ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

DOSSIER PPGEM 40 YEARS

Geometric Transformations in Geometry Teaching: different appropriations in modern mathematics

Transformações Geométricas no Ensino de Geometria: diferentes apropriações na matemática moderna

Maria Célia **Leme da Silva**[*]

ORCID iD 0000-0001-6029-0490

Ana Paula **Jahn**[**]

ORCID iD 0000-0003-0515-7536

Abstract

The present work establishes a study within the scope of a broader project on the relationships between the History of school geometry and the Modern Mathematics Movement (MMM). We analyzed how a particular knowledge – geometric transformations (GT) – circulated in the Brazilian context during the mentioned period. From a historical perspective, we investigated four textbooks produced between the 1960s and 1970s by MMM leaders. The guided questions of this study were the following: what are the similarities and differences in the approaches to GT between the different books? How did the articulation between GT and Euclidean Geometry (EG) happen? The process of insertion of GT through textbooks was diverse, highlighting the complexity of the dialogue and the prediction of reconciling demands arising from specific cultures: the academic one, as a representative of scientific production, and the school one, representing experiences of professional practices.

Keywords: History of school geometry. Mathematics textbook. Academic field. Professional field. Euclidean geometry.

Resumo

Este artigo constitui um estudo no âmbito de um projeto mais amplo acerca das relações entre a História da Geometria Escolar e o Movimento da Matemática Moderna (MMM). Analisamos como um determinado saber – as transformações geométricas (TG) – circulou no contexto brasileiro durante o referido período. Sob uma perspectiva histórica, investigamos quatro livros didáticos produzidos entre as décadas de 1960 e 1970 por líderes do MMM. As questões orientadoras deste estudo foram as seguintes: quais são as semelhanças e diferenças nas abordagens das TG entre os diferentes livros? Como ocorreu a articulação entre as TG e a Geometria Euclidiana (GE)? O processo de inserção das TG por meio dos livros didáticos foi diverso, ressaltando a complexidade do

[*] Doutora em Educação (Currículo) na Pontifícia Universidade Católica de São Paulo (PUC/SP). Professora Associada da Universidade Federal de São Paulo (UNIFESP), Diadema e do Programa de Pós-graduação da Universidade Estadual Paulista (UNESP), Bauru e Rio Claro, SP, Brasil. E-mail: celia.leme@unifesp.br.

[**] Doutora em Didática da Matemática na Universidade Joseph Fourier (UJF-Grenoble, França). Professora Doutora do Instituto de Matemática e Estatística da Universidade de São Paulo (IME-USP), São Paulo, SP, Brasil e Pós-doutoranda do Programa de Pós-graduação em Educação Matemática da UNESP de Rio Claro, SP. E-mail: anajahn@ime.usp.br.

diálogo e a viabilidade de conciliar demandas oriundas de culturas distintas: a acadêmica, como representante da produção científica, e a escolar, representante de experiências de práticas profissionais.

Palavras-chave: História da Geometria Escolar. Livro didático de Matemática. Campo acadêmico. Campo profissional. Geometria Euclidiana.

1 Initial considerations

We begin our article by inviting readers to recall some considerations about a period of significant changes in the teaching of Mathematics that, in Brazil, became known as the Modern Mathematics Movement (MMM). This is a movement to renew the teaching of Mathematics that mobilized, among others, mathematicians, Mathematics teachers, pedagogues, psychologists, epistemologists, and educators in general, from numerous countries, to discuss changes in the teaching of Mathematics in Basic Education. Conferences were held, seminars organized by the Organization for Economic Cooperation and Development (OECD), curricular programs for teaching Mathematics were created, and study groups were formed to disseminate and subsidize teachers with training courses, all to collectively build a new proposal for teaching Mathematics, for producing teaching materials with new content and new methodological approaches. Agreeing with Bloch (2001) on the difficulties and risks in defining a beginning and an end to a historical movement, we prefer to historically situate the MMM between the 1950s and 1980s.

In Brazil, some milestones justify the delimited period, such as the National Congresses on Mathematics Teaching[1], with the first edition in 1955, in Salvador/Bahia, under the leadership of Martha Dantas[2]; s; the 1960s, as the peak of the MMM, with the first publications of modern textbooks, in particular the books by Osvaldo Sangiorgi[3], launched from 1963 onwards; the 1970s, in which criticism of the MMM intensified, in particular, the assessment by mathematician Elon Lages Lima[4], at the 9th Brazilian Mathematics Colloquium, pointing out “Brazilian teaching following foreign models that did not have satisfactory approval in the countries of origin” (Búrigo, 1989, p. 215), as well as the translation into Portuguese of Morris

[1] The 1st Brazilian Congress on Mathematics Teaching was held in 1955, in Salvador/Bahia; the 2nd was held in 1957, in Porto Alegre/RS; the 3rd was held in 1959, in Rio de Janeiro/RJ; the 4th was held in 1962, in Belém/PA; the 5th and last Congress was held in 1966, in São José dos Campos/SP.

[2] Martha Maria de Souza Dantas, will be presented later in the text.

[3] Osvaldo Sangiorgi, will be presented later in the text.

[4] Elon Lages de Lima (1929-2017), mathematician, researcher and professor at IMPA – Institute for Pure and Applied Mathematics, he completed his master's and doctorate (1958) at the University of Chicago, in the area of algebraic topology.

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

Kline's[5] book, entitled *The Failure of Modern Mathematics*, in 1976.

Internationally, there are also some signs indicating the beginning of the movement, both in America and in Europe. As an example, we can mention the CIEAEM – *Comission Internationale pour l'étude et l'amélioration de l'enseignement des mathématiques*, created in 1950, which proposed to coordinate psychological, methodological, and practical work, intending to improve the teaching of mathematics, in different countries; the SMSG – *School Mathematics Study Group* (USA), in 1958, which produced so-called modern didactic texts, later translated into Portuguese; the Royaumont Seminar, in France, in 1959, which resulted in the publication of the book *Un programme moderne de mathématiques pour l'enseignement secondaire*, in 1961, also translated into Portuguese by GEEM – *Grupo de Estudos do Ensino da Matemática*[6]*;* and the holding of the First Inter-American Conference on Mathematics Education (CIAEM), held in Colombia, in 1961.

In any case, one of the central characteristics of the MMM was the attempt to bring school mathematics closer to the academic mathematics of the 20th century, so that students would arrive at the University more familiar with the concepts, notions, and language produced by mathematicians of the period. Initially, the focus was on secondary education (currently the final years of elementary school and high school), a preparatory course for higher education, however, it quickly spread to primary education (currently the initial years of elementary school) and basic education in general (Kilpatrick, 2012).

