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BOGOLIUBOV QUASIAVERAGES: SPONTANEOUS SYMMETRY BREAKING AND THE ALGEBRA OF FLUCTUATIONS

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We present arguments supporting the use of the Bogoliubov method of quasiaverages for quantum systems. First, we elucidate how it can be used to study phase transitions with spontaneous symmetry breaking (SSB). For this, we consider the example of Bose–Einstein condensation in continuous systems. Analysis of different types of generalized condensations shows that the only physically reliable quantities are those defined by Bogoliubov quasiaverages. In this connection, we also solve the Lieb–Seiringer–Yngvason problem. Second, using the scaled Bogoliubov method of quasiaverages and considering the example of a structural quantum phase transition, we examine a relation between SSB and critical quantum fluctuations. We show that the quasiaverages again provide a tool suitable for describing the algebra of critical quantum fluctuation operators in both the commutative and noncommutative cases.

Keywords: quasiaverages, generalized condensation, critical quantum fluctuations

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In memory of Dmitrii Nikolaevich Zubarev on the centennial of his birth

1. Introduction and summary

The concept of spontaneous symmetry breaking (SSB) is central in quantum physics, both in statistical mechanics and in quantum field theory and particle physics. The definition of SSB was already well known in the mid-1960s (see Sec. 6.5.2 in [1] and Sec. 4.3.4 in [2]). We recall that the starting point is a (ground or thermal) state assumed to be invariant under a symmetry group G but with a nontrivial decomposition into extremal states, which can be interpreted physically as pure thermodynamic phases (states). These extremal states are not invariant under the action of G, only under the action of a proper subgroup H of G.

There are two basic ways to define extremal states:

1. by choosing boundary conditions for the Hamiltonians H_Λ in finite regions $\Lambda \subset \mathbb{R}^d$ and then taking the thermodynamic limit ($\Lambda \uparrow \mathbb{Z}^d$ or $\Lambda \uparrow \mathbb{R}^d$) of expectations over the corresponding local states or

2. by replacing $H_\Lambda \to H_\Lambda + hB_\Lambda$, where B_Λ is a suitable extensive operator and h is a real parameter, and then taking first $\Lambda \uparrow \mathbb{Z}^d$ or $\Lambda \uparrow \mathbb{R}^d$ and second $h \to +0$ (or $h \to -0$). We here assume that the states considered are locally normal or locally finite (see, e.g., [3] and the references therein).

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Method 2 is known as the Bogoliubov method of *quasiaverages* [4], [5].

We note that although the method of boundary conditions is quite clear, for example, for classical lattice systems, it is unsatisfactory for continuous systems and even worse for quantum systems. Here, we argue for using the Bogoliubov method of quasiaverages for quantum systems.

First, we elucidate how to use the Bogoliubov method of quasiaverages to study phase transitions with SSB (see Sec. 2). For this, we consider the quantum phase transition that is the conventional one-mode Bose–Einstein condensation (BEC) of the perfect Bose gas. In this simplest case, condensation occurs into a single zero mode, which implies a spontaneous breaking of G, the gauge group of transformations (GSB). We then consider the case where the condensation is dispersed over infinitely many modes. Analysis of different types of this generalized condensation (gBEC) shows that the only physically reliable quantities are those defined by the Bogoliubov method of quasiaverages (see Remark 2.3 and Theorem 2.1).

We extend this analysis to an imperfect Bose gas. It follows from the obtained results that we can elucidate a general question posed by Lieb, Seiringer, and Yngvason [6] about the equivalence between BEC and GSB defined via the one-mode Bogoliubov quasiaverage for any type of gBEC *à la* van den Berg–Lewis–Pulé [7] and [8] (see Remark 2.6, where we also stress that quasiaverages lead to ergodic states and this clarifies an important conceptual aspect of the quasiaverage trick).

Second, using the Bogoluibov method of quasiaverages and considering the example of a structural quantum phase transition, we examine the relation between SSB and critical quantum fluctuations (see Sec. 3). The analysis in Sec. 4 shows that the Bogoliubov quasiaverages again provide a tool suitable for describing the algebra of fluctuation operators on the critical line of transitions. We study both the commutative and noncommutative cases of this algebra (see Theorems 4.1 and 4.2).

We here note that Dmitrii Nikolaevich Zubarev was the first to indicate the relevance of Bogoliubov quasiaverages in the theory of nonequilibrium processes [9]. In this case, the infinitesimal external sources serve to break the time invariance of the Liouville equation for the statistical operator. Although well known in mathematical physics as the limit-absorption principle, this approach was extended to many-body problems in [9]. This elegant extension is now called the Zubarev nonequilibrium statistical operator method [10], [11]. This interesting aspect of the Bogoliubov method of quasiaverages is outside the scope of this paper.

2. Continuous boson systems

2.1. Conventional or generalized condensations and off-diagonal long-range order. We note that the existence of gBEC makes boson systems more relevant than, for example, spin lattice systems for demonstrating the effectiveness of Bogoliubov quasiaverages. This becomes clear even on the level of the perfect Bose gas (PBG).

Therefore, we first consider the BEC of the PBG in a three-dimensional anisotropic parallelepiped $\Lambda := V^{\alpha_1} \times V^{\alpha_2} \times V^{\alpha_3}$ with a periodic boundary condition and $\alpha_1 \geq \alpha_2 \geq \alpha_3$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$, i.e., the volume $|\Lambda| = V$. In the boson Fock space $\mathcal{F}_\Lambda := \mathcal{F}_{\text{boson}}(\mathcal{L}^2(\Lambda))$, the Hamiltonian of this system for the grand canonical ensemble with a chemical potential $\mu < 0$ is defined as

$$H_{0,\Lambda,\mu} = T_\Lambda - \mu N_\Lambda = \sum_{k\in\Lambda^*} (\varepsilon_k - \mu) b_k^* b_k, \qquad \text{dom}(H_{0,\Lambda,\mu}) = \text{dom}(T_\Lambda). \tag{2.1}$$

Here, one-particle kinetic-energy operator spectrum is $\{\varepsilon_k = k^2\}_{k\in\Lambda^*}$, where the set Λ^* is dual to Λ:

$$\Lambda^* := \left\{k_j = \frac{2\pi}{V^{\alpha_j}} n_j \colon\ n_j \in \mathbb{Z}\right\}_{j=1}^{d=3}, \qquad \varepsilon_k = \sum_{j=1}^{d} k_j^2. \tag{2.2}$$

We let $b_k := b(\varphi_k^\Lambda)$ and $b_k^* = (b(\varphi_k^\Lambda))^*$ denote the k-mode boson annihilation and creation operators in the Fock space $\mathcal{F}_\Lambda$. They are indexed by the orthonormal basis $\{\varphi_k^\Lambda(x) = e^{ikx}/\sqrt{V}\}_{k\in\Lambda^*}$ in $\mathcal{L}^2(\Lambda)$ generated by the eigenfunctions of the self-adjoint one-particle kinetic-energy operator $-\Delta$ with a periodic boundary condition in $\mathcal{L}^2(\Lambda)$. These operators formally satisfy the canonical commutation relations (CCR) $[b_k, b_{k'}^*] = \delta_{k,k'}$ for $k, k' \in \Lambda^*$. Then $N_k = b_k^* b_k$ is the occupation-number operator of the one-particle state φ_k^Λ, and $N_\Lambda = \sum_{k\in\Lambda^*} N_k$ denotes the total-number operator in Λ.

