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Packing large balanced trees into bipartite graphs ☆

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ABSTRACT

We prove that for every $\gamma > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ any family of up to $n^{\frac{1}{2}-\gamma}$ trees having at most $(1-\gamma)n$ vertices in each bipartition class can be packed into $K_{n,n}$. As a tool for our proof, we show an approximate bipartite version of the Komlós–Sárközy–Szemerédi Theorem, which we believe to be of independent interest.

1. Introduction

A family $\mathcal{H}$ of graphs *packs* into a graph G if G contains pairwise edge-disjoint copies of all the members of $\mathcal{H}$. The collection of these copies is called a *packing* of $\mathcal{H}$ in G. Such a packing is *perfect* if it uses all edges of G. A graph H *decomposes* G if there is a perfect packing of a family consisting of multiple copies of H.

Packing and decomposition problems have a long history, naturally relating to Euler's 1782 question on the existence of orthogonal latin squares, and to the existence of designs. We are interested in packing trees into complete bipartite graphs. We start our exposition with an overview of related problems and results, and then we describe our results, situating them within the literature.

1.1. Ringel-type conjectures

In 1963, Ringel [35] conjectured that a perfect packing of any tree into the complete graph K_m for an appropriate m should always exist.

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Conjecture 1 *(Ringel [35]). Every tree of order $n+1$ decomposes K_{2n+1}.*

Ringel's conjecture was solved in 2021 for large n by Montgomery, Pokrovskiy, and Sudakov [33], with a different proof given by Keevash and Staden [26]. The following generalization was suggested in 2016.

Conjecture 2 *(Böttcher, Hladký, Piguet and Taraz [8]). Each family of trees of individual orders at most $n+1$ and total number of edges at most $\binom{2n+1}{2}$ packs into K_{2n+1}.*

Conjecture 2 remains open. Recently it has been confirmed for large trees with maximum degree $O(n/\log n)$ [2].

Much earlier, in 1989, Graham and Häggkvist also generalized Ringel's conjecture, but in another way, changing the complete host graph to a regular graph. Namely, they conjectured, and this is still open, that every tree of order $n+1$ decomposes any $2n$-regular graph [20, Conjecture 2.12]. This would imply Ringel's conjecture. The authors of [20] also proposed a generalization of Conjecture 2 to regular graphs [20, Conjecture 2.17].

We are interested in a variant of the problem where the host graph is the complete bipartite graph. Graham and Häggkvist conjectured that every tree of order $n+1$ decomposes any n-regular bipartite graph [20, Conjecture 2.13].[5] This would imply the following for complete bipartite hosts.

Conjecture 3 *(Graham and Häggkvist [20]). Any tree of order $n+1$ decomposes the complete bipartite graph $K_{n,n}$.*

In Conjecture 3, we cannot replace the decomposition into copies of a fixed tree with a decomposition into any given family of trees of order $n+1$ each, since for instance the family consisting of one path and $n-1$ stars does not decompose $K_{n,n}$ if $n \geq 3$.

Evidence for the conjectures from [20] was given in [14,20,30], and a related conjecture considering unbalanced complete bipartite graphs appears in [39]. Recently, Matthes [31] showed an approximate version of Conjecture 3 by proving that, for every $\gamma > 0$ and sufficiently large n, the collection of n copies of any tree of order $(1-\gamma)n$ packs into $K_{n,n}$.

1.2. Gyárfás' Tree Packing Conjecture

In 1976, Gyárfás (see [19]) put forward the following conjecture which is now widely known as the Tree Packing Conjecture. For ease of notation, let T_i denote an arbitrary tree of order i.

Conjecture 4 *(Gyárfás' Tree Packing Conjecture [19]). Any family of trees $T_1, \ldots, T_n$ packs into K_n.*

Interestingly, Gyárfás [18] proved that Conjecture 4 is equivalent to a reformulation where K_n is replaced with an arbitrary n-chromatic graph, which had been conjectured in [17]. Partial results on Conjecture 4 were given in [11–13,16,17,22,37], and (asymptotic versions of) extensions of the conjecture can be found in [2,3,8,15,27,32].

Three results are of particular interest in our context: Bollobás [6] showed that, the first $\Omega(n)$ trees of any sequence $T_1, \ldots, T_n$ pack into K_n; specifically, one can pack $T_1, \ldots, T_s$ into K_n if $s < \frac{1}{\sqrt{2}}n \approx 0.7n$. Naturally, packing the larger trees is harder. Balogh and Palmer [4] proved that, for sufficiently large n, one can pack $T_n, \ldots, T_{n-t+1}$ into K_n if $t \leq \frac{1}{4}n^{1/4}$ and none of the T_i is a star; or if $t \leq \frac{1}{4}n^{1/3}$ and the maximum degree of each T_i is at least $2n^{1/3}$. Only recently Janzer and Montgomery [24] improved this by showing that the $\Omega(n)$ largest trees of any sequence $T_1, \ldots, T_n$ pack into K_n.

