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Homogeneous involutions on graded division algebras and their polynomial identities

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In this paper, we describe the so-called homogeneous involution on finite-dimensional graded-division algebra over an algebraically closed field. We also compute their graded polynomial identities with involution. As pointed out by Fonseca and Mello, a homogeneous involution naturally appears when dealing with graded polynomial identities and a compatible involution.

Keywords: Homogeneous involution; polynomial identities.

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1. Introduction

The main purpose of this paper is to investigate the homogeneous involution on finite-dimensional graded-division algebras over an algebraically closed field of characteristic zero and their graded polynomial identities with involution.

It is known that graded algebras play a fundamental role in several branches of Mathematics, being the main topic of research or being an important tool to understand an object. The monograph [8] compiles the state of art of the theory, where understanding the gradings is the central objective. It is worth mentioning that the classification of involutions that are compatible with a given grading, the so-called *degree-preserving involution* or *graded involution*, is crucial in the approach.

On the other hand, an involution that inverts the degree appears naturally on several contexts, for instance, the Leavitt path algebras (see [11]) and matrix algebras endowed with an elementary grading and the transpose involution (see [5] for details, and see also [9]).

Considering group gradings on the upper triangular matrices (a classification is obtained in [7, 13]), one sees that the natural transpose with respect to the secondary diagonal sends a homogeneous element to a homogeneous element. If the grading is adequate, the natural involution will map a whole homogeneous component in another homogeneous component. So, de Mello studied and described the so-called *homogeneous involution* on upper triangular matrix algebra [4].

Following the paper by de Mello, we shall investigate the homogenous involutions on finite-dimensional graded division algebras over an algebraically closed field.

Next, it is known the relevance of the description of the T-ideal of all polynomial identities satisfied by a given algebra, and the related main problems of the theory. We shall describe the graded polynomial identities with homogeneous involution on the graded division algebras endowed with a homogeneous involution. The graded version is done in [1] and in [10] the graded identities with an involution were computed for certain types of gradings.

Finally, it is worth mentioning the following phenomena (see [6]) when dealing with the free graded algebra with a homogeneous involution (we shall give the precise construction in the next section). Let $X_G = \bigcup_{g\in G} X_g$, where $X_g = \{x_1^{(g)}, x_2^{(g)}, \ldots\}$, for each $g \in G$. Let $\mathbb{F}\langle X, \iota\rangle$ be the free G-graded algebra with an involution. Then, since the unary operation ι is an antiautomorphism, one obtains $\iota(x^{(g)}x^{(h)}) = \iota(x^{(h)})\iota(x^{(g)})$. If we require that ι preserves the homogeneous degree, then this equation makes sense only if the grading group is abelian. Otherwise, there should be an antihomomorphism $\tau : G \to G$ such that $\deg_G \iota(x^{(g)}) = \tau(g)$. Thus, in general, it is seems to be natural to consider homogeneous involutions in the context of graded polynomial identities endowed with a compatible involution.

2. Preliminaries

2.1. *Graded algebra*

Let G be any group. We use the multiplicative notation for G, and denote its neutral element by 1. We say that an algebra $\mathcal{A}$ is G-graded if there exists a vector-space decomposition $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ such that $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$, for all $g, h \in G$. The choice of the decomposition is called a *G-grading*, and one usually denotes by Γ. The subspace $\mathcal{A}_g$ is called *homogeneous component of degree g*. A nonzero element $x \in \mathcal{A}_g$ is called a homogeneous element of degree g. We denote $\deg_G x = g$. The *support* of the grading Γ is $\operatorname{Supp}\Gamma = \{g \in G \mid \mathcal{A}_g \neq 0\}$. By abuse of language, we shall denote the support by $\operatorname{Supp}\mathcal{A}$. A *graded division algebra* is an associative algebra $\mathcal{D}$ with 1, where each nonzero homogeneous element $x \in \mathcal{D}$ is invertible.

Finally, we provide a precise definition of the following definition.

Definition. Let $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ be a G-graded algebra, and let $\tau : G \to G$ be a map. An involution ψ on $\mathcal{A}$ is a *homogeneous involution* with respect to τ or a τ-involution if $\psi(\mathcal{A}_g) \subseteq \mathcal{A}_{\tau(g)}$, for all $g \in G$.

In this paper, *involution* will mean an $\mathbb{F}$-linear map involution. Also, we are specially interested in the case where the map τ is an anti-automorphism of order 2 of the grading group.

