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ORIGINAL PAPER

On the Structure of a Smallest Counterexample and a New Class Verifying the 2-Decomposition Conjecture

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Abstract
The 2-Decomposition Conjecture, equivalent to the 3-Decomposition Conjecture stated in 2011 by Hoffmann-Ostenhof, claims that every connected graph G with vertices of degree 2 and 3, for which $G \setminus E(C)$ is disconnected for every cycle C, admits a decomposition into a spanning tree and a matching. In this work we present two main results focused on developing a strategy to prove the 2-Decomposition Conjecture. One of them is a list of structural properties of a minimum counterexample for this conjecture. Among those properties, we prove that a minimum counterexample has girth at least 5 and its vertices of degree 2 are at distance at least 3. Motivated by the class of smallest counterexamples, we show that the 2-Decomposition Conjecture holds for graphs whose vertices of degree 3 induce a collection of cacti in which each vertex belongs to a cycle. The core of the proof of this result may possibly be used in an inductive proof of the 2-Decomposition Conjecture based on a parameter that relates the number of vertices of degree 2 and 3 in a minimum counterexample.

Keywords 2-Decomposition conjecture · Spanning tree · Matching

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1 Introduction

A *Homeomorphically Irreducible Spanning Tree* (HIST) in a graph G is a spanning tree of G without vertices of degree 2. The problem of deciding whether a graph contains a HIST is NP-complete [3], even for subcubic graphs [9]. This topic has been studied by many researchers [3, 7, 8, 10, 21] and it is related to the topic addressed in this paper, as we shall explain.

Let G be a connected cubic graph, T be a spanning tree of G, and let G' be the graph obtained from $G \setminus E(T)$ by removing the isolated vertices. Each component of G' is either a path or a cycle, so every connected cubic graph can be decomposed into a spanning tree, a collection of cycles, and a collection of paths. Moreover, T is a HIST if and only if G' is a collection of cycles. Thus deciding whether a cubic graph G contains a HIST is equivalent to deciding whether G admits a decomposition into a spanning tree and a collection of cycles. Not all cubic graphs admit such a decomposition; necessary conditions for its existence have been shown by Hoffmann-Ostenhof et al. [14]; and the following more relaxed decomposition has been conjectured by Hoffmann-Ostenhof [11].

Conjecture 1.1 (Hoffmann-Ostenhof [11]) *Every connected cubic graph can be decomposed into a spanning tree, a collection of cycles, and a (possibly empty) matching.*

Conjecture 1.1 is known as the 3-*Decomposition Conjecture* (3DC, for short) and has attracted the attention of many researchers. Although the general problem remains open, it has been verified for some classes of cubic graphs. Clearly, Conjecture 1.1 holds for every graph that contains a HIST. Liu and Li [17] verified it for cubic traceable graphs; and Ozeki and Ye [19] verified it for 3-connected planar cubic graphs and 3-connected cubic graphs on the projective plane. Later, Hoffmann-Ostenhof et al. [13] extended the result of Ozeki and Ye by verifying Conjecture 1.1 for all planar cubic graphs. Bachtler and Krumke [4] verified the 3DC for a superclass of Hamiltonian cubic graphs. Recently Xie et al. [20] verified Conjecture 1.1 for cubic graphs containing a 2-factor consisting of three cycles and, independently, Hong et al. [15] and Aboomahigir et al. [1] verified the conjecture for claw-free cubic graphs.

In addition, some weaker forms of Conjecture 1.1 have been verified. Akbari et al. [2] proved that every cubic graph can be decomposed into a spanning forest, a collection of cycles, and a matching. Li and Cui [16] showed that every cubic graph can be decomposed into a spanning tree, one cycle, and a collection of paths with length at most 2. Lyngsie and Merker [18] proved that every connected (not necessarily cubic) graph can be decomposed into a spanning tree, an even graph, and a star forest, which implies the result of Li and Cui [16] when applied to cubic graphs.

A cycle C in a connected graph G is *separating* if $G \setminus E(C)$ is disconnected. Let $\mathcal{S}$ be the class of connected graphs in which every cycle is separating, and let $\mathcal{S}_{p,q} \subseteq \mathcal{S}$ be the class of graphs in $\mathcal{S}$ with minimum degree at least p, and the maximum degree at most q. It is known that Conjecture 1.1 is equivalent to the following conjecture, known as the 2-*Decomposition Conjecture* (2DC, for short)—see [13, Proposition 14].

Conjecture 1.2 (Hoffmann-Ostenhof [12]) *Every graph in $\mathcal{S}_{2,3}$ can be decomposed into a spanning tree and a matching.*

Conjecture 1.2 is fairly new. At the best of our knowledge, the only work addressing directly Conjecture 1.2 is the one conducted by Hoffmann-Ostenhof et al. [13], where the authors verify Conjecture 1.2 for the planar case and show that this result implies that Conjecture 1.1 also holds for planar graphs.

Throughout this paper, given a graph G, we denote by $V_k(G)$ the set of vertices of G with degree k. We recall that a graph is a *cactus* if it is connected and every edge is contained in at most one cycle. We say that a cactus G is *thick* if every vertex in G belongs to a cycle. The first contribution of this paper is the following theorem, proved in Sect. 2, that verifies 2DC on graphs $G \in \mathcal{S}_{2,3}$ whose subgraph induced by $V_3(G)$ is a collection of thick cacti. This result was presented in a preliminary version of this work at LAGOS 2021 [6].

Theorem 1.3 *Every graph $G \in \mathcal{S}_{2,3}$ for which $G - V_2(G)$ is a collection of thick cacti can be decomposed into a spanning tree and a matching.*

Let G be a cubic graph and let M be a perfect matching in G. Note that $G - M$ is a collection of cycles, and let G_M be the graph obtained from G by shrinking each cycle of $G - M$ into a vertex. It seems that a natural path to tackle the 3DC, and now the 2DC, is to organize the vertices of the studied graph in a structure of cycles. For example, it is a trivial task to verify the 3DC for Hamiltonian graphs, for which G_M is a single vertex. Bachtler and Krumke [4] verified Conjecture 1.1 for 3-connected graphs containing a matching M such that G_M is a star; Xie et al. [20] verified Conjecture 1.1 for graphs containing a 2-factor consisting of three cycles, and here in this paper, we verify Conjecture 1.2 for graphs G for which $G[V_3]$ is a collection of cacti in which every vertex belongs to a cycle.

Another interesting and natural approach to explore these conjectures is to study properties and forbidden structures in a minimum counterexample. Here, we show that if G is a minimum counterexample for the 2DC, then G is simple, 2-edge connected, has girth at least 5, that the distance between any pair of vertices in $V_2(G)$ is at least 3, and that its subgraph induced by $V_3(G)$ is connected and contains a cycle. This is precisely the result stated in Theorem 1.5 (see below), the proof of which is given in Sect. 3.

For a graph G in $\mathcal{S}_{2,3}$ let

$$\varphi(G) = |V(G)| + |E(G)|. \tag{1.4}$$

We say that a graph $G \in \mathcal{S}_{2,3}$ is a *counterexample* to the 2DC if G cannot be decomposed into a spanning tree and a matching. Moreover, we say that a counterexample G is *minimum* if $\varphi(G)$ is minimum among all counterexamples to the 2DC.

Theorem 1.5 (Structure of a Minimum Counterexample to 2DC) *Every minimum counterexample G to Conjecture 1.2 satisfies the following properties:*

(1) *G is simple and 2-edge-connected;*
(2) *the girth of G is at least 5;*

(3) *the distance between vertices in* $V_2(G)$ *is at least 3;*
(4) $G - V_2(G)$ *is connected;*
(5) *There is a cycle* $C \subseteq G$ *for which* $V(C) \subseteq V_3(G)$.

At a first glance, Theorems 1.3 and 1.5 seem unrelated. In what follows we explain a connection between them. Consider the graph parameter ρ defined over the graphs $G \in \mathcal{S}_{2,3}$ by

$$\rho(G) = |V_3(G)| - 2|V_2(G)|.$$

By Theorem 1.5 (3), if G is a minimum counterexample to Conjecture 1.2, then $\rho(G) \geq 0$. The main tool in the proof of Theorem 1.3 is Proposition 2.2, which implies that graphs G for which $\rho(G) = 0$ admit a decomposition into a forest and a matching, and hence may possibly be used as a base case in a proof of Conjecture 1.2 by induction on ρ.

Notation and Terminology

The terminology used in this is work is standard and we refer the reader to [5] for missing definitions. All graphs considered in here are finite and have no loops (but may contain parallel edges). Let $G = (V, E)$ be a graph. We define $V(G) = V$ and $E(G) = E$. Given a vertex $u \in V(G)$, we denote the degree of u by $d_G(u)$ and the neighborhood of u in G by $N_G(u)$ (when G is clear from the context, we may simply write $d(u)$ and $N(u)$). We denote the minimum degree of G by $\delta(G)$. We write $H \subseteq G$ to denote that H is a subgraph of G. Given two graphs G and H, we write $G \cup H$ to denote the graph $(V(G) \cup V(H), E(G) \cup E(H))$.

