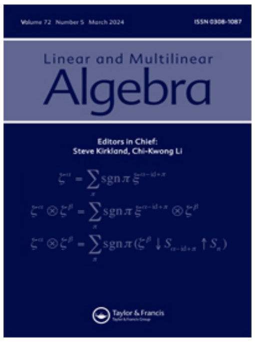

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Principal angles in pseudo-euclidean spaces of index 1

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ABSTRACT

In this paper, we introduce the notion of principal angles between subspaces of the same signature in a (real, complex or quaternionic) pseudo-euclidean space of index 1. We show that these determine the relative position of an important class of pairs of hyperbolic and elliptic subspaces (Theorems 3.9 and 4.10). Also, as an application, we will see that these angles can be used to study the relative position of pairs of an important class of totally geodesic submanifolds of the (real, complex or quaternionic) hyperbolic space (Theorem 6.3). As a consequence we obtain a nice interpretation of these principal angles as geometric invariants of the hyperbolic space.

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1. Introduction

The study of the principal angles between two subspaces is both important and natural for the understanding the geometry of a Euclidean space E. These were first introduced by C. Jordan [1] for real subspaces and were considered by many authors for real and complex subspaces (e.g. [2–6]); and more recently for quaternionic (right) vector spaces in [7,8]. Due to their relevance, the principal angles are being investigated by different authors in several areas. We highlight the importance of these in the study of Clifford algebras (e.g. [9–11]), Grassmann manifolds (e.g. [12,13]), statistic (e.g. [14]), among others.

Principal angles arise naturally when dealing with the relative position of a pair of subspaces in a Euclidean space E. Indeed, they determine the relative position of a pair (U,V) of subspaces of E in the following sense: two ordered pairs (U,V) and $(\widetilde{U},\widetilde{V})$ of subspaces of E have the same principal angles if and only if there is an isometry f of E such that $f(U)=\widetilde{U}$ and $f(V)=\widetilde{V}$ (see [8,Proposition 3.7]). So, any parameter that determines the relative position of a pair of subspaces must be a function of the principal angles. The relative position of two objects, in the sense above, is studied in different contexts. For example, this was considered for pairs of Lagrangians in a Hermitian complex vector space in [15], where the author showed that the relative position of a pair of Lagrangian

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subspaces (L_1, L_2) is determined by the characteristic polynomial of a Souriau matrix associated to (L_1, L_2). Also, it was investigated for pairs of totally geodesic submanifolds of the real hyperbolic space in [8,Section 4], where the authors showed that the relative position of a pair of non-asymptotic totally geodesic submanifolds (h_1, h_2) is determined by the distance between h_1 and h_2 and the principal angles between two specific tangent spaces.

In light of many results on relative position of two objects in several context, and more specially in linear spaces, one may wonder if we can obtain similar results in pseudo-euclidean spaces of index 1 (vector spaces endowed with an indefinite Hermitian form, where the maximal dimension of a negative subspace is 1). Motivated by this, we present a brief study of the relative position of pairs of subspaces with the same signature in a pseudo-euclidean space of index 1.

We denoted by $\mathbb{F}$ the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$ or the algebra of quaternions $\mathbb{Q}$. The structure of paper is as follows.

In Section 2, we review some elementary facts concerning $\mathbb{F}$-Hermitian forms on (right) $\mathbb{F}$-vector spaces. In particular, we review the properties of the E. Moore determinant for Hermitian matrices that we will use throughout the article. It is necessary because we include quaternionic vector spaces in our consideration.

Let X be a pseudo-euclidean $\mathbb{F}$-space of index 1 and (U, V) a pair of non-degenerate subspaces of X with the same signature. Sections 3 and 4 deal with the relative position of a pair of such subspaces. In particular, we explore the singular value decomposition of the orthogonal projection $P : U \to V$ of U onto V, more precisely, we investigate the diagonalization of the $\mathbb{F}$-linear map $P^*P : U \to U$. If X is a Euclidean $\mathbb{F}$-space then P^*P is diagonalizable, as it is self-adjoint, and the singular values always exist and are reals. But when X is a pseudo-euclidean $\mathbb{F}$-space some difficulties appear. For instance, if U is a hyperbolic subspace of X and $\mathbb{F} = \mathbb{C}$, then a self-adjoint map $f : U \to U$ is not necessarily diagonalizable, and if it is diagonalizable it may have non-real eigenvalues or a degenerate eigenspace (see [16,Section 4.2]). The situation is even more critical if we consider $\mathbb{F} = \mathbb{Q}$. So, our first objective is to characterize the pairs of subspaces (U, V), with the same signature, such that $P^*P : U \to U$ is diagonalizable and their eigenvalues are real. Next, we introduce some parameters for such subspaces, which we also call *principal angles* because the difference with the principal angles in Euclidean spaces is minimal. Then, we prove that these principal angles can obtained from the singular values of $P : U \to V$ (Propositions 3.7 and 4.8). The importance of the principal angles is reflected in Theorems 3.9 and 4.10, where we show that they determine the relative position of an important class of pairs of hyperbolic and elliptic subspaces of X. It is pivotal to notice that the last assertion is not true in general, which is an important difference with the Euclidean case. For instance, in Example 4.9 we have pairs (U, V) and $(\widetilde{U}, \widetilde{V})$ of elliptic subspaces of X with the same principal angles, but in the absence of an isometry $f : X \to X$ such that $f(U) = \widetilde{U}$ and $f(V) = \widetilde{V}$.

In Section 5 we study the duality between principal angles of two subspaces of X and the principal angles of their orthogonal complements. In particular, we show that the non-zero principal angles between two subspaces coincide with the non-zero principal angles between their orthogonal complements. Finally, Section 6 is dedicated to the study of the relative position of pairs of totally geodesic submanifolds in the $\mathbb{F}$-hyperbolic space $\mathbb{H}^n_{\mathbb{F}}$. If U is a hyperbolic $\mathbb{F}$-subspace of the pseudo-euclidean space $\mathbb{F}^{n,1}$ (see Section 6), then U determines a totally geodesic submanifold Σ_U of $\mathbb{H}^n_{\mathbb{F}}$. We show that if U and V are two hyperbolic $\mathbb{F}$-subspaces such that $U \cap V$ is non-degenerate, then the principal angles between U and V determine the relative position of Σ_U and Σ_V in $\mathbb{H}^n_{\mathbb{F}}$ (Theorem 6.3). Also, we give conditions for the existence and uniqueness of a common perpendicular $\mathbb{F}$-geodesic to two such totally geodesic submanifolds Σ_U and Σ_V (see Proposition 6.6). Furthermore, we show that the first principal angle between U and V coincides with the distance between the totally geodesic submanifolds Σ_U and Σ_V, and that the other principal angles coincide with the (usual) principal angles between two specific tangent spaces of Σ_U and Σ_V. This provides a nice interpretation of the principal angles as geometric invariants of the hyperbolic space $\mathbb{H}^n_{\mathbb{F}}$.

2. Preliminaries

Given $a \in \mathbb{F}$, we denote by $\overline{a}$ the conjugate of a in $\mathbb{F}$, which is the same as a if $a \in \mathbb{R}$, the complex conjugate if $a \in \mathbb{C}$ and the quaternion conjugate if $a \in \mathbb{Q}$.

2.1. Hermitian forms

Let X be a finite-dimensional (right) $\mathbb{F}$-vector space, an $\mathbb{F}$-Hermitian form in X is a map $\langle \cdot,\cdot \rangle : X \times X \to \mathbb{F}$ such that

$$\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle, \quad \langle u, v\lambda\rangle = \langle u, v\rangle\lambda, \quad \langle u, v\rangle = \overline{\langle v, u\rangle},$$

for $u, v, w \in X$ and $\lambda \in \mathbb{F}$. Associated to an $\mathbb{F}$-Hermitian form $\langle \cdot,\cdot \rangle$ on X we have a symmetric $\mathbb{R}$-bilinear form $\mathrm{Re}\langle \cdot,\cdot \rangle$ on the underlying real vector space of X. Moreover, $\mathrm{Re}\langle \cdot,\cdot \rangle$ is non-degenerate if $\langle \cdot,\cdot \rangle$ is.

An *isometry* (or automorphism) of a vector space X endoweed with an $\mathbb{F}$-Hermitian form $\langle \cdot,\cdot \rangle$, is an $\mathbb{F}$-linear map $f : X \to X$ satisfying $\langle fu, fv\rangle = \langle u, v\rangle$ for every $u, v \in X$. The set of all isometries of X form a group. Let $\alpha = (u_1, \dots, u_k)$ be a k-tuple of elements of X, then the matrix whose entries are given by $\langle u_i, u_j\rangle$ is called the *Gram matrix* of α, and it is denoted by $G(\alpha)$. The following result is a well-known consequence of a theorem by Höfer [17,Theorem 1].

Lemma 2.1: *Let $\langle \cdot,\cdot \rangle$ be a non-degenerate $\mathbb{F}$-Hermitian form on X. Suppose that $\alpha = (u_1, \dots, u_k)$ and $\beta = (v_1, \dots, v_k)$ are k-tuples of elements in X, such that the subspaces $U = span\{u_1, \dots, u_k\}$ and $V = span\{v_1, \dots, v_k\}$ are non-degenerate. Then, there exists an isometry f of X such that $f(u_j) = v_j$, for $1 \le j \le k$, if and only if $G(\alpha) = G(\beta)$.*

Let $M_n(\mathbb{F})$ be the set of all $n \times n$ matrices with entries in $\mathbb{F}$. A matrix $B = [b_{ij}] \in M_n(\mathbb{F})$ is called Hermitian if it coincides with its conjugate transpose $B^* = [\overline{b_{ji}}]$. Let $\langle \cdot,\cdot \rangle$ be an $\mathbb{F}$-Hermitian form on X and $\{e_1, \dots, e_n\}$ a basis of X. The matrix $A = [a_{ij}]$ in $M_n(\mathbb{F})$ whose entries are $a_{ij} = \langle e_i, e_j\rangle$ is called the matrix of $\langle \cdot,\cdot \rangle$ relative to this basis. Clearly A is a Hermitian matrix.

At this point it is necessary to digress briefly on the determinant of a matrix. Because the quaternion multiplication is not commutative, it is impossible to define a function that extend the determinant on real and complex matrices to quaternionic matrices (e.g. [18]). But it is possible to define some functions, which also are called determinants, that preserve several rules for manipulation and computation of determinants of real and complex matrices (e.g. [18,19]). In the special case of quaternionic Hermitian matrices there is a determinant, introduced by E.H. Moore [20], which associates a real number to each quaternionic Hermitian matrix. This determinant coincides with the usual determinant of real symmetric and complex Hermitian matrices and has similar properties to this. For more details on the Moore determinant, we refer the reader to the papers [18,19,21]. We will denote the Moore determinant of a Hermitian matrix $A \in M_n(\mathbb{F})$ by $\det(A)$. In the following lemma we recall two elementary results that we will use frequently in the next sections, where part (1) is immediate and part (2) follows from the Sylvester criterion (see [21,Theorem 1.1.13]).

Lemma 2.2: *Let A be the matrix of an $\mathbb{F}$-Hermitian $\langle \cdot, \cdot \rangle$ relative to a basis of X. Then the following holds*

(1) $\langle \cdot, \cdot \rangle$ *is non-degenerate if and only if* $\det(A) \neq 0$;
(2) $\langle \cdot, \cdot \rangle$ *is positive if and only if the principal minors of order k of A are positive, for $k = 1, \dots, n$.*

Let $f : X \to X$ be an $\mathbb{F}$-linear map and $\lambda \in \mathbb{F}$. Then λ is called a right eigenvalue of f if $fv = v\lambda$ for some $v \neq 0$ in X. In this situation v is called eigenvector of f corresponding to λ. The set $\mathbb{F} - \{0\}$ acts by conjugation on the set of all right eigenvalues. The orbit of a right eigenvalue λ is a singleton if $\lambda \in \mathbb{R}$, in this case we simply say that λ is an *eigenvalue* of f.

If $\langle \cdot, \cdot \rangle$ is a non-degenerate $\mathbb{F}$-Hermitian form on X and $f : X \to X$ is $\mathbb{F}$-linear, then there exists a unique $\mathbb{F}$-linear map $f^* : X \to X$ such that

$$\langle fu, v \rangle = \langle u, f^* v \rangle$$

for every $u, v \in X$. The map f^* is called *adjoint* of f respect to $\langle \cdot, \cdot \rangle$. A map f is called *self-adjoint* if $f = f^*$. It is easy to see that when we consider f and f^* as $\mathbb{R}$-linear maps on the underlying real vector space of X, then f^* is the adjoint (transpose) of f respect to $\mathrm{Re}\langle \cdot, \cdot \rangle$.