Examples. (1) If G is an abelian group, then every degree-preserving involution is a homogeneous involution with respect to the identity map of G.

(2) A degree-inverting involution is a homogeneous involution with respect to the inversion of G. It is worth mentioning that the degree-inverting involution on matrix algebras and upper triangular matrices were described in [5, 9].

(3) If $\mathcal{D} = \bigoplus_{g\in G} \mathcal{D}_g$ is a graded-division algebra (where $G = \operatorname{Supp}\mathcal{D}$) and ι is a τ-involution on $\mathcal{D}$, then τ is an involution of G. Indeed, for any g, $h \in G$, let x_g and x_h be nonzero homogeneous elements of G-degrees g and h, respectively. Then $xy \neq 0$ and

$$\tau(gh) = \deg_G \iota(x_g x_h) = \deg_G(\iota(x_h)\iota(x_g)) = \deg_G \iota(x_h) \deg_G \iota(x_g) = \tau(h)\tau(g),$$

$$g = \deg_G x_g = \deg_g \iota(\iota(x_g)) = \tau(\tau(g)).$$

Thus, τ is an anti-automorphism of order 2.

2.2. *Factor sets*

We let T be a finite group, and $\mathbb{F}^\times$ denote the set of invertible elements of $\mathbb{F}$.

A map $\sigma : T \times T \to \mathbb{F}^\times$, is called a 2-*cocycle* or a *factor set* if

$$\sigma(u, v)\sigma(uv, w) = \sigma(u, vw)\sigma(v, w), \quad \forall u, v, w \in T.$$

We denote the set of all 2-cocycles by $Z^2(T, \mathbb{F}^\times)$. Note that, using the usual multiplication, $Z^2(T, \mathbb{F}^\times)$ is an abelian group.

For each map $\mu : T \to \mathbb{F}^\times$, we define $\delta\mu : T \times T \to \mathbb{F}^\times$ by

$$\delta\mu(u, v) = \mu(u)\mu(v)\mu(uv)^{-1}, \quad u, v \in T.$$

We define $B^2(T, \mathbb{F}^\times) = \{\delta\mu \mid \mu : T \to \mathbb{F}^\times\}$. An easy exercise shows that $B^2(T, \mathbb{F}^\times)$ is a subgroup of $Z^2(T, \mathbb{F}^\times)$. The second cohomological group of T is the quotient $H^2(T, \mathbb{F}^\times) = Z^2(T, \mathbb{F}^\times)/B^2(T, \mathbb{F}^\times)$.

We can construct algebras from factor sets. Given an arbitrary map $\sigma : T \times T \to \mathbb{F}^\times$ denote by $\mathbb{F}^\sigma T$ the following algebra: $\mathbb{F}^\sigma T$ has a basis $\{X_u \mid u \in T\}$, and the product is defined by $X_u X_v = \sigma(u, v) X_{uv}$. Note that $\mathbb{F}^\sigma T$ is associative if and only if $\sigma \in Z^2(T, \mathbb{F}^\times)$. For instance, if $\sigma = 1$ (the constant function), then $\mathbb{F}^\sigma T$ is the group algebra of T. Clearly such algebra have a natural T-grading. It is known that $\mathbb{F}^\sigma T \cong \mathbb{F}^{\sigma'} T$ if and only if $[\sigma] = [\sigma']$ (equality in $H^2(T, \mathbb{F}^\times)$).

2.3. *Graded division algebra*

Assume that $\mathbb{F}$ is algebraically closed and let $\mathcal{D} = \bigoplus_{g\in G} \mathcal{D}_g$ be a finite-dimensional graded division algebra over $\mathbb{F}$. Let $T = \{g \in G \mid \mathcal{D}_g \neq 0\}$ be its support. Then it is easy to see that T is a subgroup of G. We use multiplicative notation for the product of T, and denote by 1 its neutral element.

Moreover, $\mathcal{D}_1 \supseteq \mathbb{F}$ is a division algebra. So $\mathcal{D}_1 = \mathbb{F}$, since $\mathbb{F}$ is algebraically closed and $\dim_{\mathbb{F}} \mathcal{D}_1 < \infty$. This also implies $\dim \mathcal{D}_g = 1$, for all $g \in T$. Let $\{X_u \mid u \in T\}$ be a homogeneous basis of $\mathcal{D}$. Then $X_u X_v = \sigma(u,v) X_{uv}$, for some $\sigma(u,v) \in \mathbb{F}^\times$. Since $\mathcal{D}$ is associative, from $(X_u X_v) X_w = X_u (X_v X_w)$, we derive that σ is a 2-cocycle. Hence, $\mathcal{D} \cong \mathbb{F}^\sigma T$, the twisted group algebra of T by σ. Conversely, for any finite group T and any $\sigma \in Z^2(T, \mathbb{F}^\times)$, the natural T-grading on $\mathbb{F}^\sigma T$ turns it into a graded division algebra.

