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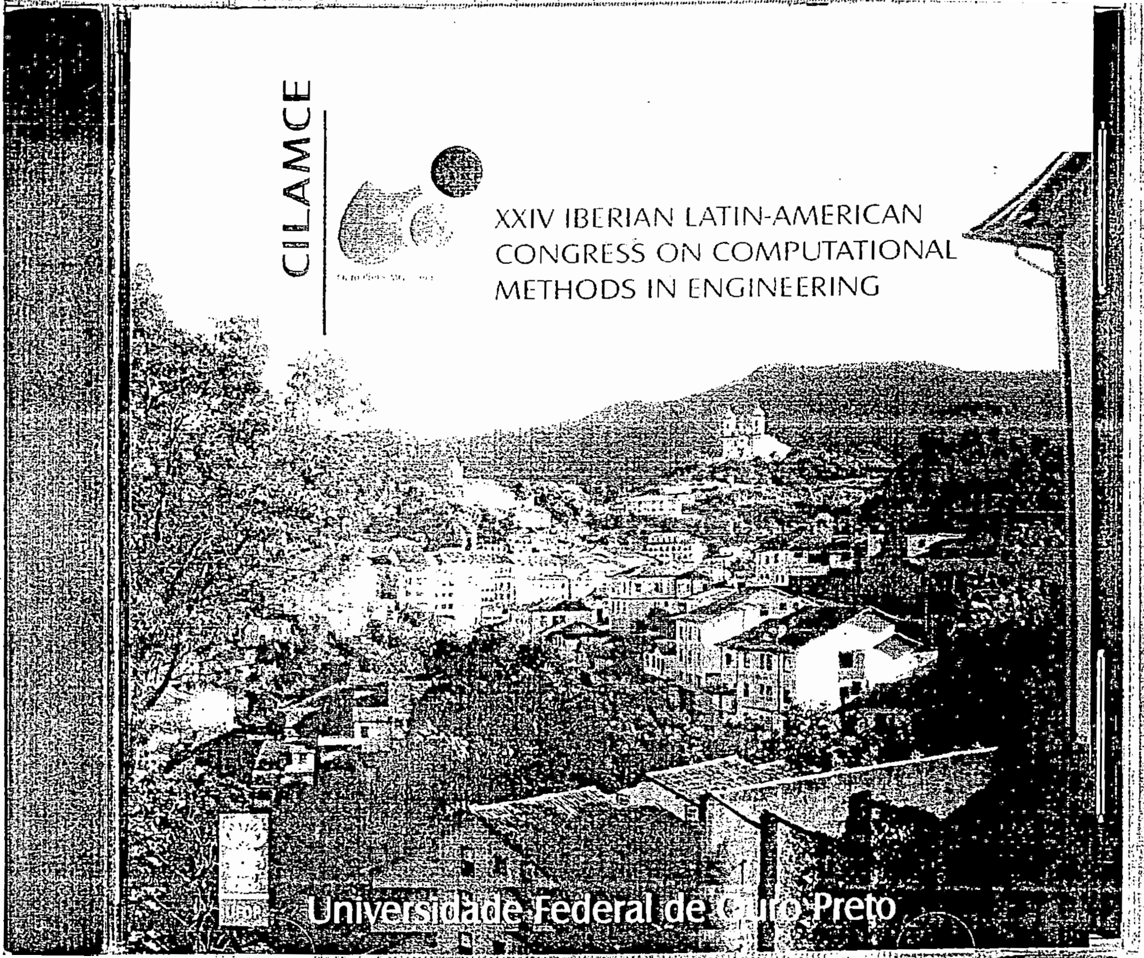
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A NEW POSITION DESCRIPTION FOR GENERAL GEOMETRIC NON-LINEAR STRUCTURAL ANALYSIS BY FEM

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Abstract. *This work presents a simple formulation to treat large deflections by the Finite Element Method (FEM). The present formulation does not use the concept of displacements; it considers position as the real variable of the problem. The strain determination is done directly from the proposed position concept. A non-dimensional space is created, relative curvature and fibers length are calculated for both reference and deformed configurations, and used to directly calculate the strain energy at general points. The initial configuration is assumed as the basis of calculation, i.e., Hooke's law relates reference stress and a non-linear engineering strain measurement. This point of view is very precise as demonstrated in the example section where numerical results are compared with analytical solutions and other important works. The technique is applied for 2D frame problems and can be easily extended to more general situations.*

Keywords: *FEM, Non-linear, Structural analysis, Frames*

1. INTRODUCTION

The analysis of structures that exhibit large deflections is of great importance in nowadays engineering. The crescent search for economy and optimal material application leads to the conception of very flexible structures and the equilibrium analysis in the non-deformed position is no more acceptable for most applications. In this sense, a lot of important works has been presented in this area. For example, the analytical solution for slender bars and simple composition of them has bee developed by various authors, such as Bisshopp & Drucker (1945) and Mattiasson (1981). This approach is quite complicate because the superposition of effects is not valid for non-linear applications.