Following the terminology on [22,Definition 9.17, p. 264], we say that an $\mathbb{F}$-vector space X endowed with a non-degenerate indefinite Hermitian form $\langle \cdot, \cdot \rangle$ is *pseudo-euclidean* space. A non-zero vector $v \in X$ is called *positive, negative or isotropic* if $\langle u, u \rangle > 0$, $\langle u, u \rangle < 0$ or $\langle u, u \rangle = 0$, respectively. The sets of all positive, negative or isotropic vectors of X are denoted by X_+, X_-, X_0, respectively. A subspace U of a pseudo-euclidean space X is called *elliptic, hyperbolic or parabolic* if the restriction of $\langle \cdot, \cdot \rangle$ to U is positive, non-degenerate and indefinite or degenerate, respectively. Note that if U is an elliptic subspace, then the restriction of $\langle \cdot, \cdot \rangle$ to U defines a (Hermitian) inner product itself.

The following lemma describes some situations in which self-adjoint maps are diagonalizable. We will use this frequently in the next sections.

Lemma 2.3: *Let $\langle \cdot, \cdot \rangle$ be a Hermitian $\mathbb{F}$-form on X and $f : X \to X$ a self-adjoint map. Then the following holds.*

(1) *If $\langle \cdot, \cdot \rangle$ is positive on X, then there exists a orthonormal basis $\{u_1 \dots, u_n\}$ such that $fu_i = u_i\lambda_i$ for some $\lambda_i \in \mathbb{R}$, $i = 1, \dots, n$.*

(2) *Suppose that* $\mathbb{F} = \mathbb{R}$ *and that* X *endowed with* $\langle \cdot, \cdot \rangle$ *is pseudo-euclidean. If* $\langle fv, v \rangle \neq 0$ *for every isotropic vector* $v \in X$*, then there exists an orthonormal basis* $\{v_1, \ldots, v_n\}$ *of* X *such that* $fv_i = v_i \lambda_i$ *for some* $\lambda_i \in \mathbb{R}$*,* $i = 1, \ldots, n$*.*

Proof: Part (1) follows from the standard theorems for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and from [23,Theorem 4.6] for $\mathbb{F} = \mathbb{Q}$. For part (2) see [22,Section 9.24, p. 271]. ∎

The *index* of a pseudo-euclidean space X is the maximum of the dimensions of the negative subspaces of X. By the Sylvester's Law of Inertia (see [24,Satz 12.46, p. 423]), the index coincide with the number of negative vectors in an orthonormal basis of X. In the following sections we will study the pairs of subspaces of pseudo-euclidean spaces of index 1. Let us see an example of such spaces. Let $\mathbb{F}^{n+1}$ be the (right) $\mathbb{F}$-vector space consisting of $(n+1)$-tuples of elements of $\mathbb{F}$. In $\mathbb{F}^{n+1}$ we consider the $\mathbb{F}$-Hermitian form

$$\langle v, w \rangle = \bar{v}_1 w_1 + \bar{v}_2 w_2 + \cdots + \bar{v}_n w_n - \bar{v}_{n+1} w_{n+1} \tag{1}$$

The pseudo-euclidean space thus obtained is denoted by $\mathbb{F}^{n,1}$. Again by Sylvester's Law of Inertia, it follows that every pseudo-eulidean $\mathbb{F}$-space X of index 1 is isometrically equivalent to $\mathbb{F}^{n,1}$.

For a practical purpose we introduce a notation. Let X be a pseudo-euclidean space and $u \in X$. Then $\|u\|$ will denote the non-negative real number such that $\|u\|^2 = \langle u, u \rangle$ if $\langle u, u \rangle \geq 0$, and $\|u\|^2 = -\langle u, u \rangle$ if $\langle u, u \rangle < 0$. We will say that $u \in X$ is unitary if $\|u\|^2 = 1$.

2.2. Pairs of subspaces of $\mathbb{F}^{n,1}$

Let $\mathbb{F}^{n,1}$ be the pseudo-euclidean $\mathbb{F}$-vector space consisting of the space of $(n+1)$-tuples of elements of $\mathbb{F}$ endowed with the $\mathbb{F}$-Hermitian form given in (1). The isometry group of $\mathbb{F}^{n,1}$ will be denoted by $U(n, 1; \mathbb{F})$. Note that the orthogonal complement $U^{\perp}$ of a elliptic, hyperbolic or parabolic subspace U of $\mathbb{F}^{n,1}$ is hyperbolic, elliptic or parabolic, respectively.

From now on the letters U and V will be reserved to denote two non-degenerate (right) subspaces of $\mathbb{F}^{n,1}$, with $\dim_{\mathbb{F}} U = k+1$, $\dim_{\mathbb{F}} V = r+1$ and $k \leq r$. Since V is non-degenerate we have $\mathbb{F}^{n,1} = V \oplus V^{\perp}$. Also, we denote by $P : U \to V$ the Hermitian orthogonal projection from U onto V, that is, the $\mathbb{F}$-linear map sending $u \in U$ to the unique element $Pu \in V$ such that $u - Pu \in V^{\perp}$. Let P^* the adjoint of P respect to $\langle \cdot, \cdot \rangle$. It is easy to see that P^* coincides with the orthogonal projection of V onto U. Considering P as a $\mathbb{R}$-linear map, we have that P^* is the adjoint of P respect to $\mathrm{Re}\langle \cdot, \cdot \rangle$.

Lemma 2.4: *Let* U*,* V *be two non-degenerate subspaces of* $\mathbb{F}^{n,1}$ *and* $P : U \to V$ *the orthogonal projection of* U *onto* V*. Suppose that* $u \in U$ *is an eigenvector of* P^*P*. If* $W = span\{u, Pu\}$*, then the following holds.*

(1) *The subspaces* $u^{\perp} \cap U$ *and* $(Pu)^{\perp} \cap V$ *are contained in* $W^{\perp}$*.*
(2) $P(u^{\perp} \cap U) \subseteq (Pu)^{\perp} \cap V$*. Moreover, if* $u^{\perp} \cap U$*,* $(Pu)^{\perp} \cap V$ *and* W *are non-degenerate, then the restriction of* P *to* $u^{\perp} \cap U$ *coincides with the orthogonal projection of* $u^{\perp} \cap U$ *onto* $(Pu)^{\perp} \cap V$*, considering these as subspaces of* $W^{\perp}$*.*

Proof: (1) Suppose that $P^*Pu = u\lambda$ for some $\lambda \in \mathbb{F}$. If u and Pu are linearly dependent, we are done. Therefore, let us assume that u and Pu are linearly independent. First we prove that $u^\perp \cap U \subseteq W^\perp$. For this notice that $\{u, Pu - u\lambda\}$ is an orthogonal basis for W. Thus, for $v \in u^\perp \cap U$ we have that

$$\langle v, Pu - u\lambda\rangle = \langle v, Pu\rangle = \langle Pv, Pu\rangle = \langle v, P^*Pu\rangle = \langle v, u\rangle\lambda = 0.$$

This equality implies that $u^\perp \cap U \subseteq W^\perp$. Now, in order to show $(Pu)^\perp \cap V \subseteq W^\perp$ notice that $\{Pu, u - Pu\}$ is an orthogonal basis for W. Then, for every $v \in (Pu)^\perp \cap V$, we have that

$$\langle v, u - Pu\rangle = \langle v, u\rangle = \langle v, Pu\rangle = 0,$$

yielding $(Pu)^\perp \cap V \subseteq W^\perp$.

(2) For convenience we set $U_1 = u^\perp \cap U$ and $V_1 = (Pu)^\perp \cap V$. If $v \in U_1$ then

$$\langle Pv, Pu\rangle = \langle v, P^*Pu\rangle = \langle v, u\rangle\lambda = 0;$$

showing $P(U_1) \subseteq V_1$.

For the last assertion of (2) note that due to non-degeneration $W^\perp = V_1 \oplus (V_1^\perp \cap W^\perp)$. By part (1), $v - Pv \in W^\perp$ for all $v \in U_1$. Thus, $v - Pv \in V^\perp \cap W^\perp \subseteq V_1^\perp \cap W^\perp$ for all $v \in U_1$. Therefore, given $v \in U_1$ we have that $Pv \in V_1$ and $v - Pv \in V_1^\perp \cap W^\perp$. In this way, the restriction of P to U_1 coincides with the orthogonal projection of U_1 onto V_1, considering these as subspaces of $W^\perp$. ■

In the following sections, we will investigate the singular value decomposition of the $P : U \to V$. As a first step we characterize the pairs of subspaces U, V such that U possesses an orthonormal basis of eigenvectors of $P^*P : U \to U$. For this we will study separately the map P^*P for pairs of hyperbolic and elliptic subspaces.

Let E be a Euclidean $\mathbb{F}$-space, that is, a $\mathbb{F}$-vector space endowed with a positive $\mathbb{F}$-Hermitian form. At the end of this preliminary section, we recall the definition of principal angles between subspaces of E, see [8,Definition 3.1].

Definition 2.5: Let S, T be two (right) subspaces of a Euclidean $\mathbb{F}$-space E. Suppose that $\dim_{\mathbb{F}} S = k$, $\dim_{\mathbb{F}} T = r$ with $k \leq r$. The *principal angles* between S and T are recursively defined by

$$\cos^2\theta_1 = \max\{|\langle x, y\rangle|^2 \mid x \in S, y \in T, \|x\| = \|y\| = 1\} = |\langle x_1, y_1\rangle|^2$$

and

$$\cos^2\theta_i = \max\{|\langle x, y\rangle|^2 \mid x \in S_i, y \in T_i, \|x\| = \|y\| = 1\} = |\langle x_i, y_i\rangle|^2,$$

where $S_i = \{x \in S \mid \langle x, x_j\rangle = 0, \forall j \leq i-1\}$ and $T_i = \{y \in T \mid \langle y, y_j\rangle = 0, \forall j \leq i-1\}$ for $1 < i \leq k$. Also, the vectors x_i, y_i are called principal vectors associated to θ_i.

3. Pairs of hyperbolic subspaces in $\mathbb{F}^{n,1}$

In this section we suppose that the subspaces U and V of $\mathbb{F}^{n,1}$ are hyperbolic. We explore the relative position of a pair (U, V) in $\mathbb{F}^{n,1}$. For this, first we investigate the singular value decomposition of $P : U \to V$, and then we introduce the principal angles between an important class of hyperbolic subspaces. Thus, we show that the principal angles determine the relative position of a pair (U, V) of such hyperbolic subspaces.

Observe that if u is negative vector in U, then Pu is negative vector of V. Also, notice that $u-Pu$ is a non-negative vector for all $u \in U$, since $V^\perp$ is elliptic. The following result is an elementary observation.

Lemma 3.1: *Suppose that U and V are two hyperbolic subspaces of* $\mathbb{F}^{n,1}$. *Then* $U \cap V = \{u \in U \mid P^*Pu = u\}$.

Proof: Suppose that $u \in U$ is such that $P^*Pu = u$, then

$$\langle u - Pu, u - Pu\rangle = \langle u, u\rangle - \langle Pu, Pu\rangle = \langle u, u\rangle - \langle P^*Pu, u\rangle = 0,$$

and since $u - Pu \in V^\perp$ and $V^\perp$ is elliptic, we obtain $u-Pu = 0$. Hence, $u = Pu \in U \cap V$. For the converse inclusion suppose $u \in U \cap V$, which is equivalent to $u = Pu$. Let $v \in U$, then we see that

$$\langle v, P^*Pu\rangle = \langle v, P^*u\rangle = \langle Pv, u\rangle = \langle v, u\rangle,$$

and since U is non-degenerate we obtain $P^*Pu = u$. ■

We remark that since U is hyperbolic then it is possible that $U \cap V$ will be degenerate or non-degenerate (hyperbolic or elliptic). A first goal in this section is to characterize the pairs of subspaces U, V such that $U \cap V$ is non-degenerate.

Notice that if there is an orthonormal basis $\{u_0, \dots, u_k\}$ of U such that $P^*Pu_i = u_i\lambda_i$, for some real numbers $\lambda_0 \geq \cdots \geq \lambda_k$, then $U \cap V$ is non-degenerate. Indeed, it is easy to see that the set $\{u_i \mid \lambda_i = 1\}$ generates $U \cap V$, and since none of these vectors is isotropic it follows that $U \cap V$ is non-degenerate. In the next paragraphs we will see that the converse of this assertion is also true. The case $\mathbb{F} = \mathbb{R}$ is essentially contained in [8,Lemma 4.4], and the proof of the general case follows from a modification of this argument. As $U \cap V$ is non-degenerate, it follows that $U = (U \cap V) \oplus \widetilde{U}$, where $\widetilde{U}$ is the orthogonal complement of $U \cap V$ in U. Thus P^*P induces a self-adjoint map on $\widetilde{U}$. We analyze the cases when $U \cap V$ is hyperbolic and elliptic separately.