2.4. *Free graded algebra with homogeneous involution*

We shall provide a construction of the free algebra endowed with a homogeneous involution. This is done using a particular case of the (relatively) free universal algebra in an adequate variety (see, for instance [12] for a general discussion, and [2, 3] as well for a particular graded version). Let G be any group, and $X^G = \bigcup_{g \in G} X^{(g)}$, where $X^{(g)} = \{x_1^{(g)}, x_2^{(g)}, \ldots\}$. Let $\tau : G \to G$ be an involution, that is, an anti-automorphism of order 2. We define the *free G-graded associative algebra with a homogeneous involution* with respect to τ, $\mathbb{F}\langle X^G, \iota\rangle$, in the following way. First, we let $\mathbb{F}\{X^G, \iota\}$ denote the absolute free G-graded binary algebra endowed with an unary operation (denote by ι). We define the following polynomial identities:

$$\begin{aligned} x_1^{(g_1)}(x_2^{(g_2)} x_3^{(g_3)}) - (x_1^{(g_1)} x_2^{(g_2)}) x_3^{(g_3)} &= 0, \\ \iota(x_1^{(g_1)} x_2^{(g_2)}) - \iota(x_2^{(g_2)}) \iota(x_1^{(g_1)}) &= 0, \\ \iota(\iota(x^{(g)})) - x^{(g)} &= 0, \\ \deg_G \iota(x^{(g)}) &= \tau(\deg_G x^{(g)}). \end{aligned} \tag{1}$$

The first relation is the associativity while the second and third indicate that ι acts as an involution. The last relation is also a polynomial identity, but to see this we need to describe the G-grading in terms of the projections (see, for instance [2]). For each $g \in G$, let π_g denote the unary operation on a G-graded algebra $\mathcal{A}$ given by the projection and inclusion $\pi_g : \mathcal{A} \to \mathcal{A}$. Then, the absolute free G-graded algebra $\mathbb{F}\{X^G, \iota\}$ is a quotient of the absolute free Ω-algebra, where Ω contains one binary operation and $|G| + 1$ unary operations (corresponding to each projection, and the involution). The quotient is given by the relations that define the G-grading, that is, $\pi_g(\pi_h(x)) = \delta_{gh} \pi_h(x)$ and $\pi_g(\pi_{g_1}(x) \pi_{g_2}(y)) = \delta_{g, g_1 g_2} \pi_g(xy)$. Hence, in the language of this Ω-algebra, the last equation of (1) is equivalent to

$$\iota(\pi_g(x)) - \pi_{\tau(g)}(\iota(x)) = 0.$$

Using either the absolute free G-graded algebra or the (relatively) free Ω-algebra, the free G-graded algebra with a τ-involution $\mathbb{F}\langle X^G, \iota\rangle$ is the quotient of $\mathbb{F}\{X^G, \iota\}$ by the identities (1). As mentioned before, the forth identity is natural in the context of graded polynomial identities with an involution.

As discussed in [6], in the special case where ι is a degree-preserving involution, then we can define the new variables $x_+^{(g)} := x^{(g)} + \iota(x^{(g)})$ and $x_-^{(g)} = x^{(g)} - \iota(x^{(g)})$ (the symmetric and skew symmetric variables). Then we get the classical construction of the free (graded) $*$-algebra. Since ι does not necessarily preserve the homogeneous degree, we cannot use such technique, since these variables would not be homogeneous.

Given a G-graded algebra $\mathcal{A}$ with a homogeneous involution ι, we denote by $\mathrm{Id}_G(\mathcal{A})$ its ideal of graded polynomial identities, and by $\mathrm{Id}_{G,\iota}(\mathcal{A})$ the set of all of its graded polynomial identities with involution ι.

3. Homogeneous Involution on Graded Division Algebra

We let $\mathbb{F}$ be a field of characteristic not 2. Let T be a finite group. Given $\sigma \in Z^2(T, \mathbb{F}^*)$, we consider the natural T-grading on $\mathbb{F}^\sigma T$.