The MMM moment constitutes a privileged space to examine the articulations between professionalization/knowledge/teacher training, in which the tensions established between the academic field and the professional field are evidenced (Valente; Bertini; Morais, 2021). One challenge posed in the modernizing proposals was to seek a dialogue between mathematicians (representatives of the academic field) and mathematics teachers (representatives of the professional field).

Within the proposal to study this movement from the perspective of the History of Mathematics Education, we are interested in a specific knowledge of Geometry in high school: that of geometric transformations (GT), aiming to analyze the insertion of this content in teaching and its participation and/or contribution in the movement of production of school Geometry in the MMM period. This is because, although this orientation of the study of GT

[5] Morris Kline (1908-1992), was a mathematician and historian of mathematics, professor of mathematics at the Courant Institute of Mathematical Sciences, New York University.

[6] GEEM was created in 1961, in São Paulo, and operated until 1976, always under the leadership of Osvaldo Sangiorgi.

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

was already present in the Campos reform of 1931, based on the ideas of Félix Klein[7] from the beginning of the 20th century – in particular with the focus on the notion of function as an integrating concept of the different fields of mathematics – with the circulation of modernizing ideas in the 1960s, GT once again had a prominent role, especially due to the possibility of articulating its study with the algebraic approach, a central orientation in Modern Mathematics.

If on the one hand, the TG returns intending to redefine the teaching of Geometry, on the other hand, the complexity of inserting new content, added to its articulation with Algebra, resulted in intense debates, criticisms, and different appropriations among the MMM leaders in Brazil, some of them verbalized by its characters, among them Benedito Castrucci and Martha Dantas:

> So, the process was to develop geometry through an algebraic structure. Then they studied geometry in high school through vector spaces, which is an algebraic structure. [...]. And the other path was *through transformation groups*, an algebraic structure, an idea of Klein's, but now refined so that it could work.
> We began to feel a failure, and the failure for me was Geometry [...] so I taught a course on vector spaces and in all my courses, I was very successful with the student-teachers. And this time I failed, that is, the students did not react well, so they ended up not doing well on the tests [...] I think that the point where I felt the decline here was geometry. And other teachers felt it too (Castrucci, 1988 *apud* Búrigo, 1989, p. 171, 209, our emphasis)
> In the 1950s, when I started teaching in high school, the abandonment of Euclidean Geometry was already notorious. The main cause was, without a doubt, the lack of preparation of teachers. [...] It was necessary to present it in a new form and that is what we did in 1964 [...]. From the recommendations of eminent mathematicians, we deduced that the study of Geometry, through *geometric transformations*, allows abstract notions to be based on simpler and more solid intuitive bases, making them better understood and facilitating the demonstration of properties that involve them (Dantas, 2002 *apud* Duarte, 2007, p. 279, our emphasis).

In this context, the objective of this study is to analyze, from a historical perspective, four textbooks produced between the 1960s and 1970s – three of them published for the middle school course (before Law 5692/7[8]) and the last one for the elementary course – that included the teaching of TG for 13-year-old students (3rd grade of middle school or 7th grade of elementary school). The main guiding questions of our study are: how were GT's treated in the four works? What are the similarities and differences in the approach to GT between the books? How was knowledge from the academic field and knowledge from the professional field guided in these didactic productions? In what follows, before explaining the theoretical-methodological foundations of the study, we briefly discuss the main ideas of the modernizing proposal for the teaching of Geometry.

---

[7] Felix Klein (1849-1925), was a German mathematician who led the first International Commission for the Study of Mathematics Teaching (ICES) in order to propose teaching reforms.

[8] The Law 5692/71 proposed a new structure for Basic Education, which became: 1st grade education, with eight years of schooling, and 2nd grade education, with three years (Brazil, 1971).

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

2 The proposals of a Modern Geometry

The teaching of geometry was the subject of great debate and conflict among Brazilian leaders. The same can be seen internationally, and one example was the famous phrase by mathematician Jean Dieudonné[9], during the Royaumont Seminar: “If I wanted to summarize in one sentence the entire program I have in mind, I would do so as a slogan: Down with Euclid!” (OECE, 1961, p. 35). Different proposals were formulated for the modernization of geometry, seeking a closer connection with 20th-century mathematics. Professor Howard Fehr[10], one of the defenders of the MMM in the USA, pointed out two trends in the teaching of geometry in 1962. The first, developed by G. D. Birkhoff[11], proposed a modification of Euclid's axioms, following Hilbert’s[12] general form. Birkhoff's axioms were modified by Edwin C. Moise and used in experimental texts in schools in the United States. He emphasizes that this trend sought to preserve Euclid's Geometry, correcting its defects through the introduction of real numbers. The second trend began in Germany and recommended the development of Geometry through GT, based on the ideas of F. Klein. In this approach, Euclidean Geometry (EG) was conceived as the study of the properties of figures that do not change under the application of transformations from the similarity group. It is important to emphasize that this group includes isometries, which are transformations that preserve distances and angles.

As we pointed out, the inclusion of GT in secondary course programs in Brazil has occurred since, at least, the Francisco Campos Reform, in 1931, but in the following reforms, in 1942 and 1951, they lost strength, since they are punctually mentioned, without emphasis and unrelated to the other notions or topics of the program (Jahn; Magalhães, 2023).

Thus, it is important to understand how the tendency to incorporate GT was appropriated in textbooks from the 1960s and 1970s, especially in books authored or supervised by leading mathematicians from the University of São Paulo (USP). In particular, we ask ourselves: What are the similarities or differences identified regarding GT? How was the articulation between GT and the *classical* EG made?

---

[9] Jean-Alexandre Dieudonné (1906-1992), was a French mathematician, one of the founders of the Bourbaki group. Nicholas Bourbaki was a pseudonym used by a group of mathematicians who advocated an evolution and internal revolution in Mathematics based on the development and study of the notion of structures (Camargo, 2009, p. 43).

[10] Howard Fehr (1901-1982), was an American mathematician.

[11] George David Birkhoff (1884-1944), was an American mathematician.

[12] David Hilbert (1862-1943), was a German mathematician.

