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A new simple class of superpotentials in SUSY quantum mechanics

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Abstract

In this work, we introduce the class of quantum mechanics superpotentials $W(x) = g\mathcal{E}(x)x^{2n}$ and study in detail the cases $n = 0$ and 1. The $n = 0$ superpotential is shown to lead to the known problem of two supersymmetrically related Dirac delta potentials (well and barrier). The $n = 1$ case results in the potentials $V_{\pm}(x) = g^2x^4 \pm 2g|x|$. For V_- , we present the exact ground-state solution and study the excited states by a variational technique. Starting from the ground state of V_- and using logarithmic perturbation theory, we study the ground states of V_+ and also of $V(x) = g^2x^4$ and compare the result obtained in this new way with other results for this last potential in the literature.

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1. Introduction

Supersymmetric quantum mechanics (SUSY QM) was first introduced by Witten [1, 2], as a simplified model (a (0+1)-dimensional field theory) to study the possibility of SUSY breaking. Soon it became a research branch in itself, a way of obtaining new solutions to problems in QM [3–6]. Of particular interest to our work, we cite many papers in the literature [7–14] devoted to the development of techniques for treating the anharmonic oscillator $V(x) = \omega^2x^2 + g^2x^4$, and other related potentials, which in general do not have exact solutions.

In this work, we present a new simple class of superpotentials in SUSY QM, in the form $W(x) = g\mathcal{E}(x)x^{2n}$ with $n = 0, 1, 2, \dots$. The first example of this class, i.e. the case $n = 0$, was studied long ago in [15] and revisited in [16, 17]. One of our results is an analytic solution for the ground-state wavefunction of the potential $V(x) = g^2x^4 - 2g|x|$, an amazing result, considering that analytic solutions do not exist for anharmonic oscillators. Another result is a new perturbative solution for the ground state of the potential $V(x) = gx^4$, starting from the solution for the potential $V(x) = g^2x^4 - 2g|x|$. The excited states of the potentials $V_{\pm}(x) = g^2x^4 \pm 2g|x|$ are also studied by a variational approximation.

This paper is organized as follows. In section 2, we make a brief introduction to the well-known case of superpotentials, which are monomials of odd powers of x , as well as to

the SUSY breaking ones, which are monomials of even powers of x . More details of these solutions can be found in [3, 18]. In section 3, we study solutions related to the class of simple superpotentials of the form $W(x) = g\varepsilon(x)x^{2n}$ ($n = 0, 1, 2, \dots$), where $\varepsilon(x)$ is the sign function. The simple analytical solution for the ground state of the corresponding SUSY system is shown, the already known case $n = 0$ is revised and the case $n = 1$ is studied in more detail. The first one is the illustrative example of the Dirac delta well and barrier potentials, which are shown to be SUSY partner potentials associated with the superpotential $W(x) = g\varepsilon(x)$. The second one, $W(x) = g\varepsilon(x)x^2$, allows us to find an analytical solution for the ground state of the potential $V(x) = g^2x^4 - 2g|x|$. In sections 3.1 and 3.2, we study the excited states of the potentials $V_{\pm}(x) = g^2x^4 \pm 2g|x|$, which are derived from $W(x) = g\varepsilon(x)x^2$. After discussing that exact solutions for the excited states cannot be obtained, we apply a variational method (section 3.1) to find approximate solutions for the energy levels and the wavefunctions. In section 3.2, a new perturbative approach to the ground state of the potentials $V(x) = gx^4$ and $V(x) = g^2x^4 + 2g|x|$ is presented. Finally, a discussion of the results is presented in section 4.

2. Our notation and definitions on SUSY QM

Let us briefly summarize some main concepts in SUSY QM. For simplicity, we will work in a system of units with Planck's constant set as $\hbar = 1$ and the particle mass set as $m = 1/2$ (i.e. $2m = 1$). We start by defining the operators A^\dagger and A :

$$A^\dagger = W(x) - ip \quad \text{and} \quad A = W(x) + ip, \quad (1)$$

where $W(x)$ is a given function of x and $p = -i d/dx$ is the momentum operator. From these operators, we can construct two Hamiltonians:

$$H_- = A^\dagger A \quad \text{and} \quad H_+ = AA^\dagger, \quad (2)$$

which in terms of p and $W(x)$ result in $H_{\pm} = p^2 + V_{\pm}$. The potentials V_{\pm} are given by the equations ($W' \equiv dW/dx$)

$$V_{\pm}(x) = W(x)^2 \pm W'(x), \quad (3)$$

which are Riccati's equations.

These equations can be understood in two ways. One way is: given $W(x)$, we can define the Hamiltonians H_{\pm} with potentials $V_{\pm}(x)$. The other is: given the potential $V_-(x)$ (or $V_+(x)$), by solving one of the Riccati equations, $W(x)$ can be found, the operators A and A^\dagger can be constructed and the partner potential $V_+(x)$ (or $V_-(x)$) can be found.