First we assume that $U \cap V$ is hyperbolic. Then $\widetilde{U}$ is elliptic. Thus, it follows from Lemma 2.3(1) that $\widetilde{U}$ has an orthonormal basis consisting of eigenvectors of P^*P. Putting together it with an orthonormal basis of $U \cap V$, we obtain an orthonormal basis of U formed by eigenvectors of P^*P.

Now we assume that $U \cap V$ is elliptic or zero. Then $\widetilde{U}$ is hyperbolic, since it is the orthogonal complement of an elliptic subspace. Considering P as an $\mathbb{R}$-linear map, we have that P^*P is a self-adjoint map, respect to $\mathrm{Re}\langle\cdot,\cdot\rangle$, on the underlying real vector space of $\widetilde{U}$. Suppose that $v \in \widetilde{U}$ is such that $\mathrm{Re}\langle v, v\rangle = 0$ and $\mathrm{Re}\langle P^*Pv, v\rangle = 0$, then

$$\langle v - Pv, v - Pv\rangle = \langle v, v\rangle - \langle Pv, Pv\rangle = \langle v, v\rangle - \langle P^*Pv, v\rangle = \mathrm{Re}\langle v, v\rangle - \mathrm{Re}\langle P^*Pv, v\rangle = 0.$$

Since $v - Pv \in V^\perp$, and $V^\perp$ is elliptic, we must have $v = Pv$. Hence, it follows that $v \in (U \cap V) \cap \widetilde{U} = 0$. Thus, if $v \in \widetilde{U}$ is isotropic we have that $\mathrm{Re}\langle P^*Pv, v\rangle \neq 0$. Applying Lemma 2.3(2) to the underlying real space of $\widetilde{U}$ endowed with the $\mathbb{R}$-bilinear form $\mathrm{Re}\langle\cdot,\cdot\rangle$, we obtain an orthonormal $\mathbb{R}$-basis $\{u_1, \ldots, u_l\}$ of $\widetilde{U}$, such that $P^*Pu_j = u_j\lambda_j$ for some real numbers $\lambda_1, \ldots, \lambda_l$. Putting together this with an orthonormal $\mathbb{R}$-basis of $U \cap V$ we obtain an orthonormal $\mathbb{R}$-basis $\{u_0, \ldots, u_k\}$ of U such that $P^*Pu_j = u_j\lambda_j$ for the real numbers $\lambda_0, \ldots, \lambda_k$. Suppose that $\lambda_0 > \lambda_1 > \cdots > \lambda_s$ are all different eigenvalues of the $\mathbb{R}$-linear map P^*P, and let

$$U_{\lambda_j} = \{u \in U \mid P^*Pu = u\lambda_j\}$$

be the eigenspace corresponding to λ_j. Since P^*P is $\mathbb{F}$-linear it follows that U_{λ_j} is an $\mathbb{F}$-vector space, and each $u \in U_{\lambda_j}$ is an eigenvector of the $\mathbb{F}$-linear map P^*P. Then U is the direct sum of the $\mathbb{F}$-subspaces $U_{\lambda_1}, \ldots, U_{\lambda_s}$. It is easy to see that U_{λ_i} is orthogonal, with respect to $\langle \cdot, \cdot \rangle$, to U_{λ_j} for $i \neq j$. Additionally, note that U_{λ_j} is non-degenerate, as it is generated by an orthonormal set of eigenvectors corresponding to λ_j, and these are not isotropic. Choosing an orthonormal $\mathbb{F}$-basis for each U_{λ_j}, and putting them together we obtain an orthonormal $\mathbb{F}$-basis of eigenvectors of P^*P for U. We emphasize this analysis in the following proposition.

Proposition 3.2: *Suppose that U, V are two hyperbolic (right) subspaces of $\mathbb{F}^{n,1}$. Then, $U \cap V$ is non-degenerate if and only if exists an orthonormal $\mathbb{F}$-basis of U such that P^*P is diagonal and real, that is, P^*P is diagonalizable. Moreover, in this case, every eigenvalue of P^*P is real.*

Let U, V be two hyperbolic subspaces such that $U \cap V$ is non-degenerate, and let $\{u_0, \ldots, u_k\}$ be an orthonormal $\mathbb{F}$-basis of U such that $P^*Pu_i = u_i\lambda_i$ for some $\lambda_i \in \mathbb{R}$, $i = 0, \ldots, k$. In this way $\lambda_i\langle u_i, u_i\rangle = \langle P^*Pu_i, u_i\rangle = \langle Pu_i, Pu_i\rangle$, and then

$$\lambda_i = \|Pu_i\|^2.$$

As the signature of U is $(k, 1)$, this basis contains a unique negative vector. Reordering, if necessary, we can assume that u_0 is negative. As $V^\perp$ is elliptic it follows that

$$0 \leq \langle u_0 - Pu_0, u_0 - Pu_0\rangle = \langle u_0, u_0\rangle - \langle Pu_0, Pu_0\rangle = -1 + \lambda_0,$$

and thus $\lambda_0 \geq 1$. Setting $v_0 = Pu_0/\|Pu_0\|$ we have that $\|v_0\| = 1$ and that $Pu_0 = v_0\sqrt{\lambda_0}$. Moreover, u_0, v_0 are (negative) unitary vectors such that

$$\langle u_0, v_0\rangle = -\|Pu_0\| = -\sqrt{\lambda_0}.$$

Let $W = span\{u_0, v_0\}$. Then $W^\perp$ is elliptic, as it is the orthogonal complement of a hyperbolic space. By Lemma 2.4, $u_0^\perp \cap U$ and $v_0^\perp \cap V$ are contained in $W^\perp$, and the restriction of P to $u_0^\perp \cap U$ is the orthogonal projection of $u_0^\perp \cap U$ onto $v_0^\perp \cap V$, considering these as subspaces of $W^\perp$. In particular, $\|Pu\| \neq 0$ for every non-zero vector u in $u_0^\perp \cap U$. This allows estimating λ_i for $i \geq 1$. More precisely, for $i \geq 1$ we have that

$$1 = \|Pu_i\|^2 + \|u_i - Pu_i\|^2 = \lambda_i + \|u_i - Pu_i\|^2,$$

as $u_i - Pu_i$ is non-negative. Thus, we concluded that $0 \leq \lambda_i \leq 1$ for $i = 1, \ldots, k$.

If $\lambda_i = 0$ for some $i = 0, \ldots, k$, then $Pu_i = 0$, or equivalently that $u_i \in V^\perp$. On the other hand, if $\lambda_i \neq 0$ then we set $v_i = Pu_i/\|Pu_i\|$. It is easy to see that the set consisting of all the vectors v_i is an orthonormal subset of V, and that the subspace generated by it is non-degenerate, as it contains v_0. Thus, we can complete this set to obtain an orthonormal basis $\{v_0, \dots, v_r\}$ of V. In this way, we obtain orthonormal bases $\{u_0, \dots, u_k\}$ of U and $\{v_1, \dots, v_r\}$ of V satisfying $Pu_i = v_i\sqrt{\lambda_i}$ for $i = 0, \dots, k$.

Proposition 3.3: *Let U, V be two hyperbolic subspaces of $\mathbb{F}^{n,1}$ such that $U \cap V$ is non-degenerate. Suppose that $\lambda_0 \geq \cdots \geq \lambda_k$ are the eigenvalues of P^*P, and that $\{u_0, \dots, u_k\}$ and $\{v_0, \dots, v_r\}$ are orthonormal bases of U and V, respectively, satisfying*

$$Pu_i = v_i\sqrt{\lambda_i} \quad \text{for} \quad i = 0, \dots, k.$$

Then

(1) $\langle u_0, v_0\rangle = -\sqrt{\lambda_0}$ *and* $\langle u_i, v_i\rangle = \sqrt{\lambda_i}$ *for* $i = 1, \dots, k$;
(2) *if* $W = \mathrm{span}\{u_0, v_0\}$, *then* $u_0^\perp \cap U, v_0^\perp \cap V \subseteq W^\perp$;
(3) $\langle u_i, v_j\rangle = \delta_{ij}\sqrt{\lambda_i}$ *for* $1 \leq i \leq k$ *and* $1 \leq j \leq r$, *where* δ_{ij} *is the Kronecker delta.*

Moreover, if $\{\widetilde{u}_0, \dots, \widetilde{u}_k\}$ and $\{\widetilde{v}_0, \dots, \widetilde{v}_r\}$ are others orthonormal bases of U and V, respectively, satisfying

$$P\widetilde{u}_i = \widetilde{v}_i\sqrt{\lambda_i} \quad \text{for} \quad i = 1, \dots, k,$$

then there exists an isometry $f \in U(n, 1; \mathbb{F})$ such that $f(u_i) = \widetilde{u}_i$ and $f(v_j) = \widetilde{v}_j$.

Proof: Part (1) follows immediately. Now, for all $j = 0, \dots, k$ we have that

$$\langle P^*Pu_i - u_i\lambda_i, u_j\rangle = \langle P^*Pu_i, u_j\rangle - \lambda_i\langle u_i, u_j\rangle = \langle Pu_i, Pu_j\rangle - \lambda_i\langle u_i, u_j\rangle = 0,$$

which implies that u_i is an eigenvector of P^*P. Hence, part (2) follows from Lemma 2.4(1). To show part (3) we see

$$\langle u_i, v_j\rangle = \langle Pu_i, v_j\rangle = \langle v_i\sqrt{\lambda_i}, v_j\rangle = \sqrt{\lambda_i}\langle v_i, v_j\rangle = \delta_{ij}\sqrt{\lambda_i},$$

for $1 \leq i \leq k$ and $1 \leq j \leq r$. Finally, for the last assertion let

$$\alpha = (u_0, \dots, u_k, v_1, \dots, v_r) \quad \text{and} \quad \widetilde{\alpha} = (\widetilde{u}_0, \dots, \widetilde{u}_k, \widetilde{v}_1 \dots, \widetilde{v}_r).$$

By the relations (1)–(3) we have that the Gram matrices $G(\alpha)$ and $G(\widetilde{\alpha})$ (see Section 2.1) coincide. As the subspaces generated by the vector in α and $\widetilde{\alpha}$ are non-degenerate (these coincide with $U + V$), Lemma 2.1 produces an isometry f of $\mathbb{F}^{n,1}$ such that $f(u_i) = \widetilde{u}_i$ and $f(v_j) = \widetilde{v}_j$. ■

Now we have achieved our first goal, that is, we characterize the hyperbolic subspaces U, V for which there is the singular value decomposition of $P : U \to V$. Moreover, Proposition 3.3 describes explicitly this decomposition. In the rest of this section we will focus on our second objective, namely, the introduction of the principal angles between two hyperbolic subspaces U, V and their relationship with the eigenvalues of P^*P.

Consider U as a real smooth manifold, and define the function $F : U \to \mathbb{R}$, sending $u \in U$ to $-\mathrm{Re}\langle Pu, Pu\rangle$. We will study the restriction of F to the set $\mathbb{S} = \{u \in U \mid \langle u, u\rangle = -1\}$. Let us observe that $\mathbb{S}$ is a real smooth manifold and that its tangent space at point u coincides with the real space $T_u\mathbb{S} = \{v \in U \mid \mathrm{Re}\langle u, v\rangle = 0\}$, that is, $T_u\mathbb{S}$ is the orthogonal complement of u in U respect to the $\mathbb{R}$-bilinear form $\mathrm{Re}\langle\cdot,\cdot\rangle$. Additionally, note that $T_u\mathbb{S}$ contains the elliptic $\mathbb{F}$-subspace $u^\perp \cap U$. Moreover, U is the direct sum of $T_u\mathbb{S}$ and the real subspace generated by u. If $u \in \mathbb{S}$ then we see that $F(u) = \|Pu\|^2$. Since F is the composition of the maps $u \mapsto (Pu, Pu)$ and $(v, w) \mapsto -\mathrm{Re}\langle v, w\rangle$, F is differentiable. The derivative of F at $u \in U$ is the $\mathbb{R}$-linear transformation $F'(u) : U \to \mathbb{R}$ defined by $F'(u)v = -2\,\mathrm{Re}\langle Pu, Pv\rangle$. By definition, $u \in \mathbb{S}$ is a critical point of $F|_{\mathbb{S}}$ if $F'(u)v = 0$ for every $v \in T_u\mathbb{S}$, that is, if $\mathrm{Re}\langle Pu, Pw\rangle = 0$ for every $w \in T_u\mathbb{S}$. The following is a technical result that will help us analyze the function F. Before this result is stated we recall that the set of all negative vectors of a pseudo-euclidean space X is denoted by X_-.