Given a set $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by the vertices in S and denote by $G - S$ the graph $(V(G) \setminus S, F)$, where $F = E(G) \setminus \{uv \in E(G) : u \in S\}$. When $S = \{u\}$, we may simple write $G - u$ instead of $G - \{u\}$. Given a set $F \subseteq E(G)$, we denote by $G \setminus F$ the graph $(V(G), E(G) \setminus F)$, and by $G + F$ the graph $(V(G), E(G) \cup F)$. When $F = \{e\}$, we may simply write $G \setminus e$ and $G + e$ instead of $G \setminus \{e\}$ and $G + \{e\}$, respectively.

A *path* in G is a sequence $u_1 u_2 \cdots u_\ell$ in which $u_i \neq u_j$ for all $i \neq j$ and $u_i u_{i+1} \in E(G)$ for $i = 1, 2, \ldots, \ell - 1$. A *cycle* in G is a sequence $u_1 u_2 \cdots u_\ell u_1$ with $\ell \geq 2$ in which $u_i \neq u_j$ for all $i \neq j$, $u_i u_{i+1} \in E(G)$ for $i = 1, 2, \ldots \ell$, where $u_{\ell+1} = u_1$, and $u_i u_{i+1} \neq u_j u_{j+1}$ for all $i \neq j$. Note that this includes cycles with two edges. Let $W = \{u_1, u_2, \ldots, u_\ell\}$ and $F = \{u_i u_{i+1} : i = 1, 2, \ldots, \ell - 1\}$. When convenient, we treat a path $P = u_1 u_2 \cdots u_\ell$ and a cycle $C = u_1 u_2 \cdots u_\ell u_1$ as being the subgraphs $P = (W, F)$ and $C = (W, F \cup \{u_\ell u_1\})$, respectively. Equivalently, a path is an acyclic connected graph with maximum degree at most 2 and a cycle is a connected 2-regular graph.

As usual, we say that a graph is *cubic* (resp. *subcubic*) if all its vertices have degree 3 (resp. at most 3).

2 Proof of Theorem 1.3

Let $\mathcal{H} \subset \mathcal{S}_{2,3}$ be the set of all simple graphs H in which $V_2(H)$ is a stable set and every vertex in $V_3(H)$ has precisely one neighbor in $V_2(H)$. One may regard a graph H in $\mathcal{H}$ as a graph obtained from a cubic graph H' containing a perfect matching M' by subdividing each edge of M' precisely once. In particular, $H - V_2(H)$ is 2-regular, as we state in the next remark.

Remark 2.1 If H is a graph in $\mathcal{H}$, then each component of $H - V_2(H)$ is a cycle.

Let H be a graph in $\mathcal{H}$. We refer to the cycles in $H - V_2(H)$ as the *basic cycles* of H. Note that the vertices of basic cycles of H define a partition of $V_3(H)$. Let $u \in V_2(H)$, and note that the neighbors x and y of u belong to basic cycles, say C and C', of H. If $C = C'$, then we say that the path $P = xuy$ is a *2-chord* of C. In this case, x and y are called the *ends* of P and u the *inner* vertex. If $C \neq C'$, then we say that u is a *connector*. In this case, we say that u *joins* C and C'. Moreover, we say that two connectors are *parallel* if they join the same pair of basic cycles; and a collection of connectors $\mathcal{C}$ of H is called *simple* if it contains no pair of parallel connectors.

We define the *basic cycles graph* (BC-*graph*, for short) of $H \in \mathcal{H}$, which we denote by $\tilde{H}$, as the graph whose vertices are the basic cycles of H and in which two vertices C, C' are adjacent whenever the graph H has a connector joining C and C'. Note that $\tilde{H}$ is connected because H is connected. Also, note that this definition ignores parallel connectors in the sense that a pair of parallel connectors yields only one edge in $\tilde{H}$. Given a collection $\mathcal{C}$ of connectors of H, we define the *underlying* BC-*graph* of $\mathcal{C}$, denoted by $\tilde{H}_{\mathcal{C}}$, as the spanning subgraph of $\tilde{H}$ in which two vertices C and C' are adjacent whenever there is a connector in $\mathcal{C}$ joining C and C'.

We refer to a decomposition of a graph G into a spanning forest F and a matching M as a 2-*decomposition* of G and we denote it by the ordered pair (F, M). Note that if a graph $G \in \mathcal{S}_{2,3}$ admits a 2-decomposition (F, M), then Conjecture 1.2 holds for G since we can complete F to a tree using edges of M. Given a 2-decomposition $\mathcal{D} = (F, M)$ of a graph $G \in \mathcal{S}_{2,3}$, we say that a vertex $u \in V(G)$ is a *full vertex* in $\mathcal{D}$ if every edge of G incident to u belongs to F.

The main result of this section (Theorem 1.3) is a consequence of the following result.

Proposition 2.2 *Let $\mathcal{C}$ be a simple collection of connectors of a graph $H \in \mathcal{H}$. If $\tilde{H}_{\mathcal{C}}$ is a forest, then H admits a 2-decomposition $\mathcal{D} = (F, M)$ such that each $u \in \mathcal{C}$ is either a full vertex in $\mathcal{D}$ or is adjacent to a full vertex in $\mathcal{D}$.*

Before we present the proof of Proposition 2.2, we show how it implies Theorem 1.3.

Proof of Theorem 1.3 The proof follows by induction on $|E(G)|$. Let $G \in \mathcal{S}_{2,3}$ so that $G - V_2(G)$ is a collection of thick cacti. The statement clearly holds for $|E(G)| \leq 3$, so we may assume that $|E(G)| \geq 4$.

First, suppose that G is not a simple graph. If there are three copies of an edge, then the cycle containing any two of these copies is not a separating cycle, a contradiction. Thus, we may assume that G has precisely two copies, say e and e', of an edge xy.

If $d(x) = 2$ and $d(y) = 3$, then y has degree 1 in $G - V_2(G)$, and hence $G - V_2(G)$ is not a collection of thick cacti, a contradiction. Thus, by symmetry, we may assume that $d(x) = d(y) = 3$. Let u (resp. v) be the neighbor of x (resp. y) distinct from y (resp. x). Since e and e' form a cycle, say C, the graph $G \setminus E(C)$ is disconnected. Note that $G' = G - \{x, y\} + \{uv\}$ is a graph in $\mathcal{S}_{2,3}$ and $G' - V_2(G)$ is a collection of thick cacti. Since $|E(G')| < |E(G)|$, by the induction hypothesis G' admits a decomposition into a spanning tree T' and a matching M'. We may assume that $uv \in E(T')$ because uv is a cut edge of G'. Therefore $\big(T' \setminus \{uv\} + \{ux, e, yv\}, M' \cup \{e'\}\big)$ is a 2-decomposition of G, as desired. Therefore, from now on, we assume that G is a simple graph.

In what follows, we obtain a graph in $\mathcal{H}$ and a simple collection of connectors satisfying the hypothesis of Proposition 2.2. Let H be the graph obtained from G by the following operations.

(1) replacing every path $P_j = ux_1x_2 \cdots x_kv$ with $k \geq 1$, $u, v \in V_3(G)$, and $x_i \in V_2(G)$ for $1 \leq i \leq k$, by a path uw_jv, where w_j is a new vertex (if $k = 1$ we just rename x_1); and
(2) subdividing once every edge of $G - V_2(G)$ that does not belong to a cycle of $G - V_2(G)$.

It is straightforward that $H \in \mathcal{H}$. Now, if $\mathcal{C}$ is the set of vertices added in step (2), then, since $G - V_2(G)$ is a collection of thick cacti, $\mathcal{C}$ is a simple collection of connectors of H whose underlying BC-graph $\tilde{H}_{\mathcal{C}}$ is a forest. By Proposition 2.2, the graph H admits a 2-decomposition $\mathcal{D} = (F, M)$ in which every vertex $u \in \mathcal{C}$ is a full vertex in $\mathcal{D}$ or is adjacent to a full vertex in $\mathcal{D}$.

Now, from (F, M), we obtain a 2-decomposition (F^*, M^*) of G. Let xy be an edge in M. If $d_H(x) = d_H(y) = 3$, then we put xy in M^*. Thus, we may assume that $d_H(x) = 2$. If x was added in (1), then, without loss of generality, there is a path $P_j = ux_1x_2 \cdots x_ky$ (with $k \geq 1$, $u, y \in V_3(G)$, $x_i \in V_2(G)$ for $1 \leq i \leq k$) that was replaced by the path uw_jy, where $w_j = x$. In this case, we put x_ky in M^*. If x was added in (2), then $x \in \mathcal{C}$. Since x is not a full vertex in $\mathcal{D}$, the vertex x must be adjacent to a full vertex in $\mathcal{D}$, say z, and so we put yz in M^*. Since M is a transversal of the cycles of H, i.e., M contains an edge in each cycle of H, by construction M^* is also a transversal of the cycles of G, which implies that $F^* = G \setminus E(M^*)$ is a forest.

Finally, to obtain a decomposition of G into a spanning tree and a matching from (F^*, M^*), one may find a minimal subset $S \subset M^*$ such that $F^* + S$ is connected. □

For the next result, we use the following notation to refer to the successor of a vertex in a cycle. Given a cycle $C = w_1w_2 \cdots w_pw_1$, for each $i \in \{1, \ldots, p\}$, we denote the successor w_{i+1} (where $w_{p+1} = w_1$) by w_i^+. Now we prove Proposition 2.2 by proving the following stronger statement.

