We are able to prove the existence of a minimal energy solution by showing an approximating procedure that can be adapted depending on the value of the parameter p, which is in the nonlinearity.

Keywords: Schrödinger-type equation; critical point theory; variational methods; ground state solution

MSC: 35E05; 35J10; 35J20; 35J50; 35Q60

1. Introduction

In the following paper, we discuss a couple of results obtained in [1,2] concerning a system of elliptic partial differential equations of interest in physics that present interesting mathematical challenges.

Over the past years, there has been great interest in the study of systems of equations that model the interaction of matter with electromagnetic fields. In fact many theories have been considered that incorporate matter equations, such as the Klein–Gordon or Schrödinger equation, coupled with equations of the electromagnetic field, for example the Maxwell equations, or even equations that provide a more accurate description of the electromagnetic field, like the Bopp–Podolsky or Born–Infeld equation. We do not enter into physical details here, but it is worth noting that the well-known Maxwell and Bopp–Podolsky equations can be seen as approximations of the more sophisticated Born–Infeld theory introduced in the seminal papers [3–6].

In essence, the coupling mentioned above of a matter field with the electromagnetic field is a model for a physical situation in which a charged particle interacts with its own electromagnetic field generated by the motion. This interaction can be rigorously described within the framework of gauge theories and involves, from a practical point of view, the replacing of the usual derivatives that appear in the Lagrangian with the so-called covariant derivatives or Weyl derivatives. Then, looking for a standing wave solution in equilibrium with its own electromagnetic field in a purely electrostatic situation, one arrives at an elliptic system of two coupled equations: the first equation is related to the matter field

and the second one is related to the electrostatic potential. In fact, according to the theory considered for the electromagnetic field, different equations appear.

Recently, we have studied these types of systems and demonstrated the existence and multiplicity of solutions in various situations. The interested reader may refer to the pioneering paper [7], starting from which many papers appeared: for example, the Maxwell and Klein–Gordon equations are combined in [8–11], the Born–Infeld and Klein–Gordon equations are considered in [12,13], the Schrödinger and Maxwell equations are considered in [14–24], and the Schrödinger and Bopp–Podolsky equations are considered in [25–29]. These are just few references, since it is almost impossible to cite all the interesting papers on the subject, but they are sufficient to give to the reader an idea of the mathematics involved, the techniques developed and the surprising results obtained.

In this paper we treat the Schrödinger equation coupled with the electromagnetic field described by the Born–Infeld equation. The derived system has a Lagrangian $\mathcal{L}_{BI}$, which provides a more accurate description of the physical situation compared to the Lagrangian employed in the Maxwell theory of the electromagnetic field, $\mathcal{L}_M$. In contrast, the electrostatic potential equation in the Born–Infeld theory is nonlinear, unlike the Maxwell theory, where the Poisson equation appears and is more studied and understood mathematically.

More specifically, our aim involves the search for standing waves of the Schrödinger equation that are in equilibrium with the electrostatic field generated by the motion within the Born–Infeld theory for the electric field. Then we are reduced to looking for solutions $u, \phi : \mathbb{R}^3 \to \mathbb{R}$ to the following system of partial differential equations:

$$\begin{cases} -\Delta u + u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3 \\[2mm] -\nabla \cdot \left(\dfrac{\nabla \phi}{\sqrt{1 - |\nabla \phi|^2}} \right) = u^2 & \text{in } \mathbb{R}^3, \\[2mm] u(x) \to 0, \ \phi(x) \to 0, & \text{as } |x| \to +\infty. \end{cases} \quad (1)$$

This system is referred to as the *Schrödinger–Born–Infeld system*. Here $p > 1$ represents the nonlinearity, evidently of power type, which simulates the interaction between numerous particles. Of course $u = \phi = 0$ solves the system, but we do not care about the zero solution.

In our discussions we will highlight the major difficulties in dealing with the above system.

Let us comment, however, a little bit on the meaning of the equations.

The first one is a nonlinear Schrödinger equation for u, the modulus of the wave function, in the presence of a potential ϕ. The point is that this electrostatic potential is not assigned and is not known a priori; in other words it cannot be classified as an *external potential*. In fact, it is generated by the wave function due to the fact that its "gradient" field has u^2 as a source: this is the second equation. Moreover the second equation is not linear in ϕ since a mean curvature operator in the Minkowski metric appears and is related, evidently, to the structure of the Born–Infeld Lagrangian. This is the first apparent difficulty. The system is accompanied with boundary conditions at infinity, which are quite natural and mean that the solutions vanish at large distances, as should be the case in any reasonable physical model.