Lemma 3.4: *Suppose that U and V are two arbitrary hyperbolic subspaces of $\mathbb{F}^{n,1}$, and that $P : U \to V$ is the orthogonal projection of U onto V. Let $F : U \to \mathbb{R}$ defined by $F(u) = -\mathrm{Re}\langle Pu, Pu\rangle$. Then the following holds.*

(1) *Suppose that $U \cap V$ is non-degenerate. Then $u \in \mathbb{S}$ is a critical point of $F|_{\mathbb{S}}$ if and only if it is an eigenvector of the $\mathbb{F}$-linear map P^*P.*
(2) $\min F|_{\mathbb{S}}$ *exists if and only if $U \cap V$ is non-degenerate.*
(3) *If* $\min F|_{\mathbb{S}}$ *exists, then* $\min F|_{\mathbb{S}} = \min\{|\langle u, v\rangle|^2 \mid u \in U_-, v \in V_-, \|u\| = \|v\| = 1\}$. *Moreover, if* $|\langle \widetilde{u}_0, \widetilde{v}_0\rangle|^2 = \min\{|\langle u, v\rangle|^2 \mid u \in U_-, v \in V_-, \|u\| = \|v\| = 1\}$ *then the set* $\{\widetilde{v}_0, P\widetilde{u}_0\}$ *is linearly dependent and* $\min F|_{\mathbb{S}} = F(\widetilde{u}_0)$.

Proof: (1) If u is a critical point of $F|_{\mathbb{S}}$, then

$$0 = \mathrm{Re}\langle Pu, Pv\rangle = \mathrm{Re}\langle P^*Pu, v\rangle,$$

for every $v \in T_u\mathbb{S}$. Hence, $P^*Pu = u\lambda$ for some $\lambda \in \mathbb{R}$. For the converse, note that as $U \cap V$ is non-degenerate, then Proposition 3.2 implies that $P^*Pu = u\lambda$ for some $\lambda \in \mathbb{R}$. Thus, for $w \in U$ we have

$$\lambda\,\mathrm{Re}\langle u, w\rangle = \mathrm{Re}\langle P^*Pu, w\rangle = \mathrm{Re}\langle Pu, Pw\rangle.$$

In particular, if $w \in T_u\mathbb{S}$ we obtain $0 = \lambda\,\mathrm{Re}\langle u, w\rangle = \mathrm{Re}\langle Pu, Pw\rangle$, and it follows that u is a critical point of $F|_{\mathbb{S}}$. One should observe that we only use that $U \cap V$ is non-degenerate in the converse.

(2) Suppose that $\min F|_{\mathbb{S}} = F(u_0)$, for some $u_0 \in \mathbb{S}$. As the minimum of $F|_{\mathbb{S}}$ is a critical point, part(1) implies that $P^*Pu_0 = u_0\lambda$, for some $\lambda \in \mathbb{R}$. Let $v \in u_0^\perp \cap U$, it follows from the equality

$$\langle P^*Pv, u_0\rangle = \langle v, P^*Pu_0\rangle = \langle v, u_0\rangle\lambda = 0$$

that P^*P induces a self adjoint map on $u_0^\perp \cap U$. As $u_0^\perp \cap U$ is an elliptic subspace, it follows from Lemma 2.3(1) that there exists an orthonormal basis of eigenvectors of P^*P for $u_0^\perp \cap U$. Putting together this basis with u_0, we obtain an orthonormal basis of eigenvectors of P^*P for U. Thus, by Proposition 3.2 $U \cap V$ is non-degenerate.

Now we show the converse. As $U \cap V$ is non-degenerate, again by Proposition 3.2, there is an orthonormal basis $\{u_0, \dots, u_k\}$ of U, such that $P^*Pu_i = u_i\lambda_i$ for $\lambda_i \in \mathbb{R}$. As the signature of U is $(k, 1)$, this basis contains a unique negative vector. Reordering, if necessary, we can assume that u_0 is negative. Thus, we must have $\lambda_0 \geq 1$ and $\lambda_0 \geq \lambda_i$

for $i \geq 1$. We claim that $F(u_0) = \lambda_0 = \min F|_{\mathbb{S}}$. For this we will show that $\|Pu\|^2 \geq \lambda_0$ for $u \in \mathbb{S}$. Given $u \in \mathbb{S}$ we write it as $u = u_0\alpha_0 + \cdots + u_k\alpha_k$, for some $\alpha_i \in \mathbb{F}$ such that $-|\alpha_0|^2 + |\alpha_1|^2 + \cdots + |\alpha_k|^2 = -1$. Then $Pu = (Pu_0)\alpha_0 + \cdots + (Pu_k)\alpha_k$, and as

$$\langle Pu_i, Pu_j\rangle = \langle P^*Pu_i, u_j\rangle = \lambda_i\langle u_i, u_j\rangle,$$

we obtain $\|Pu_i\|^2 = \lambda_i$ and $\langle Pu_i, Pu_j\rangle = 0$ for $i \neq j$. Thus

$$\begin{aligned} -\|Pu\|^2 &= -\lambda_0|\alpha_0|^2 + \lambda_1|\alpha_1|^2 + \cdots + \lambda_k|\alpha_k|^2 \\ &\leq \lambda_0(-|\alpha_0|^2 + |\alpha_1|^2 + \cdots + |\alpha_k|^2) = -\lambda_0. \end{aligned}$$

Hence $\|Pu_0\|^2 \geq \lambda_0$, as claimed.

(3) Suppose that $\min F|_{\mathbb{S}} = \|Pu_0\|^2$ for some $u_0 \in \mathbb{S}$. Let $u \in U_-$ and $v \in V_-$ with $\|u\| = \|v\| = 1$. Let us consider an orthonormal basis of V of the form $\{v, v_1, \dots, v_r\}$, then

$$Pu = v\langle v, u\rangle + v_1\langle v_1, u\rangle + \cdots + v_r\langle v_r, u\rangle.$$

Therefore

$$-\|Pu_0\|^2 \geq -\|Pu\|^2 = -|\langle u, v\rangle|^2 + |\langle u, v_1\rangle|^2 + \cdots + |\langle u, v_r\rangle|^2 \geq -|\langle u, v\rangle|^2.$$

Now, setting $v_0 = Pu_0/\|Pu_0\|$ we have that

$$\|v_0\| = 1 \quad \text{and} \quad |\langle u_0, v_0\rangle|^2 = \|Pu_0\|^2,$$

showing that $\|Pu_0\|^2 = \min\{|\langle u, v\rangle|^2 \mid u \in U_-, v \in V_-, \|u\| = \|v\| = 1\}$.

It remains to show the last assertion. If $P\widetilde{u}_0 = 0$ we are done, so we assume that $P\widetilde{u}_0 \neq 0$. Let $\widetilde{w}_0 = P\widetilde{u}_0/\|P\widetilde{u}_0\|$. Then $\widetilde{w}_0$ is unitary and $\widetilde{u}_0 + \widetilde{w}_0\langle\widetilde{w}_0, \widetilde{u}_0\rangle = \widetilde{u}_0 - \widetilde{w}_0\|P\widetilde{u}_0\|$ is in $V^\perp$. Thus

$$\langle\widetilde{u}_0 + \widetilde{w}_0\langle\widetilde{w}_0, \widetilde{u}_0\rangle, \widetilde{v}_0 + \widetilde{w}_0\langle\widetilde{w}_0, \widetilde{v}_0\rangle\rangle = 0.$$

But this equality implies that $\langle\widetilde{u}_0, \widetilde{v}_0\rangle = -\langle\widetilde{u}_0, \widetilde{w}_0\rangle\langle\widetilde{w}_0, \widetilde{v}_0\rangle$. Hence

$$\min F|_{\mathbb{S}} = |\langle\widetilde{u}_0, \widetilde{v}_0\rangle|^2 = |\langle\widetilde{u}_0, \widetilde{w}_0\rangle|^2|\langle\widetilde{w}_0, \widetilde{v}_0\rangle|^2.$$

If $\{\widetilde{v}_0, P\widetilde{u}_0\}$ is linearly independent, then so is $\{\widetilde{v}_0, \widetilde{w}_0\}$. Thus, as $\{\widetilde{v}_0, \widetilde{w}_0\}$ is hyperbolic, $|\langle\widetilde{w}_0, \widetilde{v}_0\rangle|^2 > 1$. Therefore

$$\min F|_{\mathbb{S}} > |\langle\widetilde{u}_0, \widetilde{w}_0\rangle|^2 = \|P\widetilde{u}_0\|^2,$$

which is a contradiction. Hence $\{\widetilde{v}_0, P\widetilde{u}_0\}$ is linearly dependent. Then $|\langle\widetilde{w}_0, \widetilde{v}_0\rangle| = 1$, and therefore $\min F|_{\mathbb{S}} = \|P\widetilde{u}_0\| = F(\widetilde{u}_0)$. ■

Lemma 3.4 motivates the following definition.

Definition 3.5: Let U, V be two (right) hyperbolic subspaces of $\mathbb{F}^{n,1}$ such that $U \cap V$ is non-degenerate. Suppose that $\dim_{\mathbb{F}} U = k + 1$, $\dim_{\mathbb{F}} V = r + 1$ and $k \leq r$. The *principal* *angles* between U and V are recursively defined by

$$\cosh^2\theta_0 = \min\{|\langle u, v\rangle|^2 \mid u \in U_-, v \in V_-, \|u\| = \|v\| = 1\} = |\langle u_0, v_0\rangle|^2$$

and

$$\cos^2\theta_i = \max\{|\langle u, v\rangle|^2 \mid u \in U_i, v \in V_i, \|u\| = \|v\| = 1\} = |\langle u_i, v_i\rangle|^2$$

where $U_i = \{u \in U \mid \langle u, u_j\rangle = 0, \forall\, j \le i-1\}$ and $V_i = \{v \in V \mid \langle v, v_j\rangle = 0, \forall\, j \le i-1\}$, for $1 \le i \le k$. Also, the vectors u_i, v_i are called *principal vectors* associated to θ_i.

Let U, V be two hyperbolic subspaces of $\mathbb{F}^{n,1}$ as in Definition 3.5. Note that due Lemma 3.4 the principal angle θ_0 always exists. Besides, since the sets U_i and V_i of Definition 3.5 are compact, there are also $\theta_1, \ldots, \theta_k$.

Lemma 3.6: *Let U, V be hyperbolic subspaces of $\mathbb{F}^{n,1}$ such that $U \cap V$ is non-degenerate and let $\theta_0, \ldots, \theta_k$ be the principal angles between U and V. Suppose that $\lambda_0 \ge \cdots \ge \lambda_k$ are the eigenvalues of P^*P. Then $\cosh^2\theta_0 = \lambda_0$ and $\cos^2\theta_i = \lambda_i$ for $i = 1, \ldots, k$. Consequently, we have a non-decreasing sequence $\theta_1 \le \cdots \le \theta_k$.*

Proof: By Lemma 3.4(2) $\min F|_{\mathbb{S}}$ exists, and if $\min F|_{\mathbb{S}} = F(u_0)$ then u_0 is an eigenvector of P^*P. As u_0 is negative then we have that $P^*Pu_0 = u_0\lambda_0$. Then

$$\lambda_0 = \|Pu_0\|^2 = F(u_0) = \min F|_{\mathbb{S}}.$$

Thus, by Lemma 3.4(3),

$$\lambda_0 = \min\{|\langle u, v\rangle|^2 \mid u \in U_-, v \in V_-, \|u\| = \|v\| = 1\} = \cosh^2\theta_0.$$

Let $W = span\{u_0, Pu_0\}$. By Lemma 2.4(2), the restriction of P to $u_0^\perp \cap U$ coincides with the orthogonal projection of $u_0^\perp \cap U$ onto $(Pu_0)^\perp \cap V$, considering these as subspaces of the elliptic subspace $W^\perp$. Therefore, the principal angles $\theta_1, \ldots, \theta_k$ coincide with the (usual) principal angles between $u_0^\perp \cap U$ and $(Pu_0)^\perp \cap V$ as subspaces of $W^\perp$, see Definition 2.5. Thus, using [8,Proposition 3.4] for the subspaces $u_0^\perp \cap U$ and $(Pu_0)^\perp \cap V$, we obtain

$$\cos^2\theta_i = \lambda_i \quad \text{for} \quad i = 1, \ldots, k,$$

and the lemma follows. ■

Combining Lemma 3.6 and Proposition 3.3 (and its proof), we obtain the following result.