We shall denote by $\mathrm{Aut}(T)$ the group of all automorphisms of T, and by $\overline{\mathrm{Aut}}(T)$ the group of all automorphism and antiautomorphism of T. For each $\varphi \in \mathrm{Aut}(T)$, let $\varphi\sigma : T \times T \to \mathbb{F}^*$ be defined by

$$\varphi\sigma(u, v) = \sigma(\varphi(u), \varphi(v)).$$

On the other hand, if $\psi \in \overline{\mathrm{Aut}}(T)$ is an antiautomorphism, then let $\psi\sigma : T \times T \to \mathbb{F}^\times$ be defined by

$$\psi\sigma(u, v) = \sigma(\psi(v), \psi(u)).$$

The next lemma is an elementary exercise. We include the proof for completeness.

Lemma 1. *For each $\theta \in \overline{\mathrm{Aut}}(T)$ and $\sigma' \in Z^2(T, \mathbb{F}^\times)$, $\theta\sigma' \in Z^2(T, \mathbb{F}^\times)$.*

Proof. Assume that θ is an antiautomorphism. Then, for any u, v, $w \in T$, we have

$$\begin{aligned} \theta\sigma'(u, v)\theta\sigma'(uv, w) &= \sigma'(\theta(v), \theta(u))\sigma'(\theta(w), \theta(uv)) \\ &= \sigma'(\theta(w), \theta(v))\sigma'(\theta(w)\theta(v), \theta(u)) \\ &= \theta\sigma'(v, w)\theta\sigma'(u, vw). \end{aligned}$$

In an analogous way we show that $\theta\sigma' \in Z^2(T, \mathbb{F}^\times)$ if θ is an automorphism of T. □

Hence, it is easy to see that we have an action of $\overline{\mathrm{Aut}}(T)$ on $Z^2(T, \mathbb{F}^\times)$. This action factors through $B^2(T, \mathbb{F}^\times)$:

Lemma 2. *If $\sigma' \in B^2(T, \mathbb{F}^\times)$ and $\theta \in \overline{\mathrm{Aut}}(T)$, then $\theta\sigma' \in B^2(T, \mathbb{F}^\times)$.*

Proof. Let $\sigma' = \delta\mu$. Then $\theta\delta\mu = \delta(\mu \circ \theta) \in B^2(T, \mathbb{F}^\times)$. □

Thus, we get the following corollary.

Corollary 3. *There is an action of* $\overline{\mathrm{Aut}}(T)$ *on* $H^2(T,\mathbb{F}^\times)$.

Now, the next result states the conditions so that we do have a homogeneous involution on $\mathbb{F}^\sigma T$. For this, we need maps satisfying the following condition.

Definition. Let $\sigma \in Z^2(T,\mathbb{F}^\times)$ and $\tau \in \overline{\mathrm{Aut}}(T)$. We say that the pair (σ,τ) is *compatible* if there is a map $\mu : T \to \mathbb{F}^\times$ such that $\sigma = \delta\mu \cdot \tau\sigma$ (where $\cdot$ is the product of $Z^2(T,\mathbb{F}^\times)$) and $\mu(u\tau(u)) = 1$, for all $u \in T$.

For instance, let $\tau : u \in T \mapsto u^{-1} \in T$ be the inversion. Then, for any $\sigma \in Z^2(T,\mathbb{F}^\times)$ such that $[\sigma]^2 = 1$, the pair (σ,τ) is compatible (see the proof of [9, Proposition 4]).

Theorem 4. *Let* $\tau : T \to T$. *Then there exists a* τ*-homogeneous involution on* $\mathbb{F}^\sigma T$ *if and only if* τ *is an antiautomorphism of order* 2 *and* (σ,τ) *is compatible.*

Proof. Let $\{X_u \mid u \in T\}$ be a homogeneous basis of $\mathbb{F}^\sigma T$. First, assume that ι is a τ-homogeneous involution on $\mathbb{F}^\sigma T$. Let $\tau : T \to T$ and $\mu : T \to \mathbb{F}^*$ be the maps such that

$$\iota(X_u) = \mu(u) X_{\tau(u)}, \quad u \in T.$$

Since $X_u = \iota(\iota X_u) = \mu(\tau(u))\mu(u) X_{\tau\tau(u)}$, one obtains $\tau^2 = 1$. It also implies that τ is a bijection. Since