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

3 Theoretical-methodological foundations of the research

The four books examined were important works in the 1960s and 1970s for various reasons. Among them, all of them have authors or supervisors who are considered authorities in the process of appropriation of the MMM's ideals in Brazil, particularly in the states of São Paulo and Bahia, and who maintained close relations with the GEEM[13]. Created in 1961, the GEEM, under the coordination of Osvaldo Sangiorgi, was a decisive milestone in the constitution of the movement in Brazil, allowing the new proposal to be widely disseminated beyond the restricted circles of educators and promoting the implementation of experiments supported by an articulated discussion (Búrigo, 1989). Table 1 provides the references for the sources analyzed.

Title	Author	Publisher	Year
Matemática – Curso Moderno para os ginásios – 3º vol.	Osvaldo Sangiorgi	Cia. Editora Nacional São Paulo	1967
Matemática – Curso Moderno – 3 Ciclo Ginasial	Alcides Bóscolo, Benedito Castrucci	Editora FTD São Paulo	1970
Ensino Atualizado da Matemática – curso ginasial – vol. 3	Omar Catunda, Martha M. de Souza Dantas, Eliana Costa Nogueira, Norma Coelho de Araújo, Eunice da C. Guimarães e Neide C. de Pinho e Souza	Edart São Paulo	1971
Curso Moderno de Matemática para ensino de primeiro grau – 7	Anna Averbuch, Franca Cohen Gottlieb, Lucília Bechara Sanchez, Manhúcia Liberman e Jacy Monteiro (supervisão)	Cia. Editora Nacional São Paulo	1975

Table 1- Textbooks selected for analysis
Source: elaborated by the authors based on: Sangiorgi (1967), Bóscolo and Castrucci (1970), Catunda *et al*. (1971) e Averbuch *et al*. (1975)

The proposal is to understand how GT reached school culture through textbooks. According to Badanelli and Cigales (2020), textbooks have been relevant sources for research since the historiography of education focused on school culture, being considered a fundamental device for the transmission of knowledge and the organization of school practices. Furthermore, in times of turbulence, when new approaches are introduced, textbooks gain a prominent role, as they are the first to translate new proposals for pedagogical practices. Chervel (1990, p. 191-192) points out that:

> Things happen differently when schools are entrusted with new purposes, or when the evolution of purposes disrupts the course of old disciplines. [...] On the one hand, the new objectives imposed by the political situation or by the renewal of the educational system become the subject of clear and detailed declarations. On the other hand, each

[13] A more in-depth study on GEEM can be read in Lima and Passos (2008).

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

> teacher is forced to set out on his own paths not yet taken or to experiment with the solutions that are advised.

Munakata (2012, p. 122) reiterates the emphasis on textbooks as peculiar configurations specific to the school and inherent to school culture:

> [...] the notion of school culture refers not only to norms and rules, explicit or not, symbols and representations, in addition to prescribed knowledge, but also, and above all, to practices, appropriations, attributions of new meanings, resistances, which produce multiple and varied configurations, which occur topically in school. [...] One of these things peculiar to school is precisely the textbook.

Thus, the examination of four distinct collections allows us to compare how each author, or group of authors, understands, appropriates, and consumes the different ideas that circulated, in order to create and construct a didactic proposal specific to each book.

Following the presentation of the theorists who support the study, as well as the analysis trajectory, we present the reflections of Fernandes and Garnica (2020) on the methodological path of research in the History of Mathematics Education, and we chose to adopt a qualitative approach, dialoguing with the object of investigation – the GT.

In this way, we focused on the bibliographic review of the theme and the MMM. Jahn's thesis (1998) presents a historical approach to the genesis of the concept of TG, Bkouche's study (1991) on the different internal meanings of the concept of TG itself and external ones with regard to the relations of the discipline and other areas of knowledge, in addition to reviewing the numerous studies already developed on the MMM, among which we highlight: Búrigo (1989); Duarte (2007); Camargo (2009); Rios (2010); Silva (2014).

After conducting a broad review, we decided to delve into the sources and extract from them the characteristics of the GT of the four books compared with the problems identified in the bibliographic review in order to construct the three categories of analysis that guide us in the examination of each book: (1) Approach to GT as teaching objects – the objective was to analyze how the concept of TG is defined in the books, for example, if and how it is related to the concept of function and which properties are highlighted; (2) Articulation between GT and EG – the objective was to analyze how the proposal to include GT was integrated (or not) with the EG configured for schools, for example, if and how it is related to the concept of congruence of figures; (3) Sequence of concepts studied about GT– the objective was to analyze whether there was a standardization in the choice of TG to be presented and in the order of the concepts studied. Before moving on to the analysis of the books, we will briefly introduce their authors.

4 A brief contextualization of the authors of the analyzed books

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

All textbooks in Table 1 have at least one teacher who actively participated in GEEM as their authors or supervisors. However, we can distinguish two groups: the Mathematicians (PhDs and/or Professors of Mathematics at USP[14]) who represent the academic field, the knowledge of 20th-century mathematics; and the Mathematics teachers who represent the professional field, the knowledge of school culture and its dynamics. The first group was formed by Omar Catunda (1906-1986), Professor of Analysis; Benedito Castrucci (1909-1995), Professor of Geometry, and Luiz Henrique Jacy Monteiro (1921-1975), PhD in Algebra, all men[15], USP professors, who represented national authorities in the mathematical production of the period, all linked to the Brazilian Mathematical Society of São Paulo. Among other activities, they were responsible for teaching courses on Modern Mathematics (MM) at GEEM, that is, training teachers regarding the new concepts to be incorporated into teaching, among them, GT. The participation of these three mathematicians in the preparation of the textbooks was aimed at ensuring the appropriation of MM in a correct, precise way, with the rigor and language appropriate for high school students (Duarte, 2007). The second group consisted of Mathematics teachers with degrees in Mathematics, such as Osvaldo Sangiorgi (1921-2017), Alcides Bóscolo, Lucília Bechara Sanchez and Manhúcia Perelberg Liberman (1927-2017). These teachers also taught training courses at GEEM, but their recognition in the professional field was linked to the pedagogical practices of good Mathematics teachers. Lucília Bechara Sanchez (2023), for example, told us that she learned MM in GEEM[16] courses because, during her training at the Pontifical Catholic University of Campinas, she had no contact with MM topics.

The other authors are a group of women only, but not from São Paulo and without any connection to GEEM. Two of them were from Rio de Janeiro: Anna Averbuch and Franca Cohen Gottlieb, and were invited by Manhúcia Liberman to participate in the production of the book; and four of them were from Salvador: Eliana Costa Nogueira, Norma Coelho de Araújo, Eunice da C. Guimarães and Neide C. de Pinho e Souza, all graduated in Mathematics from the then Faculty of Philosophy of the University of Bahia and Mathematics professors in the

[14] The University of São Paulo was created in 1934, establishing the Faculty of Philosophy, Sciences, and Languages (FFCLUSP) and hiring foreign professors to train the first generation of Brazilian mathematicians. Catunda was an assistant to the Italian professor Luigi Fantappié in Mathematical Analysis; Castrucci was an assistant in Geometry in the chair of professor Giacomo Albanese and Jacy Monteiro worked as a teaching assistant in the Modern Algebra course taught by Jean Dieudonné (Duarte, 2007).