The ground state of a SUSY system is defined as the zero energy state of H_- (this is a choice; changing the function $W(x) \rightarrow -W(x)$ will change the roles of H_- and H_+). As $H_- = A^\dagger A$, its ground-state wavefunction $\psi_0^-(x)$ can be obtained by imposing that it is annihilated by the operator A :

$$A\psi_0^-(x) = \left(W(x) + \frac{d}{dx}\right)\psi_0^-(x) = 0.$$

The solution is given by

$$\psi_0^-(x) = \mathcal{N} \exp \left\{ - \int^x W(y) dy \right\}. \quad (4)$$

This is a good, physically meaningful solution, provided that the function (4) is normalizable. Otherwise, a zero energy solution does not exist and SUSY is said to be broken. As is easy to see, superpotentials obeying the rule of being positive ($W(x) > 0$) for $x > 0$ and negative ($W(x) < 0$) for $x < 0$ shall manifest SUSY.

Then, starting from $W(x)$, we have two partner Hamiltonians H_- and H_+ , one of them (H_- , in our choice) having a ground state ψ_0^- with energy $E_0^- = 0$ and a tower of other states: bound states with energies $E_n^- > 0$, $n = 1, 2, 3, \dots$, or scattering states with energies $E^- > 0$. The Hamiltonian H_+ has bound energy levels E_{n-1}^+ , $n = 1, 2, 3, \dots$, with energies related to the energies of H_- by the relation: $E_{n-1}^+ = E_n^-$ or scattering energies $E^+ > 0$. Moreover, the eigenfunctions of H_- and H_+ are related according to

$$\psi_{n-1}^+ = (E_n^-)^{-1/2} A \psi_n^-, \quad (5)$$

$$\psi_n^- = (E_{n-1}^+)^{-1/2} A^\dagger \psi_{n-1}^+. \quad (6)$$

The simplest class of superpotentials manifesting supersymmetry are monomials of odd power in x :

$$W(x) = gx^{2n+1}, \quad n = 0, 1, 2, \dots \quad (7)$$

Using the Riccati equation (3), we have, for the partner potentials,

$$V_\pm(x) = W(x)^2 \pm W'(x) = g^2 x^{4n+2} \pm g(2n+1)x^{2n}. \quad (8)$$

The ground (normalizable) state of $H_- = p^2 + V_-$, with the energy $E_0^- = 0$ (see equation (4)) is given by

$$\psi_0^-(x) = \mathcal{N} \exp \left\{ \frac{-gx^{2n+2}}{(2n+2)} \right\}. \quad (9)$$

The first example of a superpotential of the class (7) is $W(x) = gx$. In this case, the associated partner potentials are

$$V_\pm(x) = g^2 x^2 \pm g, \quad (10)$$

which are simply the potentials of two harmonic oscillators of the same frequency, with a constant energy shift g added or subtracted. The ground state of H_- have $E_0^- = 0$. Its excited states and the states of H_+ are given by $E_n^- = E_{n-1}^+ = 2ng$, for $n = 1, 2, 3, \dots$

We will not pursue the study of this class of superpotentials because they are well known. We only mention that the next example of this class, $W(x) = gx^3$, corresponds to the potentials $V_\pm(x) = g^2 x^6 \pm 3gx^2$ and their ground-state solution is given by (9) with $n = 1$.

On the other hand, the class of superpotentials that are monomials in even powers of x , does not give a normalizable zero energy solution to (4) and SUSY is broken. However, we can introduce the sign function $\varepsilon(x)$ and consider superpotentials of the form $W(x) = g\varepsilon(x)x^{2n}$. For this class of superpotentials, a normalizable ground state exists and SUSY is not broken. Thus, in the following, we study this class of superpotentials, specially the $n = 0$ and $n = 1$ cases.

3. The class of superpotentials of the form $W(x) = g\varepsilon(x)x^{2n}$

The case $n = 0$ must be treated separately. So, let us consider the superpotential

$$W(x) = g\varepsilon(x), \quad (11)$$

where g is a positive constant. For this superpotential (11), the Riccati equations (3) give the following SUSY partner potentials:

$$V_\pm(x) = W(x)^2 \pm W'(x) = \pm 2g\delta(x) + g^2, \quad (12)$$

where $\delta(x)$ is the Dirac delta function. V_- is a delta well, while V_+ is a delta barrier, with the energy of the ground state displaced by g^2 . The corresponding Schrödinger equations are

$$-\psi^{\pm''}(x) \pm 2g\delta(x)\psi^\pm(x) = (E^\pm - g^2)\psi^\pm(x). \quad (13)$$

Their solutions are well known [15–17]. The well (V_-) has a single bound state with energy level $E_0^- = 0$, binding energy g^2 and wavefunction given by

$$\psi_0^-(x) = \sqrt{g} e^{-g|x|}. \quad (14)$$

All the other eigenstates are plane waves in continuous spectra of energies, the lowest one starting with $E = g^2$. Simple scattering solutions of the well V_- and the barrier V_+ can be written as

$$\psi_I^\pm(x) = \mathcal{A}_\pm e^{ikx} + \mathcal{B}_\pm e^{-ikx}, \quad x \leq 0, \quad (15)$$

$$\psi_{II}^\pm(x) = \mathcal{C}_\pm e^{ikx} + \mathcal{D}_\pm e^{-ikx}, \quad x \geq 0, \quad (16)$$

where $k = \sqrt{E^\pm - g^2}$ with $E^\pm > g^2$ and the respective constants are related according to the boundary conditions $\psi_{II}(0) = \psi_I(0)$ and $\psi'_{II}(0) = \psi'_I(0) \pm 2g\psi(0)$ required by the Dirac delta potential.