For a temperature $\beta^{-1} := k_{\mathrm{B}}T$ (where k_{B} is the Boltzmann constant) and a chemical potential μ, we let $\omega^0_{\beta,\mu,\Lambda}(\,\bullet\,)$ denote the grand canonical Gibbs state of the PBG generated by (2.1):

$$\omega^0_{\beta,\mu,\Lambda}(\,\bullet\,) = \frac{\mathrm{Tr}_{\mathcal{F}_\Lambda}(\exp(-\beta H_{0,\Lambda,\mu})\,\bullet\,)}{\mathrm{Tr}_{\mathcal{F}_\Lambda}\exp(-\beta H_{0,\Lambda,\mu})}. \tag{2.3}$$

The problem of the existence of a BEC is then related to the solution of the equation

$$\rho = \frac{1}{V}\sum_{k\in\Lambda^*}\omega^0_{\beta,\mu,\Lambda}(N_k) = \frac{1}{V}\sum_{k\in\Lambda^*}\frac{1}{e^{\beta(\varepsilon_k-\mu)}-1} \tag{2.4}$$

for a given total particle density ρ in Λ. We note that by (2.2), the thermodynamic limit $\Lambda \uparrow \mathbb{R}^3$ in the right-hand side of (2.4)

$$\mathcal{I}(\beta,\mu) = \lim_\Lambda \frac{1}{V}\sum_{k\in\Lambda^*}\omega^0_{\beta,\mu,\Lambda}(N_k) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k\,\frac{1}{e^{\beta(\varepsilon_k-\mu)}-1} \tag{2.5}$$

exists for any $\mu < 0$. At $\mu = 0$, it attains its (finite) maximum value $\mathcal{I}(\beta,\mu)|_{\mu=0} = \rho_{\mathrm{c}}(\beta)$, which is called the *critical particle density* for a given temperature.

We recall that the existence of a finite critical density $\rho_{\mathrm{c}}(\beta)$ (via the saturation mechanism) triggers a zero-mode BEC: $\rho_0(\beta) := \rho - \rho_{\mathrm{c}}(\beta)$, where the total particle density is $\rho_{\mathrm{c}}(\beta)$. We note that for $\alpha_1 < 1/2$, all the condensate is indeed in the one-particle ground state mode $k = 0$:

$$\rho_0(\beta) = \rho - \rho_{\mathrm{c}}(\beta) = \lim_\Lambda \frac{1}{V}\omega^0_{\beta,\mu_\Lambda(\beta,\rho),\Lambda}(b_0^* b_0) = \lim_\Lambda\left[\frac{1}{V}\frac{1}{e^{-\beta\mu_\Lambda(\beta,\rho)}-1}\right]_{\rho\geq\rho_{\mathrm{c}}(\beta)}, \tag{2.6}$$

$$\mu_\Lambda(\beta,\rho)\big|_{\rho\geq\rho_{\mathrm{c}}(\beta)} = -\frac{1}{V}\frac{1}{\beta(\rho-\rho_{\mathrm{c}}(\beta))} + o\left(\frac{1}{V}\right), \tag{2.7}$$

$$\lim_\Lambda \frac{1}{V}\omega^0_{\beta,\mu_\Lambda(\beta,\rho),\Lambda}(b_k^* b_k) = 0, \tag{2.8}$$

where $\mu_\Lambda(\beta,\rho)$ is a unique solution of Eq. (2.4) and $k = 0$.

Following [7], we introduce the gBEC.

Definition 2.1. The *total amount* $\rho_{\mathrm{gBEC}}(\beta,\mu)$ of the *gBEC* is defined by the double limit

$$\rho_{\mathrm{gBEC}}(\beta,\mu) := \lim_{\delta\to+0}\lim_\Lambda \frac{1}{V}\sum_{k\in\Lambda^*:\,\|k\|\leq\delta}\omega_{\beta,\mu,\Lambda}(b_k^* b_k). \tag{2.9}$$

Here, $\omega_{\beta,\mu,\Lambda}(\,\bullet\,)$ denotes the corresponding finite-volume grand canonical Gibbs state.

According to the nomenclature proposed in [7], the zero-mode BEC in the PBG is just the gBEC of type I. Indeed, it follows from (2.6) and (2.9) that a nonzero BEC implies a nontrivial gBEC: $\rho_{\mathrm{gBEC}}(\beta,\rho) > 0$. We write this relation as

$$\text{BEC} \implies \text{gBEC}. \tag{2.10}$$

Moreover, (2.6) and (2.9) yield $\rho_0(\beta,\rho) = \rho_{\text{gBEC}}(\beta,\rho)$.

We also recall that for the boson field

$$b(x) = \sum_{k\in\Lambda^*} b_k \varphi_k^{\Lambda}(x), \tag{2.11}$$

we have BEC $\Rightarrow$ ODLRO.[1] Indeed, by the definition of ODLRO [12], the value of the off-diagonal spatial correlation LRO(β,ρ) of the Bose field is

$$\begin{aligned}\text{LRO}(\beta,\rho) &= \lim_{\|x-y\|\to\infty} \lim_{\Lambda} \omega^0_{\beta,\mu_\Lambda,\Lambda}(b^*(x)b(y)) = \\ &= \lim_{\Lambda} \omega^0_{\beta,\mu_\Lambda,\Lambda}\left(\frac{b_0^*}{\sqrt{V}}\frac{b_0}{\sqrt{V}}\right) = \rho_0(\beta,\rho)\end{aligned} \tag{2.12}$$

and coincides with the zero-mode spatial average correlation of local observables (2.11). We recall that the p-mode spatial average $\eta_{\Lambda,p}(b)$ of (2.11) is equal to

$$\eta_{\Lambda,p}(b) := \frac{1}{V}\int_{\Lambda} dx\, b(x) e^{-ipx} = \frac{b_p}{\sqrt{V}}, \quad p \in \Lambda^*. \tag{2.13}$$

As is known, for the PBG, the value LRO(β,ρ) of the ODLRO coincides with the BEC (and hence also with the type-I gBEC) condensate density $\rho_0(\beta,\rho)$ [7].

To appreciate the importance of the gBEC compared with quasiaverages, we study a more anisotropic thermodynamic limit in the case $\alpha_1 = 1/2$, known as the Casimir box. We then observe infinitely many macroscopically occupied states, known as the gBEC of type II defined by (2.9). The total amount $\rho_0(\beta,\rho)$ of this condensate is asymptotically distributed between infinitely many low-energy microscopic states $\{\varphi_k^{\Lambda}\}_{k\in\Lambda^*}$ such that

$$\begin{aligned}\rho_0(\beta,\rho) = \rho - \rho_{\text{c}}(\beta) &= \lim_{\delta\to+0}\lim_{\Lambda}\frac{1}{V}\sum_{k\in\Lambda^*:\ \|k\|\le\delta}\{e^{\beta(\varepsilon_k-\mu_\Lambda(\beta,\rho))}-1\}^{-1} = \\ &= \sum_{n_1\in\mathbb{Z}}\frac{1}{(2\pi n_1)^2/2+A}, \quad \rho > \rho_{\text{c}}(\beta).\end{aligned} \tag{2.14}$$

Here, the parameter $A = A(\beta,\rho) \ge 0$ is a unique root of Eq. (2.14). The amount of the zero-mode BEC is then

$$\lim_{\Lambda}\frac{1}{V}\,\omega^0_{\beta,\mu,\Lambda}(b_0^* b_0) = A^{-1}(\beta,\rho).$$

We note that in contrast to the case of type I in (2.6), the zero-mode BEC $A^{-1}(\beta,\rho)$ is smaller than the gBEC of type II in (2.6). Therefore, the relation between BEC and gBEC is nontrivial.