In order to create automatic, general and reliable tools for the analysis of largely deflected structures, various researchers have presented important contributions along time regarding finite element procedures, James *et al.* (1974), Argyris *et al.* (1978), Risks (1979), Gadala (1984) and Wriggers & Simo (1990). These works are also very important to the development of the human knowledge on the subject, clarifying and opening the understanding of the present researchers. It is difficult to put together all works in this area, identifying them regarding their approach and classifying them by their importance. However the authors would like to make a citation of the consulted works that helped the understanding and inspired the present formulation development.

In the specialized literature there are several types of formulations based on FEM to solve geometrical non-linear problems. These formulations present differences on the coordinates description. The Lagrangian descriptions measure nodal displacements regarding a fixed Cartesian system of coordinates, and can be total (Mondkar & Powell, 1977), measure of displacements are done considering the initial reference, or updated (Peterson & Petersson, 1985 and Wong & Tinloi,1990) measure of displacements are done considering the last equilibrium position reference. The Eulerian description (Oran & Kassimali, 1976 and Izzuddin & Elnashai, 1993) follows the structure movement, measuring displacements considering the nodal position changes. Another efficient formulation to deal with geometrical non-linear problems is the co-rotational (Crisfield, 1990 and Behdinan *et al.*, 1998) that uses local coordinates systems for the finite elements making possible the consideration of curvature effects. Some formulations consider structural pos-buckling behaviour (Pai & Palazotto, 1996 and Simo *et al.*, 1986).

In this work it is proposed a formulation based on the Principle of Stationary Potential Energy. The novelties are based in two main points. The first is the identification, on all consulted references, that the definitions of bodies' kinematics necessarily pass trough the explicit definition of the concept of displacement. In this work the word displacement is not mentioned, to define the kinematics of the body, only the concept of position is assumed. The strain determination is done directly from the position concept.

The second difference is that in all consulted works the deformation function is achieved by differentiating the deformed configuration regarding the reference one. In this work a non-dimensional space is created and no direct deformation function is created, but relative curvature and fibers length are calculated for both reference and deformed configurations and used to directly calculate the strain at general points.

As a secondary consequence of these considerations one does not think (necessarily) in the words, increment, linearization, prevision, correction, tangent matrix etc., largely used in literature. Of course, that these concepts are right and important, but their use is not necessary for the development of the proposed formulation. In the end of the work various examples are shown in order to demonstrate the precision of the proposed formulation.

2. POSITIONAL NON-LINEAR FORMULATION

To state the proposed methodology one should start from the Minimum Principle of Total Potential Energy, stated from position considerations (not displacements):

$$\Pi = U_t - P \tag{1}$$

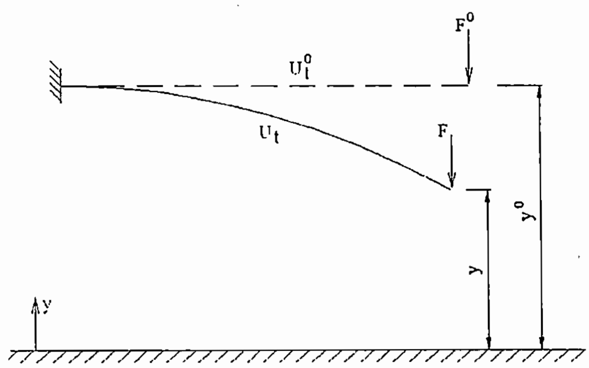

Figure 1 - Total potential energy written for a body in two different positions

The strain energy is written (linear elasticity) as:

$$U_t = \int_V \frac{1}{2} \sigma \varepsilon dV \tag{2}$$

Where stresses are evaluated in a reference configuration and strains are given by non-linear engineering (conjugate). The strain energy is assumed to be zero in a reference position (called non-deformed). The potential of loads is written as:

$$P = \sum FX \tag{3}$$

Where X is the set of positions, independent from each other, which a chosen point of a body can occupy. As a principle the potential energy may be not zero in the reference configuration. The total potential energy is then:

$$\Pi = \int_V \frac{1}{2} \sigma \varepsilon dV - \sum FX \tag{4}$$

In order to perform the integral indicated in Eq. (4) it is necessary to know the geometry of the studied body (the accepted geometric approximation) and its relation with the adopted strain measurement. Fig. 2 gives the general geometry of a curve over a plane.