Proposition 2.3 *Let $\mathcal{C}$ be a simple collection of connectors of a graph $H \in \mathcal{H}$. If $\tilde{H}_{\mathcal{C}}$ is a forest, then H admits a 2-decomposition $\mathcal{D} = (F, M)$ such that the following holds.*

(1) $|M \cap E(C)| = 1$ *for every basic cycle C of H; and*
(2) *each $u \in \mathcal{C}$ is either a full vertex in $\mathcal{D}$ or is adjacent to a full vertex in $\mathcal{D}$.*

Proof Let $\mathcal{C}$ and $H \in \mathcal{H}$ be as in the statement. The proof follows by induction on $n = |V(H)|$. Since $H \in \mathcal{H}$, it follows that $V_2(H)$ is a stable set and every vertex in $V_3(H)$ has exactly one neighbor in $V_2(H)$. Thus, we have $|V_3(H)| = 2|V_2(H)|$. If $|V_2(H)| = 1$, then $|V_3(H)| = 2$, and hence H has parallel edges, a contradiction. Therefore, we may assume $|V_2(H)| \geq 2$, which implies $|V_3(H)| \geq 4$. By Remark 2.1, $H - V_2(H)$ is a collection of (basic) cycles. First, suppose that H has exactly one basic cycle, say C. In this case, H has no connectors which implies that $\mathcal{C} = \emptyset$ and that every vertex in C is the end of a 2-chord. Let x_1 and x_2 be two adjacent vertices in C which are the ends of two distinct 2-chords in H. Let $x_1y_1z_1$ and $x_2y_2z_2$ be the 2-chords containing x_1 and x_2, respectively. For each $y \in V_2(H)\setminus\{y_1, y_2\}$, let e_y be an arbitrary edge incident to y, and let

$$M = \{x_1x_2, y_1z_1, y_2z_2\} \cup \left\{e_y : y \in V_2(H) \setminus \{y_1, y_2\}\right\} \text{ and } F = G \setminus M.$$

Note that (F, M) is a 2-decomposition of H as desired. Therefore, we may assume that H has at least two basic cycles.

In what follows, we say that a basic cycle C is of *type 1* if no 2-chord has both ends in C; otherwise, we say C is of *type 2*. The following claim on basic cycles of type 1 arises naturally. □

Claim 1 *If C is a basic cycle of type 1 in H, then C is a cut vertex of $\tilde{H}$.*

Proof Since C is of type 1, there are no 2-chords with ends in C, and since $H \in \mathcal{H}$, the graph $H \setminus E(C)$ is disconnected. Let H_1 and H_2 be two distinct components of $H \setminus E(C)$. Note that there is no connector joining a vertex in H_1 to a vertex in H_2, and hence if C_1 is a basic cycle in H_1 and C_2 in H_2, the edge $C_1C_2 \notin \tilde{H}$. Therefore, the vertex C is a cut vertex of $\tilde{H}$. □

Since $\tilde{H}$ is connected and $\tilde{H}_{\mathcal{C}} \subseteq \tilde{H}$ is a forest, there is a spanning tree T of $\tilde{H}$ such that $\tilde{H}_{\mathcal{C}} \subseteq T$. By Claim 1, the leaves of T are basic cycles of type 2. Now, let $\mathcal{C}^*$ be the collection of connectors so that $\tilde{H}_{\mathcal{C}^*} = T$. Since $\tilde{H}_{\mathcal{C}} \subseteq T = \tilde{H}_{\mathcal{C}^*}$, we may assume $\mathcal{C} \subseteq \mathcal{C}^*$. In what follows we prove that H admits a 2-decomposition $\mathcal{D} = (F, M)$ such that (a) $|M \cap E(C)| = 1$ for every basic cycle C of H; and (b) each $u \in \mathcal{C}^*$ is either a full vertex in $\mathcal{D}$ or is adjacent to a full vertex in $\mathcal{D}$. Note that, since $\mathcal{C} \subseteq \mathcal{C}^*$, (b) implies (2), and hence the result follows.

Let $V_2(H) = \{y_1, \ldots, y_\ell\}$ and, for each $y_i \in V_2(H)$, let x_i and z_i be the neighbors of y_i. Note that $V(H)$ is the disjoint union of the sets $\{x_i, y_i, z_i\}$ for $i \in \{1, \ldots, \ell\}$. Let C be a leaf of $T = \tilde{H}_{\mathcal{C}^*}$, and put

$$I = \{y_i \in V_2(H) : |\{x_i, z_i\} \cap V(C)| = 1\} \quad \text{and} \quad J = \{y_i \in V_2(H) : |\{x_i, z_i\} \cap V(C)| = 2\}.$$

We may assume, without loss of generality, that $x_i \in V(C)$ and $z_i \notin V(C)$ for every $y_i \in I$. Note that $I \neq \emptyset$, otherwise either T is disconnected or T has only one vertex, namely C, which implies that H has only one basic cycle, a contradiction. Thus, we may assume, without loss of generality, that $y_1 \in \mathcal{C}^*$ and $x_1 \in V(C)$. Let $C = u_1u_2 \ldots u_ku_1$, where $u_1 = x_1$. Now, we split the proof into two cases depending on whether the vertex x_1 is adjacent to a vertex with a neighbor that belongs to J.

Case 1. x_1 *is adjacent to a vertex with a neighbor that belongs to* J (see Fig. 1A).

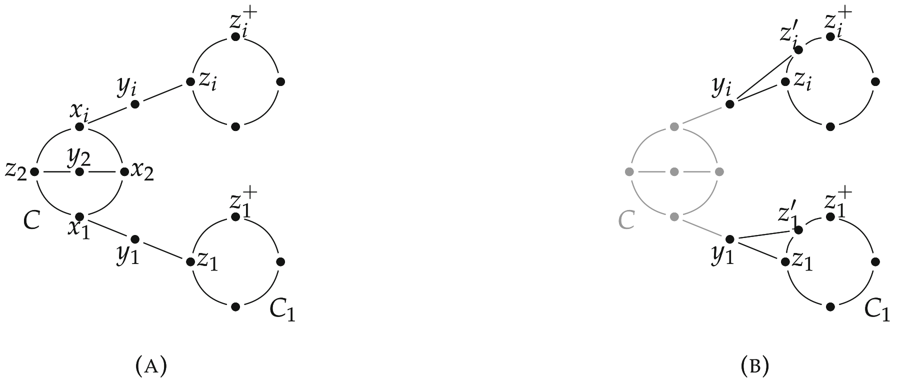


Fig. 1 Reduction from a graph H (**A**) to the graph H' (**B**) in Case 1. In (**B**), we use gray to indicate the elements from H that we removed to create H'

Suppose, without loss of generality, that $u_2 = x_2$ and $y_2 \in J$. In what follows, we obtain a graph $H' \in \mathcal{H}$ such that $|V(H')| < |V(H)|$. Let H' be the graph obtained from $H - V(C) - J$ by subdividing, for every $y_i \in I$, the edge $z_i z_i^+$, obtaining the vertex z_i', and adding the edge $y_i z_i'$ (see Fig. 1B). Note that $V_2(H') = V_2(H) \setminus J$.

Let $B \neq C$ be a basic cycle of H. Note that if $BC \notin E(\tilde{H})$, then B is a (basic) cycle in H'. On the other hand, if $BC \in E(\tilde{H})$, then B is not a cycle in H but a subdivision B' of B is a (basic) cycle in H'. Let $\varphi(B) = B$, if $BC \notin E(\tilde{H})$, and $\varphi(B) = B'$, otherwise. Now we show that φ is an isomorphism between $\tilde{H} - C$ and $\tilde{H}'$. It is not hard to see that, by the construction of H', the function φ is bijective. Now, if $XY \in E(\tilde{H} - C)$, then there is a connector, say y, joining the basic cycles X and Y in H. The only connectors affected by the construction of H' are those that contain an end in C, and since $X \neq C$ and $Y \neq C$, it follows that y is a connector joining $\varphi(X)$ to $\varphi(Y)$ in H'. Thus $\varphi(X)\varphi(Y) \in E(\tilde{H}')$. Now, suppose that $\varphi(X)\varphi(Y) \in E(\tilde{H}')$. By the construction of H', no connector is created. This implies that the set of connectors of H' is a subset of the set of connectors of H. Thus, if $\varphi(X)\varphi(Y) \in E(\tilde{H}')$, then there is a connector, say y, joining the basic cycles $\varphi(X)$ and $\varphi(Y)$. Since y is also a connector in H, $X \neq C$, and $Y \neq C$, it follows that $XY \in E(\tilde{H} - C)$. Therefore, $\tilde{H} - C$ and $\tilde{H}'$ are isomorphic.