In [22] a variant of Conjecture 4 for bipartite host graphs was suggested.

Conjecture 5 *(Hobbs, Bourgeois, and Kasiraj [22]). Any family of trees $T_1, \ldots, T_n$ packs into $K_{\lceil \frac{n}{2} \rceil, n-1}$.*

After previous work in [9,41], Yuster [40] showed that if $s \leq \left\lfloor \sqrt{\frac{5}{8}}\,n \right\rfloor \approx 0.79n$, then any family of trees $T_1, \ldots, T_s$ packs into $K_{\frac{n}{2}, n-1}$ for n even, and into $K_{\frac{n-1}{2}, n}$ for n odd. Kim, Kühn, Osthus, and Tyomkyn [27, Corollary 8.6] gave a packing into $K_{\lfloor \frac{n}{2} \rfloor, n-1}$ of almost all trees $T_1, \ldots, T_n$ (leaving out the first few trees), if their degree is bounded by a constant, and n is large. Böttcher, Hladký, Piguet, and Taraz [8, Theorem 44] generalized this result by showing that the specific orders of the trees do not matter as long as the sum of the number of edges in the trees is bounded by $e(K_{\lceil \frac{n}{2} \rceil, n-1})$.

In 2013, Hollingsworth [23] presented a variant of Conjecture 4 for balanced trees. Let $T_{i,i}$ denote an arbitrary tree with i vertices in each partition class.

Conjecture 6 *(Hollingsworth [23]). Any family of trees $T_{1,1}, \ldots, T_{n,n}$ packs into $K_{n,n}$.*

[5] They also observed that the analogue of Conjecture 2 for regular bipartite graphs is not true, however their argument to support this claim is not accurate, because there is a decomposition of a 4-regular bipartite graph with 13 vertices in each side, into a P_4 and twelve 4-stars.

Using Yuster's [40] result, Hollingsworth [23] proved[6] that $T_{1,1}, \ldots, T_{s,s}$ pack into $K_{n,n}$, for $n \geq 3$ and $s \leq \frac{1}{\sqrt{2}}n$.

1.3. Ringel-type tree-packing for balanced trees

Our first result, Theorem 7 below, is a step towards a variant of Conjecture 2 for balanced trees and complete bipartite host graphs, along the lines of Conjecture 6, where we manage to pack trees of roughly the same order as the host graph's.

Theorem 7. *For every $\gamma > 0$, there exists n_0 such that for every $n \geq n_0$ any family of at most $n^{\frac{1}{2}-\gamma}$ rooted trees, each with at most $(1-\gamma)n$ vertices in either partition class, packs into $K_{n,n}$, with all tree roots embedded in the same part of $K_{n,n}$.*

Note that the packing from Theorem 7 is only possible because we are far from fully decomposing $K_{n,n}$. Indeed, $T_{n,n}$ cannot possibly decompose $K_{n,n}$. Furthermore, for every $\varepsilon > 0$, there exists a balanced tree $T_{(1-\varepsilon)n,(1-\varepsilon)n}$ which does not approximately decompose $K_{n,n}$. To see this, consider a balanced double star D with $(1-\varepsilon)n$ vertices in either partition class and note that any vertex of $K_{n,n}$ that accommodates one of the two central vertices of some copy of D can only accommodate εn leaves of other copies of the star. If we wish to decompose a complete bipartite graph $K_{m,m}$ with copies of a tree $T_{n,n}$, then m should be larger than n. See Section 5.1 for more discussion.

Theorem 7 can be applied to the setting of Conjecture 3, as we can transform any tree $T := T_{(1-\gamma)n}$ into a balanced tree $T' := T_{(1-\gamma)n,(1-\gamma)n}$ by adding an appropriate edge between two copies of T. Theorem 7 then allows us to pack about $\sqrt{n}$ copies of T' and thus about $2\sqrt{n}$ copies of T into $K_{n,n}$, for large n. However, this is superseded by the result from Matthes [31] we mentioned above.

Returning to Conjecture 6, an immediate consequence of Theorem 7 is a result similar to the one from [4] for Gyárfás' Tree Packing Conjecture. Namely, that, for any $\gamma > 0$ and sufficiently large n, any family of trees $T_{n,n}, \ldots, T_{n-\sqrt{n}+1,n-\sqrt{n}+1}$ packs into $K_{(1+\gamma)n,(1+\gamma)n}$. However, one can do better than this, packing the $\Omega(n)$ largest trees, using heavy machinery from [27]. Indeed, it is easy to check that, along the lines of the proof of Theorem 8.5 from [27], one can deduce from Theorem 8.1 of [27] the following.