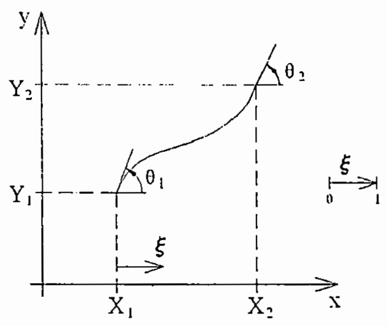

Figure 2 - Curve in a 2D space

This generic curve represents a configuration of the body. It can be parameterized as a function of a non-dimensional variable ξ (varying from 0 to 1). It should be noted that for two-dimensional problems one states a linear approximation for position in x direction and a cubic approximation for y direction. When this is done an additional care should be taken when the curve stop to be a one valued function. Adopting the above described approximation one writes:

$$x = X_1 + l_x \xi \tag{5a}$$

$$x = X_1 + (X_2 - X_1)\xi \tag{5b}$$

$$x = X_1(1-\xi) + X_2\xi \tag{5c}$$

Where:

$$l_x = (X_2 - X_1) \tag{5d}$$

Relating ξ to y, following cubic approximation, one writes:

$$y = c\xi^3 + d\xi^2 + e\xi + f \tag{6}$$

It is necessary to solve the generalized parameters c, d, e and f of Eq. (6) in order to write as a function of the nodal parameters, i.e., positions $X_1, Y_1, X_2, Y_2, \theta_1, \theta_2$. It is interesting to note that the last two parameters will appear as arguments of tangent functions, i.e., $tg(\theta_1)$ and $tg(\theta_2)$. So one has:

$$y_{(\xi=0)} = f = Y_1 \tag{7a}$$

$$\left.\frac{dy}{d\xi}\right| = 3c\xi^2 + 2d\xi + e \tag{7b}$$

$$\left.\frac{dy}{d\xi}\right|_{\xi=0} = e = \left.\frac{dy}{dx}\frac{dx}{d\xi}\right|_{\xi=0} = tg(\theta_1)\, l_x \tag{7c}$$

$$\left.\frac{dy}{d\xi}\right|_{\xi=1} = 3c + 2d + tg(\theta_1)l_x = \left.\frac{dy}{dx}\frac{dx}{d\xi}\right|_{\xi=1} = tg(\theta_2)l_x \tag{7d}$$

Then,

$$3c + 2d = [tg(\theta_2) - tg(\theta_1)]l_x \tag{7e}$$

$$y_{(\xi=1)} = c + d + tg(\theta_1)l_x + Y_1 = Y_2 \tag{7f}$$

Or:

$$c + d = l_y - tg(\theta_1)l_x \tag{7g}$$

Solving (7e) and (7g) results:

$$c = [tg(\theta_2) + tg(\theta_1)]l_x - 2l_y \tag{8}$$

$$d = 3l_y - [tg(\theta_2) + 2tg(\theta_1)]l_x \tag{9}$$

The following strain evaluation is adopted:

$$\varepsilon = \frac{ds - ds_0}{ds_0} \tag{10a}$$

And it is a Lagrangian strain measurement.

In Eq. (10a) ds is the length of a fiber inside the domain (in this case it is parallel to the central line) in any position. ds_0 is the length of this same fiber for the reference configuration. At this point the novelty is to identify that the proposed strain determination can be achieved by relative length referred to the non-dimensional space represented here by the variable ξ, i.e.:

$$\varepsilon = \frac{ds - ds_0}{ds_0} = \frac{ds/d\xi - ds_0/d\xi}{ds_0/d\xi} \tag{10b}$$

It is interesting to note that in majority of references this strain measurement is called linear strain. Any strain measurement can be used to solve geometrical non-linear problems if one takes its proper conjugate stress in order to calculate energy.

In this work it will be considered the reference configuration as a straight line. For this situation the reference approximation can be taken as:

$$x^0 = X_1^0 + l_x^0\xi \tag{11}$$

$$y^0 = Y_1^0 + l_y^0\xi \tag{12}$$

For the central line (passing trough the mass center of the bar) in the initial configuration one has:

$$\frac{ds^0}{d\xi}=\sqrt{\left(\frac{dx^0}{d\xi}\right)^2+\left(\frac{dy^0}{d\xi}\right)^2}=\sqrt{(l_x^0)^2+(l_y^0)^2}=l_0 \tag{13a}$$

Or:

$$ds^0=l_0 d\xi \tag{13b}$$

In the same way for another general configuration one has:

$$\frac{ds}{d\xi}=\sqrt{\left(\frac{dx}{d\xi}\right)^2+\left(\frac{dy}{d\xi}\right)^2}=\sqrt{(l_x)^2+(3c\xi^2+2d\xi+e)^2} \tag{14a}$$