In summary, the Hamiltonian H_- has one ground state with energy $E_0^- = 0$ and continuum of states with energies $E^- > g^2$ and H_+ has a continuum of states with $E^+ > g^2$.

To see the role of the supersymmetry in this system, let us consider a particle crossing the well (or hitting the barrier), coming from $x = -\infty$, such that we can choose $\mathcal{D}_\pm = 0$. With the appropriate boundary conditions through $x = 0$, we can determine \mathcal{B}_\pm and \mathcal{C}_\pm , getting the scattered and the transmitted solutions as functions of the incident amplitudes \mathcal{A}_\pm . The results can be written as

$$\psi_I^\pm(x) = \mathcal{A}_\pm \left\{ e^{ikx} + i \frac{(\mp g)}{k} \frac{1}{(1 - i \frac{(\mp g)}{k})} e^{-ikx} \right\}, \quad x \leq 0 \quad (17)$$

$$\psi_{II}^\pm(x) = \mathcal{A}_\pm \frac{1}{(1 - i \frac{(\mp g)}{k})} e^{ikx}, \quad x \geq 0. \quad (18)$$

It is easy to verify that the solutions ψ^- and ψ^+ are related by the supersymmetry equations (5) and (6). For example, by applying the operator A to $\psi_I^-(x)$, we obtain

$$\begin{aligned} A\psi_I^-(x) &\propto \left(g\varepsilon(x) + \frac{d}{dx} \right) \left\{ e^{ikx} + i \frac{\frac{g}{k}}{(1 - i \frac{g}{k})} e^{-ikx} \right\} \\ &\propto \left\{ e^{ikx} + i \frac{\frac{-g}{k}}{(1 - i \frac{-g}{k})} e^{-ikx} \right\} \propto \psi_I^+(x), \end{aligned}$$

explicitly showing the manifestation of the supersymmetry of the system.

Let us now consider the superpotential

$$W(x) = g\varepsilon(x)x^2, \quad (19)$$

where g is a positive constant. The two partner potentials are given by

$$V_\pm(x) = W(x)^2 \pm W'(x) = g^2 x^4 \pm 2g|x|. \quad (20)$$

In these potentials, a term $\delta V = \pm 2gx^2\delta(x)$ has been dropped. The reason is that for the wavefunctions involved in this problem its action is null. As the potentials $V_\pm(x) \rightarrow \infty$ for $x \rightarrow \pm\infty$, the spectra of $H_\pm = p^2 + V_\pm$ are discrete and their eigenfunctions are normalizable. If δV is treated as a perturbative correction to H_\pm , its action would be non-null only if $\int_{-\infty}^{\infty} dx x^2 \delta(x) |\psi(x)|^2 \neq 0$. But this condition requires a wavefunction that near $x = 0$ behaves like $f(x)/x$ with $f(0) \neq 0$, which is non-normalizable and is not in the spectra of H_\pm . On the other side, treated as part of H_\pm , the term δV could give non-trivial boundary conditions for $d\psi/dx$ at $x = 0$. To study this possibility, we must integrate the Schrödinger

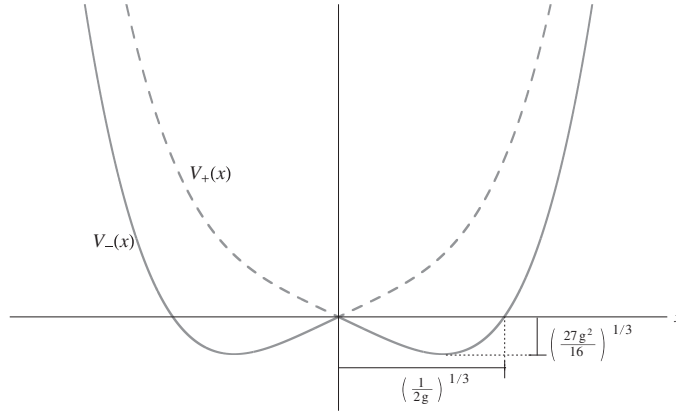


Figure 1. Partner potentials $V_-(x)$ and $V_+(x)$ associated with the superpotential $W(x) = g\varepsilon(x)x^2$.

equation in the interval $x = (-\epsilon, \epsilon)$, for $\epsilon \rightarrow 0$. A non-null effect of δV only comes if $\int_{-\epsilon}^{\epsilon} dx x^2 \delta(x) \psi(x) \neq 0$, which would require $\psi(x)$ behaving like $f(x)/x^2$ with $f(0) \neq 0$ that is also, out of the spectra of H_{\pm} .