Let $W = V(H) \setminus (I \cup J \cup V(C))$, and note that $V(H) = W \cup I \cup J \cup V(C)$. Moreover, note that $V(H') = W \cup I \cup \{z_i' : y_i \in I\}$. Since $J \neq \emptyset$, it follows that $|V(H')| < |V(H)|$. Now we show that $H' \in \mathcal{H}$. First, since C is a leaf of the spanning tree T of $\tilde{H}$, the graph $T - C$ is connected, and since $\tilde{H}'$ is isomorphic to $\tilde{H} - C$, we have that $T - C$ is a spanning tree of $\tilde{H}'$, and hence H' is connected. Also, by construction, H' is simple, each vertex in $V_3(H')$ has exactly one neighbor in $V_2(H')$, and $V_2(H')$ is a stable set. It remains to prove that every cycle in H' is a separating cycle. First, note that every basic cycle C' adjacent to C in $\tilde{H}$ yields a basic cycle of type 2 in $\tilde{H}'$. Moreover, note that a cycle containing a vertex with degree 2 or a basic cycle of type 2 is a separating cycle. Thus, we can focus on the basic cycles of type 1 in H'. Let C' be such a cycle. By Claim 1 the cycle C' is a cut vertex of $\tilde{H}$. By the construction of H', the graph $\tilde{H} - C$ is isomorphic to $\tilde{H}'$. Thus, C' is a cut vertex of $\tilde{H}'$, which implies that C' is a separating cycle of H'. Therefore, we conclude that $H' \in \mathcal{H}$.

By the induction hypothesis, the graph H' has a 2-decomposition (F', M') satisfying (1) and (2) with respect to the simple collection of connectors $\mathcal{C}^* \setminus \{y_1\}$. We now describe in four steps how to obtain the desired 2-decomposition (F, M) of H from (F', M') (see Fig. 2):

(a) We put x_1x_2 in M and all the edges of $C \setminus \{x_1x_2\}$ in F. We put y_2x_2 in F, y_2z_2 in M and, for each $y_i \in J \setminus \{y_2\}$, we put x_iy_i and y_iz_i in distinct elements of (F, M).
(b) We put x_1y_1 and y_1z_1 in F. In addition, for each $y_i \in I \setminus \{y_1\}$, we put x_iy_i in M and y_iz_i in F.
(c) We put each edge

$$e \in E(G) \setminus \Big(E(C) \cup \{y_ix_i, y_iz_i : y_i \in I \cup J\} \cup \{z_iz_i^+ : y_i \in I\}\Big)$$

in F if $e \in F'$. Otherwise, we put e in M.
(d) Finally, for each $y_i \in I$, we put $z_iz_i^+ \in E(G)$ in M if $z_i'z_i^+ \in M'$. Otherwise, we put $z_iz_i^+$ in F.

We claim that in steps (a)–(d), each edge in $E(H)$ has been put either in M or F. Indeed, in steps (a) and (b) we cover the edges in $E(C) \cup \{y_iz_i, y_ix_i : y_i \in I \cup J\}$, while the edges in $\{z_iz_i^+ : y_i \in I\}$ are covered in step (d), and all the remaining edges are covered in step (c). The following claim is useful.

Claim 2 *Edge $z_iz_i' \in E(F')$ for all $y_i \in I$.*

Proof Let B be the basic cycle of H' that contains z_iz_i'. If z_iz_i' belongs to M', then due to (1) all the edges in $E(B) \setminus \{z_iz_i'\}$ belong to F'. Hence, the cycle $\{z_iy_i, y_iz_i'\} \cup (E(B) \setminus \{z_iz_i'\})$ is contained in F', a contradiction. □

It is straightforward from the assignments in steps (a)-(d) that F is a forest and M is a matching (for the edges that were subdivided, item (1) with respect to M' and Claim 2 ensure that M is a matching and that F has no cycles). We now check that items (1) and (2) hold for (F, M). Due to step (a), we have that $|M \cap E(C)| = 1$ and due to steps (c)–(d), we have that $|M \cap E(C')| = 1$ for every other basic cycle C' of H with $C' \neq C$. Hence, (1) holds. Due to step (b), we have that y_1 is a full vertex in (F, M). Let $y \in \mathcal{C}^* \setminus \{y_1\}$. Since C is a leaf of $T = \tilde{H}_{\mathcal{C}^*}$, it follows that $y \notin I$. Thus, if y is full in (F', M'), then due to the step (c), it is also full in (F, M). So, suppose that y is not full in (F', M') and hence, it has a neighbor x which is a full vertex in (F', M'). Let C' be the basic cycle of H that contains x. If $x \neq z_i^+$ for each $y_i \in I \setminus \{y_1\}$, then due to step (c), we have that x is a full vertex in (F, M) as well. Suppose that $x = z_i^+$ for some $i \in I \setminus \{y_1\}$. Since x is a full vertex in (F', M'), we have that xy, xz_i' and xx' belong to F', where x' is the neighbor of x distinct of z_i' in the basic cycle C'. Due to step (c), xy and xx' belong to F, and due to step (d) the edge $z_iz_i^+$ belongs to F (since $z_i'x \in E(F')$). Therefore item (2) holds. This finishes the proof of Case 1.
Case 2. *x_1 is not adjacent to a vertex with a neighbor in J.*

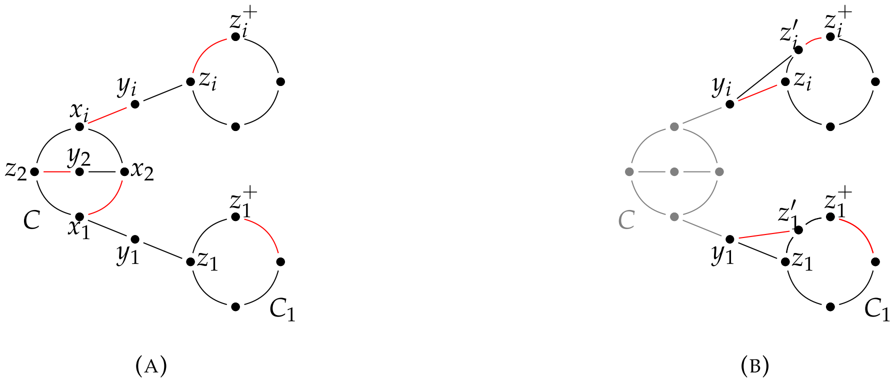


Fig. 2 Reduction from a graph H (**A**) to the graph H' (**B**) in Case 1. In both figures, for a 2-decomposition (F, M), the edges in F (resp. M) are colored black (resp. red). In (**B**), we use gray to indicate the elements from H that we removed to create H'

In this case, both neighbors of x_1 in C, namely u_2 and u_k, are adjacent to a vertex in I. Let C_1 be the basic cycle that contains z_1 and let ℓ be the smallest $i \in \{1, \ldots, k\}$ for which u_{i+1} is an end of a 2-chord. We may assume, without loss of generality, that $u_\ell = x_2$ and $u_{\ell+1} = x_3$. Let C_2 be the basic cycle that contains z_2 (possibly $C_1 = C_2$).

In what follows, analogously to Case 1, we obtain a graph $H' \in \mathcal{H}$ such that $|V(H')| < |V(H)|$. Let H' be the graph obtained from $H - V(C) - J$ by (i) identifying the vertices y_1 and y_2 into a new vertex y, and (ii) for every $y_i \in I \setminus \{y_1, y_2\}$, subdividing the edge $z_i z_i^+$, obtaining the vertex z_i', and adding the edge $y_i z_i'$ (see Fig. 3). Now, note that $V_2(H') = \{y\} \cup \big(V_2(H) \setminus (J \cup \{y_1, y_2\})\big)$.

Let $B \neq C$ be a basic cycle of H. Note that if $BC \notin E(\tilde{H})$, then B is a (basic) cycle in H'. On the other hand, if $BC \in E(\tilde{H})$, then B is not a cycle in H', but one subdivision B' of B is. Let $\varphi(B) = B$, if $BC \notin E(\tilde{H})$, and $\varphi(B) = B'$, otherwise. Now we show that, if $C_1 = C_2$, then φ is an isomorphism between $\tilde{H}'$ and $\tilde{H} - C$; otherwise, we show that φ is an isomorphism between $\tilde{H}'$ and $\tilde{H} - C + C_1C_2$ (here, we only add the edge C_1C_2 to $\tilde{H} - C$ if this action results in a simple graph). It is not hard to check that, by the construction of H', the function φ is bijective. If $XY \in E(\tilde{H} - C)$, then there is a connector, say y', joining the basic cycles X and Y in H. The only connectors affected by the construction of H' are those that contain an end in C and, since $X \neq C$ and $Y \neq C$, it follows that y' is a connector joining $\varphi(X)$ to $\varphi(Y)$ in H'. If $C_1 \neq C_2$, then, by the construction of H', the vertex y is a connector in H' joining the basic cycles $\varphi(C_1)$ and $\varphi(C_2)$ and, as result, $\varphi(C_1)\varphi(C_2) \in E(\tilde{H}')$. Now, suppose that $\varphi(X)\varphi(Y) \in E(\tilde{H}')$, and hence there exists a connector y' in H' joining $\varphi(X)$ and $\varphi(Y)$. By the construction of H', the vertex y is the only connector that we can create, so the set of connectors of H' distinct from y is a subset of the set of connectors of H. If $|\{X, Y\} \cap \{C_1, C_2\}| < 2$, then $y' \neq y$, and hence y' is a connector in H joining X and Y, and hence $XY \in E(\tilde{H} - C)$. Now, $|\{X, Y\} \cap \{C_1, C_2\}| = 2$, then, by construction, $XY \in \tilde{H} - C + C_1C_2$. Therefore, φ is an isomorphism between $\tilde{H}'$ and $\tilde{H} - C$, if $C_1 = C_2$, or between $\tilde{H}'$ and $\tilde{H} - C + C_1C_2$, otherwise.