Or:

$$ds=\sqrt{(l_x)^2+(3c\xi^2+2d\xi+e)^2}\,d\xi \tag{14b}$$

Applying linear stress measurement, Eq. (10b), for the central line one calculates:

$$\varepsilon^{m}=\frac{1}{l_0}\sqrt{(l_x)^2+(3c\xi^2+2d\xi+e)^2}-1 \tag{15}$$

Following the curvilinear coordinate s one can define an orthogonal co-ordinate to s, called z, where one defines the strain dependence (Euler-Bernoulli hypothesis). Following this hypothesis one writes the strain as a function of the curvature difference plus the central line value. As the initial curvature is zero one has (initial shape is a straight line):

$$\varepsilon=\varepsilon^{m}+\frac{1}{r}z \tag{16}$$

The exact curvature is given by:

$$\frac{1}{r}=\frac{\dfrac{dx}{d\xi}\dfrac{d^2y}{d\xi^2}-\dfrac{d^2x}{d\xi^2}\dfrac{dy}{d\xi}}{\left(\sqrt{\left(\dfrac{dx}{d\xi}\right)^2+\left(\dfrac{dy}{d\xi}\right)^2}\right)^3} \tag{17}$$

Or, replacing the known expressions (approximation for positions):

$$\frac{1}{r}=\frac{l_x(6c\xi+2d)}{\left(\sqrt{l_x^{\,2}+(3c\xi^2+2d\xi+e)^2}\right)^3} \tag{18}$$

Taking the reference stress, linear elasticity, and the strain determination one writes the specific strain energy as:

$$u_t(espec)=\frac{E}{2}(\varepsilon^m+\frac{1}{r}z)^2=\frac{1}{2}\left(\left(\varepsilon^m\right)^2+2\varepsilon^m\frac{1}{r}z+\left(\frac{1}{r}z\right)^2\right) \tag{19}$$

In order to calculate the strain energy, it is necessary to integrate the specific strain energy ($u_t(espec)$) over the initial volume of the analyzed body. For reference stress and the Lagrangian strain measurement proposed here this is done referring to the reference volume, as it is the zero energy situation and any change of energy should be referred to it. A numerical proof for this affirmation is in example three, where large strain is present.

To integrate the specific strain energy over the volume one starts by integrating it over the transverse section as follows:

$$u_t=\frac{EA}{2}\left(\varepsilon^m\right)^2+\frac{EI}{2}\left(\frac{1}{r}\right)^2 \tag{20}$$

Now one integrates the result Eq. (20) along the length of the bar, i.e..

$$U_t=\int_0^1\frac{EA}{2}\left(\varepsilon^m\right)^2+\frac{EI}{2}\left(\frac{1}{r}\right)^2 l_0dt=l_0\int_0^1 u_t dt \tag{21}$$

As the strain energy is known (written as a function of nodal parameters) it is necessary to differentiate the Total Potential Energy, expressions (1) or (4), regarding the nodal parameters, in order to obtain the equilibrium statement. In order to do that one should reorganize previous equations as follows:

$$\Pi=l_0\int_0^1 u_t d\xi-F_{x1}X_1-F_{y1}Y_1-M_1\theta_1-F_{x2}X_2-F_{y2}Y_2-M_2\theta_2 \tag{22}$$

As there are no singularities in the strain energy integral one can write:

$$\frac{\partial\Pi}{\partial X_1}=l_0\int_0^1\frac{\partial u_t}{\partial X_1}d\xi-F_{x1}=0 \tag{22a}$$

$$\frac{\partial\Pi}{\partial Y_1}=l_0\int_0^1\frac{\partial u_t}{\partial Y_1}d\xi-F_{y1}=0 \tag{22b}$$

$$\frac{\partial\Pi}{\partial\theta_1}=l_0\int_0^1\frac{\partial u_t}{\partial\theta_1}d\xi-M_1=0 \tag{22c}$$

$$\frac{\partial\Pi}{\partial X_2}=l_0\int_0^1\frac{\partial u_t}{\partial X2}d\xi-F_{x2}=0 \tag{22d}$$

$$\frac{\partial\Pi}{\partial Y_2}=l_0\int_0^1\frac{\partial u_t}{\partial Y_2}d\xi-F_{y2}=0 \tag{22f}$$

$$\frac{\partial\Pi}{\partial\theta_2}=l_0\int_0^1\frac{\partial u_t}{\partial\theta_2}d\xi-M_2=0 \tag{22g}$$