A representation of these potentials is given in figure 1. As can be seen, V_+ is a single-well potential and V_- is a double-well potential symmetric in x . The corresponding Schrödinger equations read

$$\left(-\frac{d^2}{dx^2} + g^2x^4 \pm 2g|x|\right) \psi^{\pm}(x) = E^{\pm} \psi^{\pm}(x). \quad (21)$$

The wavefunction for the ground state of the double-well potential $V_-(x) = g^2x^4 - 2g|x|$ has the energy $E_0^- = 0$ and is easily obtained from the equation

$$0 = A\psi_0 = \left(g\varepsilon(x)x^2 + \frac{d}{dx}\right) \psi_0.$$

The result (already normalized) is given by

$$\psi_0(x) = \left(\frac{3}{2}\right)^{1/3} \frac{g^{1/6}}{\Gamma(1/3)^{1/2}} e^{-g|x|^3/3}. \quad (22)$$

This is an interesting result. As is well known, exact analytic solutions for the ground (or any excited) state of the potentials $V(x) = g^2x^4$ or $V(x) = \omega^2x^2 + g^2x^4$ cannot be obtained. So, this exact solution for the potential V_- is somewhat surprising. Another characteristic of this solution is that it represents a single-lump centered at $x = 0$ (which is a local maximum of V_-) and it is not in the form, as naïvely expected, of two lumps centered at the two symmetric minima, $x = \pm(1/2g)^{1/3}$, of V_- notwithstanding the fact that, in one dimension, any attractive well supports at least a bound state. This happens because the ‘volume’ of each well is not big enough to support a bound state. (This can be seen in a WKB analysis of the potential, or even more simply, by the Heisenberg uncertainty principle. We should only observe that this well size $\Delta x(\Delta E)^{1/2}$ is independent of g .)

Let us now look for the excited states solutions. Inspired by the analytic method to solve the one-dimensional simple harmonic oscillator and by the form of the solution (22), we try a solution of the form¹

$$\psi(x) = F(x) e^{-g|x|^3/3}. \quad (23)$$

¹ In the case of the simple harmonic oscillator, we suppose that the solutions are of the form $H(x)e^{-x^2/2}$ and, imposing that these solutions are square integrable, the functions $H(x)$ become restricted to be the Hermite polynomials $\mathcal{H}_n(x^2)$.

Substituting (23) in the Schrödinger equation (21), it becomes

$$F'' - 2g\varepsilon(x)x^2F'(x) + EF(x) = 0. \quad (24)$$

For the simple harmonic oscillator, the same steps would lead us to the Hermite equation. In our case, we obtain equation (24), which is, for a particular choice of parameters, the triconfluent Heun equation [19].

We can go on to look for solutions to equation (24) through a power series method. Assuming that $F(x)$ can be written as

$$F(x) = \sum_{j=0}^{\infty} a_j x^j \quad (25)$$

and substituting this expression for $F(x)$ in the differential equation (24), we find

$$\sum_{j=0}^{\infty} j(j-1)a_j x^{j-2} - 2g\varepsilon(x) \sum_{j=0}^{\infty} j a_j x^{j+1} + E \sum_{j=0}^{\infty} a_j x^j = 0.$$

Renaming indices and rearranging terms, we have

$$2a_2 + Ea_0 + \sum_{j=1}^{\infty} [(j+2)(j+1)a_{j+2} - 2g\varepsilon(x)(j-1)a_{j-1} + Ea_j] = 0.$$

Then, given a_0 and a_1 , this equation is satisfied if the coefficients a_j , $j \geq 2$, are given by the three-term recursion relations:

$$a_2 = -\frac{E}{2}a_0, \quad j = 2, \quad (26)$$

$$a_j = \frac{2g\varepsilon(x)(j-3)a_{j-3} - Ea_{j-2}}{j(j-1)}, \quad j \geq 3. \quad (27)$$

The corresponding recursion relation for the the harmonic oscillator potential is a simple two-term recursion relation. To obtain a normalizable solution, we choose the values of E so as to terminate the series in a polynomial. In this way, we get the set of discretized values of the energy spectrum and the corresponding wavefunctions that turn up to be the Hermite polynomials (see footnote 1).

In our case, the recurrence relation (27) is a three-term recurrence relation and there is no way of choosing a subset of values of E to terminate the series in polynomials, so as to have a normalizable solution. Then, no analytic solution can be found, and in the next sections, we move to looking for approximate solutions. In section 3.1, a variational approximation is studied, and in section 3.2, a perturbative approximation is discussed, which will also allow us to study solutions for the potential $V(x) = gx^4$.

3.1. Looking for approximate solutions by a variational method

Let us first apply a variational method. The trial function that we are going to use is

$$\phi(x) = \sum_{j=1}^m \alpha_j f_j(x), \quad (28)$$

where $j = 1, 2, \dots, m$ and the coefficients $\alpha_j \in \mathbb{C}$ are the variational parameters. The functions $f_j(x)$ are chosen to be

$$f_j(x) = x^{j-1} e^{-g|x|^3/3}. \quad (29)$$

This trial function corresponds to the one previously used in the power series method, with the additional restriction of being a finite polynomial of degree $m - 1$, instead of an infinite series in x .