To elucidate this point, we consider $\alpha_1 > 1/2$ (the so-called van den Berg–Lewis–Pulé box [7]). We then obtain

$$\lim_{\Lambda}\omega^0_{\beta,\mu,\Lambda}\left(\frac{b_k^* b_k}{V}\right) = \lim_{\Lambda}\frac{1}{V}\{e^{\beta(\varepsilon_k-\mu_\Lambda(\beta,\rho))}-1\}^{-1} = 0, \quad k\in\Lambda^*, \tag{2.15}$$

i.e., there is no macroscopic occupation of any mode $k \in \Lambda^*$ for any value of the particle density ρ. Therefore, the density of the zero-mode BEC is zero, but the gBEC (called type III) does exist in the same sense as defined by (2.9):

$$\rho - \rho_{\text{c}}(\beta) = \lim_{\delta\to+0}\lim_{\Lambda}\frac{1}{V}\sum_{k\in\Lambda^*:\ \|k\|\le\delta}\{e^{\beta(\varepsilon_k-\mu_\Lambda(\beta,\rho)}-1\}^{-1} > 0, \quad \rho > \rho_{\text{c}}(\beta), \tag{2.16}$$

[1] ODLRO means off-diagonal long-range order.

with the total density the same as for types I and II.

We note that calculating the ODLRO in the case of type-II and type-III gBECs is a nontrivial problem even for the PBG. In particular, this concerns the regime where there is the second critical density $\rho_{\mathrm{m}}(\beta) > \rho_{\mathrm{c}}(\beta)$ separating different types of gBEC (see [13] and [8]). It is also clear that the zero-mode BEC is a more restrictive concept than the gBEC.

The fact that the gBEC differs from the BEC is not necessarily due to a special anisotropy ($\alpha_1 > 1/2$) or any other geometric feature of the PBG [8]. In fact, the phenomenon of the type-III gBEC occurs as a result of a repulsive interaction. The Hamiltonian [14]

$$H_\Lambda = \sum_{k\in\Lambda^*} \varepsilon_k b_k^* b_k + \frac{a}{2V} \sum_{k\in\Lambda^*} b_k^* b_k^* b_k b_k, \quad a > 0, \tag{2.17}$$

is a simple example.

In summary, we note that the concept of gBEC (2.9) covers the cases (e.g., (2.17)) where calculating the conventional BEC gives a trivial value: gBEC $\not\Rightarrow$ BEC (cf. (2.10)). We also conclude that the relations between the BEC, gBEC, and ODLRO are subtle. This motivates and supports the relevance of the Bogoliubov method of quasiaverages [4], [5], which we also consider in connection with the SSB of the gauge invariance for the Gibbs states (we call such SSB GSB in what follows).

2.2. Condensates, Bogoliubov quasiaverages, and pure states. We now study the states of boson systems and assume (see Sec. 4.3.2 in [12]) that they are analytic in the sense of the definition in [15] (see Sec. 5.2.3).

We start with the Hamiltonian with periodic boundary conditions for bosons in a cubic box $\Lambda \subset \mathbb{R}^3$ of side L and volume $V = L^3$:

$$H_{\Lambda,\mu} = H_{0,\Lambda,\mu} + V_\Lambda, \tag{2.18}$$

where the interaction term has the form

$$V_\Lambda = \frac{1}{2V} \sum_{k,p,q\in\Lambda^*} \nu(p) b_{k+p}^* b_{q-p}^* b_q b_k \tag{2.19}$$

and $\nu(p)$ is the Fourier transform in $\mathbb{R}^3$ of the two-body potential $v(x)$ with the bound

$$|\nu(k)| \le \nu(0) < \infty. \tag{2.20}$$

We define the group G of (global) gauge transformations $\{\tau_s\}_{s\in[0,2\pi)}$ by the Bogoliubov canonical maps of the CCR:

$$\tau_s(b^*(f)) = b^*(e^{is} f) = e^{is} b^*(f), \qquad \tau_s(b(f)) = b(e^{is} f) = e^{-is} b(f), \tag{2.21}$$

where $b^*(f)$ and $b(f)$ are the creation and annihilation operators smeared over test-functions f from the Schwartz space. We note that for $f = \varphi_k^\Lambda$, they coincide with b_k^* and b_k in (2.11) and $\tau_s(\,\bullet\,) = e^{isN_\Lambda}(\,\bullet\,)e^{-isN_\Lambda}$ (see (2.1)). By definition (2.21) and by virtue of (2.1) and (2.19), Hamiltonian (2.18) is gauge invariant:

$$H_{\Lambda,\mu} = e^{isN_\Lambda} H_{\Lambda,\mu} e^{-isN_\Lambda}. \tag{2.22}$$

We note that this property obviously implies that Gibbs state (2.3) in the case of Hamiltonian (2.18) of an imperfect Bose gas is gauge invariant:

$$\omega_{\beta,\mu,\Lambda}(\,\bullet\,) = \omega_{\beta,\mu,\Lambda}(\tau_s(\,\bullet\,)) = \frac{\mathrm{Tr}_{\mathcal{F}_\Lambda}(\exp(-\beta H_{\Lambda,\mu})\tau_s(\,\bullet\,))}{\mathrm{Tr}_{\mathcal{F}_\Lambda}\exp(-\beta H_{\Lambda,\mu})}. \tag{2.23}$$

This symmetry is a source of selection rules. For example,

$$\omega_{\beta,\mu,\Lambda}(A_{n,m}) = 0 \quad \text{for } A_{n,m} = \prod_{i=1}^{n}\prod_{j=1}^{m} b_{k_i}^* b_{k_j}, \quad n \neq m. \tag{2.24}$$

We take the quasi-Hamiltonian corresponding to (2.18) with GSB sources in the form

$$H_{\Lambda,\mu,\lambda_\phi} = H_{\Lambda,\mu} + H_\Lambda^{\lambda_\phi}. \tag{2.25}$$

Here, the sources are switched on only in the zero mode ($k = 0$):

$$H_\Lambda^{\lambda_\phi} = \sqrt{V}(\bar{\lambda}_\phi b_0 + \lambda_\phi b_0^*) \tag{2.26}$$

for

$$\lambda_\phi = \lambda e^{i\phi}, \quad \lambda \geq 0, \qquad \arctan \lambda_\phi = \phi \in [0, 2\pi). \tag{2.27}$$