The algebraic strategy is to develop the derivatives inside the integrals and after that integrate it numerically along the non-dimensional space. As it can be noted the numerical integral result is not linear regarding the nodal parameters. Therefore, one writes the above system of equations in the following general way:

$$g_1(X_1,Y_1,\theta_1,X_2,Y_2,\theta_2)=f_1(X_1,Y_1,\theta_1,X_2,Y_2,\theta_2)-F_{x1}=0 \tag{23a}$$

$$g_2(X_1,Y_1,\theta_1,X_2,Y_2,\theta_2)=f_2(X_1,Y_1,\theta_1,X_2,Y_2,\theta_2)-F_{y1}=0 \tag{23b}$$

$$g_3(X_1,Y_1,\theta_1,X_2,Y_2,\theta_2)=f_3(X_1,Y_1,\theta_1,X_2,Y_2,\theta_2)-M_1=0 \tag{23c}$$

$$g_4(X_1,Y_1,\theta_1,X_2,Y_2,\theta_2)=f_4(X_1,Y_1,\theta_1,X_2,Y_2,\theta_2)-F_{x2}=0 \tag{23d}$$

$$g_5(X_1,Y_1,\theta_1,X_2,Y_2,\theta_2)=f_5(X_1,Y_1,\theta_1,X_2,Y_2,\theta_2)-F_{y2}=0 \tag{23e}$$

$$g_6(X_1,Y_1,\theta_1,X_2,Y_2,\theta_2)=f_6(X_1,Y_1,\theta_1,X_2,Y_2,\theta_2)-M_2=0 \tag{23f}$$

Or, in a compact notation:

$$g_i(X_j,F_i)=f_i(X_j)-F_i=0 \tag{24}$$

The numerical representation of nodal parameters $(X_1,Y_1,\theta_1,X_2,Y_2,\theta_2)=(1,2,3,4,5,6)$ is adopted. Following a vector notation representation one has:

$$g(X,F)=0 \tag{25}$$

Or:

$$f(X)-F=0 \tag{26}$$

Note that in this work the external forces have been assumed not dependent regarding space. Space dependent forces are easily implemented if desired. The vectorial function g(x) is non-linear regarding nodal parameters (X and F), but Eq. (25) represents the minimum potential energy situation and therefore the equilibrium of the analyzed body. To solve Eq. (25) one can use the Newton-Raphson procedure, i.e.:

$$g(X)=0=g(X^0)+\nabla g(X^0)\Delta X \tag{27}$$

Or:

$$\Delta X=-\left[\nabla g(X^0)\right]^{-1} g(X^0) \tag{28}$$

At this point all usual words of non-linear analysis could be introduced, but the reader is invited to understand the procedure as a simple non-linear system solver. One can calculate the Hessian matrix $\nabla g(X^0)$ from expressions (22), (23) and (24), as:

$$\nabla g(X^0)=g_{i,k}(X^0)=f_{i,k}(X^0)-F_{i,\ell} \tag{29}$$

Where i=1,6; k=1-6 for parametric positions and ℓ=7-12 for external forces. It is easy to achieve the following representation:

$$\nabla g\left(X^0\right)=l_0\int_0^l u_{l,ik}\,d\xi\Big|_{X^0}-\delta_{i\ell} \tag{30}$$

For Eq. (28) we need to calculate $g(X^0)$

$$g\left(X^0\right)=l_0\int_0^l u_{l,i}\,d\xi\Big|_{X^0}-F_i \tag{31}$$

The iterative (Newton-Raphson) process is summarized as:

1) Assume initially X^0 as the initial configuration (non-deformed). Calculate $g\left(X^0\right)$ following Eq. (31)
2) For this same X^0, calculate the Hessian matrix by unity of length, $u_{l,ik}\big|_{X^0}$. Integrate this value as indicated in (30) and results the gradient of g at X^0
3) Solves the system of Eq. (28) and determines ΔX
4) Update position $X^0=X^0+\Delta X$. Goes back to step 1 until ΔX is sufficiently small

Theoretically the process is not incremental, however to divide the total loading (or prescribed position) in cumulative steps helps to start the iterative procedure at a position nearer to the final desired result. Introducing this step division results:

a) X^0 (initial position)
b) $X^0=X^0+\Delta f$, where Δf is a load increment into a single vector X^0
c) {1,2,3,4} iterations
d) Goes back to item b)

2.1 Final comment about the implementation of the formulation

One should note that if the element presents l_x very near zero the Hessian matrix loses its objectivity. To solve this problem one creates the auxiliary co-ordinate shown in Fig. 3. Now l_x will be always far from zero.