For interesting papers about the mean curvature operator in the Minkowski metric as well the Born–Infeld equations, see, e.g., [12,30–38].

A second mathematical difficulty related to the problem is that the problem is situated in the entire space $\mathbb{R}^3$, where compact embeddings of Sobolev spaces are not valid anymore. From a variational point of view, this presents certain challenges in establishing the compactness of Palais–Smale sequences for the energy functional: it can happen that

"quasi-solutions" escape at infinity. Indeed an indirect approach in order to control these quasi-solutions is implemented.

Despite these challenges, we find some existence results, and depending on the values of p two different approaches are developed.

2. Materials and Methods

As with many interesting physical phenomena, the problem under study is variational: this means that the equations are obtained as critical points of a certain functional. This is something more general than Maupertuis's principle or the minimal action principle, according to which natural phenomena appear and are stable when the energy is minimal. In fact the solutions could be also other types of critical points of the energy functional: not only minima, but also saddle-like, and they give rise in this case to "less stable" phenomena.

The advantage of working with a variational system is that its solutions are exactly the critical points of the energy functional; then a problem in partial differential equations is transferred to a problem in Critical Point Theory. This theory deals with very abstract and general functionals and a robust theory is established, although continuous advances are being made by many authors around the world.

Also our system is variational: we can write a functional of two variables whose critical points give the solutions. What is not immediate is how to deal with a functional that is unbounded, so no notion of the minimum can be given. Nevertheless it can be reduced using a general method applied to this kind of problem to a functional I of a single variable, which is easier to deal with and for which we will find a minimum; in other words we will find a minimal-energy solution for the problem, or ground state. In order to apply the general theory to this new functional, some aspects about its geometry and compactness have to be explored. Generally speaking, the minimum, even for a functional that is bounded from below, is not guaranteed.

To this aim, an interesting notion is introduced in Critical Point Theory, which is the Palais–Smale condition: a smooth functional I is said to satisfy the Palais–Smale compactness condition if any sequence $\{u_n\}$ is such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ admits a convergence subsequence (eventually up to subsequences). From this it is very easy to check that, for example, the functional $I(u) = e^u, u \in \mathbb{R}$, although bounded below, does not satisfy the Palais–Smale condition: the sequence $\{-n\}$ is such that $I(-n) = e^{-n}$ is bounded, $I'(-n) = e^{-n}$ goes to zero, but $\{-n\}$ is unbounded so cannot have any convergent subsequence.

Now this study is divided into two cases depending on the values of p. In fact the geometry of the functional is very different, and the estimate we find for $p \in (5/2, 5)$ is not valid for $p \in (2, 5/2]$.

We say here that the range $p \in (2, 5)$ is natural for this kind of problem in order to have a well-defined energy functional. So we are able to cover the entire range of p's; this is not trivial: many interesting problems are solved just for suitable values of the parameter p.

In both cases, we will argue by studying not directly our functional I, but a nearby one. More specifically, we perform the following:

- We introduce a parameter $\lambda > 0$, which measures in some sense the distance from the original functional to the new one;
- We find critical points of this new functional, which then solve not exactly our problem but a near one;
- We take these approximating solutions and show that they actually form a sequence converging to a solution of the original problem, or to a minimum of the original functional.

In showing the steps above we need to find suitable estimates, prove the right geometry of the functionals, and prove the right compactness properties in order to avoid, as we said before, the escaping of the quasi solutions. All these concepts will be made rigorous in the next sections, where the implementation of suitable modifications of the classical Mountain Pass theorem is carried out.

3. Results

Before stating the results, we show in which spaces the unknowns have to be found. They seems to be the right ones since they combine and merge the needs of the mathematical process (the rigor of the statement and the justification of the steps) and the physical result (its reasonability).

Looking at the first equation in the system, we see that the unknown u has to be found in the usual Sobolev space $H^1(\mathbb{R}^3)$, namely the functions are square summable joint with their gradients. This is quite reasonable even from a physical point of view since the Hilbert space appears in many problems in Quantum Mechanics. The norm in this space is denoted by $\| \cdot \|$.