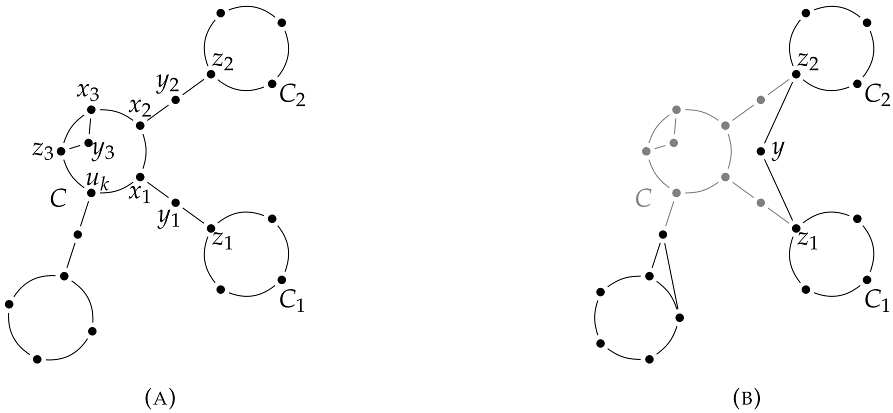
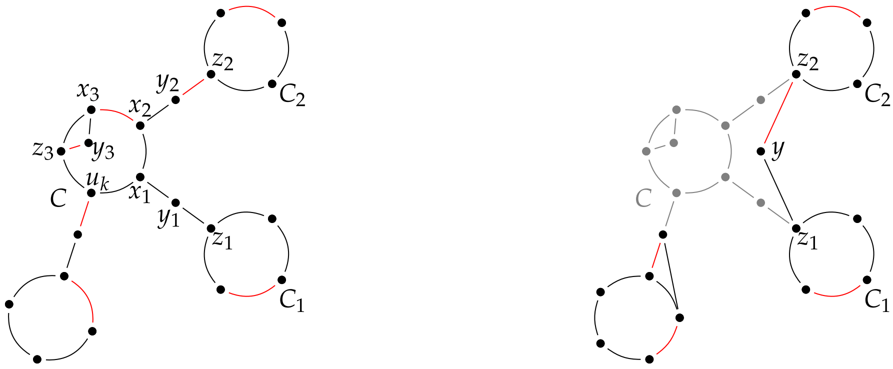


Fig. 3 Reduction from a graph H (**A**) to the graph H' (**B**) in Case 2. In (**B**), we use gray to indicate the elements from H that we removed to create H'

Let $W = V(H)\backslash(I \cup J \cup V(C))$, and hence $V(H) = W \cup I \cup J \cup V(C)$. Moreover, note that $V(H') = W \cup (I\backslash\{y_1, y_2\}) \cup \{z'_i : y_i \in I\backslash\{y_1, y_2\}\} \cup \{y\}$. It follows that $|V(H')| < |V(H)|$. Now we claim that $H' \in \mathcal{H}$. First, since C is a leaf of a spanning tree T of $\tilde{H}$, and either $\tilde{H} - C$ or $\tilde{H} - C + C_1C_2$ is isomorphic to $\tilde{H}'$, it follows that $T - C$ is a spanning tree of $\tilde{H}'$, and hence H' is connected. Also, by construction, H' is simple, each vertex in $V_3(H')$ has exactly one neighbor in $V_2(H')$, and $V_2(H')$ is a stable set. It remains to prove that every cycle in H' is a separating cycle. Again, if a cycle $C' \subset H'$ contains a vertex in $V_2(H')$ or has a 2-chord, then C' is a separating cycle. Thus, we can assume that $V(C') \subseteq V_3(H')$ and that C' has no 2-chords. By the construction of H', C' must be a basic cycle of type 1 in H, and so by Claim 1, C' is a cut vertex of $\tilde{H}$. Now we show that C' is a cut vertex in $\tilde{H}'$. Let H_1 be the component in $\tilde{H} - C'$ containing the vertex C. Note that, if $C_i \in V(\tilde{H} - C')$ for $i \in \{1, 2\}$, then C_i belongs to H_1. Since $\tilde{H}'$ is isomorphic either to $\tilde{H} - C$ or to $\tilde{H} - C + C_1C_2$, to show that C' is a cut vertex in $\tilde{H}'$ it is sufficient to show that C' has neighbor in $V(H_1 - C)$ in the graph $\tilde{H}'$. If $C'C \notin E(\tilde{H})$, then clearly C' has neighbor in $V(H_1 - C)$. Thus, we may assume that $C'C \in E(\tilde{H})$. Now, note that $C' \in \{C_1, C_2\}$ and $C_1 \neq C_2$, otherwise, by the construction of H', the cycle C' would be a basic cycle of type 2. Suppose, without loss of generality, that $C' = C_1$. Therefore, by the construction of H', the edge $C'C_2 \in E(\tilde{H}')$. Hence C' has a neighbor in $V(H_1 - C)$ in the graph $\tilde{H}'$, which implies that C' is a cut vertex in $\tilde{H}'$ and, consequently, that C' is a separating cycle in H'.

By induction hypothesis, the graph H' admits a 2-decomposition (F', M') satisfying (1) and (2) with respect to the simple collection of connectors $\mathcal{C}^* \setminus \{y_1\}$. In what follows, we obtain from (F', M') a 2-decomposition (F, M) of G as desired (see Fig. 4).

(a) We put x_2x_3 in M and all the edges of $C \setminus x_2x_3$ in F. We put y_3x_3 in F, y_3z_3 in M and, for each $y_i \in J\backslash\{y_3\}$, we put x_iy_i and y_iz_i in distinct elements of (F, M).
(b) We put x_1y_1 and x_2y_2 in F. In addition, for each $y_i \in I \setminus \{y_1, y_2\}$, we put x_iy_i in M and y_iz_i in F.



(A) (B)

Fig. 4 Reduction from a graph H (**A**) to the graph H' (**B**) in Case 2. In both figures, for a 2-decomposition (F, M), the edges in F (resp. M) are colored black (resp. red). In (**B**), we use gray to indicate the elements from H that we removed to create H'

(c) We put each edge

$$e \in E(G) \setminus \left(E(C) \cup \left\{z_i z_i^+ : y_i \in I \setminus \{y_1, y_2\}\right\} \cup \{y_i z_i, y_i x_i : y_i \in I \cup J\}\right)$$

in F if $e \in E(F')$. Otherwise, we put e in M.

(d) For each $y_i \in I \setminus \{y_1, y_2\}$, we put $z_i z_i^+$ in M if $z_i' z_i^+ \in M'$. Otherwise, we put $z_i z_i^+$ in F.

(e) Finally, for each $y_i \in \{y_1, y_2\}$, we put $y_i z_i$ in F if $y z_i \in E(F')$. Otherwise, we put $y_i z_i$ in M.

We now show that (F, M) is a 2-decomposition of H as desired. It is straightforward that each edge of $E(H)$ is either in F or M. Analogously to **Case 1**, the following claim arises (the same proof applies).

Claim 3 *Edge* $z_i z_i' \in E(F')$ *for all* $y_i \in I \setminus \{y_1, y_2\}$.

From the assignments in steps (a)–(e), since item (1) holds for H', and by Claim 3, it is clear that M is a matching. We now check that F is a forest. Due to the assignments in steps (a)–(d), it is clear that F is a forest in the graph $G \setminus \{y_1 z_1, y_2 z_2\}$. The assignments in step (e) ensure that edges $y_1 z_1, y_2 z_2$ are assigned to F or M without creating cycles in F: both $y_1 z_1, y_2 z_2$ are put in F if $y z_1, y z_2$ are in F', and one of them $y_1 z_1$, or $y_2 z_2$ is in M whenever its copy in H', namely $y z_1$ or $y z_2$, is in M'. Note that item (1) holds due to the assignments in steps (a) and (c), and that item (2) follows from the assignments in steps (a)–(c) and Claim 3. This finishes the proof of Case 2, and concludes the proof of Proposition 2.3. □

3 Structural Properties of a Minimum Counterexample

In this section, we study the structure of a minimum counterexample to Conjecture 1.2. In fact, in order to avoid technical issues, we work with the following conjecture, which is equivalent to Conjecture 1.2, as we show in Proposition 3.2.

Conjecture 3.1 *Every graph in $\mathcal{S}_{1,3}$ can be decomposed into a spanning tree and a matching.*

Conjecture 3.1 is more convenient in a proof by induction or minimum counterexample, since it allows vertices with degree 1 without leaving the studied class. However, the properties of a minimum counterexample to Conjecture 3.1 can be easily transferred to a minimum counterexample to Conjecture 1.2 (see *Proof of Theorem* 1.5 below).