For the harmonic oscillator with a very similar choice of the trial function we would find exact solutions. In that case, the variational parameters would be, except for the normalization, the coefficients of the Hermite polynomials $\mathcal{H}_n(x^2)$.

Before proceeding, let us consider a convenient change of variables. As can easily be seen, by making the rescaling $x \rightarrow g^{-1/3}x$, it is possible to factor out of the Hamiltonians H_{\pm} the constant $g^{2/3}$:

$$H_{\pm} = g^{2/3} \left(-\frac{d^2}{dx^2} + x^4 \pm 2|x| \right). \quad (30)$$

So, in the rest of this section, we will work with $g = 1$ and after finding the energy eigenvalues, we can restore the dependence of the energy levels in g by multiplying the results by a factor of $g^{2/3}$. The restoration of the corresponding wavefunctions (or trial functions) can also be obtained by rescaling $x \rightarrow g^{1/3}x$ in the results.

To go on with the variational method, we construct the expectation value of the energy with these trial functions:

$$E = \frac{\langle \phi | H_{\pm} | \phi \rangle}{\langle \phi | \phi \rangle} = \frac{\sum_{k=1}^m \sum_{l=1}^m \alpha_k \alpha_l \langle f_k | H_{\pm} | f_l \rangle}{\sum_{k=1}^m \sum_{l=1}^m \alpha_k \alpha_l \langle f_k | f_l \rangle} \quad (31)$$

and minimize E with respect to the parameters α_l . This condition gives the system of linear equations:

$$\sum_{l=1}^m ((H_{\pm})_{kl} - ES_{kl}) \alpha_l = 0, \quad (32)$$

where we used the notation $H_{kl} = \langle f_k | H | f_l \rangle$ and $S_{kl} = \langle f_k | f_l \rangle$. The values of E that minimize the above system of equations are the eigenvalues of the matrix

$$M_{kl} = (ES_{kl} - (H_{\pm})_{kl}) \quad (33)$$

and are obtained by solving the equation $\det M = 0$. The wavefunctions corresponding to each of these eigenvalues are obtained by substituting the value of E in the linear system above and solving for the parameters α_k . The matrix elements that we need to construct M_{kl} are

$$S_{kl} = \langle f_k | f_l \rangle = \int_{-\infty}^{+\infty} dx e^{-\frac{2}{3}|x|^3} x^{k+l-2}, \quad (34)$$

$$(H_{\pm})_{kl} = \langle f_k | H_{\pm} | f_l \rangle = \int_{-\infty}^{+\infty} dx e^{-\frac{2}{3}|x|^3} [-(l-1)(l-2)x^{k+l-4} + 2(l \pm 1)\varepsilon(x)x^{k+l-1}]. \quad (35)$$

For $(k+l)$ odd, the integrands in (34) and (35) are odd functions and $S_{kl} = (H_{\pm})_{kl} = 0$. Otherwise, for $(k+l)$ even, we find

$$S_{kl} = \left(\frac{3}{2}\right)^{\frac{k+l-4}{3}} \Gamma\left(\frac{k+l-1}{3}\right) \quad (36)$$

$$(H_{\pm})_{kl} = -2 \left(\frac{3}{2}\right)^{\frac{k+l-3}{3}} \left[\frac{(l-1)(l-2) - (l \pm 1)(k+l-3)}{(k+l-3)} \right] \Gamma\left(\frac{k+l}{3}\right). \quad (37)$$

Table 1. Energy values associated with H_- calculated for different numbers of variational parameters.

m	E_0^- ^a	E_1^-	E_2^-	E_3^-	E_4^-	E_5^-	E_6^-	E_7^-
1	0.000 00							
2	0.000 00	2.044 41						
3	0.000 00	2.044 41	5.765 41					
4	0.000 00	1.978 52	5.765 41	10.001 91				
5	0.000 00	1.978 52	5.541 35	10.001 91	14.941 74			
6	0.000 00	1.971 15	5.541 35	9.494 46	14.941 74	20.370 28		
7	0.000 00	1.971 15	5.513 02	9.494 46	14.065 58	20.370 28	26.299 53	
8	0.000 00	1.969 91	5.513 02	9.413 70	14.065 58	19.029 62	26.299 53	32.643 99
9	0.000 00	1.969 91	5.508 42	9.413 70	13.901 48	19.029 62	24.431 94	32.643 99
10	0.000 00	1.969 63	5.508 42	9.398 68	13.901 48	18.734 98	24.431 94	30.187 55

^a For this level, the variational method provides the exact solution.**Table 2.** Energy values associated with H_+ calculated for different numbers of variational parameters.