The difficult point is the understanding of the right space in which to find the second unknown ϕ. In fact, looking at the second equation, it appears in a nonlinear fashion, although it is interesting from a mathematical point of view since it is the mean curvature operator in the Minkowski metric. In fact we are forced to find a different space in which to look at the unknown ϕ. The right setting for the second unknown has to be

$$\mathcal{X} := \mathcal{D}^{1,2}(\mathbb{R}^3) \cap \{\phi \in C^{0,1}(\mathbb{R}^3) : \|\nabla\phi\|_\infty \leq 1\}$$

where $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the L^2—norm of the gradient. We denote by $\| \cdot \|_q$ the norm in $L^q(\mathbb{R}^3)$ for $q \in [1, +\infty]$.

Some interesting properties of the space $\mathcal{X}$ are listed below:

1. $\mathcal{X}$ has continuous embedding in $W^{1,p}(\mathbb{R}^3) = \{u \in L^p(\mathbb{R}^3) : |\nabla u| \in L^p(\mathbb{R}^3)\}$ for every $p \in [6, +\infty)$;
2. $\mathcal{X}$ has continuous embedding in $L^\infty(\mathbb{R}^3)$;
3. If $\phi \in \mathcal{X}$, then $\lim_{|x|\to\infty} \phi(x) = 0$;
4. $\mathcal{X}$ is weakly closed;
5. If $\{\phi_n\}_n \subset \mathcal{X}$ is bounded, there exists $\bar{\phi} \in \mathcal{X}$ such that, up to a subsequence, $\phi_n \rightharpoonup \bar{\phi}$ weakly in $\mathcal{X}$ and uniformly on compact sets.

For a proof see Lemma 2.1 in [33].

3.1. Statement of the Results

Once we have the functional framework, the definition of weak solution we give is the following. A couple $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{X}$ is a weak solution of (1) if

$$
\begin{cases}
\displaystyle\int_{\mathbb{R}^3} (\nabla u \nabla v + uv + \phi uv) = \int_{\mathbb{R}^3} |u|^{p-1} uv \\[2mm]
\displaystyle\int_{\mathbb{R}^3} \frac{\nabla\phi\nabla\psi}{\sqrt{1 - |\nabla\phi|^2}} = \int_{\mathbb{R}^3} u^2 \psi.
\end{cases}
$$

for all $(v, \psi) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3)$.

It would be interesting, however, to find such a solution, namely working in the spaces $H^1(\mathbb{R}^3)$ and $\mathcal{X}$. In fact, due to technical difficulties, we are able to find solutions in some subspace of them, where the functions have a symmetry property, which helps in the computations.

Then we introduce the spaces of radial functions

$$H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) \mid u \text{ is radial }\},$$

$$\mathcal{X}_r = \{\phi \in \mathcal{X} \mid \phi \text{ is radial}\}.$$

The main result we are going to discuss is the following.

Theorem 1. *Let $p \in (2,5)$. Problem (1) possesses a radial ground state solution, that is, a solution $(u^*, \phi^*) \in H_r^1(\mathbb{R}^3) \times \mathcal{X}_r$ whose energy is minimal among all the other nontrivial radial solutions.*

We point out that different approaches are used according to the values of $p \in (5/2, 5)$ or $p \in (2, 5/2]$. In any case, the solutions we find, u and ϕ, satisfy the integral identities as stated above. But actually they are classical solutions, namely of class $C^2(\mathbb{R}^3)$, and then the equations are also solved pointwise. The proof of this last fact relies on classical bootstrap arguments.

This paper is organized as follows. In Section 3.2 some preliminaries are given. They are used to give the right variational framework for the problem. In Section 3.3 we recall some well-known arguments that are used to prove Theorem 1 in case $p \in (5/2, 5)$. In Section 3.4 we give some ideas of how to obtain the result stated in the case $p \in (2, 5/2]$. All the details, for the interested reader, are in [1,2].

3.2. Functional Setting and Preliminary Results

Our problem is variational in the sense that it comes from the action functional F defined by

$$F(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) + \frac{1}{2} \int_{\mathbb{R}^3} \phi u^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}$$
$$- \frac{1}{2} \int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla \phi|^2}\right).$$

The verification of this is straightforward, and it happens that the partial derivatives of F give the operators appearing in the two equations. So we need to find critical points of F, and a first variational principle holds.