Proposition 3.7: *Let U, V be two hyperbolic subspaces of $\mathbb{F}^{n,1}$ such that $U \cap V$ is non-degenerate, and suppose $\theta_0, \theta_1, \ldots, \theta_k$ are the principal angles between U and V. Then, there exist orthonormal bases $\{u_0, \ldots, u_k\}$ of U and $\{v_0, \ldots, v_r\}$ of V such that*

(1) *$Pu_0 = v_0 \cosh\theta_0$ and $Pu_i = v_i \cos\theta_i$ for $i = 1, \ldots, k$;*
(2) *$\langle u_0, v_0\rangle = -\cosh\theta_0$ and $\langle u_i, v_i\rangle = \cos\theta_i$ for $i = 1, \ldots, k$; that is, u_i, v_i are principal vectors associated with θ_i;*

(3) $\langle u_i, v_j \rangle = \delta_{ij} \cos\theta_i$ *for* $1 \leq i \leq k$ *and* $1 \leq j \leq r$;
(4) *if* $W = span\{u_0, v_0\}$, *then* $u_0^{\perp} \cap U, v_0^{\perp} \cap V \subseteq W^{\perp}$. *Moreover,* $\theta_1, \dots, \theta_k$ *coincide with the principal angles between* $u_0^{\perp} \cap U$ *and* $(Pu_0)^{\perp} \cap V$, *considering these as subspaces of* $W^{\perp}$.

The following corollary is an immediate consequences of Proposition 3.7, which summarize some important properties of the principal angles.

Corollary 3.8: *Let* U, V *be two hyperbolic subspaces of* $\mathbb{F}^{n,1}$ *such that* $U \cap V$ *is non-degenerate. Suppose that* $\theta_0, \theta_1, \dots, \theta_k$ *are the principal angles between* U *and* V, *then*

(1) $U \cap V$ *is hyperbolic if and only if* $\theta_0 = 0$.
(2) $U \cap V$ *is elliptic if and only if* $\theta_0 > 0$ *and* $\theta_i = 0$ *for some* $i \geq 1$.
(3) $U \cap V = 0$ *if and only if* $\theta_i > 0$ *for all* $i = 0, \dots, k$.
(4) $\dim(U \cap V) = \#\{\theta_i \mid \theta_i = 0\}$.
(5) $\dim(U \cap V^{\perp}) = \#\{\theta_i \mid \theta_i = \pi/2, i \geq 1\}$.

The following result reflects the importance of the principal angles between two hyperbolic subspaces U, V. This says that the principal angles between U and V, such that $U \cap V$ is non-degenerate, determine the relative position of the pair (U, V) in $\mathbb{F}^{n,1}$.

Theorem 3.9: *Let* (U, V) *and* $(\widetilde{U}, \widetilde{V})$ *be two pairs of hyperbolic subspaces of* $\mathbb{F}^{n,1}$ *such that* $\dim_{\mathbb{F}} U = \dim_{\mathbb{F}} \widetilde{U}, \dim_{\mathbb{F}} V = \dim_{\mathbb{F}} \widetilde{V}$ *and* $U \cap V$, $\widetilde{U} \cap \widetilde{V}$ *are non-degenerate. Then,* (U, V) *and* $(\widetilde{U}, \widetilde{V})$ *have the same principal angles if and only if there exists an isometry* $f \in U(n, 1; \mathbb{F})$, *such that* $f(U) = \widetilde{U}$ *and* $f(V) = \widetilde{V}$.

Proof: It is clear that if there exists an isometry f of $\mathbb{F}^{n,1}$ such that $f(U) = \widetilde{U}$ and $f(V) = \widetilde{V}$, then the principal angles of (U, V) and $(\widetilde{U}, \widetilde{V})$ coincide. Conversely, let $\theta_0, \dots, \theta_k$ be the principal angles between U and V. By Proposition 3.7 there exist orthonormal bases $\{u_0, \dots, u_k\}$ and $\{v_0, \dots, v_r\}$ of U and V, respectively, and $\{\widetilde{u}_0, \dots, \widetilde{u}_k\}$ and $\{\widetilde{v}_0, \dots, \widetilde{v}_r\}$ of $\widetilde{U}$ and $\widetilde{V}$, respectively, which verify (1)–(4) of the last proposition. Thus, setting

$$\alpha = (u_0, \dots, u_k, v_0, \dots, v_r) \quad \text{and} \quad \widetilde{\alpha} = (\widetilde{u}_0, \dots, \widetilde{u}_k, \widetilde{v}_0, \dots, \widetilde{v}_r),$$

we have that the Gram matrices (see Section 2.1) of these tuples are given by

$$G(\alpha) = G(\widetilde{\alpha}) = \left(\begin{array}{c|c} I_{k+1} & M \\ \hline M^t & I_{r+1} \end{array} \right),$$

where $I_{k+1} = \mathrm{diag}(-1, 1, \dots, 1)$, $I_{r+1} = \mathrm{diag}(-1, 1, \dots, 1)$ and M is the matrix of order $(k+1) \times (r+1)$ given by $\mathrm{diag}(-\cosh\theta_0, \cos\theta_1, \dots, \cos\theta_k)$. Let us observe that the subspaces generated by the elements of α and $\widetilde{\alpha}$, respectively, are non-degenerate. Thus, Lemma 2.1 produces an isometry $f \in U(n, 1; \mathbb{F})$ such that $f(u_i) = \widetilde{u}_i$ and $f(v_j) = \widetilde{v}_j$, for $i = 0, \dots, k$ and $j = 0, \dots, r$. Hence $f(U) = \widetilde{U}$ and $f(V) = \widetilde{V}$, and the theorem follows. ∎

4. Pairs of elliptic subspaces in $\mathbb{F}^{n,1}$

In this section we investigate the relative position of elliptic subspaces U and V of $\mathbb{F}^{n,1}$. Firstly, we saw in Lemma 4.1 that when the subspaces U and V are hyperbolic, then $U \cap V$

coincides with the eigenspace corresponding to the eigenvalue $\lambda = 1$ of the $\mathbb{F}$-linear map $P^*P : U \to U$. It is interesting to notice that when the subspaces U and V are elliptic, this is non-necessarily true.

Lemma 4.1: *Suppose that U and V are two elliptic subspaces of $\mathbb{F}^{n,1}$. We have that $U \cap V \subseteq \{u \in U \mid P^*Pu = u\}$, and the equality holds only if $U + V$ is non-degenerate.*

Proof: For convenience let us denote the subspace on the right side of the inclusion by U_1. Let $u \in U \cap V$, which amounts to $Pu = u$. Then, for $v \in U$ we have that

$$\langle v, P^*Pu\rangle = \langle v, P^*u\rangle = \langle Pv, u\rangle = \langle v, u\rangle.$$

This forces to $P^*Pu = u$, and thus $U \cap V \subseteq U_1$. For the last assertion note that $(U + V) \cap (U^\perp \cap V^\perp) = 0$, as $U + V$ is non-degenerate. Therefore, in order to show the converse inclusion it is enough to verify that $u - Pu \in (U + V) \cap (U^\perp \cap V^\perp)$ for every $u \in U_1$. Given $u \in U_1$ and $v \in U$, we have that

$$\langle u - Pu, v\rangle = \langle u, v\rangle - \langle Pu, v\rangle = \langle P^*Pu, v\rangle - \langle Pu, v\rangle = \langle Pu, Pv\rangle - \langle Pu, v\rangle = 0$$

since $Pu \in V$ and $Pv - v \in V^\perp$. Thus $u - Pu \in U^\perp \cap V^\perp$, and as clearly $u - Pu \in U + V$ it follows that $u - Pu \in (U + V) \cap (U^\perp \cap V^\perp)$. ∎

The following result is a version of Proposition 3.3 for elliptic subspaces. Before to state this proposition we recall that by Lemma 2.3(1) the eigenvalues of $P^*P : U \to U$ are real, since U and V are elliptic.

Proposition 4.2: *Let U, V be two elliptic subspaces of $\mathbb{F}^{n,1}$. Suppose that $\lambda_0 \geq \cdots \geq \lambda_k$ are the eigenvalues of P^*P, then there exist orthonormal bases $\{u_0, \dots, u_k\}$ of U and $\{v_0, \dots, v_r\}$ of V, such that*

(1) $Pu_i = v_i\sqrt{\lambda_i}$ *for* $i = 0, \dots, k$;
(2) $\langle u_i, v_i\rangle = \sqrt{\lambda_i}$ *for* $i = 0, \dots, k$;
(3) *if* $W = span\{u_0, v_0\}$, *then* $u_0^\perp \cap U, v_0^\perp \cap V \subseteq W^\perp$;
(4) $\langle u_i, v_j\rangle = \delta_{ij}\sqrt{\lambda_i}$ *for* $0 \leq i \leq k$ *and* $0 \leq j \leq r$.

Proof: By Lemma 2.3(1) there exists an orthonormal basis $\{u_0, \dots, u_k\}$ of U such that $P^*Pu_i = u_i\lambda_i$. If $0 = \lambda_i = \|Pu_i\|^2$ for some i, then $Pu_i = 0$, which amounts to $u_i \in V^\perp$. On the other hand, if $\lambda_i \neq 0$ then we set $v_i = Pu_i/\sqrt{\lambda_i}$. Clearly the vectors v_i form an orthonormal subset of V. Let us complete this set to an orthonormal basis $\{v_0, \dots, v_r\}$ of V. In this way

$$Pu_i = v_i\sqrt{\lambda_i} \quad \text{and} \quad \langle u_i, v_i\rangle = \sqrt{\lambda_i} \quad \text{for} \quad i = 0, \dots, k;$$

showing (1) and (2). Part (3) follows from Lemma 2.4(1). Finally, for $0 \leq i \leq k$ and $0 \leq j \leq r$ we see that

$$\langle u_i, v_j\rangle = \langle Pu_i, v_j\rangle = \langle v_i\sqrt{\lambda_i}, v_j\rangle = \sqrt{\lambda_i}\langle v_i, v_j\rangle = \delta_{ij}\sqrt{\lambda_i},$$

showing part (4). ∎

Corollary 4.3: *Let U, V two elliptic subspaces of $\mathbb{F}^{n,1}$. Suppose that $\lambda_0 \geq \cdots \geq \lambda_k$ are the eigenvalues of P^*P. Then the following holds*

(1) *if $U+V$ is elliptic, then $\lambda_0 \leq 1$;*
(2) *if $U+V$ is not elliptic, then $\lambda_0 \geq 1$.*

Proof: (1) Suppose that $U+V$ is elliptic. Let $u_0 \in U$ an be unitary vector such that $P^*Pu_0 = \lambda_0 u_0$. Then $u_0 - Pu_0$ is a non-negative vector. Thus,

$$0 \leq \langle u_0 - Pu_0, u_0 - Pu_0 \rangle = 1 - \|Pu_0\|^2 = 1 - \lambda_0.$$

(2) Suppose that $U+V$ is not elliptic, and let us assume that $\lambda_0 < 1$. Then $\lambda_i < 1$ for $i = 0, \ldots, k$. In this situation the sum $U+V$ is direct, as if $u \in U \cap V$ then $P^*Pu = u$. Let $\{u_0, \ldots, u_k\}$ and $\{v_0, \ldots, v_r\}$ be the orthonormal bases of U and V, respectively, satisfying (1)–(4) from Proposition 4.2. Then, setting $W_i = span\{u_i, v_i\}$ for $i = 0, \ldots, k$, and $W_{k+1} = span\{v_{k+1}, \ldots, v_r\}$, we have that the subspaces W_i are pairwise orthogonal, for $i = 0, \ldots, k+1$. Since the determinant of $\langle \cdot, \cdot \rangle$ restrict to W_i coincides with $1 - \lambda_i$, it follows from Lemma 2.2(2) that the subspace W_i is elliptic, for $i = 0, \ldots, k$. Furthermore, as W_{k+1} is clearly elliptic, and

$$U \oplus V = W_0 \oplus W_1 \oplus \cdots \oplus W_{k+1},$$

we conclude that $U+V$ is elliptic, which contradicts the assumption on $U+V$. ■

Suppose that $U+V$ is elliptic and that $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_k$ are the eigenvalues of P^*P. Then we can view $P : U \to U$ as the orthogonal projection of U onto V in $U+V$. Let $\theta_0, \ldots, \theta_k$ (usual) principal angles between U and V, considering these as subspace of $U+V$ (see Definition 2.5). Applying Proposition [8,Proposition 3.4] we have that

$$\cos^2 \theta_i = \lambda_i \quad \text{for} \quad i = 0, \ldots, k. \tag{2}$$

Let us see that in this case the principal angles between U and V, determine the relative position of the pair (U, V) in $\mathbb{F}^{n,1}$.