$$\begin{aligned} \mu(uv)\sigma(u,v)X_{\tau(uv)} &= \iota(X_u X_v) = \iota(X_v)\iota(X_u) \\ &= \sigma(\tau(v),\tau(u))\mu(v)\mu(u)X_{\tau(v)\tau(u)}, \end{aligned} \tag{2}$$

we obtain $\tau(uv) = \tau(v)\tau(u)$. Hence, τ is an antiautomorphism. Note that (2) also shows that $\sigma = \delta\mu \cdot (\tau\sigma)$, thus $[\sigma] = [\tau\sigma]$. Finally, from $\iota(X_u X_{\tau(u)}) = \iota(X_{\tau(u)})\iota(X_u)$, we get

$$\mu(u\tau(u))\sigma(u,\tau(u)) = \mu(u)\mu(\tau(u))\sigma(u,\tau(u)),$$

thus $\mu(u\tau(u)) = \mu(u)\mu(\tau(u))$. Since $X_u = \iota(\iota(X_u)) = \mu(\tau(u))\mu(u)X_u$, we get that $\mu(u\tau(u)) = 1$, for all $u \in T$.

On the other hand, assume that τ is an antiautomorphism of order 2 such that (σ,τ) is compatible. So let $\mu : G \to \mathbb{F}^\times$ satisfy $\sigma = \delta\mu \cdot \tau\sigma$. We claim that

$$\iota(X_u) := \mu(u) X_{\tau(u),}$$

is a τ-homogeneous involution on $\mathbb{F}^\sigma T$. Indeed,

$$\begin{aligned} \iota(X_u X_v) &= \sigma(u,v)\mu(uv)X_{\tau(uv)} \\ &= \sigma(u,v)\mu(uv)\sigma(\tau(v),\tau(u))^{-1} X_{\tau(v)} X_{\tau(u)} \\ &= \mu(u)\mu(v) X_{\tau(v)} X_{\tau(u)} = \iota(X_v)\iota(X_u). \end{aligned}$$

Now,

$$\iota(\iota(X_u)) = \mu(u)\mu(\tau(u)) X_{\tau^2(u)} = \mu(u)\mu(\tau(u)) X_u.$$

It remains to prove that $\mu(u)\mu(\tau(u)) = 1$. Since

$$\mu(u)\mu(\tau(u))\mu(u\tau(u))^{-1} = \sigma(u,\tau(u))\tau\sigma(u,\tau(u))^{-1} = 1,$$

we obtain that $\mu(u)\mu(\tau(u)) = \mu(u\tau(u)) = 1$, since (σ,τ) is compatible. □

Problem. Classify the compatible pairs (σ,τ).

4. Polynomial Identities of Graded Division Algebra

4.1. *Codimension sequence*

First, we recall some definitions concerning codimension sequence. Let T be a finite group and consider the natural T-grading on $\mathbb{F}^\sigma T$. Given a sequence $s = (u_1,\dots,u_m) \in T^m$, we let

$$P_m^s = \operatorname{Span}\{x_{\pi(1)}^{(u_{\pi(1)})}\cdots x_{\pi(m)}^{(u_{\pi(m)})} \mid \pi\in\mathcal{S}_m\},\quad P_m^s(\mathbb{F}^\sigma T) = P_m^s/P_m^s \cap \operatorname{Id}_T(\mathbb{F}^\sigma T),$$

where $\mathcal{S}_m$ is the symmetric group on the set of m elements. The *graded codimension sequence* of $\mathbb{F}^\sigma T$ is

$$c_m^T(\mathbb{F}^\sigma T) = \dim \sum_{s\in T^m} P_m^s(\mathbb{F}^\sigma T).$$

Now, given $I = (j_1,\dots,j_m) \in \{0,1\}^m$, let

$$P_{s,I}^T = \operatorname{Span}\{\iota^{j_1}(x_{\pi(1)}^{(u_{\pi(1)})})\cdots\iota^{j_m}(x_{\pi(m)}^{(u_{\pi(m)})}) \mid \pi\in\mathcal{S}_m\}.$$

As before, we let

$$P_{s,I}^T(\mathbb{F}^\sigma T) = P_{s,I}^T/P_{s,I}^T \cap \operatorname{Id}_{T,\iota}(\mathbb{F}^\sigma T),\quad c_m^{T,\iota}(\mathbb{F}^\sigma T) = \dim\sum_{s,I} P_{s,I}^T(\mathbb{F}^\sigma T).$$