Proposition 1. *The pair $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{X}$ is a weak solution of (1) if and only if it is a critical point of F.*

In this way we are changing the problem of finding solutions to the system into one of finding critical points of a function of two variables that is differentiable. Then Critical Point Theory enters the game, and we can take advantage of the abstract results, which ensures the existence of critical points for functionals.

Some difficulties, however, arise now. First of all, the space $\mathcal{X}$ is not a vector space, implying among other things that not all the directions are admissible for computing the derivative of the functional; even the notion of the critical point of F requires some care. Note that for this aim, radial symmetry is not yet necessary.

Moreover the functional F is unbounded from above and from below, in the sense that there is a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that $F(u_n, 0) \to +\infty$ and, with a fixed function $u_* \in H^1(\mathbb{R}^3) \setminus \{0\}$, a sequence $\{\phi_n\} \subset \mathcal{X}$ such that $F(u_*, \phi_n) \to -\infty$.

The next step is then trying to deal with a better functional. This is performed by solving the second equation for every fixed $u \in H^1(\mathbb{R}^3)$. In fact considering the functional

$$E : H^1(\mathbb{R}^3) \times \mathcal{X} \to \mathbb{R}$$

defined as

$$E(u, \phi) = \int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla \phi|^2} \right) - \int_{\mathbb{R}^3} \phi u^2.$$

the following holds.

Lemma 1. *For any fixed $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{X}$ such that the following properties hold:*

1. *ϕ_u is the unique minimizer of $E(u, \cdot) : \mathcal{X} \to \mathbb{R}$ and $E(u, \phi_u) \leq 0$, namely*

$$\int_{\mathbb{R}^3} \phi_u u^2 \geq \int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla \phi_u|^2} \right); \tag{2}$$

2. *$\phi_u \geq 0$ and $\phi_u = 0$ if and only if $u = 0$;*
3. *If ϕ is a weak solution of the second equation of system (1), then $\phi = \phi_u$, and it satisfies the following equality:*

$$\int_{\mathbb{R}^3} \frac{|\nabla \phi_u|^2}{\sqrt{1 - |\nabla \phi_u|^2}} = \int_{\mathbb{R}^3} \phi_u u^2.$$

Moreover, if $u \in H^1_r(\mathbb{R}^3)$, then $\phi_u \in \mathcal{X}_r$ is the unique weak solution of the second equation of system (1).

As we can see, here enters radial symmetry: only in the radial case are we able to prove that the minimum of the functional $E(u, \cdot)$ (which also exists in the non-radial setting) solves the second equation of (1). Due to this fact, u has to be radial (and in this case ϕ_u also is).

By Lemma 1, we are able to define the following functional on $H^1(\mathbb{R}^3)$ of a single variable by

$$
\begin{aligned}
I(u) &= F(u, \phi_u) \\
&= \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 \right) + \frac{1}{2} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla \phi_u|^2} \right) \\
&= \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 \right) - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} - \frac{1}{2} E(u, \phi_u).
\end{aligned}
$$

to show that the functional C^1 is not immediate since we do not know if the map $u \mapsto \phi_u$ is. And in fact the smoothness of I is proved by hand using the definition. See [1] for the details.

The advantage now is that this functional is no longer strongly indefinite.

As we have seen, we have to restrict the functional to the subspace of radial functions $H^1_r(\mathbb{R}^3)$. The important point now is that $H^1_r(\mathbb{R}^3)$ is a natural constraint for I in the sense that

$$I'(u)[v] = 0 \quad \forall v \in H^1_r(\mathbb{R}^3) \implies I'(u)[v] = 0 \quad \forall v \in H^1(\mathbb{R}^3).$$

Indeed I is invariant under the action of $O(3)$ on $H^1(\mathbb{R}^3)$, that is

$$T_g : u \in H^1(\mathbb{R}^3) \mapsto u \circ g \in H^1(\mathbb{R}^3), \quad g \in O(3).$$

This can be seen by using the previous Lemma, which gives

$$E(u, T_{g^{-1}} \phi_{T_g u}) = E(T_g u, \phi_{T_g u}) \leq E(T_g u, T_g \phi_u) = E(u, \phi_u)$$

and so $\phi_u = T_{g^{-1}}\phi_{T_g u}$ by virtue of the uniqueness of the minimizer of $E(u, \cdot)$. Once we have $\phi_{T_g u} = T_g \phi_u$ the invariance of I is easy to check. The Palais Principle of Symmetric Criticality can be applied and implies that we can find critical points of I on $H_r^1(\mathbb{R}^3)$. In other words we arrive at a second variational principle:

Proposition 2. *If $(u, \phi) \in H_r^1(\mathbb{R}^3) \times \mathcal{X}_r$ is a weak nontrivial solution of* (1), *then $\phi = \phi_u$ and u is a critical point of I. On the other hand, if $u \in H_r^1(\mathbb{R}^3) \setminus \{0\}$ is a critical point of I, then (u, ϕ_u) is a weak nontrivial (radial) solution of* (1).