In this case, the corresponding Gibbs state is not gauge-invariant state (2.24), because, for example,

$$\omega_{\beta,\mu,\Lambda,\lambda_\phi}(b_k) = \frac{\mathrm{Tr}_{\mathcal{F}_\Lambda}(\exp(-\beta H_{\Lambda,\mu,\lambda_\phi}) b_k)}{\mathrm{Tr}_{\mathcal{F}_\Lambda} \exp(-\beta H_{\Lambda,\mu,\lambda_\phi})} \neq 0 \quad \text{for } k = 0. \tag{2.28}$$

The GSB of state (2.28), induced by the sources in (2.25), persists in the thermodynamic limit for the state $\omega_{\beta,\mu,\lambda_\phi}(\,\bullet\,) := \lim_{V\to\infty} \omega_{\beta,\mu,\Lambda,\lambda_\phi}(\,\bullet\,)$. But it can occur in this limit spontaneously without external sources. We set

$$\omega_{\beta,\mu}(\,\bullet\,) := \lim_{V\to\infty} \omega_{\beta,\mu,\Lambda,\lambda_\phi}(\,\bullet\,)\big|_{\lambda_\phi=0}. \tag{2.29}$$

Definition 2.2. We say that the state $\omega_{\beta,\mu}$ undergoes a spontaneous breaking of the G invariance (GSB) if

1. $\omega_{\beta,\mu}$ is G-invariant and
2. $\omega_{\beta,\mu}$ has a nontrivial decomposition into ergodic states $\omega'_{\beta,\mu}$, which means that at least two such distinct states occur in the representation

$$\omega_{\beta,\mu}(\,\bullet\,) = \int_0^{2\pi} d\nu(s)\, \omega'_{\beta,\mu}(\tau_s \,\bullet\,)$$

and $\omega'_{\beta,\mu}(\tau_s \,\bullet\,) \neq \omega'_{\beta,\mu}(\,\bullet\,)$ for some s.

We note that ergodic states are characterized by the clustering property, which implies a decorrelation of zero-mode spacial averages (2.13) for the PBG and also in general for the imperfect Bose gas.

We initially take $\lambda \geq 0$ and consider PBG (2.1) to define the Hamiltonian

$$H_{0,\Lambda,\mu,\lambda_\phi} = H_{0,\Lambda,\mu} + H_\Lambda^{\lambda_\phi}, \tag{2.30}$$

which is not globally gauge invariant. To separate the symmetry-breaking term H_0, we rewrite (2.30) as $H_{0,\Lambda,\mu,\lambda_\phi} = H_0 + H_k$ (with $k \neq 0$), where

$$H_0 = -\mu b_0^* b_0 + \sqrt{V}(\bar{\lambda}_\phi b_0 + \lambda_\phi b_0^*) = -\mu\left(b_0 - \frac{\sqrt{V}\lambda_\phi}{\mu}\right)^*\left(b_0 - \frac{\sqrt{V}\lambda_\phi}{\mu}\right) + \frac{V|\lambda_\phi|^2}{\mu}.$$

We recall that the grand canonical partition function $\Xi_{0,\Lambda}$ for the PBG splits into a product over the zero mode and the remaining modes. We introduce the canonical shift transformation

$$\hat{b}_0 := b_0 - \frac{\lambda_\phi \sqrt{V}}{\mu} \tag{2.31}$$

without altering the nonzero modes. Because $\mu < 0$, we thus obtain

$$\Xi_{0,\Lambda}(\beta,\mu,\lambda_\phi) = (1-e^{\beta\mu})^{-1} \exp\biggl(-\frac{\beta|\lambda_\phi|^2}{\mu}V\biggr)\Xi'_{0,\Lambda}(\beta,\mu) \tag{2.32}$$

for the grand canonical partition function, where

$$\Xi'_{0,\Lambda}(\beta,\mu) := \prod_{k\neq 0}(1-e^{-\beta(\epsilon_k-\mu)})^{-1}, \qquad \epsilon_k = k^2. \tag{2.33}$$

We recall that the grand canonical state for the PBG is (see Sec. 2.1)

$$\omega^0_{\beta,\mu,\Lambda,\lambda_\phi}(\,\bullet\,) := \frac{1}{\Xi_{0,\Lambda}(\beta,\mu,\lambda_\phi)}\,\mathrm{Tr}_{\mathcal{F}_\Lambda}[e^{-\beta H_{0,\Lambda,\mu,\lambda_\phi}}(\,\bullet\,)]. \tag{2.34}$$

It then follows from (2.32)–(2.34) that the mean density is

$$\rho = \omega^0_{\beta,\mu,\Lambda,\lambda_\phi}\biggl(\frac{N_\Lambda}{V}\biggr) = \frac{|\lambda_\phi|^2}{\mu^2} + \frac{1}{V}\frac{1}{e^{-\beta\mu}-1} + \frac{1}{V}\sum_{k\neq 0}\frac{1}{e^{\beta(\epsilon_k-\mu)}-1}. \tag{2.35}$$

Equation (2.35) is the starting point of our analysis. Because the critical density $\rho_{\mathrm{c}}(\beta) = \mathcal{I}(\beta,\mu)|_{\mu=0}$ is finite, we have the following statement.

Proposition 2.1. *Let $0 < \beta < \infty$ be fixed. Then for each $\rho_{\mathrm{c}}(\beta) < \rho < \infty$ and each $\lambda > 0$, $V < \infty$, there exists a unique solution of* (2.35) *of the form*

$$\mu_\Lambda(\rho,|\lambda_\phi|) = -\frac{|\lambda_\phi|}{\sqrt{\rho-\rho_{\mathrm{c}}(\beta)}} + \alpha(|\lambda_\phi|,V), \tag{2.36}$$

where $\alpha(|\lambda_\phi|,V) \geq 0$ for all $|\lambda_\phi|$ and V, and such that

$$\lim_{|\lambda_\phi|\to 0}\lim_{V\to\infty}\frac{\alpha(|\lambda_\phi|,V)}{|\lambda_\phi|} = 0. \tag{2.37}$$

Proof. The proof of this statement is straightforward and follows from Eq. (2.35).

Remark 2.1. Proposition 2.1 holds not only for the cube Λ but also for a three-dimensional anisotropic parallelepiped $\Lambda := V^{\alpha_1}\times V^{\alpha_2}\times V^{\alpha_3}$, with periodic boundary conditions and $\alpha_1 \geq \alpha_2 \geq \alpha_3$, $\alpha_1+\alpha_2+\alpha_3 = 1$, i.e., when we have type-II or type-III condensations for $\lambda = 0$.

Because $|\lambda_\phi|^2 = \lambda_\phi\bar{\lambda}_\phi = \lambda^2$, we find that the limit of the expectation is related to the derivative of the grand canonical pressure with respect to the symmetry-breaking sources (2.25):

$$\lim_{\lambda\to+0}\lim_{V\to\infty}\frac{\partial}{\partial\lambda_\phi}p_{\beta,\mu,\Lambda,\lambda_\phi} = -\lim_{\lambda\to+0}\lim_{V\to\infty}\omega^0_{\beta,\mu,\Lambda,\lambda_\phi}\biggl(\frac{b_0^*}{\sqrt{V}}\biggr), \tag{2.38}$$

where

$$p_{\beta,\mu,\Lambda,\lambda_\phi} := \frac{1}{\beta V} \log \Xi_{0,\Lambda}(\beta, \mu, \lambda_\phi). \tag{2.39}$$

We recall that the left-hand side of (2.38) is in fact the Bogoliubov quasiaverage of the operator of the observable $b_0^*/\sqrt{V}$.