Theorem 8. *Suppose $0 < 1/n \ll \varepsilon \ll \alpha, 1/\Delta$. Let $H_1, \ldots, H_s$ be $2n$-vertex balanced bipartite graphs with $\Delta(H_i) \leq \Delta$, for each $i \in [s]$. Let G be an (ε, d)-super-regular balanced bipartite graph. If $\sum_{i=1}^{s} e(H_i) \leq (1-\alpha)e(G)$, then $H_1, \ldots, H_s$ pack into G.*

Then the following is an immediate consequence of Theorem 8.

Corollary 9. *Suppose $0 < 1/n \ll \alpha, 1/\Delta$. If $T_{\lceil\alpha n\rceil,\lceil\alpha n\rceil}, \ldots, T_{n,n}$ are balanced trees such that $T_{i,i}$ has $2i$ vertices, and $\Delta(T_{i,i}) \leq \Delta$, for each $\alpha n \leq i \leq n$, then $T_{\lceil\alpha n\rceil,\lceil\alpha n\rceil}, \ldots, T_{n,n}$ pack into $K_{n,n}$.*

1.4. A bipartite Komlós–Sárközy–Szemerédi Theorem

Our proof of Theorem 7 is given in Section 4 and relies on the following idea. We embed the high degree vertices of each tree into distinct vertices of $K_{n,n}$, and pack the rest of the trees into the remainder. For the packing, we develop a balanced tree embedding result for dense bipartite graphs, namely Theorem 11 below.

We believe Theorem 11 may be of independent interest and useful for other embedding problems with bipartite host graphs. It can be viewed as a bipartite approximate version of the following well-known and widely used result.

Theorem 10 *(Komlós, Sárközy and Szemerédi [28]). For each $\gamma > 0$ there are $c, n_0 > 0$ such that for all $n \geq n_0$ every graph on n vertices with minimum degree at least $(\frac{1}{2}+\gamma)n$ contains a copy of every tree on n vertices with maximum degree at most $\frac{cn}{\log n}$.*

Böttcher, Heinig, and Taraz [7] proved a variant of Theorem 10 where the host graph is bipartite and trees are replaced by bipartite graphs with bounded maximum degree and sublinear bandwidth. Since every tree with bounded maximum degree has sublinear bandwidth [10, Theorem 3.9], their result implies a bipartite version of Theorem 10 for trees whose maximum degree is bounded by a constant. Our balanced bipartite version of Theorem 10 allows for trees of linear maximum degree.

Theorem 11. *For each $\gamma > 0$, there are $c, n_0 > 0$ such that the following holds for every $n \geq n_0$. If $G = (A, B, E)$ is a balanced bipartite graph on $2n$ vertices with $\delta(G) \geq (\frac{1}{2}+\gamma)n$, and T is a balanced rooted tree on $2(1-\gamma)n$ vertices with $\Delta(T) \leq cn$, then T embeds in G with the root of T embedded in A.*

Note that in Theorem 11, the bound on $\Delta(T)$ is better than in Theorem 10 and in the result from [7]. However, our trees are only almost spanning, while in these other results the trees are spanning. Moreover, one does not expect to have a bound as in our theorem for spanning trees as we will explain in Section 5.

[6] A slightly weaker bound for s is stated in [23], but one can infer the bound we state here from the proof in [23].

The proof of Theorem 11 is given in Section 3, and relies on the regularity method and embedding results for trees which are discussed in Section 2. Possible extensions of Theorem 11 are discussed in Section 5.

2. Preliminaries

2.1. Regularity

Let A, B be nonempty disjoint sets of vertices of a graph G. Define their *density* as $d(A,B) := \frac{e(A,B)}{|A||B|}$. For $\varepsilon > 0$ we call $A' \subseteq A$ ε*-significant* if $|A'| \geq \varepsilon|A|$, and analogously for $B' \subseteq B$. We say (A, B) is ε*-regular* if $|d(A,B) - d(A',B')| \leq \varepsilon$ for all ε-significant subsets $A' \subseteq A$, $B' \subseteq B$. If furthermore $d(A,B) \geq d$ for some $d \geq 0$, we call (A,B) (ε, d)*-regular*. A vertex $v \in A$ is called ε*-typical* to an ε-significant set $B' \subseteq B$ if $\deg(v, B') \geq (d(A,B) - \varepsilon)|B'|$, and uses an analogous definition for $v \in B$. For $\varepsilon > 0$ we write $x = y \pm \varepsilon$ if $x \in [y - \varepsilon, y + \varepsilon]$.