The graded exponent and the homogeneous-involution exponent (if exists) are respectively defined by

$$\exp^T(\mathbb{F}^\sigma T) = \lim_{m\to\infty}\sqrt[m]{c_m^T(\mathbb{F}^\sigma T)},\quad \exp^{T,\iota}(\mathbb{F}^\sigma T) = \lim_{m\to\infty}\sqrt[m]{c_m^{T,\iota}(\mathbb{F}^\sigma T)}.$$

4.2. *Polynomial identities of* $\mathbb{F}^\sigma T$

In this section, we describe the graded polynomial identities with involution of $\mathbb{F}^\sigma T$. Let $\mathbb{F}$ be a field of characteristic zero and ι a τ-homogeneous involution on $\mathbb{F}^\sigma T$. Let $U = (u_1,\dots,u_m) \in T^m$, $I = (i_1,\dots,i_m), I' = (i'_1,\dots,i'_m) \in \{0,1\}^m$ and $\theta,\theta' \in \mathcal{S}_m$ be such that

$$\tau^{i'_{\theta'(1)}}(u_{\theta'(1)})\cdots\tau^{i'_{\theta'(m)}}(u_{\theta'(m)}) = \tau^{i_{\theta(1)}}(u_{\theta(1)})\cdots\tau^{i_{\theta(m)}}(u_{\theta(m)}). \tag{3}$$

Then we can find a constant $\alpha_{U,I,I',\theta,\theta'} \in \mathbb{F}$ such that

$$\iota^{i'_{\theta'(1)}}(X_{u_{\theta'(1)}})\cdots\iota^{i'_{\theta'(m)}}(X_{u_{\theta'(m)}}) = \alpha_{U,I,I',\theta,\theta'}\iota^{i_{\theta(1)}}(X_{u_{\theta(1)}})\cdots\iota^{i_{\theta(m)}}(X_{u_{\theta(m)}}).$$

Hence, denoting $z_i = x_i^{(u_i)}$,

$$f_{U,I,I',\theta,\theta'} = \iota^{i'_{\theta'(1)}}(z_{u_{\theta'(1)}})\cdots\iota^{i'_{\theta'(m)}}(z_{u_{\theta'(m)}}) - \alpha_{U,I,\theta}\iota^{i_1}(z_{\theta(1)})\cdots\iota^{i_m}(z_{\theta(m)}),$$

is a G-graded polynomial identity with involution of $\mathbb{F}^\sigma T$, which shall be called an *elementary* identity (following the graded case of [1]). Let $\mathscr{T}_U$ be the set of all triples $(U, I, I', \theta, \theta')$ such that (3) holds valid.

Theorem 5. $\mathrm{Id}_{T,\iota}(\mathbb{F}^\sigma T)$ *is generated by* $\{f_{U,I,I',\theta,\theta'} \mid (I, I', \theta, \theta') \in \mathscr{T}_U,\ |U| \le |T|\}$.

Proof. First, we shall prove that any multilinear polynomial identity is a consequence of the elementary ones. Then, we show that the elementary identities follow from the ones having length at most $|T|$. Let $\mathcal{I}$ be the $T_{T,\iota}$-ideal generated by $\{f_{U,I,I',\theta,\theta'} \mid (I, I', \theta, \theta') \in \mathscr{T}_U,\ |U| \le |T|\}$, and $\mathcal{J}$ be the $T_{T,\iota}$-ideal generated by $\{f_{U,I,I',\theta,\theta'} \mid (I, I', \theta, \theta') \in \mathscr{T}_U\}$.

Let $f \in \mathrm{Id}_{T,\iota}(\mathbb{F}^\sigma T)$ be a G-homogeneous polynomial. Since $\operatorname{char} \mathbb{F} = 0$, we may assume that f is multilinear. Write

$$f = \sum_{\substack{\theta \in \mathcal{S}_m \\ I=(i_1,\ldots,i_m)}} \alpha_{\theta,I} p_{\theta,I}, \tag{4}$$

where $p_{\theta,I} = \iota^{i_{\theta(1)}}(z_{\theta(1)}) \cdots \iota^{i_{\theta(m)}}(z_{\theta(m)})$. Since every monomial in (4) satisfies $\tau^{i_{\theta(1)}}(u_{\theta(1)}) \cdots \tau^{i_{\theta(m)}}(u_{\theta(m)}) = \deg_T f$, we see that $(I, I', \theta, \theta') \in \mathscr{T}_U$ for every pair of (I, θ) and (I', θ') appearing with nonzero coefficient in (4). Here, $U = (\deg_T z_1, \ldots, \deg_T z_m)$. Hence, modulo $\mathcal{J}$, f is equal to a single monomial, up to a scalar. As $\mathbb{F}^\sigma T$ is a graded division algebra, a monomial cannot be a graded polynomial identity of it. Thus, $f = 0$ modulo $\mathcal{J}$, so $f \in \mathcal{J}$.