[15] We can identify gender difference as a determining factor in the formative process established in GEEM, however, this is not our object of study.

[16] Lucília was a Mathematics teacher, hired by the São Paulo Education Department, in the city of Conchas and came to São Paulo in the 2nd semester of 1961, to participate in the MM course at GEEM, as a commissioner.

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

Scientific Section of Mathematics of the Bahia Science Teaching Center (CECIBA) of UFBA. Finally, still from Bahia, we have Professor Martha de Souza Dantas (1923-2011), a graduate in Mathematics, professor of the *Special Didactics of Mathematics* subject of the Mathematics course at the University of Bahia, and coordinator of CECIBA.

5 Analysis of the books by Sangiorgi and Bóscolo & Castrucci

The books by Osvaldo Sangiorgi (OS) and Bóscolo and Castrucci (BC) include the GT as appendices in order not to integrate the GT with the EG inserted in the body of the works and, for this reason, we performed a joint analysis of both. Both books have practically the same title, indicating the presence of the MM in the collection, and were published three years apart, in the 1960s, when secondary education was still organized as primary (first 4 grades), middle school (4 grades) and high school (3 grades).

The book by OS contains 314 pages, organized into 4 chapters plus the appendix, with chapters 3, and 4 and the aforementioned appendix dedicated to Geometry, totaling 63.7% of the book. The book by BC has 297 pages, containing 19 chapters and the appendix. Among the chapters, the part related to the teaching of Geometry is seen from chapters 12 to 19 and in the appendix, which corresponds to 51% of the book. We can see that both dedicate more than half of their books to Geometry; and the GT, in both books, are in the appendix, with the OS book dedicating 14 pages and the BC book 15 pages. The presence of the GT in the appendix denotes the authors' conception of not integrating their study with the one of EG. Everything indicates that the GT was inserted only to meet the recommendations of a modern program defended by GEEM[17].

However, the approach and sequence of concepts for teaching GT in the two books differ considerably, especially in the order and way in which GTs are addressed, as we can see in Table 2.

[17] Sangiorgi indicates in the preface that the book is in accordance with the Program that was approved by the Secondary Education Board of the Ministry of Education and Culture, from November 25 to 30, 1963 and with Suggestions for a program outline for the mathematics subject, secondary course – 1st cycle – published in the Federal Register of January 19, 1965.

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

Geometric plane transformations - OS	***Transformations of the plane –*** BC
Translation Group: oriented segment, algebraic measure, equipollent segment, translation in the plane, translation of plane figures, addition of translation, verification that translations in the plane, in relation to the addition operation, have a commutative group structure.	***Plane transformations***: identity and isometry.
Rotation Group: rotation at a point, in the figures, co-terminal arcs, addition of rotations and check the four properties to justify the structure of the commutative group.	***Translation:*** defines translation at a point, highlights the properties of the transformation – isometry, that a straight line not parallel to the amplitude corresponds to a straight line parallel to it, a straight line parallel to the amplitude corresponds to the straight line itself, an angle corresponds to an angle congruent to it –, equal and inverse translations, the composition of translations and verifies the properties of the transformation with the composition.
Axial Symmetry: defines a symmetrical point, axis of symmetry, axial symmetry, presents the symmetry of a quadrilateral and a triangle.	***Central symmetry***: defines a point, highlights the properties of the transformation, and presents the composition of two central symmetries of center O and O' obtaining the translation of amplitude OO' and parallel to OO'.
Central Symmetry: defines a symmetrical point, central symmetry, and presents as an example the central symmetry of a triangle.	***Rotation:*** Sets rotation at a point, highlights the transformation properties and checks the transformation properties with the composition.
Practical application of **axial symmetry** – shortest path problem.	***Axial Symmetry:*** defines a point, highlights the properties of the transformation, and presents two possibilities for composing axial symmetries: one with parallel axes that result in a translation and another with perpendicular axes that result in central symmetry of the center at the intersection of the axes.
Attention test: set of ten exercises: 4 translations, 1 rotation, 1 translation followed by rotation, 1 axial symmetry, 1 central symmetry, and 2 applications of TG to shortest path problems.	***Isometria Direta e Inversa***: relaciona com a ideia de deslizamento e sobreposição.
	Transformation Group: revisits the examples of addition and multiplication in the numerical sets studied previously (Z, Q, and R) as commutative groups, and announces that some transformations have the same properties with the composition operation, exemplifying the case of the commutative group of translations in relation to composition and says that it is possible to verify the same for rotations.

Table 2 –Appendix title and order of contents on GT in OS and BC books
Source: elaborated by the authors based on: Sangiorgi (1967) and Bóscolo and Castrucci (1970)

The analysis of Table 2 shows several differences: the first of these concerns the way of beginning the study of GT. OS begins the study with the Translations Group, defining translation as: "the translation T_a also called a transformation of the points of the plane, because it "transforms" a point A into a point A', by means of the oriented segment AA'" (Sangiorgi, 1967, p. 303), without, however, previously defining what is a transformation in the plane. The use of the word *transform* by OS, in quotation marks, may indicate an intuitive way of approaching GT, a *concern with the pedagogical*, indicating aspects of the professional field, without necessarily establishing a relationship with the concept of function, that is, the academic knowledge of GT is approached without a formal definition.

On the other hand, BC begins the study of GT by defining transformation in the plane as “a biunivocal correspondence between the points of the plane π” (Bóscolo; Castrucci, 1970, p. 283) and then presents the identity transformation and the concept of isometry, with its properties. Considering its formation and main activity, we infer that the option is made by the formal aspect of the knowledge in question, with the academic field prevailing. This difference indicates distinct approaches, that is, OS does not relate GT to the study of functions, while BC does so and identifies and examines the properties resulting from the respective functions.

Regarding the choice and sequence of contents, the authors once again do not follow the same order: while OS begins by presenting the commutative groups of translations and rotations, BC chooses to work first with transformations in general and their respective properties, presenting the structure of the commutative group only at the end of the Appendix.