Moreover, u is a ground state of I if and only if (u, ϕ_u) is a ground state of F.

Put in other terms, to find solutions of (1) we are reduced to finding critical points of I on $H_r^1(\mathbb{R}^3)$. Moreover the solution with minimal energy of the system is exactly the ground state of I.

Observe that searching for critical points of I means solving

$$-\Delta u + u + \phi_u u = |u|^{p-1}u \quad \text{in } \mathbb{R}^3. \tag{3}$$

With the aim of looking at critical points of I, we immediately observe that, even in presence of an acceptable power nonlinearity, the behavior of the functional is not clear the under the rescaling

$$t \in (0, +\infty) \mapsto u_t := t^\alpha u(t^\beta \cdot) \in H_r^1(\mathbb{R}^3).$$

This means that classical scaling arguments are not applicable (as, e.g., for the Schrödinger–Maxwell system). This is indeed another difficulty in the problem.

However some advantages of working with variational problems is that we can find a relation that any solution has to satisfy. In Physics this is also called *Virial Theorem* and in Mathematics the Pohozaev (or Derrick's) Theorem: if (u, ϕ) is a smooth solution of (1), then the following identity is satisfied:

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2}\int_{\mathbb{R}^3} u^2 + 2\int_{\mathbb{R}^3} \frac{|\nabla \phi|^2}{\sqrt{1-|\nabla \phi|^2}}$$
$$- \frac{3}{2}\int_{\mathbb{R}^3}\left(1-\sqrt{1-|\nabla \phi|^2}\right) = \frac{3}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}.$$

The proof of this identity is carried out by multiplying the first equation by $x \cdot \nabla u$, the second one by $x \cdot \nabla \phi$, integrating on a ball B_R, and making standard computations. Of course the boundary terms that appear have to be carefully estimated. From a purely formal point of view, it can also be derived by the equation

$$\frac{d}{dt}I(u(tx))\Big|_{t=1} = 0.$$

We omit the technical details here.

Let us describe now the approach we use to find critical points of I. In fact it is based on variants and on suitable modifications of Mountain Pass-type arguments.

3.3. Sketch of Proof of Theorem 1: The Case $p \in (5/2, 5)$

In order to find critical points of I we have to study its properties, especially with regard to the geometry and the compactness. The next technical lemma is useful for studying the shape and the behavior of the functional I. We also present the short but interesting proof. This result permits control of the gradient of ϕ_u with the norm of the same u. This will be useful later; however it is an independent result that is interesting on its own.

Lemma 2. *Let q be in $[2, 3)$. Then there exist positive constants C and C' such that, for any $u \in H^1(\mathbb{R}^3)$, we have*

$$\|\nabla \phi_u\|_2^{\frac{q-1}{q}} \leq C\|u\|_{2(q^*)'} \leq C'\|u\|.$$

Here we denote by q^* the critical Sobolev exponent related to q, and $(q^*)'$ is its conjugate exponent, namely

$$q^* = \frac{3q}{3-q} \quad and \quad (q^*)' = \frac{3q}{4q-3}.$$

Proof. Since $\|\nabla \phi_u\|_\infty \leq 1$ and $q \in [2, 3)$ we have

$$\|\phi_u\|_{q^*} \leq C\|\nabla \phi_u\|_q = C\left(\int_{\mathbb{R}^3} |\nabla \phi_u|^2 |\nabla \phi_u|^{q-2}\right)^{1/q} \leq C\|\nabla \phi_u\|_2^{2/q},$$

so, by (2) and $2(q^*)' \in [2, 6]$, we have

$$\|\nabla \phi_u\|_2^2 \leq C\int_{\mathbb{R}^3}\left(1 - \sqrt{1 - |\nabla \phi_u|^2}\right) \leq C\int_{\mathbb{R}^3} \phi_u u^2$$

$$\leq C\|\phi_u\|_{q^*}\|u\|_{2(q^*)'}^2 \leq C\|\nabla \phi_u\|_2^{2/q}\|u\|_{2(q^*)'}^2$$

which gives the conclusion. $\square$

For the case $p \in (5/2, 5)$ the proof of the main result uses the following monotonicity trick from Struwe [39,40].