Let $\mathfrak{C}$ be the set of all counterexamples for Conjecture 3.1, i.e., the set of graphs in $\mathcal{S}_{1,3}$ that cannot be decomposed into a forest and a matching. Recall from (1.4) that $\varphi(G) = |V(G)| + |E(G)|$. Let $\varphi^* = \min\{\varphi(G) \colon G \in \mathfrak{C}\}$, and let $\mathfrak{M} = \{G \in \mathfrak{C} \colon \varphi(G) = \varphi^*\}$. So $\mathfrak{M}$ is the set of all the minimum counterexamples for Conjecture 3.1 according to the function φ.

First, we show that Conjectures 3.1 and 1.2 are equivalent.

Proposition 3.2 *Conjecture 3.1 is equivalent to Conjecture 1.2.*

Proof Clearly, since $\mathcal{S}_{2,3} \subseteq \mathcal{S}_{1,3}$, Conjecture 1.2 holds if Conjecture 3.1 holds. Thus, suppose that Conjecture 1.2 holds and, towards a contradiction, suppose that Conjecture 3.1 does not. Let $G \in \mathfrak{M}$ and suppose that G contains a vertex u with degree 1. Let $G' = G - u$, and note that G' is a connected subcubic graph in $\mathcal{S}_{1,3}$ for which $\varphi(G') < \varphi(G)$. Thus, by the minimality of G, the graph G' contains a 2-decomposition (T', M'). Let v be the only neighbor of u in G. Since T' is spanning, $v \in V(T')$. Thus $T = T' + uv$ is a spanning tree of G and (T, M') is a 2-decomposition of G, a contradiction. Thus, we may assume that $V_1(G) = \emptyset$, and so G is a graph in $\mathcal{S}_{2,3}$ that does not admit a 2-decomposition, a contradiction. □

The main theorem of this section is the following.

Theorem 3.3 (Structure of a Minimum Counterexample to Conjecture 3.1) *Every graph $G \in \mathfrak{M}$ satisfies the following properties:*

(1) *G is simple and 2-edge-connected;*
(2) *the girth of G is at least 5;*
(3) *the distance between vertices in $V_2(G)$ is at least 3;*
(4) *$G - V_2(G)$ is connected;*
(5) *There is a cycle $C \subseteq G$ for which $V(C) \subseteq V_3(G)$.*

Before proving Theorem 3.3, we show that it implies Theorem 1.5.

Proof of Theorem 1.5 Suppose G is a minimum counterexample to the 2DC and does not satisfy the properties stated in Theorem 1.5. Hence, G is in $\mathfrak{C}$ but is not in $\mathfrak{M}$. Therefore, there is a graph $G' \in \mathfrak{M}$ such that $\varphi(G') < \varphi(G)$. Due to Claim 1 (see below), G' has minimum degree at least 2, and hence G' is a minimum counterexample to the 2DC, a contradiction to the choice of G. □

Let G be a graph and let $S \subseteq V(G)$. The *shrink* of S in the graph G, denoted by G/S, is the graph obtained from G by identifying the vertices of S and removing the possible loops. Given a path P, the *length* of P is its number of edges. A *shortest* path joining vertices u and v is a path joining u and v with minimum length among all such paths; and the *distance* between u and v, denoted by $\mathrm{dist}_G(u, v)$, is the length of such a shortest path—again, when G is clear from the context, we may drop the subscript.

Proof of Theorem 3.3 Let $G \in \mathfrak{M}$. First, we show that

Claim 1 *G is 2-edge-connected.*

Proof Towards a contradiction, suppose that G contains a cut edge $e = uv$. Let $G' = G \setminus e$, and let G'_u and G'_v, respectively, be the component of G' containing the vertex u and v. Clearly, for all $x \in \{u, v\}$, the graph G'_x is a connected subcubic graph with $\varphi(G'_x) < \varphi(G)$. Moreover, note that $d_{G'_x}(x) \leq 2$, and hence every cycle containing the vertex x in G'_x is a separating cycle. Thus, it is not hard to check that $G'_x \in \mathcal{S}_{1,3}$, and hence, by the minimality of G, there are 2-decompositions (T'_u, M'_u) and (T'_v, M'_v) of G'_u and G'_v, respectively. Clearly, $(T'_u \cup e \cup T'_v, M'_u \cup M'_v)$ is a 2-decomposition of G, a contradiction to the choice of G. □

Note that since G has maximum degree at most 3, Claim 1 implies that G is 2-connected, and hence has no cut vertex. Now, we can prove that G is a simple graph.

Claim 2 *G contains no parallel edges.*

Proof Towards a contradiction, suppose that G contains parallel edges. Let $e = uv$ and $f = uv$ be two parallel edges. The graph G cannot have another parallel edge $f' = uv$, otherwise the 2-cycle $e \cup f$ would not be a separating cycle. If $d(u) = d(v) = 2$, then G is a 2-cycle, a contradiction to the choice of G. Thus, we may assume that $d(u) = 3$ and that u has a neighbor w distinct of v. Since $G \in \mathcal{S}_{1,3}$ and $e \cup f$ is a cycle, the graph $G \setminus \{e, f\}$ is disconnected, and hence uw is a cut edge, a contradiction to Claim 1. □

Claim 3 *G is triangle-free.*

Proof Towards a contradiction, suppose that G contains a triangle $H = xyzx$. By Claim 2, G is simple. We may assume that there is a vertex in $V(H)$, say x, that has degree 3 in G, otherwise G would be a triangle, a contradiction to the choice of G. If $d(y) = d(z) = 2$, then x is a cut vertex, a contradiction to Claim 1. Thus, we may assume, without loss of generality, that $d(y) = 3$. The remaining proof is divided in two cases, depending on whether $d(z) = 2$ or $d(z) = 3$.

First, suppose that $d(z) = 2$. Let $G' = G/V(H)$ and name u the vertex yielded by the shrink of $V(H)$. Note that G' is a connected subcubic graph for which $\varphi(G') < \varphi(G)$. Moreover, note that $d_{G'}(u) = 2$, and hence every cycle in G' containing u is separating. Thus, it is not hard to see that $G' \in \mathcal{S}_{1,3}$, and hence, by the minimality of G, it follows that there exists a 2-decomposition (T', M') of G'. For $v \in \{x, y\}$, let e_v be the edge in G incident to v that is not in $E(H)$, and let e'_v be the edge in G' yielded from e_v by the shrinking of $V(H)$. If $\{e'_x, e'_y\} \subseteq E(T')$, then let $T = T' \setminus \{e'_x, e'_y\} +$

$\{e_x, e_y, xy, yz\}$, and hence $(T, M' \cup \{xz\})$ is a 2-decomposition of G, a contradiction to the choice of G. Thus, we may assume, without loss of generality, that $e'_x \in M'$, and hence $e'_y \in E(T')$. Let $T = T' \setminus \{e'_y\} + \{e_y, xy, xz\}$ and $M = (M' \backslash \{e'_x\}) \cup \{e_x, yz\}$, and hence (T, M) is a 2-decomposition of G, a contradiction to the choice of G.

Now, suppose that $d(z) = 3$. Since $G \in \mathcal{S}_{1,3}$, the graph $G' = G \setminus E(H)$ is disconnected. It not hard to see that there exists a component in G' that contains only one vertex, say x, in the triangle H. Thus, x is a cut vertex of G, a contradiction to Claim 1. □

Claim 4 *If* $u, v \in V_2(G)$, *then* $\operatorname{dist}(u, v) \geq 2$.

Proof Towards a contradiction, suppose that $e = uv$ is an edge of G. By Claim 2, we have $|N(u)| \geq 2$. Thus, we may assume that $N(u) = \{v, x\}$ and $N(v) = \{u, y\}$ (possibly $x = y$). Let $G' = G/\{u, v\}$, and let w be the vertex yielded by the contraction of the edge e. Clearly G' is a connected subcubic graph for which $\varphi(G') < \varphi(G)$. Since $d_{G'}(w) = 2$, every cycle in G' containing w is separating. Thus, it is not hard to see that $G' \in \mathcal{S}_{1,3}$, and by the minimality of G, there exists a 2-decomposition (T', M') of G'. Let T be the spanning tree of G obtained from T' by replacing the edges incident to w to its corresponding edges in G (the edges incident to either u or v), and by adding the edge uv to T. Therefore (T, M') is a 2-decomposition of G, a contradiction to the choice of G. □

Claim 5 *If* $u \in V_2(G)$ *and* $N(u) = \{x, y\}$, *then* $N(x) \cap N(y) = \{u\}$.

Proof By Claim 2, we may assume that G is simple, thus $x \neq y$, and by Claim 4, we may assume that $d(x) = d(y) = 3$. By Claim 3, G is triangle-free, and hence the edge $xy \notin E(G)$.