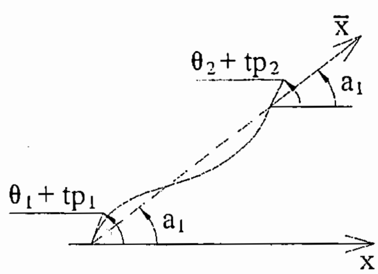

Figure 3 - Auxiliary coordinate system

The calculation becomes:

$$l_x=X_2-X_1 \tag{32a}$$

$$l_y = Y_2 - Y_1 \tag{32b}$$

$$a_1 = arctg\left(\frac{l_y}{l_x}\right) \tag{32c}$$

$$\bar{x}_{11} = cos(a_1)X_1 + sen(a_1)Y_1 \tag{32d}$$

$$\bar{y}_{11} = -sen(a_1)X_1 + cos(a_1)Y_1 \tag{32f}$$

$$\bar{x}_{21} = cos(a_1)X_2 + sen(a_1)Y_2 \tag{32g}$$

$$\bar{y}_{21} = -sen(a_1)X_2 + cos(a_1)Y_2 \tag{32h}$$

$$l_{\bar{x}} = (\bar{x}_{21} - \bar{x}_{11}) \tag{32i}$$

$$l_{\bar{y}} = (\bar{y}_{21} - \bar{y}_{11}) \tag{32j}$$

$$\bar{\theta}_1 = \theta_1 + tp_1 - a_1 \tag{32k}$$

$$\bar{\theta}_2 = \theta_2 + tp_2 - a_1 \tag{32l}$$

It is important to note that *tp1* and *tp2* are equal each other and represents the slope of the finite element in the reference configuration (non-deformed).

3. NUMERICAL EXAMPLES

3.1 Square frame subject to a pair of opposite forces

The first example is a square frame loaded at the midpoints by a pair of opposite forces. Two load cases are considered, the tension case and the compression case. To solve this problem the symmetry condition is used, as presented in Fig. 4. One half of structure is discretized into 40 finite elements. Mattiasson (1981) solved analytically this example. In the numerical analysis the load is divided into 100 steps, in order to show the results at different load levels, but the problem can be solved using three load steps.

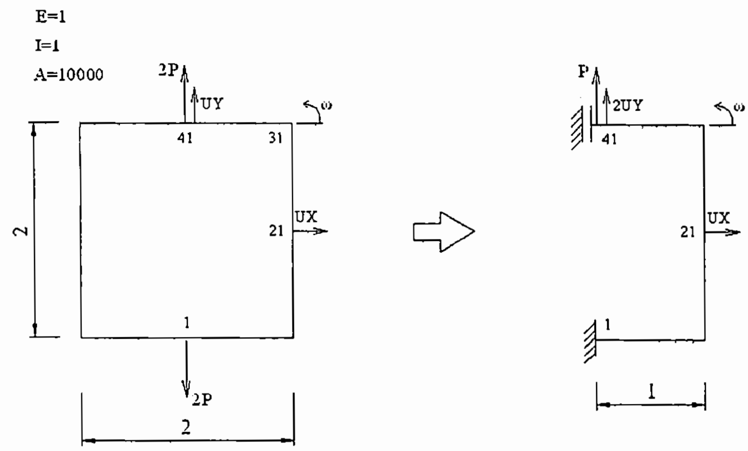

Figure 4 - Square frame input data

Following the convention of Fig. 4, in Fig. 5, 6 and 7 the responses *UX*, *UY* and ω, are compared with the analytical solution. In Fig. 8 are presented two deformed shapes for the tension case. In Fig. 9 are also presented two deformed shapes for the compression case; it is interesting to note that an inversion of top and down parts of structure occurs.