Proposition 4.4: *Let (U, V) and $(\widetilde{U}, \widetilde{V})$ be two pairs of elliptic subspaces of $\mathbb{F}^{n,1}$ such that* $\dim_{\mathbb{F}} U = \dim_{\mathbb{F}} \widetilde{U}, \dim_{\mathbb{F}} V = \dim_{\mathbb{F}} \widetilde{V}$ *and $U+V$, $\widetilde{U}+\widetilde{V}$ are elliptic. Then, (U, V) and $(\widetilde{U}, \widetilde{V})$ have the same principal angles if and only if there exists an isometry $f \in U(n, 1; \mathbb{F})$, such that $f(U) = \widetilde{U}$ and $f(V) = \widetilde{V}$.*

Proof: It is clear that if there exists an isometry f of $\mathbb{F}^{n,1}$ such that $f(W_i) = \widetilde{W}_i$, then the principal angles of (U, V) and $(\widetilde{U}, \widetilde{V})$ coincide. Conversely, let $\theta_0, \ldots, \theta_k$ be the principal angles between U and V. By Proposition 4.2, there exist orthonormal bases $\{u_0, \ldots, u_k\}$ and $\{v_0, \ldots, v_r\}$ of U and V, respectively, and $\{\widetilde{u}_0, \ldots, \widetilde{u}_k\}$ and $\{\widetilde{v}_0, \ldots, \widetilde{v}_r\}$ of $\widetilde{U}$ and $\widetilde{V}$, respectively; satisfying the relations (1)–(4) of the this proposition. Thus, setting

$$\alpha = (u_0, \dots, u_k, v_0, \dots, v_r) \quad \text{and} \quad \widetilde{\alpha} = (\widetilde{u}_0, \dots, \widetilde{u}_k, \widetilde{v}_0, \dots, \widetilde{v}_r),$$

and using Equation (2), we have that the Gram matrices (see Section 2.1) of α and $\widetilde{\alpha}$ are given by

$$G(\alpha) = G(\widetilde{\alpha}) = \left(\begin{array}{c|c} I_{k+1} & M \\ \hline M^t & I_{r+1} \end{array} \right),$$

where $I_{k+1} = \text{diag}(1, 1, \dots, 1)$, $I_{r+1} = \text{diag}(1, 1, \dots, 1)$ and M is the matrix of order $(k + 1) \times (r + 1)$ given by $\text{diag}(\cos\theta_0, \cos\theta_1, \dots, \cos\theta_k)$. Thus, Lemma 2.1 produces an isometry $f \in U(n, 1; \mathbb{F})$ such that $f(u_i) = \widetilde{u}_i$ and $f(v_j) = \widetilde{v}_j$, for $i = 0, \dots, k$ and $j = 0, \dots, r$. Hence $f(U) = \widetilde{U}$ and $f(V) = \widetilde{V}$, and the proposition follows. ∎

It remains to investigate the case which $U + V$ is not elliptic. For this we have the following definition.

Definition 4.5: Let U, V be two (right) elliptic subspaces of $\mathbb{F}^{n,1}$ such that $U + V$ is not elliptic. Suppose that $\dim_{\mathbb{F}} U = k + 1$, $\dim_{\mathbb{F}} V = r + 1$ and $k \leq r$. The *principal angles* between U and V are recursively defined by

$$\cosh^2\theta_0 = \max\{|\langle u, v\rangle|^2 \mid u \in U, v \in V, \|u\| = \|v\| = 1\} = |\langle u_0, v_0\rangle|^2$$

and

$$\cos^2\theta_i = \max\{|\langle u, v\rangle|^2 \mid u \in U_i, v \in V_i, \|u\| = \|v\| = 1\} = |\langle u_i, v_i\rangle|^2$$

where $U_i = \{u \in U \mid \langle u, u_j\rangle = 0, \forall\, j \leq i - 1\}$ and $V_i = \{v \in V \mid \langle v, v_j\rangle, \forall\, j \leq i - 1\}$, for $1 \leq i \leq k$. Also, the vectors u_i, v_i are called *principal vectors* associated to θ_i.

As in the previous section, we investigate the relation between the singular value decomposition of $P : U \to V$ and the principal angles of Definition 4.5. For this we consider U as a real smooth manifold and we will study the function $H : U \to \mathbb{R}$, $u \mapsto \text{Re}\langle Pu, Pu\rangle$, restricted to the set $S = \{u \in U \mid \langle u, u\rangle = 1\}$. Note that S is a real smooth manifold and that its tangent space at point $u \in S$ coincides with the real space $T_uS = \{v \in U \mid \text{Re}\langle u, v\rangle = 0\}$, that is, T_uS is the orthogonal complement of u in U respect to the $\mathbb{R}$-bilinear form $\text{Re}\langle\cdot,\cdot\rangle$. Since S is compact, the function $H|_S$ has a global maximum. As H is the composition of $u \mapsto (Pu, Pu)$ and $(v, w) \mapsto \text{Re}\langle v, w\rangle$, H is differentiable. The derivative of H at $u \in U$ is the $\mathbb{R}$-linear transformation $H'(u) : U \to \mathbb{R}$ defined by $H'(u)v = 2\,\text{Re}\langle Pu, Pv\rangle$. A point $u \in S$ is, by definition, a critical point of $H|_S$ if $H'(u)v = 0$ for every $v \in T_uS$, or equivalently, if $\text{Re}\langle Pu, Pv\rangle = 0$ for every $v \in T_uS$. In the following lemma we statement some properties of the function $H|_S$.

Lemma 4.6: *Suppose that U and V are two elliptic subspaces of $\mathbb{F}^{n,1}$, and that $P : U \to V$ is the orthogonal projection of U onto V. Let $H : U \to \mathbb{R}$, defined by $H(u) = \text{Re}\langle Pu, Pu\rangle$. Then the following holds.*

(1) *A point $u \in S$ is a critical point of $H|_S$ if and only if u is an eigenvector of P^*P.*

(2) $\max H|_S = \max\{|\langle u, v\rangle|^2 \mid u \in U, v \in V, \|u\| = \|v\| = 1\}$

Proof: (1) The proof is the same as in Lemma 3.4(1).

(2) Note that due to compactness there are the two maximums in item (2). Suppose that $\max H|_S = \|Pu_0\|^2$ for some $u_0 \in S$. Let $u \in U$ and $v \in V$ such that $\|u\| = \|v\| = 1$, and let us consider an orthonormal basis $\{v, v_1, \ldots, v_r\}$ of V. Then $Pu = v\langle v, u\rangle + v_1\langle v_1, u\rangle + \cdots + v_r\langle v_r, u\rangle$. Thus

$$\|Pu_0\|^2 \geq \|Pu\|^2 = |\langle u, v\rangle|^2 + |\langle u, v_1\rangle|^2 + \cdots + |\langle u, v_r\rangle|^2 \geq |\langle u, v\rangle|^2.$$

Setting $v_0 = Pu_0/\|Pu_0\|$, we have that $\|v_0\| = 1$ and $|\langle u_0, v_0\rangle|^2 = \|Pu_0\|^2$. This implies that $\|Pu_0\|^2 = \max\{|\langle u, v\rangle|^2 \mid u \in U, v \in V, \|u\| = \|v\| = 1\}$. ■

The relationship between the principal angles between U and V and the eigenvalues of P^*P is emphasized in the following lemma.

Lemma 4.7: *Let U, V be elliptic subspaces of $\mathbb{F}^{n,1}$ such that $U + V$ is not elliptic and let $\theta_0, \ldots, \theta_k$ be the principal angles between U and V. Suppose that $\lambda_0 \geq \cdots \geq \lambda_k$ are the eigenvalues of P^*P. Then $\cosh^2\theta_0 = \lambda_0$ and $\cos^2\theta_i = \lambda_i$ for $i = 1, \ldots, k$. Consequently, we have a non-decreasing sequence $\theta_1 \leq \cdots \leq \theta_k$.*

Proof: Suppose that $\max H|_S = H(u_0)$. Then, by Lemma 4.6(1), u_0 is an eigenvector of P^*P. Thus $\lambda_0 = \|Pu_0\|^2 = H(u_0)$, and by Lemma 4.6(2)

$$\lambda_0 = \max\{|\langle u, v\rangle|^2 \mid u \in U, v \in V, \|u\| = \|v\| = 1\} = \cosh^2\theta_0.$$

Now, let $U_1 = u_0^\perp \cap U$ and $V_1 = v_0^\perp \cap V$. Then, by Lemma 2.4(1), $P(U_1) \subseteq V_1$, and thus the restriction of P to U_1 coincides with the orthogonal projection of U_1 onto V_1 in $\mathbb{F}^{n,1}$. Let $S_1 = \{u \in U_1 \mid \|u\| = 1\}$. By Lemma 4.6(1), $\max H|_{S_1} = H(u_1)$ for some u_0 eigenvector of P^*P. Thus $\lambda_1 = H(u_1)$, and applying Lemma 2.4(2) to the subspaces U_1 and V_1 we have that

$$\lambda_1 = \max\{|\langle u, v\rangle|^2 \mid u \in U_1, v \in V_1, \|u\| = \|v\| = 1\} = \cos^2\theta_1.$$

Continuing with this inductive process we obtain $\cos^2\theta_i = \lambda_i$ for $i = 1, \ldots, k$. ■

Combining Lemma 4.7 and Proposition 4.2 we obtain the following result.

Proposition 4.8: *Let U, V be two elliptic subspaces of $\mathbb{F}^{n,1}$ such that $U + V$ is not elliptic, and suppose $\theta_0, \theta_1, \ldots, \theta_k$ are the principal angles between U and V. Then, there exist orthonormal bases $\{u_0, \ldots, u_k\}$ of U and $\{v_0, \ldots, v_r\}$ of V such that*

(1) *$Pu_0 = v_0\cosh\theta_0$ and $Pu_i = v_i\cos\theta_i$ for $i = 1, \ldots, k$;*
(2) *$\langle u_0, v_0\rangle = \cosh\theta_0$ and $\langle u_i, v_i\rangle = \cos\theta_i$ for $i = 1, \ldots, k$, that is, u_i, v_i are principal vectors associated with θ_i;*
(3) *if $W = span\{u_0, v_0\}$, then $u_0^\perp \cap U, v_0^\perp \cap V \subseteq W^\perp$;*
(4) *$\langle u_i, v_j\rangle = \delta_{ij}\cos\theta_i$ for $1 \leq i \leq k$ and $1 \leq j \leq r$.*

It is convenient to highlight at this point another difference between the principal angles between hyperbolic and elliptic subspaces. We should note that when $U + V$ is not elliptic, the principal angles between U and V do not necessarily determine the relative position of a pair (U, V) in $\mathbb{F}^{n,1}$ (cf. Theorem 3.9).

Example 4.9: Let U, V be two coincident one-dimensional elliptic subspaces of $\mathbb{F}^{n,1}$; that is, $U = V$, and let $\widetilde{U}, \widetilde{V}$ be two one-dimensional elliptic subspaces of $\mathbb{F}^{n,1}$ such that $\widetilde{U} + \widetilde{V}$ is degenerate. Then the (principal) angle between U and V and between $\widetilde{U}$ and $\widetilde{V}$ are $\theta = \widetilde{\theta} = 0$. Hence, there is no isometry $f \in U(n, 1; \mathbb{F})$ such that $f(U) = \widetilde{U}$ and $f(V) = \widetilde{V}$.

Example 4.9 shows that some non-degeneration condition is required to guarantee that the principal angles determine the relative position of a pair of elliptic subspaces of $\mathbb{F}^{n,1}$. This will be dealt with in the following result. The proof of this is, in essence, the same as Theorem 3.9.

Theorem 4.10: *Let (U, V) and $(\widetilde{U}, \widetilde{V})$ be two pairs of elliptic subspaces of $\mathbb{F}^{n,1}$ such that $\dim_{\mathbb{F}} U = \dim_{\mathbb{F}} \widetilde{U}, \dim_{\mathbb{F}} V = \dim_{\mathbb{F}} \widetilde{V}$ and $U + V$, $\widetilde{U} + \widetilde{V}$ are non-degenerate. Then, (U, V) and $(\widetilde{U}, \widetilde{V})$ have the same principal angles if and only if there exists an isometry $f \in U(n, 1; \mathbb{F})$, such that $f(U) = \widetilde{U}$ and $f(V) = \widetilde{V}$.*

5. Principal angles and orthogonal complements

In this section we will investigate the relation between the principal angles of U and V and their orthogonal complements $U^{\perp}$ and $V^{\perp}$. For this we suppose that U and V are hyperbolic (and thus $U^{\perp}$ and $V^{\perp}$ are elliptic) with $\dim_{\mathbb{F}} U = k + 1$, $\dim_{\mathbb{F}} V = r + 1$ and $k \leq r$. We denote by

$$P_1 : \mathbb{F}^{n,1} \to U, \quad P_2 : \mathbb{F}^{n,1} \to V, \quad \widetilde{P}_1 : \mathbb{F}^{n,1} \to U^{\perp} \quad \text{and} \quad \widetilde{P}_2 : \mathbb{F}^{n,1} \to V^{\perp}$$

the orthogonal projections of $\mathbb{F}^{n,1}$ onto $U, V, U^{\perp}$ and $V^{\perp}$, respectively. Notice that for every $u \in \mathbb{F}^{n,1}$ we have that $u = P_1 u + \widetilde{P}_1 u$ and $u = P_2 u + \widetilde{P}_2 u$. Also, if $P = P_2|_U$ and $\widetilde{P} = \widetilde{P}_1|_{V^{\perp}}$ then $P^* = P_1|_V$ and $\widetilde{P}^* = \widetilde{P}_2|_{U^{\perp}}$.