First, we show that $|N(x) \cap N(y)| < 3$. Towards a contradiction, suppose that $N(x) = N(y) = \{u, v, w\}$. Note that $vw \notin E(G)$, since G is triangle-free. Let $S = \{u, x, y, v, w\}$, and note that $G[S]$ is isomorphic to $K_{2,3}$. We may assume, without loss of generality, that $d(v) = d(w) = 3$, otherwise either G would be isomorphic to $K_{2,3}$, a contradiction to the choice of G, or G would have a cut vertex, a contradiction to Claim 1. For $z \in \{v, w\}$, let e_z be the edge incident to z not in $E(G[S])$. Let $G' = G/S$, and note that G' is a connected subcubic graph for which $\varphi(G') < \varphi(G)$. Let u' be the vertex in G' yielded by the shrink of the set S, and note that u' has degree 2 in G', and hence, every cycle in G' containing the vertex u' is separating. Thus, it is not hard to see that $G' \in \mathcal{S}_{1,3}$, and hence, by the minimality of G, there exists a 2-decomposition (T', M') of G'. Let e'_v and e'_w be the edge of G' yielded from the edge e_v and e_w, respectively, by the shrink of S. If $\{e'_v, e'_w\} \subseteq E(T')$, then let $T = T' \setminus \{e'_v, e'_w\} + \{e_v, e_w, wy, yv, wx, xu\}$, and hence $(T, M' \cup \{uy, xv\})$ is a 2-decomposition of G, a contradiction to the choice of G. Thus, we may assume, without loss of generality, that $e'_v \in E(T')$ and $e'_w \in M'$. Therefore, let $T = T' \setminus \{e'_v, e'_w\} + \{e_v, e_w, vy, yw, wx, xu\}$, and hence $\big(T, (M' \backslash \{e'_w\}) \cup \{e_w, uy, vx\}\big)$ is a 2-decomposition of G, a contradiction to the choice of G.

Now, we show that $|N(x) \cap N(y)| < 2$. Towards a contradiction, suppose that $N(x) \cap N(y) = \{u, v\}$. Let $S = \{u, x, y, v\}$, and note that $G[S]$ is isomorphic to C_4. For $z \in \{x, y, v\}$, let e_z be the edge incident to z in G not in $E(G[S])$ (when $z = v$, it is possible that the edge e_z does not exist, in this case, let e_z undefined). Let $G' = G/S$, and note that G' is a connected subcubic graph for which $\varphi(G') < \varphi(G)$. Let u' be the vertex in G' yielded by the shrink of the set S. The remaining proof is divided into two cases depending on whether u' has degree 2 or 3 in G'.

First, suppose that u' has degree 2 in G', and hence, v has degree 2 in G. Note that every cycle in G' containing the vertex u' is separating. Thus, it is not hard to see that $G' \in \mathcal{S}_{1,3}$, and hence, by the minimality of G, there exists a 2-decomposition (T', M') of G'. Let e'_x, e'_y, be the edges of G' yielded from the edges e_x, e_y, respectively, by the shrink of S. Let T be the subgraph of G obtained from T' by replacing each of its edges of the type e'_z for e_z, and by adding the edges xv, vy, and yu. Let M be the subgraph of G obtained from M' by replacing each of its edges of the type e'_z for e_z, and by adding the edge ux. It is not hard to check that (T, M) is a 2-decomposition of G, a contradiction to the choice of G.

Now, suppose that u' has degree 3 in G', and hence v has degree 3 in G. Let a, b, c be the neighbors of x, y, v, respectively, in G which is not in S. Since G is triangle-free, a, b, c are three distinct vertices. Let $G' = G - S + bc$ and note that G' is a subcubic graph for which $\varphi(G') < \varphi(G)$. Moreover, note that G' is connected, otherwise, since b and c are in the same component of G', x would be a cut vertex of G, a contradiction to Claim 1.

Now, we claim that $G' \in \mathcal{S}_{1,3}$. Towards a contradiction, suppose that G' contains a non-separating cycle C'. It is not hard to check that $bc \in E(C')$. Let $P = C' \setminus bc$. Since C' is a non-separating cycle, there is a path $Q_{z,w}$ in $G' \setminus E(C') = G - S \setminus E(P)$ for every pair of vertices in $z, w \in V(G)\setminus S$. Let $C = P \cup cvyb$ and note that C is a cycle of G. Now we show that C is a non-separating cycle in G. Since there is the path $Q_{z,w}$ in $G - S \setminus E(P) \subseteq G \setminus E(C)$ for every pair of vertices $z, w \in G - S$, to prove that $G \setminus E(C)$ is connected, it is sufficient to show that there is a path in $G \setminus E(C)$ from every vertex in S to $c \in V(G)\setminus S$. Since $G' \setminus E(C')$ is connected, there is a path R in $G' \setminus E(C') = G - S \setminus E(P)$ joining a and c. Moreover, note that $G \setminus E(C)$ contains a path from a to every vertex in S. Therefore, $G \setminus E(C)$ is connected, a contradiction to the choice of G.

Therefore, we may assume that $G \in \mathcal{S}_{1,3}$, and hence, by the minimality of G, there is a 2-decomposition (T', M') of G'. If $bc \in E(T')$, then let $T = T' \setminus \{bc\} + \{cv, vy, yb, ax, xa\}$, and hence $(T, M' \cup \{vx, yu\})$ is a 2-decomposition of G, a contradiction to the choice of G. Thus, we may assume that $bc \in M'$, and hence let $T = T' + \{ax, xv, vy, yu\}$, and hence $\left(T, (M'\setminus\{e\}) \cup \{cv, by, xu\}\right)$ is a 2-decomposition of G, a contradiction to the choice of G. □

Finally, we can prove that G has girth at least 5 and that the distance between any two vertices in $V_2(G)$ is at least 3.

Claim 6 *G has girth at least* 5.

Proof By Claim 2, G is simple, and by Claim 3, the girth of G is a least 4. Towards a contradiction, suppose that there is a cycle $C \subseteq G$ with length 4. Note that C is an induced cycle. By Claim 5, it follows that $d_G(u) = 3$ for every $u \in V(C)$.

Let $G' = G \setminus E(C)$. Since $G \in \mathcal{S}_{1,3}$, the graph G' is disconnected. If there is a component H of G' such that $V(H) \cap V(C) = \{u\}$, then u is a cut vertex in G, a

Fig. 5 Illustration of cases (1) and (2) in the proof of Claim 6.

contradiction to Claim 1. Thus, it follows that G' has precisely two components H_1 and H_2, and $|V(H_1) \cap V(C)| = |V(H_2) \cap V(C)| = 2$. Let $V(H_1) \cap V(C) = \{u_1, u_2\}$ and $V(H_2) \cap V(C) = \{u_3, u_4\}$. For $i \in \{1, 2, 3, 4\}$, let $u_i v_i \in E(G \setminus E(C))$.

Note that there are two possible cases: (i) $\text{dist}_C(u_1, u_2) = 1$ or (ii) $\text{dist}_C(u_1, u_2) = 2$. In the first case, suppose, without loss of generality, that $C = u_1u_2u_3u_4u_1$ and, in the second, that $C = u_1u_3u_2u_4u_1$ (see Fig. 5). Note that in both cases the graph G contains the paths $v_1u_1u_4v_4$ and $v_2u_2u_3v_3$.

Let $G'' = G - V(C) + \{v_1v_4, v_2v_3\}$ and note that G'' is a subcubic connected graph. To see that every cycle in G'' is separating, suppose the contrary and let C'' be a non-separating cycle of G''. Since v_1v_4, v_2v_3 is an edge cut, it follows that either C'' contains both of these edges or none of them. If it contains both of them, then clearly C'' is a separating cycle. If it contains none of them, then $C'' \subseteq G$ and it is not hard to check that C'' would be a non-separating cycle in G, a contradiction to $G \in \mathcal{S}_{1,3}$. Therefore, $G'' \in \mathcal{S}_{1,3}$. Moreover, note that $\varphi(G'') < \varphi(G)$ and hence, by the minimality of G, there is a 2-decomposition (T'', M'') of G''.

First, suppose that $|E(T'') \cap \{v_1v_4, v_2v_3\}| = 1$. Suppose, without loss of generality, that $v_1v_4 \in E(T'')$ and $v_2v_3 \in M''$. If $C = u_1u_2u_3u_4u_1$, then let $T = T'' \setminus v_1v_4 + \{u_1v_1, u_1u_2, u_2u_3, u_3u_4, u_4v_4\}$. Otherwise, we have $C = u_1u_3u_2u_4u_1$, and let $T = T'' \setminus v_1v_4 + \{u_1v_1, u_1u_3, u_2u_3, u_2u_4, u_4v_4\}$. Let $M = (M''\setminus\{v_2v_3\}) \cup \{u_1u_4, u_2v_2, u_3v_3, \}$. Therefore, (T, M) is a 2-decomposition of G, a contradiction to the choice of G.

Now, suppose that $|E(T'') \cap \{v_1v_4, v_2v_3\}| = 2$. Let

$$T = T'' \setminus \{v_1v_4, v_2v_3\} + \{u_1v_1, u_1u_4, u_4v_4, u_2v_2, u_2u_3, u_3v_3\}.$$

If $C = u_1u_2u_3u_4u_1$, then let $M = M'' \cup \{u_1u_2, u_3u_4\}$. Otherwise, we have case $C = u_1u_3u_2u_4u_1$, and let $M = M'' \cup \{u_1u_3, u_2u_4\}$. Hence, (T, M) is a 2-decomposition of G, a contradiction to the choice of G. □

Claim 7 *If* $u, v \in V_2(G)$, *then* $\text{dist}(u, v) \geq 3$.