m	E_0^+	E_1^+	E_2^+	E_3^+	E_4^+	E_5^+	E_6^+
1	2.314 47						
2	2.314 47	6.133 24					
3	2.044 93	6.133 24	10.549 40				
4	2.044 93	5.636 55	10.549 40	15.634 69			
5	1.990 66	5.636 55	9.664 70	15.634 69	21.219 33		
6	1.990 66	5.538 88	9.664 70	14.309 56	21.219 33	27.285 56	
7	1.976 66	5.538 88	9.465 67	14.309 56	19.369 16	27.285 56	33.765 58
8	1.976 66	5.516 11	9.465 67	13.981 07	19.369 16	24.867 27	33.765 58
9	1.972 35	5.516 11	9.415 24	13.981 07	18.857 87	24.867 27	30.729 24
10	1.972 35	5.510 07	9.415 24	13.893 69	18.857 87	24.136 59	30.729 24

With these results, the matrix M (33) obtains the form

$$M_{\pm} = \begin{pmatrix} (M_{\pm})_{11} & 0 & (M_{\pm})_{13} & 0 & \cdots & (M_{\pm})_{1m} \\ 0 & (M_{\pm})_{22} & 0 & (M_{\pm})_{24} & \cdots & (M_{\pm})_{2m} \\ (M_{\pm})_{31} & 0 & (M_{\pm})_{33} & 0 & \cdots & (M_{\pm})_{3m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (M_{\pm})_{m1} & (M_{\pm})_{m2} & (M_{\pm})_{m3} & (M_{\pm})_{m4} & \cdots & (M_{\pm})_{mm} \end{pmatrix}. \quad (38)$$

In this matrix, all elements in positions (k, l) , such that $(k + l)$ is odd are null, while those with $(k + l)$ even are given by (33) with S_{kl} and H_{kl} , respectively, given by (36) and (37). To find the energy values, we must solve the equation $\det M = 0$.

Tables 1 and 2 show some results found for different number (m) of parameters and for $g = 1$. For different values of g , the values in the tables must be multiplied by a factor of $g^{2/3}$, as observed above.

The results in tables 1 and 2 reflect the manifestation of SUSY in the system, at least with respect to the equality between the energy levels E_n^- and E_{n-1}^+ , $n > 0$, of H_- and H_+ . As expected, the ground-state energy of H_- is zero and it is not equal to any energy of H_+ . Moreover, for $n > 0$, increasing the number of variational parameters, we find, mainly for the first levels, energies E_n^- more and more closer to E_{n-1}^+ .

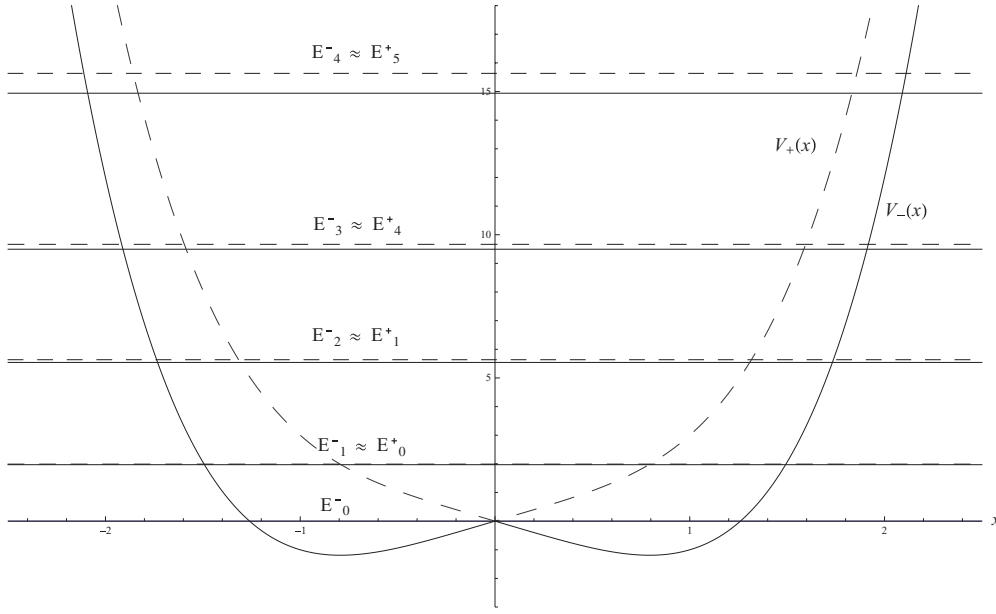


Figure 2. Scheme for the first five levels of H_- (and first five levels of H_+) using six variational parameters.

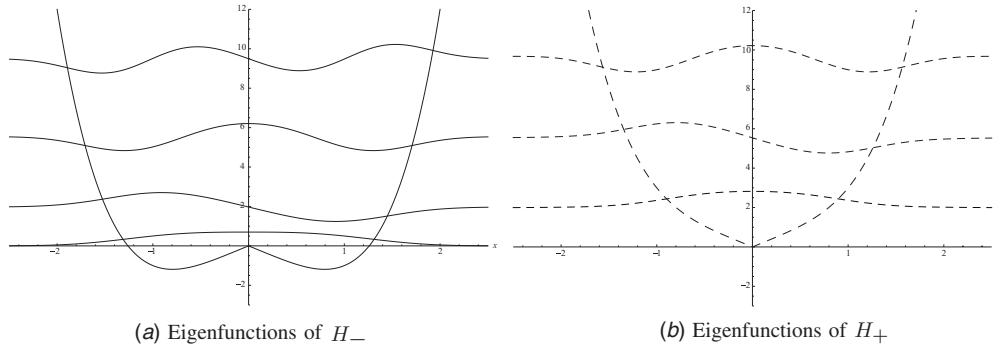


Figure 3. Eigenfunctions of the first levels of H_- and H_+ for six variational parameters.