Finally, let $p = f_{U,I,I',\theta,\theta'}$ with $|U| > |T|$. We shall prove by induction on $|U|$ that $f \in \mathcal{I}$. For the sake o simplicity, we may replace θ by the identity, and then θ' becomes $\theta'' = \theta'\theta^{-1}$, and we may assume that $I = (0, 0, \ldots, 0)$ and I' is replaced by $I'' = (i'_1 + i_1, \ldots, i'_m + i_m)$ (where the sum is taken modulo 2). Formally, we shall call $z_j = \iota^{i_j}(x_{\theta(j)}^{u_{\theta(j)}})$, and then

$$p = z_1 \cdots z_m - \iota^{i''_1}(z_{\theta''(1)}) \cdots \iota^{i''_m}(z_{\theta''(m)}).$$

To make notations cleaner, we may also suppress the double prime, so we shall write $p = f_{U,(0,\ldots,0),I,1,\theta}$. Denote also $v_i = \tau(u_{\theta(i)})$.

Consider the elements $\{v_1, v_1v_2, \ldots, v_1v_2 \cdots v_m\}$, having exactly $|U| > |T|$ elements. By the pigeonhole principle, there should be two distinct sequences whose product coincides. Thus, there exists $1 < i < j \le m$ such that $v_iv_{i+1} \cdots v_j = 1$. Since $(v_i \cdots v_{j-1})v_j = v_j(v_i \cdots v_{j-1}) = 1$, we may, modulo $\mathcal{I}$, cyclically permute the variables $\iota^{i_i}(z_i) \cdots \iota^{i_j}(z_j)$. As $(v_i \cdots v_j)v_k = v_k(v_i \cdots v_j)$ for any k, we may also move the string $\iota^{i_i}(z_i) \cdots \iota^{i_j}(z_j)$ anywhere in the last monomial, modulo $\mathcal{I}$. Finally, since $\tau(1) = 1$, we may apply ι to $\iota^{i_i}(z_i) \cdots \iota^{i_j}(z_j)$, modulo $\mathcal{I}$.

Hence, modulo $\mathcal{I}$, we may assume that there exists ℓ such that z_ℓ and $z_{\ell+1}$ appears consecutively (in this order) in the second monomial of p, and ι is applied in the product or $z_\ell z_{\ell+1}$ or not. More precisely, modulo $\mathcal{I}$, either

$$p = w_1 z_\ell z_{\ell+1} w_2 - w'_1 z_\ell z_{\ell+1} w'_2,$$

or

$$p = w_1 z_\ell z_{\ell+1} w_2 - w_1' \iota(z_\ell + 1)\iota(z_\ell) w_2'.$$

Defining $g = u_\ell u_{\ell+1}$ we have that p is a consequence of either one of the elementary identities $w_1 x^{(g)} w_2 - w_1' x^{(g)} w_2'$ or $w_1 x^{(g)} w_2 - w_1' \iota(x^{(g)}) w_2'$. In any case, p is a consequence of an elementary identity of total degree $|U| - 1$. By induction, this implies that $p \in \mathcal{I}$, proving the result. □

Finally, we shall obtain some estimates to the homogeneous-involution codimension sequence of the $\mathbb{F}^\sigma T$. We shall prove the following theorem.

Theorem 6. *Let T be a finite group, and $\sigma \in Z^2(T, \mathbb{F}^\times)$. Assume that $\mathbb{F}^\sigma T$ admits a τ-homogeneous involution ι. Then*

(1) $c_m^T(\mathbb{F}^\sigma T) \le c_m^{T,\iota}(\mathbb{F}^\sigma T) \le |T| c_m^T(\mathbb{F}^\sigma T)$, $\forall\, m \in \mathbb{N}$,
(2) *For all $m \in \mathbb{N}$,*

$$|T|^m \le c_m^T(\mathbb{F}^\sigma T) \le |T'||T|^m \quad \text{and} \quad |T|^m \le c_m^{T,\iota}(\mathbb{F}^\sigma T) \le |T'||T|^{m+1},$$

where $T' = [T, T]$.