In short, although both authors dedicate the study of GT to the Appendix, the approach taken is clearly different, both in the order in which the GTs are presented and, fundamentally, in the approach: BC proposes a functional treatment, identifies the invariants in each GT, defines direct and inverse isometry, and ends with the structures of the commutative group of translations and rotations, exactly the same themes with which OS begins the study of GTs.

The next two books insert GT into the body of the book and seek to articulate the study of GT with EG, but again, in very different ways. Therefore, we will analyze each of the books separately.

6. Analysis of the books by Catunda *et al*. (CD) and Averbuch *et al*. (GRUEMA)

The third book examined, written by Catunda, Dantas, and collaborators, which we will refer to as CD, is organized into 4 chapters, plus two initial topics: Introduction and Notions of Logic. Three chapters are dedicated to the geometry of the plane: Chapter II – *Straight line*, which is developed in an algebraic way and in which the GT on the line is addressed, namely: translations, symmetries, affine transformations, and homothety; Chapter III – *Affine geometry of the plane* and Chapter IV – *Euclidean geometry – Distances and Polygons*. The three chapters make up a total of 111 pages, which corresponds to approximately 78% of the volume, revealing a significant percentage dedicated to geometry and in all chapters, the GT is present. The aforementioned chapters and the book’s introduction clearly indicate the existence of an unusual

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

approach to teaching geometry: first Affine Geometry, then Euclidean Geometry, "only after exploring the purely linear part – two-dimensional vector and affine spaces – does the metric part of elementary geometry begin", which is a "reformulation idealized by Professor Omar Catunda" (Catunda *et al*., 1971, p. VII-VIII). Table 3 briefly presents the list and order of contents related to GT developed in the aforementioned geometry chapters.

Chapter II Straight Line	***Translation*** on the real line; ***Symmetry*** on the real line***; Set of translations and symmetries; Affine transformation*** or affinity on the line***; Homothety*** on the line.
Chapter III Affine Plane	***Translations in the plane:*** definition, composition of translations (addition of vectors), translations form a commutative group; **Dilations:** definition (multiplication of vector by scalar), composition of dilations; **Properties** of translations and dilations to define vector space.
	Transformations in the affine plane: definition of transformation in the plane (pointwise correspondence); definitions of **translation; central symmetry; homothety; Theorems** on the product of two symmetries in the plane and on the product of a symmetry by a translation**; Elementary affine group:** set of translations and pointwise symmetries**. Congruent figures:** which can be transformed into each other by a transformation of the elementary affine group; **Homothety in the Triangle.**
Chapter IV Euclidian Geometry	***Axial*** ***symmetry*** (symmetry in relation to the straight line); **Properties of axial symmetry** (fixed points; half-plane in opposite half-plane; composition of symmetry with itself is Id; symmetrical figure of a straight line is another straight line); **Definitions:** two figures, symmetrical to each other, in relation to an axis r, are said to be congruent **(congruence by symmetry); Symmetrical figure and axis of symmetry** (a figure that transforms into itself by an axial symmetry); **Perpendicular line** from axial symmetry**; Symmetrical property of orthogonality; Composition of symmetries** (with coordinate system); **Theorems:** composed of 2 axial symmetries in relation to parallel axes and in relation to orthogonal axes; **group of congruences** (isometric or Euclidean): set of transformations that decompose into axial symmetries.
	Rotation: defined by the composition of two symmetries whose axes are neither parallel nor orthogonal; Transport of figures (by transporting segments and angles); cases of congruence of triangles (by transporting segments and angles).
	Similar figures: definition: a figure F is similar to another figure F' when F is congruent to a figure F'' which is homothetic to F; **similarity is obtained by the composition of a homothety with a congruence,** properties of similarity (homothety theorems). Cases of similarity of triangles and of any figures.

Table 3 – GT-related topics in the CD book
Source: elaborated by the authors based on Catunda *et al*. (1970)

Table 3 highlights a large number of new concepts to be addressed in the high school course, in addition to the detailed study of GT – in particular: the definition of vector space, affine plane, and elementary affine group – all of which are necessary to define congruent and similar figures. EG is addressed in chapters III and IV, although the designation is only in

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

chapter IV. With Affine Geometry [18]defined, the properties that can be justified involving parallelism and collinearity are studied and some are demonstrated; for example, properties of parallelograms and trapezoids and, with homothety in triangles, Thales' Theorem and its consequences. In chapter IV, the concept of congruent figures is expanded, allowing the transport of figures by the defined TG and this transport is reduced to the transport of segments and angles. As expected in the study of EG, measures of segments and angles are conceptualized in a relatively precise manner. Transport by congruence will be the focus of the study of triangle congruence. In fact, the theorems that state these cases are demonstrated by mobilizing the transport of segments and angles, and, subsequently, allow us to resume the study of the properties of triangles and quadrilaterals in the Euclidean approach. As an example, the demonstration of the first theorem related to isosceles triangles: "the bisector of angle *A* is the axis of symmetry of the isosceles triangle" (Catunda *et al*., 1971, p. 11) is carried out by considering the symmetry in relation to the bisector of the angle opposite to the base.

Regarding the methodological choice, the book presents a traditional approach: concepts start with definitions, followed by frequently demonstrated theorems and, at the end, exercises predominantly using algebraic language. Everything suggests that the choice of studying geometry via GT, strongly structured on the notions of vector space, affine plane, transformation groups, and abstract concepts, ends up mitigating the intuitive and visual aspects of GT, making the text similar to those of textbooks on so-called Academic Mathematics.

Finally, the book by Averbuch *et al*. (GRUEMA[19]) is organized into 12 topics, 4 of which are dedicated to Geometry, which corresponds to 46% of the work. The first (topic 3), entitled *Parallelism and Direction*, represents a systematization of the parallelism relation and has the following objectives: recognizing that the parallelism relation is an equivalence relation; identifying equipollent segments; constructing the midpoint of a segment; dividing segments into equipollent parts; and grading a straight line. The last three topics (10, 11, and 12) revisit the study of Geometry: Circles, Symmetry, and Congruence, respectively. With this, we can say that the study of GT begins with the concept of Symmetry (topic 11), which refers to reflection in relation to a straight line (called axial symmetry), whose objectives are:

> 1. Recognize and construct symmetrical points in relation to a line; 2. Determine the axis of symmetry of a segment; 3. Determine the symmetrical of a plane figure in relation to an axis; 4. Determine the axis of symmetry of plane figures; 5. Identify the

[18] Affine Geometry studies the properties of figures that can be deduced from the axioms of incidence and parallelism, limiting itself to just a few axioms of congruence. In relation to these axioms, exclusively those related to the notion of equality of segments or the ratio between segments contained in the same straight line or parallel lines are considered.