The next fact contains two well known properties of regular pairs (see [29]).

Fact 12. *Let (A, B) be an ε-regular pair of density d, and let $\varepsilon, \delta > 0$. Then the following hold:*

(a) *For each ε-significant $B' \subseteq B$, all but at most $\varepsilon|A|$ vertices from A are ε-typical to B'.*
(b) *For all δ-significant $A' \subseteq A$ and $B' \subseteq B$, the pair (A', B') is $\frac{2\varepsilon}{\delta}$-regular with density $d \pm \varepsilon$.*

We also need the following fact, which has been observed before (see e.g. [34]), but for completeness we include its short proof.

Fact 13. *For all $i \in [s]$ let (X, Y_i) be (ε, d)-regular and let $Y_i' \subseteq Y_i$ be an ε-significant set. Then at least $\left(1 - \sqrt{\varepsilon}\right)|X|$ vertices in X are ε-typical to at least $(1 - \sqrt{\varepsilon})s$ sets Y_i'.*

Proof. Suppose, for a contradiction, that the statement is false. Then more than $\sqrt{\varepsilon}|X|$ vertices in X are not ε-typical to at least $\sqrt{\varepsilon}s$ of the sets Y_i'. So there are at least $\varepsilon|X|s$ pairs (v, i) such that $v \in X$ is not ε-typical to Y_i'. Thus, there is $i \in [s]$ such that more than $\varepsilon|X|$ vertices of X are atypical to Y_i', contradicting Fact 12(a). $\square$

Szemerédi's regularity lemma [38] states that every large graph has a partition into a bounded number of vertex sets, most of which are pairwise ε-regular. We will need a version of this lemma for bipartite graphs $G = (V, W, E)$. An (ε, d)*-regular partition of* G is a pair $(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{X}$ is a partition of V, $\mathcal{Y}$ is a partition of W, there exist $X_0 \in \mathcal{X}$ and $Y_0 \in \mathcal{Y}$ with $|X_0|, |Y_0| \leq \varepsilon n$, and for all $X \in \mathcal{X} \setminus \{X_0\}$ and $Y \in \mathcal{Y} \setminus \{Y_0\}$ it holds that

1. $|X| = |Y|$,
2. X and Y are independent sets, and
3. (X, Y) is ε-regular with density either $d(X,Y) > d$ or $d(X,Y) = 0$.

We often call the sets $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ *clusters*. The (ε, d)*-reduced graph* R of G with respect to $(\mathcal{X}, \mathcal{Y})$ is the graph on vertex set $(\mathcal{X} \setminus \{X_0\}) \cup (\mathcal{Y} \setminus \{Y_0\})$ having an edge between $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ whenever $d(X,Y) > d$. We will use the following variant of Szemerédi's regularity lemma.

Lemma 14 *(Bipartite Regularity Lemma [7]). For every $\varepsilon > 0$ and $k_0 \in \mathbb{N}$, there is a K_0 such that the following holds for every $d \in [0,1]$ and $n \geq K_0$. For every balanced bipartite graph G on $2n$ vertices with $\delta(G) \geq \lambda n$ for some $0 < \lambda < 1$, there exists a spanning subgraph $G' \subseteq G$, a graph R, and an integer s with $k_0 \leq s \leq K_0$ with the following properties:*

(a) *R is the (ε, d)-reduced graph with respect to an (ε, d)-regular partition of G'.*
(b) *$|V(R)| = 2s$ and $\delta(R) \geq (\lambda - (d + \varepsilon))s$.*

2.2. Tree partitioning and assigning results

Our embedding strategy relies on the embedding of small trees into regular pairs. The next result shows that any tree can be cut into small subtrees with few connecting vertices. This type of result already appeared in the literature, probably first in [1]. Here we use a version from [5]. In what follows, let $r(T)$ denote the root of a rooted tree T.

Proposition 15 *(Proposition 4.1 in [5]). For each $\beta \in (0,1)$, each $t > \beta^{-1}$ and each rooted tree T with t edges, there is a set $S \subseteq V(T)$ and a family $\mathcal{P}$ of disjoint rooted trees such that*

(a) *$r(T) \in S$ and $|S| < \frac{1}{\beta} + 2$,*

(b) $\mathcal{P}$ *consists of the components of* $T - S$*, with each tree* $P \in \mathcal{P}$ *rooted at the vertex* $r(P)$ *closest to* $r(T)$ *in* T*, and*
(c) $|V(P)| \le \beta t$ *for each* $P \in \mathcal{P}$*.*

The pair $(S, \mathcal{P})$ from Proposition 15 is called a *ß-decomposition* of T. The vertices from S are called *seeds*.