Proposition 5.1: *Suppose that $u \in U$, with $u \notin V$, is a non-isotropic eigenvector of P^*P. Then $\widetilde{P}_2 u$ is an eigenvector of $\widetilde{P}^*\widetilde{P}$ with the same eigenvalue as u. Conversely, assume that $v \in V^{\perp}$, with $v \notin U^{\perp}$, is an eigenvector of $\widetilde{P}^*\widetilde{P}$. Then $P_1 v$ is an eigenvector of P^*P with the same eigenvalue as v.*

Proof: First note that $\widetilde{P}_2 u \neq 0$, as $u \notin V$. Suppose that $P^*Pu = \lambda u$ for some $\lambda \in \mathbb{F}$. Since u is non-isotropic, the equality

$$\langle Pv, Pv \rangle = \langle v, P^*Pv \rangle = \langle v, v\lambda \rangle = \langle v, v \rangle \lambda$$

implies that $\lambda \in \mathbb{R}$. Thus, the first assertion follows from the equality

$$\widetilde{P}_2 \widetilde{P}_1 \widetilde{P}_2 u = \widetilde{P}_2 \widetilde{P}_1 (u - P_2 u) = -\widetilde{P}_2 \widetilde{P}_1 P_2 u = -\widetilde{P}_2 (P_2 u - P_1 P_2 u) = \widetilde{P}_2 (P_1 P_2 u) = \lambda \widetilde{P}_2 u.$$

For the converse note that $P_1 v \neq 0$, as $v \notin U^{\perp}$. Also, as $U^{\perp}$ and $V^{\perp}$ are elliptic, the eigenvalues self-adjoint map $\widetilde{P}^*\widetilde{P} : V^{\perp} \to V^{\perp}$ are reals. Hence, if $\widetilde{P}^*\widetilde{P}v = \mu v$ for $\mu \in \mathbb{R}$, then the second assertion follows from the equality

$$P_1P_2P_1v = P_1P_2(v - \widetilde{P}_1v) = -P_1P_2\widetilde{P}_1v = -P_1(\widetilde{P}_1v - \widetilde{P}_2\widetilde{P}_1v) = P_1(\widetilde{P}_2\widetilde{P}_1v) = \mu P_1v.$$

■

Corollary 5.2: *Let U, V be two hyperbolic subspaces of $\mathbb{F}^{n,1}$ such that $U \cap V$ is non-degenerate. Then, the eigenvalues of P^*P different from 1 coincide with the eigenvalues of $\widetilde{P}^*\widetilde{P}$ different from 1.*

Proof: We saw in Section 3 that $U \cap V$ coincides with the eigenspace for the eigenvalue $\lambda = 1$ of P^*P. Also, as $U^\perp + V^\perp$ is non-degenerate, Lemma 4.1 implies that $U^\perp \cap V^\perp$ coincides with the eigenspace for the eigenvalue $\lambda = 1$ of $\widetilde{P}^*\widetilde{P}$. Thus, the corollary follows from Proposition 5.1. ■

Corollary 5.3: *Let U, V be two hyperbolic subspaces of $\mathbb{F}^{n,1}$ such that $U \cap V$ is non-degenerate. Then, the non-zero principal angles between U and V coincide with the non-zero principal angles between $V^\perp$ and $U^\perp$. Moreover, these coincide with the principal angles between $(U \cap V)^\perp \cap U$ and $(U \cap V)^\perp \cap V$.*

Proof: The first assertion it follows at once from Corollary 5.2. The second assertion follows from to observe that $P((U \cap V)^\perp \cap U) \subseteq (U \cap V)^\perp \cap V$. ■

Remark 5.4: It is important to note that if we consider configurations of subspaces other than those studied so far, we encounter some difficulties. For instance, if U and V are respectively an elliptic and a hyperbolic subspaces of $\mathbb{F}^{n,1}$ and $P : U \to V$ is the orthogonal projection of U onto V, then $U \cap V = \{u \in U \mid P^*Pu = u\}$ and P^*P is diagonalizable. However, we have no control over the signature of the subspace $P(U)$ and this makes a general analysis very difficult. On the other hand, if U and V are respectively a hyperbolic and an elliptic subspace of $\mathbb{F}^{n,1}$, we have that $U \cap V = \{u \in U \mid P^*Pu = u\}$, but we have no guarantee that P^*P is diagonalizable. The situation is even more critical if we consider U or V parabolic, because a lot of fundamental results used for the definition of principal angles are not necessarily valid.

6. Pairs of totally geodesic submanifolds

Let $X = \mathbb{F}^{n,1}$. We recall that X is the vector space consisting of $(n+1)$-tuples of elements of $\mathbb{F}$ endowed with the $\mathbb{F}$-Hermitian form $\langle \cdot, \cdot \rangle$ given in Equation (1). In $X - \{0\}$ define the following equivalence relation: $u \sim v$ if and only if $u = v\lambda$ for some non-zero $\lambda \in \mathbb{F}$. Denote by $\mathbb{FP}^n$ the set of equivalence classes $[u]$, $u \in X$, endowed with the quotient topology, and by $\pi : X - \{0\} \to \mathbb{FP}^n$ the natural projection. The *hyperbolic space* $\mathbb{H}^n_{\mathbb{F}}$ is defined as the projection of the set of all negative vectors of X, that is, $\mathbb{H}^n_{\mathbb{F}} = \pi(X_-)$. It is easy to see that $\mathbb{H}^n_{\mathbb{F}} = \pi(X_{-1})$, where $X_{-1} = \{u \in X \mid \langle u, u \rangle = -1\}$. Note that X_{-1} is a real manifold, and that the tangent space of X_{-1} at point u is the real subspace $T_uX_{-1} = \{v \in X \mid \mathrm{Re}\langle u, v \rangle = 0\}$.

Let $\mathbb{B}^n = \{(v_1, \ldots, v_n) \in \mathbb{F}^n \mid |v_1|^2 + \cdots + |v_n|^2 < 1\}$ be the unitary open ball in $\mathbb{F}^n$. The map $\varphi : \mathbb{H}^n_{\mathbb{F}} \to \mathbb{B}^n$, sending $[u_1, \ldots, u_{n+1}]$ to $(u_1u_{n+1}^{-1}, \ldots, u_nu_{n+1}^{-1})$, is a well defined homeomorphism between $\mathbb{H}^n_{\mathbb{F}}$ and $\mathbb{B}^n$. As $\mathbb{B}^n$ is a smooth manifold, $\mathbb{H}^n_{\mathbb{F}}$ is a smooth manifold with the differential structure induced by the map φ. With this differential structure φ becomes a diffeomorphism. It is shown in [25,Section 2.3] that $\pi : X_{-1} \to \mathbb{H}^n_{\mathbb{F}}$ is smooth, and that $\pi'(u) : T_u X_{-1} \to T_{[u]}\mathbb{H}^n_{\mathbb{F}}$ maps $u^{\perp}$ isomorphically onto $T_{[u]}\mathbb{H}^n_{\mathbb{F}}$.

The metric ρ in $\mathbb{H}^n_{\mathbb{F}}$ is given by the expression

$$\cosh^2(\rho([u],[v])) = \frac{|\langle u, v\rangle|^2}{\langle u, u\rangle\langle v, v\rangle},$$

(see [25,Proposition 2.2.4]). The group $U(n,1;\mathbb{F})$ acts in the projective space $\mathbb{FP}^n$ leaving $\mathbb{H}^n_{\mathbb{F}}$ invariant. But, since this action is not faithful, $U(n,1;\mathbb{F})$ does not act by isometries on $\mathbb{H}^n_{\mathbb{F}}$. Then, passing to the quotient by the kernel of this action, we can view the projective group $PU(n,1;\mathbb{F}) = U(n,1;\mathbb{F})/Z(n,1;\mathbb{F})$, where $Z(n,1;\mathbb{F})$ is the center of $U(n,1;\mathbb{F})$, as a subgroup of the isometry group of $\mathbb{H}^n_{\mathbb{F}}$. More precisely, given $f \in U(n,1;\mathbb{F})$ we have an isometry $\widetilde{f}$ of $\mathbb{H}^n_{\mathbb{F}}$ defined by $\widetilde{f}([u]) = [f(u)]$, $u \in X_-$. Moreover, applying the chain rule to the equality $(\widetilde{f} \circ \pi)(u) = (\pi \circ f)(u)$, we obtain

$$\widetilde{f}'([u]) \circ \pi'(u) = \pi'(f(u)) \circ f'(u) = \pi'(f(u)) \circ f, \tag{3}$$

since f is linear. Thus, $\widetilde{f}'([u]) = \pi'(f(u)) \circ f \circ \pi'(u)^{-1}$ is an isomorphism between $T_{[u]}\mathbb{H}^n_{\mathbb{F}}$ and $T_{[f(u)]}\mathbb{H}^n_{\mathbb{F}}$, for $u \in X_-$.

Recall that X_0 denotes the set of all isotropic vector of X. Given w_1 and w_2 in X_0, we can assume, after normalizing, that $\langle w_1, w_2\rangle = -1$. Let $\gamma : \mathbb{R} \to X$ be the path defined by $\gamma(t) = w_1 e^{t/2} + w_2 e^{-t/2}$. Then, it follows from [26,Proposition 5.1] that $t \to [\gamma(t)]$ is a geodesic of $\mathbb{H}^n_{\mathbb{F}}$ parametrised by arc length t. On the other hand, given two point $[u],[v] \in \mathbb{H}^n_{\mathbb{F}}$ there exists a geodesic arc joining $[u]$ to $[v]$. Indeed, normalizing if necessary, we can assume that $\|u\| = \|v\| = 1$ and $\langle u, v\rangle = -\cosh\theta$, for some $\theta \in \mathbb{R}$. If $\widetilde{u}$ is the vector determined by the equation

$$\widetilde{u}\sinh\theta + u\cosh\theta = v,$$

then the path $[0,\theta] \to \mathbb{H}^n_{\mathbb{F}}$, defined by $t \mapsto [\widetilde{u}\sinh t + u\cosh t]$, is a geodesic arc joining $[u]$ to $[v]$.

A submanifold Σ of $\mathbb{H}^n_{\mathbb{F}}$ is called *totally geodesic* if it contains every geodesic of $\mathbb{H}^n_{\mathbb{F}}$ which is tangent to it. If U is a hyperbolic $\mathbb{F}$-subspace of $\mathbb{F}^{n,1}$ then $\pi(U_-)$, the projection of the negative vectors of U, is a totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{F}}$ (see [25,Proposition 2.5.1]). For practical purposes, we say that $\pi(U_-)$ is an $\mathbb{F}$-*totally geodesic submanifold* of $\mathbb{H}^n_{\mathbb{F}}$ and denote it by Σ_U. It follows from [25,Proposition 2.5.1] that if $\mathbb{F} = \mathbb{R}$, then every totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{F}}$ is an $\mathbb{F}$-totally geodesic submanifold. Observe that if $U_{-1} = \{u \in U \mid \langle u, u\rangle = -1\}$, then $\Sigma_U = \pi(U_{-1})$. For each $[u] \in \Sigma_U$ we have that $\pi'(u)$ maps $T_{[u]}\Sigma_U$ isomorphically onto $u^{\perp} \cap U$. An $\mathbb{F}$-totally geodesic submanifold Σ_U is called $\mathbb{F}$-geodesic if $\dim_{\mathbb{F}} U = 2$.

We say that an $\mathbb{F}$-geodesic Σ_U meets ortogonally a $\mathbb{F}$-totally geodesic submanifold Σ_V at $[u]$ if $T_{[u]}\Sigma_U$ is orthogonal to $T_{[u]}\Sigma_V$, with respect to the $\mathbb{F}$-Hermitian form $\langle\cdot,\cdot\rangle$.