[19]The acronym GRUEMA (Grupo de Ensino de Matemática Atualizada) became the authorship denomination used in books from 1972 onwards, with the authors listed in Table 1.

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

> invariants of a symmetry; 6. Recognize and construct the perpendicular bisector as the axis of symmetry of a segment and the bisector as the axis of symmetry of an angle; 7. Identify and apply the properties of the perpendicular bisector and the bisector; 8. Recognize and construct the medians of a triangle; 9. Construct perpendiculars to a line using symmetry (Averbuch *et al*., 1975, p. 143, our translation).

The first difference identified in the work, when compared to the others, is the study of only one GT in the book for the 7th grade[20], which is the axial symmetry (or reflection in a straight line). After exploratory exercises that allow characterizing symmetrical points, the definition is presented: "The axial symmetry of axis $\overleftrightarrow{AB}$ is the function that transforms points of the plane into its symmetrical points in relation to $\overleftrightarrow{AB}$." (Averbuch *et al*., 1975, p. 144).

We note that the definition of axial symmetry is based on the concept of function, with emphasis on its properties, namely: conservation of alignment, of distance, takes line/segment to line/segment and takes circumference to circumference (with the same radius). With this, we infer that the authors sought to characterize the main invariant properties of axial symmetry, seen as a function, in accordance with the definition presented.

Still examining the approach of GT, it is important to emphasize that the topics of the book have a common structure that can be described as follows: they begin with groups of *preliminary exercises*; followed by text boxes systematizing what was covered in such exercises (with the following names: *Write down; Did you observe that; In general; Do you remember?; We learned that*) and new groups of *application exercises* are proposed (in some of them, text boxes follow with the name *You proved that* or *You demonstrated that*). In addition, a comic strip with two characters (children) is used as a way of introducing texts with language that is more accessible to students, aiming to contextualize, problematize and/or systematize the mathematical notions being studied. It is noted that the definition of symmetry is only presented after a diverse group of preliminary exercises on symmetrical points.

After defining symmetry, the book proposes *exercises for applying the concept*, which allow students to *discover* a way to find the axis of symmetry given a point *P* and its symmetric *P'*, as well as a way to obtain the midpoint of a segment by geometric construction. We also note as a relevant characteristic of the proposal that the studies of mediatrix and bisector are based on the concept of axial symmetry, exploring and identifying the properties of this isometry. For example, the bisector of an angle is approached as the axis of symmetry of the angle, which indicates a way of developing EG concepts via (or from) TG, in this case, reflection in relation to a straight line.

---

[20] The authors and the supervisor chose not to present translation and rotation, since they were presented in the other three books analyzed.

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

The same can be said regarding the concept of congruence of figures. Based on exercises and comic strips, the idea of applying successive axial symmetries to the same figure is introduced in order to make it correspond to another figure by superimposing them, and then the definition of congruent figures is presented as follows: "Note: When it is possible to transform a figure *F* into a figure *G* by a succession of symmetries, the figures are congruent." (Averbuch *et al*., 1975, p. 175).

In summary, the GRUEMA book is characterized by proposing only one of the TG in the 7th grade book, the axial symmetry. However, through a detailed study, based on the concept of function, which had been studied in a previous topic in the book, and by mobilizing several exploratory exercises, with geometric constructions, use of comic strips, among other pedagogical resources, the book finally articulates axial symmetry with the concepts of EG and develop a deductive study of plane geometric figures, especially triangles and quadrilaterals.

7 Conclusions

We have reached the end of the examination of the four books and return to the guiding questions of our study: what are the similarities or differences identified regarding the introduction of GT? How was the articulation between GT and EG made? To answer these questions, we constructed three categories aiming at a detailed study, the summary of which is presented in Table 4 below.

Author(s)	GT Approach (definition)	Articulation between GT and EG	Contents
OS	Translation and Rotation, sets equipped with the addition operation to justify the commutative group structure.	There was not an articulation; GT's isolated in the Appendix of the book	Translations Rotations Axial Symmetry Central Symmetry
BC	Transformations in the plane as functions and study of their properties. Transformation Group, exemplifying with the commutative group of translations in relation to composition	There was not an articulation; GT's isolated in the Appendix of the book	Transformations Translations Central Symmetry, Rotation Axial Symmetry Direct and Inverse Isometry
CD	Transformations in the affine plane as functions and study of their properties. Elementary affine group, congruence group (isometric or Euclidean)	Integrates GT with EG in chapters III and IV	Translations, Dilations Transformation in the affine plane Elementary affine group Homothety Axial symmetry Congruence group Rotation
GRUEMA	Axial Symmetry as a Function and Study of its Properties	Integrates GT with EG in the topics Symmetry and Congruence	Axial symmetry

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

Table 4 – Summary of categories analyzed in selected books
Source: elaborated by the authors

When we compare the four books in a broad way, we identify certain similarities, but the differences are much more evident than the similarities. It is possible to say that we have four clearly differentiated proposals for approaching GT in the textbooks examined.

The category of articulating GT with EG in the textbook was a dividing line in the analytical process, since the OS and BC books chose not to include GT in the body of the text, leaving them isolated (or disconnected) from the study of "classical EG", present in textbooks since the beginning. Everything indicates that the choice of this approach may be linked to the authors' experiences in the professional field, as teacher trainers and secondary school teachers.

In the case of OS, his recognition was linked to good professional practices. He was the first author to introduce Modern Mathematics. He announced in his book "The Good Bit" for teaching geometry: "Do not memorize theorem demonstrations" (Jahn; Leme da Silva, 2023), a topic discussed and touted during the 1950s as one of the main problems in teaching geometry. We can infer that his objective in proposing a modernizing geometry was to favor students' understanding of the logical-deductive process present in demonstrations, by inserting exploratory exercises with experimental tasks in order to value an intuitive geometry before the deductive treatment. The approach to GT through the structure of commutative groups, as OS presented in the Appendix, would possibly mean a concept that is difficult to understand.

BC's book was written by two authors with different backgrounds: Bóscolo was a high school teacher and represented the professional field, and Castrucci was a professor at USP and had expertise in the academic field, in 20th century mathematics. In his book entitled "Lessons in Elementary Geometry", he sets out his position on the approach:

> There is a movement to replace the geometric concept in high school courses and, perhaps, in high school, with an algebraization of Geometry, treating it as a chapter of Linear Algebra. We believe that this innovation advocated by great mathematicians cannot be done immediately, because in our view it would be, at the moment, a bold step (Castrucci, 1969, s.n., author's emphasis, our translation).