Using (2.33) and (2.39), we obtain

$$\frac{\partial}{\partial \lambda_\phi} p_{\beta,\mu,\Lambda,\lambda_\phi} = -\frac{\bar{\lambda}_\phi}{\mu}. \tag{2.40}$$

The asymptotic form of the chemical potential for a given ρ is (2.36), and taking (2.38) and (2.40) into account, we therefore obtain

$$\lim_{\lambda\to+0} \lim_{V\to\infty} \omega^0_{\beta,\mu,\Lambda,\lambda_\phi}\left(\frac{b_0^*}{\sqrt{V}}\right) = \sqrt{\rho_0(\beta,\rho)}e^{-i\phi}, \tag{2.41}$$

where

$$\rho_0(\beta,\rho) = \rho - \rho_{\mathrm{c}}(\beta,\rho)$$

in accordance with (2.35) and (2.40) is the zero-mode PBG condensation. We can therefore see that the phase in (2.38) remains in (2.41) even after the limit $\lambda \to +0$.

The following definition was suggested in [6] using a more general method for studying an imperfect Bose gas (see below).

Definition 2.3. We say that a state $\omega_{\beta,\mu,\Lambda,\lambda_\phi}$ *undergoes a spontaneous GSB in the sense of Bogoliubov quasiaverages* ($(\mathrm{GSB})_{\text{q-a}}$) if limit state (2.29) remains gauge invariant while the state

$$\omega_{\beta,\mu,\phi}(\,\bullet\,) := \lim_{\lambda\to+0} \lim_{V\to\infty} \omega_{\beta,\mu,\Lambda,\lambda_\phi}(\,\bullet\,) \tag{2.42}$$

is not gauge invariant and moreover $\omega_{\beta,\mu,\phi} \neq \omega_{\beta,\mu,\phi'}$ for $\phi \neq \phi'$.

We note that $(\mathrm{GSB})_{\text{q-a}}$ is equivalent to GSB in the sense of Definition 2.2, where the ergodic states $\omega'_{\beta,\mu}$ in point 2 of the definition coincide with the set of $\omega_{\beta,\mu,\phi}$ in (2.42) (see Theorem 2.1 below). Nevertheless, the notion of $(\mathrm{GSB})_{\text{q-a}}$ is useful for comparison with the results in [6].

Remark 2.2. Using (2.35) together with Proposition 2.1 and relation (2.41), we obtain

$$\begin{aligned} \rho_0(\beta,\rho) &= \lim_{\lambda\to+0} \lim_{V\to\infty} \omega^0_{\beta,\mu,\Lambda,\lambda_\phi}\left(\frac{b_0^*}{\sqrt{V}} \frac{b_0}{\sqrt{V}}\right) = \\ &= \lim_{\lambda\to+0} \lim_{V\to\infty} \omega^0_{\beta,\mu,\Lambda,\lambda_\phi}\left(\frac{b_0^*}{\sqrt{V}}\right) \cdot \lim_{\lambda\to+0} \lim_{V\to\infty} \omega^0_{\beta,\mu,\Lambda,\lambda_\phi}\left(\frac{b_0}{\sqrt{V}}\right). \end{aligned} \tag{2.43}$$

In addition to decorrelation of the zero-mode spatial averages $\eta_{\Lambda,0}(b^*) = b_0^*/\sqrt{V}$ and $\eta_{\Lambda,0}(b) = b_0/\sqrt{V}$ given by (2.13), for the Bogoliubov quasiaverages, Eq. (2.43) also establishes an identity between the zero-mode condensation fraction $\rho_0(\beta,\rho)$ and $\mathrm{LRO}(\beta,\rho)$ given by (2.12), denoted by $(\mathrm{ODLRO})_{\text{q-a}}$. Decorrelation in the right-hand side of (2.43) shows that in the presence of a condensate, we have a nontrivial GSB for the Bogoliubov quasiaverages in the sense of Definition 2.3 (also see (2.41)).

Remarks 2.1 and 2.2 lead to the definition of the quasiaverage *states* for the PBG as

$$\omega^0_{\beta,\mu,\phi} := \lim_{\lambda\to+0}\lim_{V\to\infty}\omega^0_{\beta,\mu,\Lambda,\lambda_\phi}, \tag{2.44}$$

where the double limit along a subnet $\Lambda\uparrow\mathbb{R}^3$ exists by the $*$-weak compactness of the set of states [2]. Below, we use the notation ω for a Gibbs state in general case (2.18)–(2.20) and keep ω^0 for the PBG.

Definition 2.4. We recall that a Bose gas undergoes the zero-mode BEC if

$$\lim_{V\to\infty}\frac{1}{V}\,\omega_{\beta,\mu,\Lambda}(b_0^*b_0) = \lim_{V\to\infty}\frac{1}{V^2}\int_\Lambda\int_\Lambda dx\,dy\,\omega_{\beta,\mu,\Lambda}(b^*(x)b(y)) > 0. \tag{2.45}$$

Simultaneously, this means a nontrivial correlation (2.12),

$$\lim_{\|x-y\|\to\infty}\lim_{V\to\infty}\omega_{\beta,\mu,\Lambda}(b^*(x)b(y)) > 0, \tag{2.46}$$

of zero-mode spatial averages (2.13), denoted by *ODLRO*.

As shown in Sec. 2.1, this definition is too restrictive even for the PBG because (2.45) might be trivial although condensation does exist because of a finite critical density $\rho_c(\beta,\mu)$. We say that a Bose gas undergoes a gBEC (Definition 2.1) if

$$\lim_{\delta\to+0}\lim_{\Lambda}\frac{1}{V}\sum_{k\in\Lambda^*:\ \|k\|\le\delta}\omega_{\beta,\mu,\Lambda}(b_k^*b_k) = \rho-\rho_c(\beta,\mu) > 0. \tag{2.47}$$

To classify different types of the gBEC, we must consider the values of the limits

$$\lim_{\Lambda}\frac{1}{V}\,\omega_{\beta,\mu,\Lambda}(b_k^*b_k) =: \rho_k, \quad k\in\Lambda^*. \tag{2.48}$$

According to Sec. 2.1, we then have $\rho_0=\rho-\rho_c$ for the type-I gBEC and $\rho_0<\rho-\rho_c$ for the type-II gBEC. If we have $\{\rho_k\}_{k\in\Lambda^*}$ and a nontrivial (2.47), then the gBEC is of type III, i.e., where the condensate is zero in the zero mode.