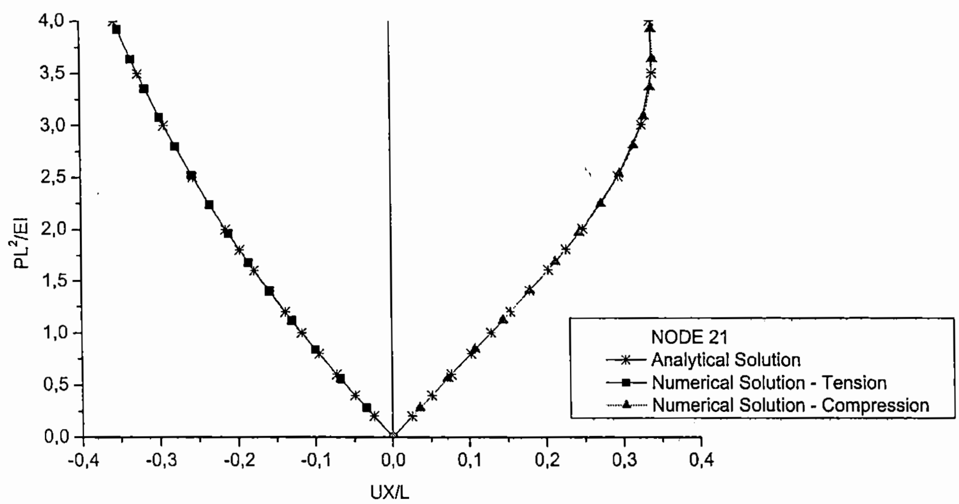

Figure 5 - Non-dimensional deflections in the direction of coordinate *UX*

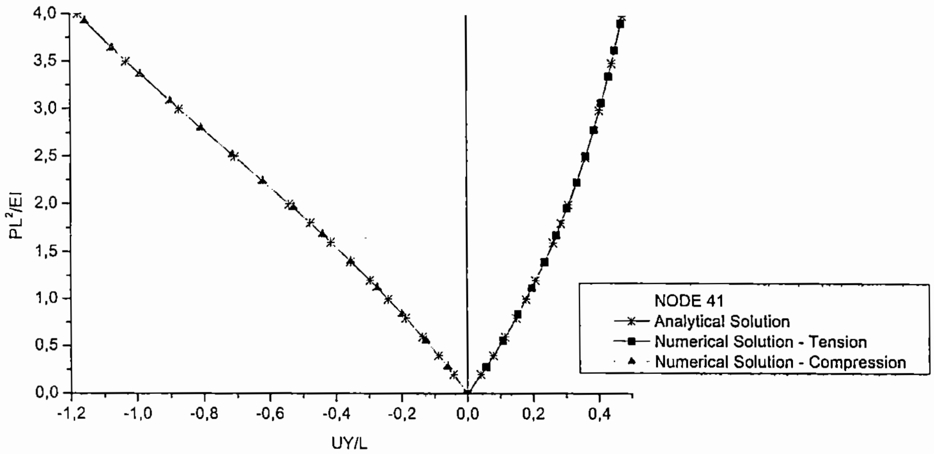

Figure 6 - Non-dimensional deflections in the direction of coordinate *UY*

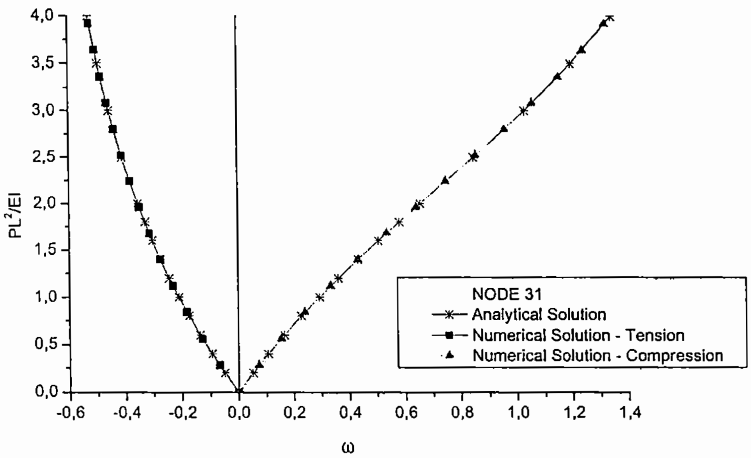

Figure 7 - Non-dimensional deflections in the direction of coordinate ω

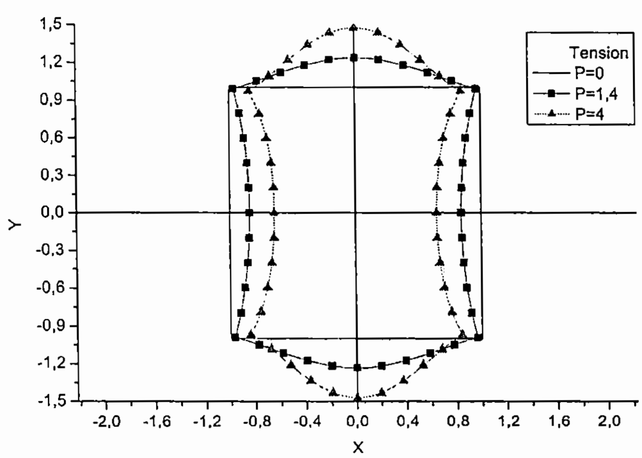

Figure 8 - Deformed shapes for the load tension case

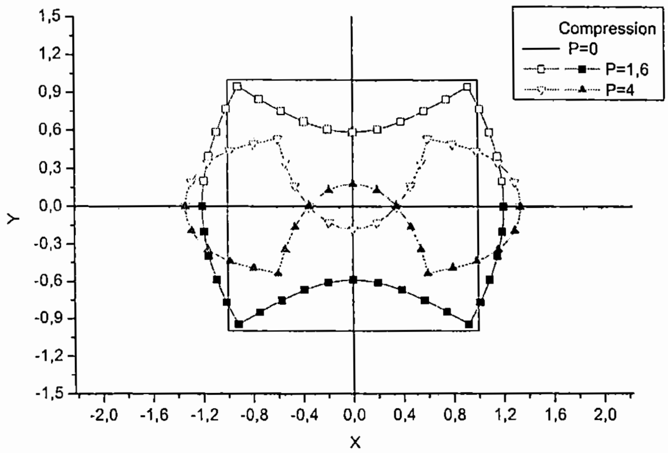

Figure 9 - Deformed shapes for the load compression case

No significant differences are found between analytical and the proposed numerical formulation.