Once the tree is decomposed, we will have to decide where to embed each of the small trees in $\mathcal{P}$. For this, we use the following result from [36].

Lemma 16 *(Lemma 3.5 in [36]). Let* $m, s \in \mathbb{N}$, $\mu > 0$ *and let* $(x_i, y_i)_{i\in I} \subseteq \mathbb{N}^2$ *be such that*

(a) $(1-\mu)\sum_{i\in I} x_i \le \sum_{i\in I} y_i \le (1+\mu)\sum_{i\in I} x_i$,
(b) $x_i + y_i \le \mu m$*, for every* $i \in I$*, and*
(c) $\max\left\{\sum_{i\in I} x_i, \sum_{i\in I} y_i\right\} < (1-10\mu)ms$*.*

Then there is a partition $\{J_1, \ldots, J_s\}$ *of* I *such that for each* $i \in [s]$*,*

$$\sum_{j\in J_i} x_j \le (1-7\mu)m \quad \text{and} \quad \sum_{j\in J_i} y_j \le (1-7\mu)m.$$

3. Embedding trees in dense balanced graphs: proof of Theorem 11

To improve readability, we omit all floors and ceilings. From the statement of the theorem, we are given a number $\gamma > 0$. We may assume that $\gamma < \frac{1}{2}$. Apply Lemma 14 (the Bipartite Regularity Lemma) with $\varepsilon = \left(\frac{\gamma}{120}\right)^2$ and $k_0 = \frac{1}{\varepsilon}$ to obtain K_0. Set

$$c = \frac{\varepsilon\gamma}{50K_0^2} \quad \text{and} \quad n_0 = \frac{2K_0^4}{\varepsilon}.$$

To prove the theorem, we need to show that for every $n \ge n_0$, if $G = (A, B, E)$ is a balanced bipartite graph on $2n$ vertices with $\delta(G) \ge (\frac{1}{2} + \gamma)n$, and T is a balanced rooted tree on $2(1-\gamma)n$ vertices with $\Delta(T) \le cn$, then T embeds in G with the root of T embedded in A.

So let $n \ge n_0$ and let G and T be as above. We proceed in five steps: regularisation of G, decomposition of T (into a β-decomposition $(S, \mathcal{P})$, for an appropriately chosen β), setting linking zones, assigning the trees in $\mathcal{P}$ to clusters of the decomposition of G, and embedding T.

Step 1. Regularization of G. By Lemma 14 with $d = 5\sqrt{\varepsilon}$ and $\lambda = \frac{1}{2} + \gamma$, there are a spanning subgraph $G' \subseteq G$, a graph R, and an integer s with $k_0 \le s \le K_0$ such that R is the (ε, d)-reduced graph with respect to an (ε, d)-regular partition $(\mathcal{X}, \mathcal{Y})$ of G', with $\mathcal{X} = \{X_1, \ldots, X_s\}$ and $\mathcal{Y} = \{Y_1, \ldots, Y_s\}$, where $X_i \subseteq A$ and $Y_i \subseteq B$ for all i, and

$$\delta(R) \ge \left(\frac{1}{2} + \frac{\gamma}{4}\right)s. \tag{1}$$

A straightforward application of Hall's Theorem (see [21]) yields a perfect matching M of R. We can assume M pairs X_i with Y_i for each $i \in [s]$. This decomposition of G will guide our embedding of T in G. In particular, most of the edges of T will be embedded into edges between clusters X_i and Y_i.

Step 2. Decomposition of T. Set $t = |E(T)|$ and $\beta = \frac{\varepsilon\gamma}{K_0^4}$. Proposition 15 provides a β-decomposition $(S, \mathcal{P})$ of T. Define the set of *linking vertices* as $L(T) = \{r(P) : P \in \mathcal{P}\}$. Because the parent of each linking vertex is in S and by our choice of c, we have

$$|S \cup L(T)| \;\le\; |S| \cdot \Delta(T) + 1 \;\le\; \frac{4K_0}{\varepsilon} \cdot cn \;\le\; \frac{\gamma}{8}|X_1|, \tag{2}$$

that is, the set of all seeds and linking vertices is not too large when compared to the size of each cluster in the decomposition of G.