Therefore, the better the trial we make, the closer we are to satisfying the equality between energy levels. Moreover, because the one parameter trial function for the ground state of H_- has the same form of the exact (analytical) solution, the value $E_0^- = 0$ found is exact and the condition of having a zero energy ground state is naturally satisfied.

Figure 2 shows the first energy levels of H_- and H_+ . We must remember that the values found are better for the increasing number of variational parameters and for the lowest levels. Thus, for instance, we are supposed to find, for the level $n = 4$, a worse approximation than for the level $n = 1$.

The graphics in figure 3 show the approximations for the first levels eigenfunctions of H_- and H_+ , respectively. These approximations were found using six variational parameters.

As expected, we note that the eigenfunctions found have well-defined parity, interchanging even and odd solutions with even solutions for the ground states.

3.2. Looking for approximate solutions by a logarithmic perturbation theory

We now apply a variant of the logarithmic perturbation theory (LPT) to our problem. The LPT is explained in more detail, e.g., in [3, 7–9, 11].

Starting from the known solution ψ_0^- of V_- , we can perturbatively obtain the ground state of V_+ , or, e.g., of the anharmonic potential $V(x) = x^4$. We start by writing

$$V(x; \delta) = V_0(x) + \delta V_1(x), \quad (39)$$

where

$$V_0(x) = V_-(x) = x^4 - 2|x| \quad (40)$$

$$V_1(x) = 4|x|. \quad (41)$$

Observe that $V(x; \delta = 1) = V_+$ and that $V(x; \delta = 1/2) = x^4$. As we only know the ground state of V_- , we cannot go beyond the first order in the Rayleigh–Schrödinger perturbation theory. To bypass this difficulty, we will use the so-called LPT, where only the knowledge ψ_0^- is required to calculate the ground-state energy level of $V(x; \delta)$ to any order in δ (at least numerically). For that aim, we consider the perturbed Schrödinger equation

$$-\Psi'' + (V_0 + \delta V_1)\Psi = E\Psi \quad (42)$$

and write the expansions

$$E = E_0 + \delta E_1 + \delta^2 E_2 + \dots \quad (43)$$

$$\Psi = \exp(S_0 + \delta S_1 + \delta^2 S_2 + \dots), \quad (44)$$

where S_1, S_2 , etc, are functions and E_1, E_2 , etc, are numbers to be determined. By substituting these expressions in the Schrödinger equation above and equating the terms of the same powers in δ , we obtain the set of equations:

$$S_0'' + S_0'^2 = -E_0 + V_0, \quad (45)$$

$$S_1'' + 2S_0'S_1' = -E_1 + V_1, \quad (46)$$

$$S_2'' + 2S_0'S_2' + S_1'^2 = -E_2, \quad (47)$$

$$S_3'' + 2S_0'S_3' + 2S_1'S_2' = -E_3, \quad (48)$$

⋮

Starting with $E_0 = 0$ and $S_0 = -|x|^3/3$ (i.e., $\Psi_0(x) = \psi_0^- = \mathcal{N} e^{-|x|^3/3}$), these equations can be recursively solved to obtain E_k and S_k to the desired order in δ .

Equation (46) can be rewritten as

$$(S_1' \exp(2S_0))' = (V_1 - E_1) \exp(2S_0). \quad (49)$$

By substituting $S_0 = -|x|^3/3$ and $V_1 = 4|x|$ in this equation, integrating both sides in the interval $x = (-\infty, +\infty)$ and observing that the integrand of the left-hand side tends exponentially to zero at both ends of the integration range, we obtain for E_1 the result

$$\begin{aligned} E_1 &= \frac{\langle \psi_0 | V_1(x) | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = \frac{\int_{-\infty}^{+\infty} dx e^{-\frac{2}{3}|x|^3} 4|x|}{\int_{-\infty}^{+\infty} dx e^{-\frac{2}{3}|x|^3}} \\ &= 4 \left(\frac{3}{2} \right)^{1/3} \frac{\Gamma(2/3)}{\Gamma(1/3)} = 2.31447. \end{aligned} \quad (50)$$

Inserting this result for E_1 back into the same equation and integrating now in the interval $y = (0, x)$ we obtain

$$\begin{aligned} S'_1(x) &= |\psi_0(x)|^{-2} \int_0^x dy |\psi_0(y)|^2 [E_1 - V_1(y)] \\ &= e^{\frac{2}{3}|x|^3} \int_0^x dy e^{-\frac{2}{3}|y|^3} \left[4 \left(\frac{3}{2} \right)^{1/3} \frac{\Gamma(2/3)}{\Gamma(1/3)} - 4|y| \right] \\ &= -2 \left(\frac{2}{3} \right)^{1/3} e^{\frac{2}{3}|x|^3} \left[\frac{\Gamma(2/3)}{\Gamma(1/3)} \Gamma(1/3, 2x^3/3) - \Gamma(2/3, 2x^3/3) \right], \end{aligned} \quad (51)$$

where $\Gamma(\alpha, x) \equiv \int_x^\infty dt e^{-t} t^{\alpha-1}$ are the upper incomplete gamma functions [20].