Definition 2.5. We say that a Bose gas *undergoes BEC of the Bogoliubov quasiaverage type* $(\mathrm{BEC})_{\text{q-a}}$ if

$$\lim_{\lambda\to+0}\lim_{V\to\infty}\omega_{\beta,\mu,\Lambda,\lambda_\phi}\left(\frac{b_0^*}{\sqrt{V}}\frac{b_0}{\sqrt{V}}\right) > 0. \tag{2.49}$$

Remark 2.3. First, the facts noted in Remark 2.2 are independent of the anisotropy, i.e., of whether the condensation for $\lambda=0$ occurs into a single mode ($k=0$) (i.e., BEC) or is extended to a larger number of modes up to the gBEC of type II or III (see Sec. 2.1). The type-I condensate occurs into the mode $k=0$ because of a special property of Hamiltonian (2.2) from which it follows that $\varepsilon_k=0$ for $k=0$.

Second, these results show that the Bogoliubov quasiaverage method for a perfect gas answers the question about the equivalence between BEC, GSB, and ODLRO if they are defined in terms of the one-mode quasiaverage $k=0$:

$$(\mathrm{BEC})_{\text{q-a}} \iff (\mathrm{GSB})_{\text{q-a}} \iff (\mathrm{ODLRO})_{\text{q-a}} \tag{2.50}$$

(here $\iff$ denotes bi-implication).

The quasiaverage for $k \neq 0$, i.e., for $\varepsilon_k > 0$, needs a certain elucidation. For this, we reconsider PBG (2.1) with symmetry-breaking sources (2.26) in a single mode $q \in \Lambda^*$, which in the general case is not a zero mode:

$$H^0_\Lambda(\mu; h) := H^0_\Lambda(\mu) + \sqrt{V}(\bar{h} b_q + h b^*_q), \quad \mu \leq 0. \tag{2.51}$$

For a fixed density ρ, Eq. (2.4) for the condensate in the grand canonical ensemble with Hamiltonian (2.51) then becomes

$$\rho = \rho_\Lambda(\beta, \mu, h) := \frac{1}{V} \sum_{k \in \Lambda^*} \omega^0_{\beta,\mu,\Lambda,h}(b^*_k b_k) =$$

$$= \frac{1}{V}(e^{\beta(\varepsilon_q - \mu)} - 1)^{-1} + \frac{1}{V} \sum_{k \in \Lambda^* \setminus q} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} + \frac{|h|^2}{(\varepsilon_q - \mu)^2}. \tag{2.52}$$

In accordance with the quasiaverage method, to investigate a possible condensation, we must first take the thermodynamic limit in the right-hand side of (2.52) and then switch off the symmetry-breaking source, $h \to 0$. We recall that the critical density, which defines the saturation threshold in the boson system, is equal to $\rho_c(\beta) = \mathcal{I}(\beta, \mu)|_{\mu=0}$ (see (2.5)), where $\mathcal{I}(\beta, \mu) = \lim_\Lambda \rho_\Lambda(\beta, \mu, h)|_{h=0}$.

Because $\mu \leq 0$, we must now distinguish two cases:

1. Let the mode $q \in \Lambda^*$ be such that $\lim_\Lambda \varepsilon_q > 0$. From (2.52), we then obtain the condensate equation and the simplest q-mode GSB expectation:

$$\rho = \lim_{h\to 0} \lim_\Lambda \rho_\Lambda(\beta, \mu, h) = \mathcal{I}(\beta, \mu), \qquad \lim_{h\to 0} \lim_{V\to\infty} \omega^0_{\beta,\mu,\Lambda,h}\left(\frac{b^*_q}{\sqrt{V}}\right) = \lim_{h\to 0} \frac{\bar{h}}{\varepsilon_q - \mu} = 0.$$

   This means that the quasiaverage coincides with the average. Hence, we return to the analysis of condensate equation (2.52) with $h = 0$. This leads to finite-volume solutions $\mu_\Lambda(\beta, \rho)$ and consequently to all possible types of condensation as a function of the anisotropy α_1 (see Sec. 2.1 for the details).

2. On the other hand, if $q \in \Lambda^*$ is such that $\lim_\Lambda \varepsilon_q = 0$, then the thermodynamic limit in the right-hand side of condensate equation (2.52) and the q-mode GSB expectation are

$$\rho = \lim_\Lambda \rho_\Lambda(\beta, \mu, h) = \mathcal{I}(\beta, \mu) + \frac{|h|^2}{\mu^2}, \qquad \lim_{V\to\infty} \omega^0_{\beta,\mu,\Lambda,h}\left(\frac{b^*_q}{\sqrt{V}}\right) = \frac{\bar{h}}{-\mu}. \tag{2.53}$$

   If $\rho \leq \rho_c(\beta)$, then the limit of the solution of (2.53) is

$$\lim_{h\to 0} \mu(\beta, \rho, h) = \mu_0(\beta, \rho) < 0,$$

   where $\mu(\beta, \rho, h) = \lim_\Lambda \mu_\Lambda(\beta, \rho, h) < 0$ is thermodynamic limit of the finite-volume solution of condensate equation (2.52). Therefore, there is no condensation in any mode, and according to (2.53), the corresponding q-mode GSB expectation for $h \to 0$ (the Bogoliubov quasiaverage) is again equal to zero. But if $\rho > \rho_c(\beta)$, then (2.52) yields $\lim_{h\to 0} \mu(\beta, \rho, h) = 0$. Therefore, by (2.53), the condensate density and the Bogoliubov quasiaverage are

$$\rho_0(\beta) = \rho - \rho_c(\beta) = \lim_{h\to 0} \frac{|h|^2}{\mu(\beta, \rho, h)^2},$$

$$\lim_{h\to 0} \lim_{V\to\infty} \omega^0_{\beta,\mu_\Lambda(\beta,\rho,h),\Lambda,h}\left(\frac{b^*_q}{\sqrt{V}}\right) = \tag{2.54}$$

$$= \lim_{h\to 0} \lim_{V\to\infty} \omega^0_{\beta,\mu_\Lambda(\beta,\rho,h),\Lambda,h}\left(\frac{b^*_0}{\sqrt{V}}\right) = \sqrt{\rho_0(\beta)} e^{-i \arg h}.$$

We consider the first case in more detail. Let $\lim_\Lambda \varepsilon_q =: \varepsilon_q > 0$. Then by virtue of (2.52), the expectation of the particle density in the q-mode in the finite-volume case is

$$\omega^0_{\beta,\mu,\Lambda,h}\left(\frac{b_q^* b_q}{V}\right) = \frac{1}{V}(e^{\beta(\varepsilon_q-\mu)}-1)^{-1} + \frac{|h|^2}{(\varepsilon_q-\mu)^2}. \tag{2.55}$$

In the one-particle spectrum $\{\varepsilon_k\}_{k\in\Lambda^*}$, all $\varepsilon_k \geq 0$ and, moreover, $\varepsilon_k = 0$ for $k = 0$ (see (2.2)). The solution of (2.52) is therefore unique and negative: $\mu_\Lambda(\beta,\rho,h) < 0$. The Bogoliubov quasiaverage of $b_q^* b_q/V$ for any particle density $\rho > \rho_c(\beta)$ is then written as

$$\lim_{h\to 0}\lim_\Lambda \omega^0_{\beta,\mu_\Lambda(\beta,\rho,h),\Lambda,h}\left(\frac{b_q^* b_q}{V}\right) =$$

$$= \lim_{h\to 0}\lim_\Lambda \frac{1}{V}(e^{\beta(\varepsilon_q-\mu_\Lambda(\beta,\rho,h))}-1)^{-1} + \lim_{h\to 0}\lim_\Lambda \frac{|h|^2}{(\varepsilon_q-\mu_\Lambda(\beta,\rho,h))^2} = 0. \tag{2.56}$$