Castrucci's (1969, s.n.) position as "a bold step" may have been decisive for Bóscolo to agree to insert the GT in *isolation* in the Appendix, given that Castrucci's role in the didactic production was to guarantee the scientific mastery of Modern Mathematics books:

> After GEEM, some colleagues started writing books on Modern Mathematics for secondary school, and I wasn't very interested. But then a colleague of mine [Bóscolo] said, no, help me write it, because I'm afraid of making mistakes in the rigorous part, so I wrote a collection of FTD (Castrucci, 1988 *apud* Duarte, 2007, p. 284, our translation).

On the other hand, even as an Appendix, the proposal of BC's book focuses on the

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

functional approach, seeking to identify invariants for each of the GT and, only at the end, presents the structure of the commutative group, which characterizes a presentation and treatment different from that made by OS. In any case, we can say that both prioritized the experiences lived in the professional field, in the practice of the teacher in the classroom, considering even the practice of teacher trainer, in which Castrucci reveals the difficulties encountered with teachers in the courses offered by GEEM.

Another characteristic highlighted in Table 4 concerns the GT selected to be studied by 13-year-old students. OS goes directly to the isometries: Translation, Rotation, Axial and Central Symmetry, while BC introduces the concept of transformation in the plane as a starting point, presents the GT in a different order from that proposed by OS and includes Direct and Inverse Isometries.

CD's book initially presents Translation and Dilation together; presents the transformations in the affine plane, defines an elementary affine group, carries out the study of homothety and intertwines the concept of congruence when defining a group of congruences, to conclude with Rotation. Central Symmetry is not done separately, it is integrated as a particular case of Rotation. Certainly, CD's book differs from other works by providing a complete treatment of TG, as close as possible to the concepts developed in the academic field, and is even compared, by Castrucci himself, to the approach of a postgraduate course:

> [...] there are remnants of Modern mathematics in Bahia, there is Martha de Souza Dantas who still believes that Modern mathematics should be taught, [...] presenting results of *geometry taught by transformations*. I only know that what she presents there, I teach in graduate school. If her students there learn it, it is very interesting, it is exceptional. But there is nothing else, no one is teaching this anymore (Castrucci, 1988 *apud* Duarte, 2007, p. 281, our emphasis, our translation).

It is important to consider that the CD book was written by a group of authors with different backgrounds: Catunda, a retired professor from USP, represented the academic field, while Dantas and the other authors represented the professional experience accumulated at CECIBA. In this case, unlike BC, everything indicates that the academic field prevailed compared to the professional field, as Martha Dantas considered, even mentioning the low repercussion of the work:

> The writing of the new texts was only possible because they had the collaboration of Omar Catunda, "who even accepted the proposal we made to him to use, in the approach to Geometry, geometric transformations, a centuries-old recommendation – made by Felix Klein, in the last century" (Dantas, 1993 *apud* Duarte, 2007, p. 23-24, our translation).
>
> [...] Catunda's original ideas did not become concrete because, as Dienes, a famous pedagogue, rightly said, Catunda was one of those who skipped the stage of concretization. Thus, the algebraism used, especially in the introduction of geometry, and the abstraction resulting from the introduction of concepts were responsible, in

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

> part, for the rejection of the books (Dantas, 1993 *apud* Duarte, 2007, p. 217, our translation).

Martha's assessment reiterates the challenges of trying to bring together an abstract and algebraic geometry from academia and a school geometry and also highlights the trust placed by the team in Catunda's academic knowledge to put the study of GT into practice.

The GRUEMA book was written by a group of teachers under the supervision of Jacy Monteiro. However, it presents a very different proposal from that made by CD, when they selected a GT, the Axial Symmetry, as the only one to be studied in the 7th grade and articulated with EG. When asked about this approach, Lucília Bechara Sanchez (2023) explains:

> *Lucília: There was a lot of this idea that when you create an innovative textbook, you need to talk to existing mathematics, so that people can understand and get involved, without getting the idea: Oh, it changed all mathematics! So when we worked on transformations in GRUEMA, which are so important, we started with the following: what is worked on in Euclidean geometry? And we saw that they were: congruences in the 3rd grade and similarities in the 4th grade, basically. So we thought: let's work on congruences and similarities through geometric transformations.*
> *Interviewer: And why were translation and rotation left out?*
> *Lucília: So, that's a question I wouldn't know how to answer... I could risk... First of all, translation... is a movement that is little perceived, theoretical, let's say... not very motivating! And rotation in space would correspond to axial symmetry in the plane and thus would help when it came to understanding symmetry in the plane, making a rotation in space. A rotation in the plan, however, is not easy, right, it is more difficult. Furthermore, in the book, we tried to make not too many changes so that it would be more acceptable to teachers.*

The justification adds pedagogical aspects to the objective of integrating GT with the central concepts of GE, and not through the exhaustive study of GT. Once again, demands from the professional field prevailed, even with Jacy Monteiro as supervisor, by removing the GT that could be complex and prioritizing an integration with the EG present in the previous textbooks. It is worth considering that Lucília Bechara, in addition to the training obtained through GEEM courses, in 1961, participated in the preparatory course for the Vocational Schools of São Paulo, which included Psychology and Didactics classes, having experienced, between 1962 and 1968, the experience of being a teacher and coordinator of mathematics at the Vocational Gymnasium of São Paulo, considered an innovative experience in public education in the state of São Paulo (Nakamura; Garnica, 2018).

In short, the analysis of how GT were inserted in the textbooks reiterates the tensions established between the academic and professional fields. The examination of the process of inserting GT into school culture, through textbooks, carried out by different authors, subjects with different backgrounds and experiences, reveals the complexity of the dialogue, of the possibility of combining demands coming from two distinct cultures: the academic, as a representative of scientific production, and the school, representing experiences of professional practices.

The MMM brought many lessons, allowed contact with different approaches, such as the four proposals for the study of GT analyzed here, explaining the diversity, the disparities and even the boldness in the face of a renewal movement. We did not find consensus for the study of GT even in a collective of leaders, which allows us to emphasize the non-existence of a single representation of GT in teaching during the MMM, but rather the existence of multiple and diverse geometries of – or with – transformations.

Acknowledgements

To FAPESP, Process 2023/04639-8, for financial support.

References

AVERBUCH, A. *et al*. **Curso Moderno de Matemática para ensino de primeiro grau – 7**. São Paulo: Companhia Editora Nacional, 1975.