Proposition 3. *Let $(X, \|\cdot\|)$ be a Banach space and $J \subset \mathbb{R}^+$ an interval. Consider a family of C^1 functionals I_λ on X defined by*

$$I_\lambda(u) = A(u) - \lambda B(u), \qquad for \ \lambda \in J,$$

with B being non-negative and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to +\infty$ such that $I_\lambda(0) = 0$. For any $\lambda \in J$, we set

$$\Gamma_\lambda := \{\gamma \in C([0, 1], X) \mid \gamma(0) = 0, \ I_\lambda(\gamma(1)) < 0\}.$$

Assume that for every $\lambda \in J$, the set Γ_λ is non-empty and

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0.$$

Then for almost every $\lambda \in J$, there is a sequence $\{v_n\} \subset X$ such that

(i) *$\{v_n\}$ is bounded in X;*
(ii) *$I_\lambda(v_n) \to c_\lambda$, as $n \to +\infty$;*
(iii) *$I'_\lambda(v_n) \to 0$ in the dual space X^{-1} of X, as $n \to +\infty$.*

The importance of the previous result is clear: it allows us to obtain a critical point of our original functional I by means of "approximation". In other words, instead of dealing with the functional I we work with a slight perturbation I_λ for which the proposition furnishes a bounded Palais–Smale sequence and then a critical point u_λ. Of course the critical point u_λ does not need to solve our equation, but just an approximated one, due to the presence of the parameter λ (see below).

The great advantage of this procedure is that our "approximating sequence" $\{u_\lambda\}$ satisfies the identity $I'_\lambda(u_\lambda) = 0$, which proves very useful during the computations.

In our specific case $X = H_r^1(\mathbb{R}^3)$, $J = [1/2, 1]$,

$$A(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \frac{1}{2} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{2} \int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla \phi_u|^2} \right),$$

$$B(u) = \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}.$$

and by (2), $A(u) \to +\infty$ as $\|u\| \to +\infty$.

Then our aim is to look for bounded Palais–Smale sequences of the following perturbed functionals:

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \frac{1}{2} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{2} \int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla \phi_u|^2} \right) - \frac{\lambda}{p+1} \int_{\mathbb{R}^3} |u|^{p+1}$$

for almost all λ near 1. The idea is that for $\lambda \to 1$ we recover the functional I.

At this point we need to show that the above abstract Proposition 3 can be applied. And this fact is a consequence of the next result.

Proposition 4. *The following facts hold:*

(i)　$\Gamma_\lambda \neq \emptyset$ for every $\lambda \in [1/2, 1]$;

(ii)　For every $\lambda \in [1/2, 1]$, there is $\alpha > 0$ and $\rho > 0$, suitably small, such that $I_\lambda(u) \geq \alpha$, for all $u \in H^1(\mathbb{R}^3)$, with $\|u\| = \rho$. It follows that $c_\lambda \geq \alpha > 0$.

Proof. We just show the first point (i). Indeed for a fixed $\lambda \in [1/2, 1]$ and $u \in H_r^1(\mathbb{R}^3) \setminus \{0\}$, by Lemma 2 and for $q \in [2, 3)$ and standard inequalities we have

$$\begin{aligned}
I_\lambda(u) &\leq \frac{1}{2} \|u\|^2 + \frac{1}{2} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{\lambda}{p+1} \|u\|_{p+1}^{p+1} \\
&\leq \frac{1}{2} \|u\|^2 + c \|\phi_u\|_6 \|u\|_{\frac{12}{5}}^2 - \frac{\lambda}{p+1} \|u\|_{p+1}^{p+1} \\
&\leq \frac{1}{2} \|u\|^2 + c \|\nabla \phi_u\|_2 \|u\|^2 - \frac{\lambda}{p+1} \|u\|_{p+1}^{p+1} \\
&\leq \frac{1}{2} \|u\|^2 + c \|u\|^{\frac{3q-2}{q-1}} - \frac{\lambda}{p+1} \|u\|_{p+1}^{p+1}.
\end{aligned}$$

So, whenever $\lambda \in [1/2, 1]$ and $t > 0$, we get

$$I_\lambda(tu) \leq c_1 t^2 + c_2 t^{\frac{3q-2}{q-1}} - c_3 \lambda t^{p+1}.$$

From the fact that $p \in (5/2, 5)$, it is possible to find $q \in [2, 3)$ such that, for t sufficiently large, $I_\lambda(tu) < 0$.　□

Then by a standard but straightforward proof we obtain the following.