Step 3. Setting linking zones. We reserve a subset of each cluster X_i, Y_i for $L(T)$. To this end, for every $i \in [s]$, we arbitrarily partition X_i (Y_i) into two sets $X_{i,L}$ and $X_{i,P}$ ($Y_{i,L}$ and $Y_{i,P}$) so that $|X_{i,L}| = |Y_{i,L}| = \frac{\gamma}{4}|X_i|$. We call these subsets the *L-slice* and the *P-slice* of X_i (Y_i). As (X_i, Y_i) is ε-regular with density $d(X_i, Y_i) > d$, and the sets $X_{i,L}$, $X_{i,P}$, $Y_{i,L}$, and $Y_{i,P}$ are $\frac{\gamma}{4}$-significant for all i, Fact 12 implies that each pair $(X_{i,J}, Y_{j,J'})$, with $J, J' \in \{L, P\}$, is $\frac{8\varepsilon}{\gamma}$-regular with density at least $d - \varepsilon > \frac{d}{2}$.

Step 4. Assigning the trees of $\mathcal{P}$ *to clusters.* We now decide which small trees $P \in \mathcal{P}$ will be embedded into each pair of clusters (X_i, Y_i). Let V_e (respectively, V_o) be the set of all vertices of T having even (resp., odd) distance to the root $r(T)$. Set $m := |X_{1,P}|$ and, for each $P \in \mathcal{P}$, set $x_P := |V_e \cap V(P)|$ and $y_P := |V_o \cap V(P)|$. As T is balanced, conditions (a) and (c) of Lemma 16 are satisfied with $\mu = c$, while (b) holds because $|V(P)| \le \beta t$ for each $P \in \mathcal{P}$. Thus, there is a partition $\{\mathcal{P}_1, \ldots, \mathcal{P}_s\}$ of $\mathcal{P}$ such that

$$\sum_{P\in\mathcal{P}_i} x_P \le (1-7c)|X_{1,P}| \text{ and } \sum_{P\in\mathcal{P}_i} y_P \le (1-7c)|Y_{1,P}| \text{ for every } i\in[s]. \tag{3}$$

Step 5. Embedding T. Recall we have a β-decomposition $(S,\mathcal{P})$ of T, where S is a set of vertices of T called seeds, and $\mathcal{P}$ is the set of (small) trees that result from the removal of S from T. The embedding of T will be done by successively embedding each seed $v\in S$ and each small tree $P\in\mathcal{P}$ in the following order. It will start by embedding the seed $r(T)$. Then it will always choose to embed next either a seed or a small tree $P\in\mathcal{P}$ which is adjacent to an already embedded vertex of T. We will ensure that each seed $v\in S$ will be embedded in an unused vertex $\varphi(v)$ in the P-slice of some cluster such that

$$\begin{aligned}&\text{if } \varphi(v)\in X, \text{ then } \varphi(v) \text{ is typical to } Y_{i,L} \text{ for all but at most } \sqrt{\varepsilon}s \text{ indices } i;\\&\text{if } \varphi(v)\in Y, \text{ then } \varphi(v) \text{ is typical to } X_{i,L} \text{ for all but at most } \sqrt{\varepsilon}s \text{ indices } i.\end{aligned} \tag{4}$$

At first, we embed the seed $v=r(T)$ in $X_{1,P}$ so that (4) holds. This is possible since there is no used vertex and, therefore, we can apply Fact 13. Now assume we are about to embed another seed $v\in S$. Embed v in a neighbor of the image of its parent in T and so that (4) holds. Again, this is possible because the number of seeds is bounded according to (2). Thus, by applying Fact 13, we can find an unused vertex satisfying (4).

Recall the partition $\{\mathcal{P}_1,\dots,\mathcal{P}_s\}$ of $\mathcal{P}$ from Step 4. Now assume we are about to embed a small tree $P\in\mathcal{P}$, and let $i\in[s]$ be such that $P\in\mathcal{P}_i$. Note that $r(P)$ is a linking vertex, and its parent is a seed that was already embedded. We choose a cluster from the opposite side to embed $r(P)$ linking the already embedded parent of $r(P)$ to the pair of clusters (X_i,Y_i) where the rest of P will be embedded. This process is explained in detail next.

Say the parent v of $r(T)$ was already embedded to a vertex $\varphi(v)\in X$ obeying (4). (The case $\varphi(v)\in Y$ is analogous.) By the minimum degree bound on the reduced graph R given in (1), there is a cluster Y_ℓ such that $X_iY_\ell\in E(R)$ and $\varphi(v)$ is typical to the L-slice $Y_{\ell,L}$. We embed $r(P)$ to a neighbor of $\varphi(v)$ in the L-slice $Y_{\ell,L}$ that is typical to the (at this time) unused part of the P-slice $X_{i,P}$. Then, we embed the remaining vertices of P into $X_{i,P}\cup Y_{i,P}$, always choosing vertices that are typical to the currently unused part of the other set. By Fact 12(a), we can find such typical vertices as long as the proportion of the number of unused vertices in the P-slice of each cluster is at least ε. Because of the bound on the number of vertices assigned to each pair $(X_{i,P},Y_{i,P})$, stated in (3), and the choice of c, this is always true. Therefore, we can embed P into the pair $(X_{i,P},Y_{i,P})$.