Definition 6.1: Let (Σ_U, Σ_V) be an ordered pair of $\mathbb{F}$-totally geodesic submanifolds of $\mathbb{H}^n_{\mathbb{F}}$. Then (Σ_U, Σ_V) is called

(1) *reverse* if $\Sigma_U \cap \Sigma_V = \emptyset$ and $U \cap V = \{0\}$;
(2) *ultra-parallel* if $\Sigma_U \cap \Sigma_V = \emptyset$ and $U \cap V$ is elliptic;
(3) *asymptotic* if $\Sigma_U \cap \Sigma_V = \emptyset$ and $U \cap V$ is parabolic;
(4) *concurrent* if $\Sigma_U \cap \Sigma_V \neq \emptyset$, that is, $U \cap V$ is hyperbolic.

Remark 6.2: The pairs of $\mathbb{F}$-totally geodesic submanifolds of $\mathbb{H}^n_{\mathbb{F}}$ called *reverse* in this paper are called *skew* in geometry.

Theorem 6.3: *Let* (Σ_U, Σ_V) *and* $(\Sigma_{\widetilde{U}}, \Sigma_{\widetilde{V}})$ *be two non-asymptotic pairs of* $\mathbb{F}$*-totally geodesic submanifolds of* $\mathbb{H}^n_{\mathbb{F}}$ *such that* $\dim_{\mathbb{F}} U = \dim_{\mathbb{F}} \widetilde{U}$ *and* $\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} \widetilde{V}$. *Then, there exists* $\widetilde{f} \in PU(n,1;\mathbb{F})$ *such that* $\widetilde{f}(\Sigma_U) = \Sigma_{\widetilde{U}}$ *and* $\widetilde{f}(\Sigma_V) = \Sigma_{\widetilde{V}}$ *if and only if* (U,V) *and* $(\widetilde{U},\widetilde{V})$ *have the same principal angles.*

Proof: First suppose that (U,V) and $(\widetilde{U},\widetilde{V})$ have the same principal angles. By Theorem 3.9 there exists an isometry $f \in U(n,1;\mathbb{F})$ such that $f(U) = \widetilde{U}$ and $f(V) = \widetilde{V}$. Thus,

$$\widetilde{f}(\Sigma_U) = \widetilde{f}(\pi(U_-)) = \pi(f(U_-)) = \pi(\widetilde{U}_-) = \Sigma_{\widetilde{U}},$$

and analogously $\widetilde{f}(\Sigma_V) = \Sigma_{\widetilde{V}}$.

Conversely, suppose that $\widetilde{f} \in PU(n,1;\mathbb{F})$ satisfies $\widetilde{f}(\Sigma_U) = \Sigma_{\widetilde{U}}$ and $\widetilde{f}(\Sigma_V) = \Sigma_{\widetilde{V}}$. Then, $\widetilde{f}'([u]) : T_{[u]}\mathbb{H}^n_{\mathbb{F}} \to T_{[f(u)]}\mathbb{H}^n_{\mathbb{F}}$ maps $T_{[u]}\Sigma_U$ isomorphically onto $T_{[f(u)]}\Sigma_{\widetilde{U}}$, for $u \in U_-$. Thus, Equation (3) implies that

$$f(u^{\perp} \cap U) = [\pi'(f(u))]^{-1} \circ \widetilde{f}'([u]) \circ \pi'(u)(u^{\perp} \cap U) = f(u)^{\perp} \cap \widetilde{U}.$$

This allows to conclude that $f(U) = \widetilde{U}$. Analogously we can show that $f(V) = \widetilde{V}$. Hence, again by Theorem 3.9, we have that (U,V) and $(\widetilde{U},\widetilde{V})$ have the same principal angles. ■

Definition 6.4: The *distance* between two $\mathbb{F}$-totally geodesic submanifolds Σ_U and Σ_V is given by $\rho(\Sigma_U, \Sigma_V) = \min\{\rho(p,q) \mid p \in \Sigma_U, q \in \Sigma_V\}$, when this minimum exists.

Note that the existence of the minimum in Definition 6.4 is equivalent to the existence of a common perpendicular geodesic to Σ_U and Σ_V. Clearly, if $\Sigma_U \cap \Sigma_V \neq \emptyset$ this geodesic exists, but in general it is not unique. We will show that if (Σ_U, Σ_V) is a non-asymptotic pair of $\mathbb{F}$-totally geodesic submanifolds, then there exists a unique common perpendicular $\mathbb{F}$-geodesic to Σ_U and Σ_V. This implies, in particular, the existence of a common perpendicular geodesic to Σ_U and Σ_V. If $\mathbb{F} = \mathbb{R}$, the existence and uniqueness of this common perpendicular $\mathbb{F}$-geodesic are well understood, see [27,Sections VIII.3 and VIII.4] for $n = 2, 3$ and [8,Section 4] for the general case.

Lemma 6.5: *Let* (Σ_U, Σ_V) *be an ordered pair of* $\mathbb{F}$*-totally geodesic submanifolds of* $\mathbb{H}^n_{\mathbb{F}}$. *Then, the distance* $\rho(\Sigma_U, \Sigma_V)$ *exists if and only if* (Σ_U, Σ_V) *is non-asymptotic. Moreover, in this case,* $\rho(\Sigma_U, \Sigma_V)$ *coincides with* θ_0*, the first principal angle between* U *and* V.

Proof: Notice that since $\Sigma_U = \pi(U_{-1})$ and $\Sigma_V = \pi(V_{-1})$, then

$$\begin{aligned}\cosh^2 \rho(\Sigma_U, \Sigma_V) &= \min\{\cosh^2 \rho([u],[v]) \mid u \in U_-, v \in V_-, \|u\| = \|v\| = 1\} \\ &= \min\{|\langle u, v\rangle|^2 \mid u \in U_-, v \in V_-, \|u\| = \|v\| = 1\}.\end{aligned}$$

By Lemma 3.4 the minimum above exists if and only if $U \cap V$ is non-degenerate, and the first assertion follows. Moreover, in this case, it follows at once from Definition 3.5 that $\rho(\Sigma_U, \Sigma_V) = \theta_0$. ■

For the following result we recall that $P : U \to V$ denotes the orthogonal projection of U onto V (see Section 2.2).

Proposition 6.6: *Let (Σ_U, Σ_V) be a reverse or ultra-parallel pair of $\mathbb{F}$-totally geodesic submanifolds of $\mathbb{H}^n_{\mathbb{F}}$. Then, there exists a unique common perpendicular $\mathbb{F}$-geodesic to Σ_U and Σ_V.*

Proof: By Lemma 6.5

$$|\langle u_0, v_0\rangle|^2 = \cosh^2(\Sigma_U, \Sigma_V) = \min\{|\langle u, v\rangle|^2 \mid u \in U, v \in V, \|u\| = \|v\| = 1\},$$

for some $u_0 \in U$ and $v_0 \in V$. Since that (Σ_U, Σ_V) is reverse or ultra-parallel, we have that $[u_0] \neq [v_0]$, that is, u_0 and v_0 are linearly independent. Setting $W = span\{u_0, v_0\}$, we have that $\Sigma = \pi(W_-)$ is an $\mathbb{F}$-geodesic. We claim that Σ meets orthogonally Σ_U at $[u_0]$ and Σ_V at $[v_0]$. For this note that

$$T_{[u_0]}\Sigma_U = u_0^{\perp} \cap U \quad \text{and} \quad T_{[u_0]}\Sigma = u_0^{\perp} \cap W.$$

Thus, since $W = span\{u_0\} + (u_0^{\perp} \cap W)$, Lemma 2.4(1) implies that

$$u_0^{\perp} \cap U \subseteq W^{\perp} \subseteq (u_0^{\perp} \cap W)^{\perp}.$$

Hence Σ meets orthogonally Σ_U at $[u_0]$. Analogously it is shown that Σ meets orthogonally with Σ_V. So, we have shown the existence.

Let us focus on the uniqueness. Suppose that $\widetilde{\Sigma}$ is another $\mathbb{F}$-geodesic which meets orthogonally Σ_U at $[\widetilde{u}]$ and Σ_V at $[\widetilde{v}]$. Then we have that

$$\begin{aligned}|\langle u_0, v_0\rangle|^2 = |\langle \widetilde{u}, \widetilde{v}\rangle|^2 &= \cosh^2(\Sigma_U, \Sigma_V) \\ &= \min\{|\langle u, v\rangle|^2 \mid u \in U_-, v \in V_-, \|u\| = \|v\| = 1\}.\end{aligned}$$

Combining Lemma 3.4 and Lemma 6.5, we have that u_0 and $\widetilde{u}$ are eigenvectors of P^*P corresponding to the eigenvalue $\lambda_0 = \cosh^2 \theta_0$. As the index of $\mathbb{F}^{n,1}$ is 1, the eigenspace $\{u \in U \mid P^*Pu = u\lambda_0\}$ has dimension 1 over $\mathbb{F}$. Then, u_0 and $\widetilde{u}$ are linearly dependent, and so $[u_0] = [\widetilde{u}]$. Moreover, it follows from Lemma 3.4(3) that $[v_0] = [Pu_0]$ and $[\widetilde{v}] = [P\widetilde{u}]$. Hence $[v_0] = [\widetilde{v}]$, and the uniqueness follows. ■

Let U, V be two hyperbolic subspaces of $\mathbb{F}^{n,1}$ and $\theta_0, \ldots, \theta_k$ the principal angles between U and V. Lemma 6.5 provides a geometric interpretation θ_0 in terms of the geometry of $\mathbb{H}^n_{\mathbb{F}}$. We will see that $\theta_1, \ldots, \theta_k$ also have a such geometric interpretation. Firstly suppose that (Σ_U, Σ_V) is reverse or ultra-parallel pair of $\mathbb{F}$-totally geodesic submanifolds, and let $u_0 \in U$ and $v_0 \in V$ such that

$$\rho(\Sigma_U, \Sigma_V) = \rho([u_0], [v_0]).$$

Then, by Lemma 3.4 it follows that u_0 is an eigenvector of P^*P corresponding to the eigenvalue $\lambda_0 = \cosh^2\theta_0$. Thus, by Proposition 2.4 it follows that $T_{[u_0]}\Sigma_U = u_0^\perp \cap U$ and $T_{[v_0]}\Sigma_V = v_0^\perp \cap V$ are contained in $W^\perp$, where $W = span\{u_0, v_0\}$. As $W^\perp$ is elliptic, $\theta_1, \ldots, \theta_k$ coincide with the (usual) principal angles between $T_{[u_0]}\Sigma_U$ and $T_{[v_0]}\Sigma_V$, considering these as subspaces of $W^\perp$ (see Definition 2.5). So, the parameters $\theta_1, \ldots, \theta_k$ depend only on the geometry of $\mathbb{H}^n_{\mathbb{F}}$. Now, if (Σ_U, Σ_V) is a concurrent pair and $u \in U \cap V$, then $T_{[u]}\Sigma_U$ and $T_{[u]}\Sigma_V$ are contained in the elliptic subspace $u^\perp$. In this situation we have the following proposition.

Proposition 6.7: *Let (Σ_U, Σ_V) be a concurrent pair of $\mathbb{F}$-totally geodesic submanifolds, and let u be a fixed (but arbitrary) point of $U \cap V$. Then, $\theta_1, \ldots, \theta_k$ coincide with the principal angles between $T_{[u]}\Sigma_U$ and $T_{[u]}\Sigma_V$, considering these as subspaces of $u^\perp$.*

Proof: As $T_{[u]}\Sigma_U \cap T_{[u]}\Sigma_V = u^\perp \cap (U \cap V)$, we have that

$$U \cap V = span\{u\} \oplus (T_{[u]}\Sigma_U \cap T_{[u]}\Sigma_V),$$

and thus $(U \cap V)^\perp = u^\perp \cap (T_{[u]}\Sigma_U \cap T_{[u]}\Sigma_V)^\perp$. This implies

$$U \cap (U \cap V)^\perp = T_{[u]}\Sigma_U \cap (T_{[u]}\Sigma_U \cap T_{[u]}\Sigma_V)^\perp$$

and

$$V \cap (U \cap V)^\perp = T_{[u]}\Sigma_V \cap (T_{[u]}\Sigma_U \cap T_{[u]}\Sigma_V)^\perp.$$

Corollary 5.3 implies that the non-zero principal angles between $T_{[u]}\Sigma_U$ and $T_{[u]}\Sigma_V$ coincide with the non-zero principal angles between U and V. On the other hand, $\dim(T_{[u]}\Sigma_U \cap T_{[u]}\Sigma_V)$ coincides with the number of principal angles between $T_{[u]}\Sigma_U$ and $T_{[u]}\Sigma_V$ equals to 0, and Corollary 3.8(4) says that $\dim(U \cap V) = \#\{\theta_i = 0 \mid i \geq 0\}$. Thus the $\#\{\theta_i = 0 \mid i \geq 1\}$ coincides with the number of principal angles between $T_{[u]}\Sigma_U$ and $T_{[u]}\Sigma_V$ equals to zero. This shows the proposition. ■

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