Condensate equation (2.52) and the q-mode GSB expectation now become

$$\rho = \lim_\Lambda \rho_\Lambda(\beta,\mu,h) = \mathcal{I}(\beta,\mu) + \frac{|h|^2}{(\varepsilon_q-\mu)^2} =: \rho(\beta,\mu,h), \tag{2.57}$$

$$\lim_{V\to\infty} \omega^0_{\beta,\mu,\Lambda,h}\left(\frac{b_q^*}{\sqrt{V}}\right) = \frac{\bar{h}}{\varepsilon_q-\mu}. \tag{2.58}$$

Remark 2.4. Hamiltonian (2.51) is the simplest example of a condensation model depending on an external source in a nonzero mode. Indeed, for the PBG with one-particle spectrum (2.2), the solution $\mu(\beta,\rho,h)$ of condensate equation (2.57) satisfies the relations

$$\lim_{\rho\to\rho_c(\beta,h)} \mu(\beta,\rho,h) = 0, \qquad \rho_c(\beta,h) := \sup_{\mu\leq 0} \rho(\beta,\mu,h) = \rho(\beta,\mu,h)\big|_{\mu=0} > \rho_c(\beta).$$

Because $\varepsilon_q > 0$ and $\varepsilon_0 = 0$, the finite saturation density $\rho_c(\beta,h)$ triggers the BEC mechanism in the zero mode of PBG (2.51) if $\rho > \rho_c(\beta,h)$. In connection with this, we note that from (2.52), (2.55), and (2.57), we obtain

$$\rho - \rho_c(\beta,h) = \lim_\Lambda \frac{1}{V}\omega^0_{\beta,\mu_\Lambda(\beta,\rho,h),\Lambda,h}(b_0^* b_0), \tag{2.59}$$

where the solution of (2.52) as $V \to \infty$ has the asymptotic form

$$\mu_\Lambda(\beta,\rho,h) = -\big(\rho - \rho_c(\beta,h)\big)V^{-1} + o(V^{-1}).$$

As a result, model (2.51) is the PBG with external sources, and its behavior is not identical to that of a Bose gas with $h = 0$ (see Sec. 2.1). For example, this concerns the higher critical density (2.57), $\rho(\beta,\mu,h)|_{\mu=0} \geq \rho_c(\beta)$, and a nonzero expectation (2.55) of the particle density in a nonzero q-mode $q \neq 0$.

We summarize the consideration of case 1. The nonzero-mode sources for the PBG and the corresponding Bogoliubov quasiaverages give the same results as for the PBG without external sources. Hence, the quasiaverages in this case have no impact and lead to the same conclusions (and problems) as the gBEC in Sec. 2.1. If we keep the nonzero-mode source, then this gBEC has a source-dependent critical density as in Remark 2.4.

We now consider case 2. We first note that by virtue of (2.52) and (2.53), we have $\mu(\beta,\rho,h) < 0$, $h \neq 0$. Moreover, for any $k \neq q$, even if $\lim_\Lambda \varepsilon_k = 0$,

$$\lim_{h\to 0}\lim_\Lambda \omega^0_{\beta,\mu_\Lambda(\beta,\rho,h),\Lambda,h}\left(\frac{b_k^* b_k}{V}\right) = \lim_{h\to 0}\lim_\Lambda \frac{1}{V}\frac{1}{e^{\beta(\varepsilon_k-\mu_\Lambda(\beta,\rho,h))}-1} = 0. \tag{2.60}$$

This means that $(\mathrm{BEC})_{\text{q-a}}$ for any anisotropy α_1 occurs only in the zero-mode (BEC type I), while the gBEC for $\alpha_1 > 1/2$ is of type III (see Sec. 2.1). Diagonalization (2.31) $b_q \to \hat{b}_q$ and (2.54) allow using the quasiaverage method to calculate a nonzero $(\mathrm{GSB})_{\text{q-a}}$ for $\rho > \rho_{\mathrm{c}}(\beta)$:

$$\lim_{h\to 0}\lim_{\Lambda}\omega^0_{\beta,\mu_\Lambda(\beta,\rho,h),\Lambda,h}\left(\frac{b_q}{\sqrt{V}}\right) = \lim_{h\to 0}\frac{h}{\mu(\beta,\rho,h)} = e^{i\arg h}\sqrt{\rho-\rho_{\mathrm{c}}(\beta)}, \tag{2.61}$$

where the limit is understood in the sense $|h|\to 0$ for $h = |h|e^{i\arg h}$. Analyzing (2.56) and (2.61), we then find that $(\mathrm{GSB})_{\text{q-a}}$ and $(\mathrm{BEC})_{\text{q-a}}$ are equivalent:

$$\lim_{h\to 0}\lim_{\Lambda}\omega^0_{\beta,\mu_\Lambda(\beta,\rho,h),\Lambda,h}\left(\frac{b^*_q}{\sqrt{V}}\right)\omega^0_{\beta,\mu_\Lambda(\beta,\rho,h),\Lambda,h}\left(\frac{b_q}{\sqrt{V}}\right) = \\ = \lim_{h\to 0}\lim_{\Lambda}\omega^0_{\beta,\mu_\Lambda(\beta,\rho,h),\Lambda,h}\left(\frac{b^*_q b_q}{V}\right) = \rho-\rho_{\mathrm{c}}(\beta). \tag{2.62}$$

We note that $(\mathrm{GSB})_{\text{q-a}}$ and $(\mathrm{BEC})_{\text{q-a}}$ are then in turn equivalent to $(\mathrm{ODLRO})_{\text{q-a}}$ by virtue of (2.12). In contrast to $(\mathrm{BEC})_{\text{q-a}}$ for the one-mode BEC, we obtain

$$\lim_{\Lambda}\omega^0_{\beta,\mu_\Lambda(\beta,\rho,0),\Lambda,0}\left(\frac{b^*_q b_q}{V}\right) = \lim_{\Lambda}\omega_{\beta,\mu^0_\Lambda(\beta,\rho,0),\Lambda,0}\left(\frac{b^*_q}{\sqrt{V}}\right)\omega^0_{\beta,\mu_\Lambda(\beta,\rho,0),\Lambda,0}\left(\frac{b_q}{\sqrt{V}}\right) = 0$$

for any ρ and $q\in\Lambda^*$ as soon as $\alpha_1 > 1/2$ (see Sec. 2.1). On the other hand, the value of gBEC coincides with $(\mathrm{BEC})_{\text{q-a}}$.

Remark 2.5. For the zero mode, the conventional BEC and the quasiaverage $(\mathrm{BEC})_{\text{q-a}}$ for the PBG are not equivalent: $(\mathrm{BEC})_{\text{q-a}} \not\Longleftrightarrow$ BEC, while $(\mathrm{BEC})_{\text{q-a}} \Longleftrightarrow$ gBEC. Equivalence (2.50) shows that the Bogoliubov quasiaverage method is definitely applicable in the PBG case.