4. Packing large balanced trees: Proof of Theorem 7

We will use the following corollary of Theorem 11.

Corollary 17. *For each $\gamma>0$, there are $c,n_0>0$ such that the following holds for every $n\ge n_0$. If $G=(A,B,E)$ is a balanced bipartite graph on $2n$ vertices with $\delta(G)\ge(\frac12+\gamma)n$, and $F=(A_F,B_F,E_F)$ is a balanced forest on at most $2(1-\gamma)n$ vertices with $\Delta(F)\le cn$, then F embeds in G, with $A_F\subseteq A$.*

Proof of Corollary 17. Let $c,n_0>0$ be given by Theorem 11 for input γ. We can assume $cn_0\ge 2$ (otherwise simply choose n_0 sufficiently larger). We use induction on the number of components of F. If F has only one component, we are done by Theorem 11.

Otherwise, F must have two components C_1, C_2 such that there are two leaves $a\in V(C_1)\cap A_F$, $b\in V(C_2)\cap B_F$. Indeed, if this is not the case then either all leaves of F are in the same component, which is absurd as all components have leaves, or one of A_F, B_F only has vertices of degree at least two, which is impossible as F is a balanced forest. We add the edge ab to F. The maximum degree of the resulting forest is $\max\{\Delta(F),2\}\le cn$, and thus we can apply induction. □

Corollary 17 enables us to prove the following lemma.

Lemma 18. *For each $\gamma>0$, there are $c,n_0>0$ such that the following holds for every $n\ge n_0$ and for all functions $t(n),d(n)$ with $c\cdot t(n)d(n)\le(\frac12-\gamma)n$. If $\mathcal{F}=\{F_1,\dots,F_{t(n)}\}$ is a family of $t(n)$ balanced forests, with $F_i=(A_i,B_i,E_i)$, each F_i on at most $2(1-\gamma)n$ vertices and of maximum degree at most $c\cdot d(n)$, then $\mathcal{F}$ packs into $K_{n,n}$, with all A_i's embedded in the same part of $K_{n,n}$.*

Proof. Let c,n_0 be given by Corollary 17 for input γ. We can assume that $\gamma\le\frac{1}{10}$ and $n_0\ge 15$. Let $n\ge n_0$ and consider a family $\mathcal{F}=\{F_1,\dots,F_{t(n)}\}$ as in the lemma. Denote one of the parts of $K_{n,n}$ by A. Set $G_1=K_{n,n}$. For $i=1,\dots,t(n)$, we will find an embedding φ of $F_i=(A_i,B_i,E_i)$ into G_i, with all vertices of A_i embedded in A, and then set $G_{i+1}=G_i-\varphi(E_i)$. This is possible by Theorem 11, as for each $i\le t(n)$, we have $\delta(G_i)\ge n-(i-1)\,c\cdot d(n)>n-c\cdot t(n)d(n)>(\frac12+\gamma)n$. □

Now we are ready to prove Theorem 7.

Proof of Theorem 7. Fix c, $n_0>0$ given by Lemma 18 for input γ. We may assume that $c,\gamma\le\frac12$ and $n_0\ge(\frac{8}{c\gamma})^{2/\gamma}$. For $n\ge n_0$, consider any family $\{T_1,\dots,T_t\}$ of $t\le n^{\frac12-\gamma}$ balanced rooted trees, each on at most $2(1-\gamma)n$ vertices. Let A,B be the color classes of $K_{n,n}$ and for $i = 1, \ldots, t$ let A_i, B_i be the color class of T_i, with A_i containing its root $r(T_i)$. Let $A' \subseteq A$ and $B' \subseteq B$ be sets of size $n' := \lfloor \gamma n \rfloor$.

Let H_i^A be the set of the $\lfloor \frac{8\sqrt{n}}{c} \rfloor$ vertices of highest degrees in A_i, for $i = 1, \ldots, t$. Define H_i^B analogously, and set $H_i := H_i^A \cup H_i^B$. Since each tree T_i has less than $2n$ edges, we know that each vertex in the balanced forest $F_i := T_i - H_i$ has degree at most $\frac{c\sqrt{n}}{2} \le c\sqrt{n-n'}$. Also, $|V(F_i)| \le 2(1-\gamma)(n-n')$. We use Lemma 18, with $t(n) = t$ and $d(n) = \sqrt{n}$, to pack the forests $F_1, \ldots, F_t$ into $K_{n,n} - (A' \cup B')$, with all the A_i's embedded in A.