3.2 Elastic contact of a ring

The last example is a circular ring pressed against a rigid surface. The force is applied at the top of the structure using 100 load increments. To solve the problem the symmetry condition is used, as presented in Fig. 10. One half of structure is discretized into 20 finite elements. Simo *et al.* (1986) solved numerically this example.

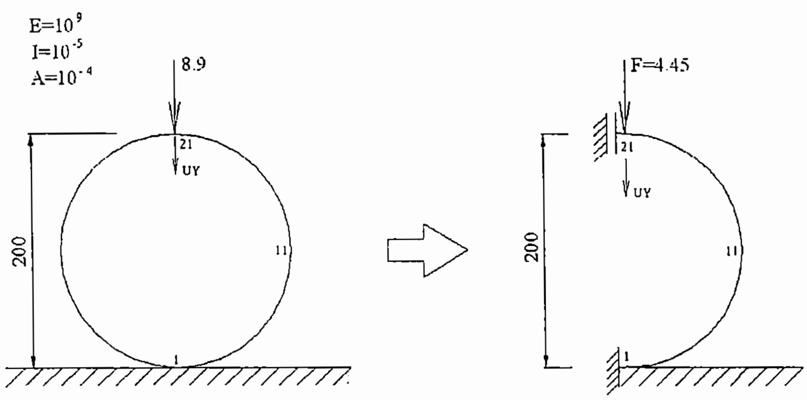

Figure 10 - Elastic ring input data

Figure 11 presents *UY* numerical solutions, comparing the present formulation and the reference one. In Fig. 12 are presented deformed shape for some load levels.

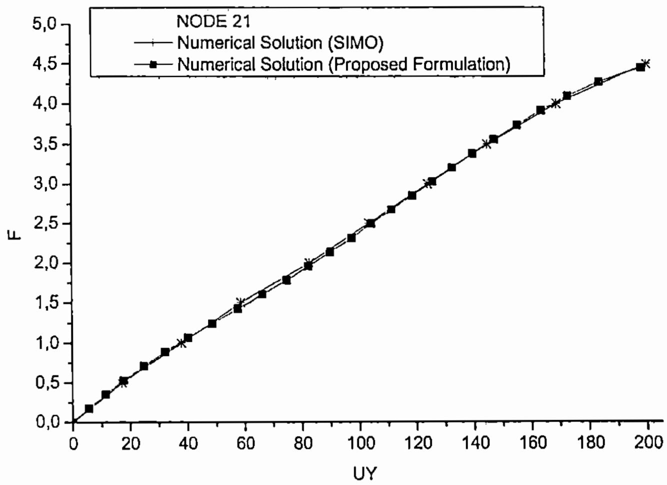

Figure 11 - Deflections in the direction of coordinate *UY*

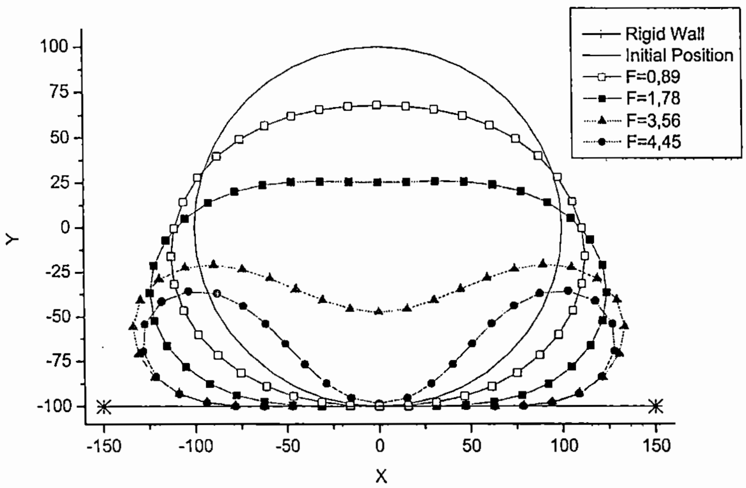

Figure 12 - Deformed shapes for some load levels

As one can see the results are in good agreement.

4. CONCLUSIONS

A new and efficient method based on the Finite Element Method for solving static nonlinear problems with large deflections and rotations has been presented. The developed formulation presents a high degree of convergence and accuracy, the number of iterations falls as the number of degree of freedoms raise. The formulation is capable to analyze severe geometrical non-linear behaviors, including structural post-buckling behaviors. Two numerical examples are presented and very good responses were obtained, compared with analytical and other numerical solutions.

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