Proposition 5. *For almost every $\lambda \in [1/2, 1]$, there exists $u_\lambda \in H_r^1(\mathbb{R}^3)$, $u_\lambda \neq 0$, such that $I_\lambda'(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_\lambda$.*

With this procedure we have found a nontrivial solution $u_\lambda \in H_r^1(\mathbb{R}^3)$ of the following perturbed (namely depending on λ) equation:

$$-\Delta u + u + \phi_u u = \lambda |u|^{p-1} u \quad \text{in } \mathbb{R}^3 \tag{4}$$

for almost any value of λ near 1. This is not exactly (3); indeed we need to pass to the limit as $\lambda \to 1$ in order to recover a nontrivial critical point for I. This is performed by using the

relevant fact that u_λ satisfies the equation $I'_\lambda(u_\lambda) = 0$ and so Nehari- and Pohozaev-type identities are available

$$\frac{d}{dt} I_\lambda(tu(x))\Big|_{t=1} = 0 \quad \text{and} \quad \frac{d}{dt} I_\lambda(u(tx))\Big|_{t=1} = 0.$$

Indeed as we write down and combine the above identities together and then pass to the limit as $\lambda \to 1$ we obtain a nontrivial critical point u^* of I.

Up to now, we do not know if it is actually a ground state for I. But we define

$$\mathcal{S}_r = \left\{ u \in H^1_r(\mathbb{R}^3) \setminus \{0\} \mid I'(u) = 0 \right\} \neq \varnothing,$$

$$\sigma_r = \inf_{u \in \mathcal{S}_r} I(u).$$

The above infimum is strictly positive. In fact, any $u \in \mathcal{S}_r$ satisfies

$$\|u\|^2 \leq \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} u^2 + \int_{\mathbb{R}^3} \frac{|\nabla \phi_u|^2}{\sqrt{1 - |\nabla \phi_u|^2}} = \int_{\mathbb{R}^3} |u|^{p+1} \leq C\|u\|^{p+1},$$

and therefore

$$\inf_{u \in \mathcal{S}_r} \|u\| > 0,$$

meaning that the set $\mathcal{S}_r$ is bounded away from zero. However since $I(u) \geq c\|u\|^2$ for all $u \in \mathcal{S}_r$, we conclude. As a final step, one easily shows that the infimum is achieved, proving the existence of a ground state solution u^* for (3). Having u^*, we recover $\phi^* := \phi_{u^*}$ as given by Lemma 1, namely as the unique minimizer of $E(u^*, \cdot)$.

3.4. Sketch of Proof of Theorem 1: The Case $p \in (2, 5/2]$

The case $p < 5/2$ is more involved and indeed a different approach is needed. In fact just the monotonicity trick proves not to be useful, since it was based on the fundamental Lemma 2 used at the end of the proof of Proposition 4.

We now use a different approximating problem: instead of (4) we consider the following one:

$$-\Delta u + u + \phi_u u + \lambda \|u\|_2 u = |u|^{p-1}u + \lambda |u|^{q-1}u, \qquad u \in H^1_r(\mathbb{R}^3)$$

where

- $\lambda \in (0, 1]$;
- $q \in (\max\{p+1, 4\}, 6)$.

As we can see, two parameters appear in the perturbation. We naturally have the associated energy functional

$$\mathcal{I}_\lambda(u) = I(u) + \frac{\lambda}{3} \left(\int_{\mathbb{R}^3} u^2 \right)^{3/2} - \frac{\lambda}{q+1} \int_{\mathbb{R}^3} |u|^{q+1}$$

where I is the original functional defined before. As a first step we aim to find a solution of the above perturbed equation (or a critical point of $\mathcal{I}_\lambda$) and then pass to the limit as $\lambda \to 0^+$, instead of $\lambda \to 1$.

For such a functional the following properties hold.