Proof. For each sequence $s = (u_1, \ldots, u_m) \in T^m$, let

$$B_s = \{u_{\pi(1)} \cdots u_{\pi(m)} \mid \pi \in \mathcal{S}_m\}$$

and let (in $\mathbb{F}\langle X\rangle / \mathrm{Id}^{T,\iota}(\mathbb{F}^\sigma T)$)

$$P_s^T(\mathbb{F}^\sigma T) = \mathrm{Span}\{x_{\pi(1)}^{(u_{\pi(1)})} \cdots x_{\pi(m)}^{(u_{\pi(m)})} \mid \pi \in \mathcal{S}_m\}.$$

Since $x_{\pi(1)}^{(u_{\pi(1)})} \cdots x_{\pi(m)}^{(u_{\pi(m)})} = x_{\theta(1)}^{(u_{\theta(1)})} \cdots x_{\theta(m)}^{(u_{\theta(m)})}$, up to a constant, if and only if $u_{\pi(1)} \cdots u_{\pi(m)} = u_{\theta(1)} \cdots u_{\theta(m)}$ (see the discussion before Theorem 5), we see that

$$\dim P_s^T(\mathbb{F}^\sigma T) = |B_s|.$$

Hence, we may conclude that

$$c_m^T(\mathbb{F}^\sigma T) = \sum_{s \in T^m} \dim P_s^T(\mathbb{F}^\sigma T) = \sum_{s \in T^m} |B_s|. \tag{5}$$

Now, for each sequence $s \in T^m$, note that the elements of B_s are constant modulo $[T, T]$ (since T/T' is an abelian group). Thus, B_s contains at most $|T'|$ elements, that is, $1 \le |B_s| \le |T'|$. From (5), this gives us

$$|T|^m \le c_m^T(\mathbb{F}^\sigma T) \le |T'||T|^m.$$

Finally, consider a basis of $P_m^T(\mathbb{F}^\sigma T)$ consisting of monomials. Given $x_{i_1}^{(u_1)} \cdots x_{i_m}^{(u_m)}$, consider the set

$$\{\iota^{j_1}(x_{i_1}^{(u_1)}) \cdots \iota^{j_m}(x_{i_m}^{(u_m)}) \mid j_1, \ldots, j_m \in \{0, 1\}\}.$$

By the discussion before Theorem 5, if two elements of the set under a nonzero evaluation assume the same value on the algebra up to a constant, then the monomials are equal modulo $\mathrm{Id}_{T,\iota}(\mathbb{F}^\sigma T)$. So, the set spans a subspace of dimension at most $|T|$. Hence, $c_m^{T,\iota}(\mathbb{F}^\sigma T) \le |T| c_m^T(\mathbb{F}^\sigma T)$. □

As a consequence, we obtain the homogeneous-involution exponent of $\mathbb{F}^\sigma T$:

Corollary 7. *Let T be a finite group, $\sigma \in Z^2(T, \mathbb{F}^\times)$ and ι a homogeneous involution on $\mathbb{F}^\sigma T$. Then* $\exp^{T,\iota}(\mathbb{F}^\sigma T) = |T|$.

Specializing Theorem 5 in the special case where T is an abelian group, we obtain the exact codimension sequence.

Corollary 8. *If T is an abelian group and ι is a homogeneous involution on $\mathbb{F}^\sigma T$, then for each $m \in \mathbb{N}$,*

$$|T|^m \le c_m^{T,\iota}(\mathbb{F}^\sigma T) \le |T|^{m+1} \quad \textit{and} \quad c_m^T(\mathbb{F}^\sigma T) = |T|^m.$$

If, moreover, ι is degree-preserving, then for each $m \in \mathbb{N}$,

$$c_m^{T,\iota}(\mathbb{F}^\sigma T) = c_m^T(\mathbb{F}^\sigma T) = |T|^m.$$

Proof. If T is abelian, then $T' = 1$. So we obtain the first statement. Now, if ι is a degree-preserving involution, then, modulo $\mathrm{Id}_{T,\iota}(\mathbb{F}^\sigma T)$ and up to a constant, $x_i^{(u)} = \iota(x_i^{(u)})$ (see the discussion before Theorem 5). Thus, we may remove the ι in each variable, and obtain that $c_m^{T,\iota}(\mathbb{F}^\sigma T) = c_m^T(\mathbb{F}^\sigma T)$. □

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