BADANELLI, A. B.; CIGALES, M. Questões metodológicas em manualística. **Revista Brasileira de História da Educação**, Maringá, v. 20, n.1, p. 1-7, 2020.

BKOUCHE, R. De la géométrie et des transformations. **Repères-IREM**, Grenoble, [s.v.], n. 4, p. 134-158, juil. 1991.

BLOCH, M. **Apologia da História ou O ofício de historiador**. Rio de Janeiro: Jorge Zahar, 2001.

BÓSCOLO, A.; CASTRUCCI, B. **Matemática – Curso Moderno** – 3 Ciclo Ginasial, São Paulo: FTD, 1970.

BRASIL. Lei nº 5.692, de 11 de agosto de 1971. Fixa Diretrizes e Bases para o ensino de 1° e 2º graus, e dá outras providências. **Diário Oficial da União**: seção 1, Brasília, DF, p. 6377, 12 ago. 1971. Disponível em: https://www2.camara.leg.br/legin/fed/lei/1970-1979/lei-5692-11-agosto-1971-357752-publicacaooriginal-1-pl.html. Acesso em: 20 jul. 2024.

BÚRIGO, E. Z. **Movimento da matemática moderna no Brasil:** estudo da ação e do pensamento de educadores matemáticos nos anos 60. 1989. 286 f. Dissertação (Mestrado em Educação) - Universidade Federal do Rio Grande do Sul, Porto Alegre,1989.

CAMARGO, K. C. **O ensino da geometria nas coleções didáticas em tempos do Movimento da matemática Moderna na capital da Bahia**. 2009. 168 f. Dissertação (Mestrado em Educação Matemática) - Universidade Bandeirantes de São Paulo, São Paulo, 2009.

CASTRUCCI, B. **Geometria Curso Moderno**. São Paulo: Livraria Nobel S. A., 1969.

CATUNDA, O. *et al*. **Ensino Atualizado da Matemática** – Curso ginasial – volume 3. São Paulo: EDART-São Paulo, 1971.

CHERVEL, A. História das disciplinas escolares: reflexões sobre um campo de pesquisa. **Teoria & Educação,** Porto Alegre, [s.v.], n. 2, p. 177-229, 1990.

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

DUARTE, A. R. S. **Matemática e Educação Matemática:** a dinâmica de suas relações ao tempo do Movimento da Matemática Moderna no Brasil. 2007. 437 f. Tese (Doutorado em Educação Matemática) - Pontifícia Universidade Católica de São Paulo, São Paulo, 2007.

FERNANDES, F. S.; GARNICA, A. V. M. Metodologia de Pesquisa em Educação Matemática: éticas e políticas na inserção de novos sujeitos, cenários e conhecimentos. **Perspectivas da Educação Matemática**, Campo Grande, v. 14, n. 34, p. 1-16, 2020.

JAHN, A. P. **Des transformations des figures aux transformations ponctuelles. Etude d'une séquence d'enseignement avec Cabri-géomètre** : relations entre aspects géomètriques et fonctionnels en classe de Seconde. 1998. 459 f. Tese (Doutorado em Didática da Matemática) - Université Jospeh Fourier, Grenoble, 1998.

JAHN, A. P.; LEME DA SILVA, M. C. Não decore demonstrações de teoremas! – A Geometria Moderna de Osvaldo Sangiorgi. **RIPEM -** Revista Internacional de Pesquisa em Educação Matemática, Brasília, v. 13, n. 4, p. 1-19, set./dez. 2023.

JAHN, A. P.; MAGALHÃES, G. R. Transformações geométricas em livros didáticos do ensino secundário: indícios da geometria escolar no período de 1930-1950. *In*: CONGRESO IBEROAMERICANO DE HISTORIA DE LA EDUCACIÓN MATEMÁTICA, 7., 2023, Heredia. **Memorias...** Heredia, Costa Rica: Universidad Nacional, 2023, p. 334-348. Disponível em: https://repositorio.una.ac.cr/handle/11056/28041?show=full. Acesso em: 20 jul. 2024.

KILPATRICK, J. The New Math as an international phenomenon. **ZDM – Mathematics Education**, Berlim, v. 44, n. 4, p. 563-571, 2012.

LIMA, F. R.; PASSOS, L. F. G.E.E.M. – Grupo de Estudos do Ensino da Matemática e o Movimento da Matemática Moderna. *In*: VALENTE, W. R. (org.). **Osvaldo Sangiorgi**: um professor moderno. São Paulo: Annablume Editora, 2008. p. 95-118.

NAKAMURA, M. E. F. P.; GARNICA, A. V. M. Aspectos do ensino de Matemática nos Ginásios Vocacionais paulistas: integração de disciplinas e Matemática Moderna. **Educação Matemática Pesquisa**, São Paulo, v. 20. n. 2, p. 85-112, 2018.

MUNAKATA, K. Livro didático como indício da cultura escolar. **História da Educação**, Porto Alegre, v. 20, n. 50, p. 119-138, 2012.

ORGANIZAÇÃO EUROPEIA PARA A COOPERAÇÃO ECONÓMICA - OECE. **Mathématiques Nouvelles**. Paris: OECE, 1961.

RIOS, M. S. B. **A proposta de ensino nos livros do GRUEMA**. 2010. 179 f. Dissertação (Mestrado em Educação Matemática) - Universidade Bandeirante de São Paulo, São Paulo, 2010.

SANGIORGI, O. **Matemática Curso Moderno**. 3º Volume para os ginásios. São Paulo: Companhia Editora Nacional, 1967.

SANCHEZ, L. B. **Entrevista concedida a Ana Paula Jahn, Maria Célia Leme da Silva e Guilherme Rodrigues Magalhães**. São Paulo, 18 de novembro de 2023.

SILVA, J. C. D. **As transformações geométricas nos currículos prescritos de matemática no ensino fundamental (1930-2010)**. 2014. 158 f. Dissertação (Mestrado em Educação Matemática) - Universidade Bandeirante de São Paulo, São Paulo, 2014.

VALENTE, W. R.; BERTINI, L. F.; MORAIS, R. S. Saber profissional do professor que ensina

ISSN 1980-4415
DOI: http://dx.doi.org/10.1590/1980-4415v38a2302891

Matemática: discussões teórico-metodológicas de uma pesquisa coletiva em perspectiva histórica. **Revista Brasileira de História da Educação**, Maringá, v. 21, [s.n.], p. 1-20, 2021.

Submetido em 21 de Dezembro de 2023.
Aprovado em 28 de Março de 2024.