- There is a Mountain Pass Geometry for $\mathcal{I}_\lambda$, which is uniform (i.e., independent) in λ. This is something like Proposition 4 of the previous case, but with a proof that is a little bit more involved. After all, two additional terms in the functional are present with respect to the previous functional.

- The Mountain Pass level $c_\lambda > 0$ satisfies

$$0 < m \le c_\lambda \le M \quad \text{and} \quad c_\lambda \to c_* \text{ as } \lambda \to 1.$$

   Indeed this is a fundamental step in order to take the limit, since we do not want c_λ to go to zero; otherwise we have the risk of finding the zero solution.
- $\mathcal{I}_\lambda$ satisfies the Palais–Smale condition: it is important to avoid escaping at infinity, as we commented before.

The result of these facts are very technical and quite long. We do not think that they are necessary here, but the interested reader can refer to [2]. We just say that in proving such properties, the "new" perturbation, which involves λ and q, has been extensively used. We suspect that also in the first case we could have used the double-parameter perturbation, but we would have obtained the same result without any real advantage and at the expense of a more complicated proof.

Then, with the Palais–Smale condition also satisfied, there is an analogous of Proposition 5.

Proposition 6. *For any $\lambda \in (0, 1]$, there is $u_\lambda \in H_r^1(\mathbb{R}^3)$, $u_\lambda \ne 0$, such that $\mathcal{I}_\lambda'(u_\lambda) = 0$ and $\mathcal{I}_\lambda(u_\lambda) = c_\lambda$.*

The nontrivial part now is to show that the family $\{u_\lambda\}_{\lambda \in (0,1]}$ gives a bounded Palais–Smale sequence for the original functional I. In fact after some computations, one effectively shows the existence of a critical point u^* for I at the level c_*.

Finally, the fact that I possesses a ground state is addressed as before.

Remark 1. *Actually in [2] we considered a nonlinearity that is still more general. Indeed we studied the problem*

$$\begin{cases} -\triangle u + u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -div\left(\dfrac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = u^2 & \text{in } \mathbb{R}^3, \\ u(x) \to 0, \ \phi(x) \to 0, & \text{as } x \to \infty, \end{cases}$$

with the following assumptions on the nonlinearity f:

1. *$f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{s\to 0} f(s)/s = 0$;*
2. *$|f(s)| \le C(1 + |s|^p)$ for $p \in (2, 5)$;*
3. *For any $s > 0$, $0 < \varrho F(s) \le f(s)s$, where $\varrho \in (3, 4)$ and $F(s) = \int_0^s f(\tau)d\tau$.*

They are quite natural when one deals with variational methods. Actually we also obtain a multiplicity result of solutions; in fact the problem admits infinitely many solutions in which the energy functional tends to infinity. In this case a symmetric version of the Mountain Pass theorem is used.

Finally it is worth noting that the critical case is also treated. However a further assumption due to compactness issues is necessary:

4. *There exist $D > 0$ and $2 < r < 6$ such that $F(t) \ge Dt^r$ for $t \ge 0$.*

We prove then that under the set of assumption (1)–(4) the system has a ground state solution if (i) $r \in (4, 6)$, or (ii) $r \in (2, 4]$ and D is sufficiently large.

4. Discussion

The results discussed here are interesting since they show how many techniques can be combined to solve an elliptic system: variational tools, fine estimates, and the perturbation approach. The results that this paper discussed are new; however some open questions remain, like the following:

- How to treat the nonradial case?
- What about the problem with a priori given the L^2 norm of u?
- What about different kinds of nonlinearity, like, e.g., of logarithmic type?

  We think they are all challenging and nothing is known in this direction.

5. Conclusions

We have seen how an approximating procedure can be useful in different contexts. Actually the type of approximation is dictated by the problem or by its difficulties. We have to say that in many problems, an approximating procedure can be implemented and usually depends on the nature of the problem. We have seen that just the range where the parameter p varies led us to choose different approximations of the original problem.

In our case, to prove the result in the case $p \in (5/2, 5)$ we have approximated in some way the problem with a parameter $\lambda > 0$, for which it was easier to find a solution. Then sending λ to 1, we recovered the original problem and its solution. However in doing that, an auxiliary result, not available for the second case, $p \in (2, 5/2]$, was used. And in fact the approximating procedure for this second case, $p \in (2, 5/2]$, is very different and indeed more complex, and for this reason we have skipped many details.

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