The second-order equation (47) can also be written in the form

$$(S'_2 \exp(2S_0))' = (-S_1'^2 - E_2) \exp(2S_0). \quad (52)$$

Integrating this equation in the interval $x = (-\infty, +\infty)$, and observing that the integrand of the left-hand side tends to zero at both ends of the integration range, we obtain E_2 as an integral over S'_1 :

$$E_2 = -\frac{\langle \psi_0 | S_1'(x)^2 | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = -\frac{3}{\Gamma(1/3)} \left(\frac{2}{3} \right)^{1/3} \int_0^\infty dx e^{-\frac{2}{3}|x|^3} S_1'(x)^2. \quad (53)$$

Substituting (51) into (53), we find

$$E_2 = -\frac{4}{\Gamma(1/3)} \left(\frac{2}{3} \right)^{2/3} \left\{ \frac{\Gamma(2/3)^2}{\Gamma(1/3)^2} I_{2/3} \left(\frac{1}{3}, \frac{1}{3} \right) + I_{2/3} \left(\frac{2}{3}, \frac{2}{3} \right) - 2 \frac{\Gamma(2/3)}{\Gamma(1/3)} I_{2/3} \left(\frac{1}{3}, \frac{2}{3} \right) \right\}, \quad (54)$$

where

$$I_\alpha(x, y) = \int_0^\infty dt e^t t^{-\alpha} \Gamma(x, t) \Gamma(y, t), \quad x > 0, \quad y > 0, \quad 0 < \alpha < 1. \quad (55)$$

Evaluating the integrals, expression (54) gives $E_2 = -0.438\,17$.

In summary, up to the second order, the ground-state energy of $V(x; \delta)$ is given by

$$E(\delta) = E_0 + \delta E_1 + \delta^2 E_2, \quad (56)$$

with $E_0 = 0$, $E_1 = 2.314\,47$ and $E_2 = -0.438\,17$.

For $\delta = 1$, we obtain for the ground-state energy of V_+ , the result $E_0^+ = 1.876\,30$.

For $\delta = 1/2$, we find the result $E_0^{x^4} = 1.047\,69$ for the ground-state energy of the quartic anharmonic potential $V(x) = x^4$. This result can be compared with the exact one given in [3], noting that our ‘coupling’ constant g is related to their constant \tilde{g} by $g^{2/3} = (\frac{1}{4})^{1/3} \tilde{g}^{1/3}$. Thus, multiplying our result by $(\frac{1}{4})^{1/3}$, we find $\tilde{E}_0^{x^4} = 0.660\,00$, differing that of [3] only by about 1.2%.

On the other hand, comparing the value of E_0^+ found here with the most accurate result of the variational method (see table 2), we see that they differ by about 4.9%, what does not seem very good. But, following the suggestion of [8] or [21], and substituting expression (56) by the corresponding $[1, 1]$ Padé approximant in δ , we find

$$E(\delta) = \frac{E_0 E_1 + (E_1^2 - E_0 E_2) \delta}{E_1 - E_2 \delta}, \quad (57)$$

which results (for $\delta = 1$) in $E_0^+ = 1.946\,05$. This result now differs from the result of table 2 only by 1.3%. Doing the same for $\delta = 1/2$ (and then multiplying by $(\frac{1}{4})^{1/3}$), we find $\tilde{E}_0^{x^4} = 0.665\,97$, differing from the result of Cooper *et al* [3] by only 0.03%. A pretty good result.

4. Conclusions

In this paper, we studied the class of superpotentials $W(x) = \varepsilon(x)x^{2n}$ in SUSY QM. After revisiting the case $n = 0$, we continued studying in detail the case $W(x) = \varepsilon(x)x^2$. As a result we obtained the exact solution for the ground state of the potential $V_-(x) = x^4 - 2|x|$, showed that exact solutions do not exist for the excited states and studied these states by a variational method. Finally, starting from the known ground state of $V(x) = x^4 - 2|x|$, we obtained the ground states for the potentials $V(x) = x^4$ and $V(x) = x^4 + 2|x|$, using the LPT. Comparison with other known results in the literature and in the paper were made.

Some other approaches and improvements can be used to study this class of superpotentials. In a forthcoming paper, we analyze the solutions for the ground states of $V_{\pm}(x) = x^4 \pm 2|x|$ by starting with the solutions of $V_{\pm} = x^2 \pm C$ and using the LPT and the δ expansion of Bender [7] and Cooper [8].

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