Proof By Claim 2, we may assume that G is simple. Let x and y be the two neighbors of u. By Claims 1 and 4, $\text{dist}(u, v) \geq 2$, and hence $d(x) = d(y) = 3$. Let $N(x) = \{u, x_1, x_2\}$ and $N(y) = \{u, y_1, y_2\}$. By Claim 5, the vertices x_1, x_2, y_1, y_2 are all distinct. Towards a contradiction, suppose that $d(u, v) = 2$. We may assume, without loss of generality, that $v = x_1$. Let $G' = (G - x)/\{x_1, u\}$, and let u' be the vertex in G' yielded by the shrink of the set $\{x_1, u\}$. Note that G' is a subcubic graph for which $\varphi(G') < \varphi(G)$. Moreover, note that G' is connected, otherwise x is a cut vertex of G, a contradiction to Claim 1. Since u' has degree 2 in G', every cycle in G' containing u' is a separating cycle. It is not hard to check that in fact $G' \in \mathcal{S}_{1,3}$. Therefore, by the minimality of G, there exists a 2-decomposition (T', M') of G'. Let e_{x_1} and e_u be the edges in G incident to x_1 and u, respectively, which is not incident to x. Let e'_{x_1} and e'_u be the edge yielded from e_{x_1} and e_u, respectively, by the shrink of $\{u, x_1\}$ in G. The remaining proof is divided into two cases depending on whether $|E(T') \cap \{e'_{x_1}, e'_u\}|$ is 1 or 2.

First, suppose that $|E(T') \cap \{e'_{x_1}, e'_u\}| = 1$. Suppose, without loss of generality, that $e'_{x_1} \in E(T')$ and $e'_u \in M'$. Thus, let $T = T' \setminus \{e'_{x_1}\} + \{e_{x_1}, x_2x, xu\}$, and hence $\big(T, (M' \backslash \{e'_u\}) \cup \{e_u, xx_1\}\big)$ is a 2-decomposition of G, a contradiction to the choice of G. Now, suppose that $|E(T') \cap \{e'_{x_1}, e'_u\}| = 2$. Thus $d_{T'}(u') = 2$, the graph $T' - u'$ has two components. Let $T' = T'_{x_1} \cup T'_u$ where $V(T'_{x_1}) \cap V(T'_u) = \{u'\}$, $e'_{x_1} \in E(T'_{x_1})$, and $e'_u \in E(T'_u)$. Suppose, without loss of generality, that $x_2 \in V(T'_{x_1})$, and let $T = T' \setminus \{e'_{x_1}, e'_u\} + \{e_{x_1}, e_u, x_2x, xu\}$. Hence, $(T, M' \cup \{x_1x\})$ is a 2-decomposition of G, a contradiction. □

In what follows we prove the last two properties, namely that $G - V_2(G)$ is connected and that G has a cycle containin only vertices with degree 3. For that, let $V_2(G) = \{v_1, v_2, \ldots, v_\ell\}$ and for each $v_i \in V_2(G)$, let $N(v_i) = \{u_i, u'_i\}$.

Claim 8 *$G - V_2(G)$ is connected.*

Proof Let $v_i \in V_2(G)$. Since G is triangle-free, $u_iu'_i \notin E(G)$. Let $G' = G - v_i + u_iu'_i$ and note that G' is a subcubic connected graph. If every cycle in G' is separating, then, by the minimality of G, there is a 2-decomposition (T', M') of G. If $u_iu'_i \in M'$, then $(T' + u_iv_i, M' \cup \{u'_iv_i\})$ is a 2-decomposition of G and, if $u_iu'_i \in E(T')$, then $(T' + \{u_iv_i, u'_iv_i\}, M')$ is a 2-decomposition of G. In both cases we have a contradiction to the choice of G.

Thus, we may assume that G' contains a non-separating cycle C. Note that $u_iu'_i \in E(C)$. Moreover, all vertices in C have degree 3 in G', and so in G, since $d_{G'}(v) = d_G(v)$ for all $v \in V(C)$. Let $P_i = C \backslash u_iu'_i$ and note that P_i is a path in $G - v_i$ joining the vertices u_i and u'_i containing only vertices in $V_3(G)$. Since v_i was arbitrarily chosen, theses properties hold for every $v_i \in V_2(G)$. If $G' = G - V_2(G)$ is disconnected, then there is $v_i \in V_2(G)$ for which u_i and u'_i, are in different components of G'. But since $V(P_i) \subseteq V_3(G)$, the path P_i is in G' and joins u_i and u'_i, a contradiction. □

Claim 9 *There is a cycle $C \subseteq G$ containing only degree 3 vertices.*

Proof By Claim 8, $G' = G - V_2(G)$ is connected. Towards a contradiction, suppose that G' is a tree. Let $T = G' + \{v_iu_i : v_i \in V_2(G)\}$ and note that T is a spanning tree of G. Let $M = \{v_iu'_i : v_i \in V_2(G)\}$ and let $v_ju'_j \in M$. Since $d_G(v_j) = 2$ and $v_ju_j \in E(T)$, there is only one edge in M incident to v_j, namely $v_ju'_j$. If there is another edge in M, say $v_ku'_k$, incident to u'_j, then $u'_j = u'_k$ and $\mathrm{dist}_G(v_j, v_k) \leq 2$, a contradiction to Claim 7. Thus, M is a matching in G and (T, M) is a 2-decomposition of G, a contradiction. Therefore, $G - V_2(G)$ is not a tree, and hence contains a cycle $C \subseteq G - V_2(G)$. By Claim 1, $\delta(G) \geq 2$, and hence $G - V_2(G) = G[V_3]$. Therefore, $d_G(u) = 3$ for all $u \in V(C)$.

□

4 Concluding Remarks

We recall that a graph is *claw-free* if it contains no induced copy of $K_{1,3}$. As mentioned before, the 3DC has been verified for claw-free cubic graphs [1, 15]. Using an idea similar to the one used by Aboomahigir, Ahanjideh and Akbari [1], one can show that the 2DC also holds for claw-free graphs. For completeness, we provide its proof below.

Theorem 4.1 *If $G \in \mathcal{S}_{1,3}$ is a claw-free graph, then G can be decomposed into a spanning tree and a matching.*

Proof Towards a contradiction, suppose the opposite, and let G be a counterexample with a minimum number of vertices. It is easy to check that the result holds for $|V(G)| \leq 2$, thus we may assume that $|V(G)| > 2$.

First, suppose that G contains a cut edge uv and let $G' = G \setminus uv$. Let G_u and G_v be the components of G' containing the vertices u and v, respectively. By the minimality of G, there are 2-decompositions (T_u, M_u) and (T_v, M_v) of G_u and G_v, respectively. Hence, $(T_u \cup T_v \cup \{uv\}, M_u \cup M_v)$ is a 2-decomposition of G, a contradiction. Thus, we may assume that G is 2-edge connected, and hence $\delta(G) \geq 2$.

Moreover, G must contain a vertex v with degree 3, otherwise G would be a cycle and the result follows. Let $N(v) = \{x, y, z\}$. Since G is claw-free, we may assume without loss of generality that $xy \in E(G)$. Let C be the cycle $xvyx$. Since $G \in \mathcal{S}_{1,3}$, the graph $G \setminus E(C)$ is disconnected. If $d(x) = d(y) = 3$, then G has a cut edge, a contradiction. Hence we may assume without loss of generality that $d(x) = 2$. If $yz \in E(G)$, then either G is the diamond (K_4 minus an edge) or G has a cut edge incident to z; in both cases, we reach a contradiction.

Thus, we may assume that $yz \notin E(G)$. Note that $d(y) = 3$ and $d(z) \geq 2$, otherwise vz would be a cut edge. Let $G' = G - \{x, v\} + yz$, and note that G' is a connected subcubic graph.

We claim that G' is claw-freehyperref. Indeed, if $d_G(z) = 2$, then it is easy to see that G' is claw-free. Thus suppose that $d_G(z) = 3$ and let $N_G(z) = \{v, a, b\}$. Note that $a, b \notin \{x, y\}$, because $yz \notin E(G)$ and $d_G(x) = 2$. Since $\{z, v, a, b\}$ does not induces a claw and $va, vb \notin E(G)$, it follows that $ab \in E(G)$. Thus, $\{z, y, a, b\}$ does not induces a claw in G', and hence G' is claw-free.

Now we show that $G' \in \mathcal{S}_{1,3}$. Let C' be a cycle in G'. If $yz \in C'$, then C' is a separating cycle, since $d_{G'}(y) = 2$. Otherwise, $yz \notin E(C')$, and hence, C' is a cycle in $G \in \mathcal{S}_{1,3}$. In this case, it is easy to check that C' is separating.

Therefore, G' is a claw-free, connected, subcubic graph in $\mathcal{S}_{1,3}$ with fewer vertices than G, and by the minimality of G, the graph G' admits a decomposition into a spanning tree T' and a matching M'.

Now, we show how to find a decomposition of G into a spanning tree T and a matching M from T' and M'. If $yz \in M'$, then let $T = T + \{yv, vx\}$ and $M = (M' \backslash \{yz\}) \cup \{vz, xv\}$ (recall that $yz \notin E(G)$). Otherwise, $yz \in T'$, and hence let $T = T' \setminus yz + \{yv, yx, vz\}$ and $M = M' \cup \{vx\}$. Note that in both cases (T, M) is a 2-decomposition of G, a contradiction. □

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Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

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