We embed $H^A = \bigcup_{1 \le i \le t} H_i^A$ into A', and $H^B = \bigcup_{1 \le i \le t} H_i^B$ into B', which is possible as

$$|H^A| = |H^B| < t \cdot \frac{8\sqrt{n}}{c} \le \lfloor \gamma n \rfloor = |A'| = |B'|,$$

by our choice of n_0. This finishes the packing. □

5. Final Remarks

5.1. Open questions for tree-packing into bipartite graphs

As discussed in the introduction, if we wish to decompose a complete bipartite graph $K_{m,m}$ with copies of a tree $T_{n,n}$, then m needs to be larger than n, as in the following conjecture, which follows from Conjecture 3.

Conjecture 19. *Any tree $T_{n,n}$ decomposes $K_{2n-1,2n-1}$.*

However, it may be possible that the host graph can be somewhat smaller. We propose the following question.

Question 20. *What is the smallest k such that every tree $T_{n,n}$ decomposes $K_{2n-1,k}$.*

Clearly, k needs to be at least n. Letting $D_{n,n}$ be the double-star with n vertices in either partition class, one can prove by induction on n that $K_{2n-1,n}$ decomposes into copies of $D_{n,n}$. So one could think that $k = n$. However, this is false for $n = 3$ and paths. Indeed, let A and B be the parts of $K_{5,3}$ such that $|A| = 5$, and $|B| = 3$. Note that we cannot embed two vertices of degree two in a vertex of A. Since each path on six vertices has two vertices of degree two in both sides of the bipartition, any decomposition of $K_{5,3}$ into three 6-vertex paths must embed two vertices of degree two in the same vertex in A. One can generalize this example for all $n \ge 3$.

Paths are not the only trees that force us to assume $k > n$ in Question 20. For instance, assume $n = 2\ell + 1$ is odd and consider the tree T that arises from taking two copies of $D_{\ell+1,\ell+1}$, deleting a leaf from each and connecting the neighbors of these leaves by an edge. Each side of T contains two vertices of degree $\ell + 1$. So T cannot decompose $K_{4\ell+1,2\ell+1} = K_{2n-1,n}$, because the larger side would receive two of the high degree vertices, which is impossible.

In a similar spirit as Question 20, we ask the following.

Question 21. *What is the smallest k such that any family of k balanced trees, each with n vertices in either partition class, packs into $K_{2n-1,k}$?*

Generalizing this even more, one could ask for the smallest k such that any family of balanced trees, each with at most n vertices in either partition class, and with a total number of edges not exceeding $(2n-1)k$, packs into $K_{2n-1,k}$.

5.2. An exact bipartite KSS theorem?

In Theorem 11 we have a γn slack on each side of the bipartition of the host graph. However, we believe the following direct analogue of Theorem 10 should be true in the bipartite setting.

Conjecture 22. *For each $\gamma > 0$, there are $c, n_0 > 0$ such that the following holds for every $n \ge n_0$. If $G = (A, B, E)$ is a balanced bipartite graph on $2n$ vertices with $\delta(G) \ge (\frac{1}{2} + \gamma)n$, and T is a balanced rooted tree on $2n$ vertices with $\Delta(T) \le \frac{cn}{\log n}$, then T embeds in G with the root of T embedded in A.*

Note that Conjecture 22 uses the same bound $\Delta(T) \le cn/\log n$ as in Theorem 10 (and not cn as in Theorem 11). This is necessary, as can be seen by considering the following adaptation of an example from [28]. Consider $\lceil \alpha \log n \rceil$ stars whose sizes differ by at most one, where α is a constant. Join a new vertex r to each of the centers of the stars. Take two copies of the obtained tree and obtain a balanced tree T by joining the two copies of r. The random balanced bipartite graph with $p = 0.9$ has w.h.p. minimum degree at least $0.8n$ but w.h.p. it does not contain T as a subgraph.

We were not able to adapt previous strategies for a proof of Conjecture 22. In Komlós, Sárközy, and Szemerédi's [28] strategy, we could not find a way to distribute the leaves of the tree into a star cover of the reduced graph. An alternative route might be to use the absorption method, as used, for example, by Kathapurkar and Montgomery [25]. Their strategy consists of splitting the tree T into two subtrees one of which is small and serves for the absorption at the very end, while the other is large and is embedded with an approximate embedding result. As it is not possible to split any balanced tree into two balanced subtrees of the adequate sizes (for instance, for the tree from the previous paragraph this is not possible), we believe that in order to use the strategy from [25], it would first be necessary to prove an unbalanced variant of Theorem 11.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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