We note that (2.41) and (2.44) show that the states $\omega_{\beta,\mu,\phi}$ are not gauge invariant. If we assume that they are ergodic states in the ergodic decomposition of $\omega_{\beta,\mu}$, then it follows for an interacting Bose gas that $(\mathrm{BEC})_{\text{q-a}} \Longleftrightarrow (\mathrm{GSB})_{\text{q-a}}$, which is similar to the equivalence for the PBG. This illuminates the explicit mechanism for the appearance of the symmetry-breaking phase ϕ connected with (2.36) in Proposition 2.1 in the PBG case. We note that the chemical potential remains proportional to $|\lambda|$ in this case even after thermodynamic limit (2.40). This property also persists for an interacting Bose gas (see the next subsection).

2.3. Interaction, quasiaverages, and the Bogoliubov *c*-number approximation. We consider an imperfect Bose gas with interaction (2.18)–(2.20). Our tool is the famous Bogoliubov approximation [4] according to which $\eta_{\Lambda,0}(b)$ and $\eta_{\Lambda,0}(b^*)$ given by (2.13) are replaced with complex numbers (also see [14], [16], [17]). The exactness of this procedure was proved by Ginibre [18] on the level of thermodynamics. Later, Lieb, Seiringer, and Yngvason [19], [6] and independently Sütö [20] improved the arguments in [18] and evaluated the exactness of the Bogoliubov approximation. In our analysis, we rely on the method in [6], which uses the Berezin–Lieb inequality [21].

We recall that the Fock space $\mathcal{F}_\Lambda \simeq \mathcal{F}_0\otimes\mathcal{F}'$, where $\mathcal{F}_0$ denotes the zero-mode subspace and $\mathcal{F}' := \otimes\mathcal{F}_k$ with $k\neq 0$ (see Sec. 2.1).

Let $z\in\mathbb{C}$ be a complex number and $|z\rangle = e^{-|z|^2/2+zb^*_0}|0\rangle$ be the Glauber coherent vector in $\mathcal{F}_0$. As in [6], let the operator $(H_{\Lambda,\mu,\lambda})'(z)$ be the lower symbol of the operator $H_{\Lambda,\mu,\lambda}$ given by (2.25). The pressure $p'_{\beta,\Lambda,\mu,\lambda}$ corresponding to this symbol is then defined by

$$\exp(\beta V p'_{\beta,\Lambda,\mu,\lambda}) = \Xi_\Lambda(\beta,\mu,\lambda)' = \int_{\mathbb{C}} d^2z\ \mathrm{Tr}_{\mathcal{F}'}\exp(-\beta(H_{\Lambda,\mu,\lambda})'(z)). \tag{2.63}$$

We consider the probability density

$$\mathcal{W}_{\mu,\Lambda,\lambda}(z) := \Xi_\Lambda(\beta,\mu,\lambda)^{-1}\,\mathrm{Tr}_{\mathcal{F}'}\langle z|\exp(-\beta H_{\Lambda,\mu,\lambda})|z\rangle. \tag{2.64}$$

As proved in [6], the density $\mathcal{W}_{\mu,\Lambda,\lambda}(\zeta\sqrt{V}\,)$ for almost all $\lambda > 0$ converges as $V \to \infty$ to a δ-density at the point

$$\zeta_{\max}(\lambda) = \lim_{V\to\infty}\frac{z_{\max}(\lambda)}{\sqrt{V}},$$

where $z_{\max}(\lambda)$ maximizes the partition function $\mathrm{Tr}_{\mathcal{F}'}\exp(-\beta(H_{\Lambda,\mu,\lambda})'(z))$. Although $\phi = 0$ was chosen in (2.27) in [6], the results in the general case can be obtained using the simple substitution $b_0 \to b_0 e^{-i\phi}$, $b_0^* \to b_0^* e^{i\phi}$ motivated by (2.25). We note that expression (34) in [6] can hence be rewritten as

$$\lim_{V\to\infty}\omega_{\beta,\mu,\Lambda,\lambda}(\eta_{\Lambda,0}(b^* e^{i\phi})) = \lim_{V\to\infty}\omega_{\beta,\mu,\Lambda,\lambda}(\eta_{\Lambda,0}(b e^{-i\phi})) =$$
$$= \frac{\partial p(\beta,\mu,\lambda)}{\partial\lambda} = \zeta_{\max}(\lambda) \tag{2.65}$$

and consequently yields (see definition (2.13))

$$\lim_{V\to\infty}\omega_{\beta,\mu,\Lambda,\lambda}(\eta_{\Lambda,0}(b^*)\eta_{\Lambda,0}(b)) = |\zeta_{\max}(\lambda)|^2. \tag{2.66}$$

Here, we let

$$p(\beta,\mu,\lambda) = \lim_{V\to\infty} p_{\beta,\mu,\Lambda,\lambda} \tag{2.67}$$

denote the grand canonical pressure of the imperfect Bose gas in (2.18)–(2.20) in the thermodynamic limit. Equality (2.65) follows from the convexity of $p_{\beta,\mu,\Lambda,\lambda}$ in $\lambda = |\lambda_\phi|$ by the Griffiths lemma [22]. It was shown in [6] that the pressure is equal to

$$p(\beta,\mu,\lambda)' = \lim_{V\to\infty} p'_{\beta,\mu,\Lambda,\lambda}. \tag{2.68}$$

Moreover, $p(\beta,\mu,\lambda)$ in (2.67) is also equal to the pressure $p(\beta,\mu,\lambda)''$, which is the thermodynamic limit of the pressure associated with the upper symbol of the operator $H_{\Lambda,\mu,\lambda}$. The crucial point [6] is the proof that all of these three pressures p', p, and p'' coincide with $p_{\max}(\beta,\mu,\lambda)$, which is the pressure associated with $\max_z \mathrm{Tr}_{\mathcal{F}'}\exp(-\beta(H_{\Lambda,\mu,\lambda})'(z))$:

$$p_{\max}(\beta,\mu,\lambda) = \lim_{V\to\infty}\frac{1}{\beta V}\log\{\max_z \mathrm{Tr}_{\mathcal{F}'}\exp(-\beta(H_{\Lambda,\mu,\lambda})'(z))\}. \tag{2.69}$$

We are now ready to prove a main result in this paper.

Theorem 2.1. *If we have* $(\mathrm{ODLRO})_{\text{q-a}}$ *or* $(\mathrm{BEC})_{\text{q-a}}$ *in the system of interacting bosons given by relations* (2.18)–(2.27), *then the limit*

$$\omega_{\beta,\mu,\phi} := \lim_{\lambda\to+0}\lim_{V\to\infty}\omega_{\beta,\mu,\Lambda,\lambda_\phi}$$

exists on the set of monomials $\{\eta_0(b^*)^m\eta_0(b)^n\}_{m,n\in\mathbb{N}\cup 0}$ *and satisfies the equalities*

$$\omega_{\beta,\mu,\phi}(\eta_0(b^*)) = \sqrt{\rho_0}e^{i\phi}, \tag{2.70}$$

$$\omega_{\beta,\mu,\phi}(\eta_0(b)) = \sqrt{\rho_0}e^{-i\phi}, \tag{2.71}$$

and we also have $(\mathrm{CSB})_{\text{q-a}}$,

$$\omega_{\beta,\mu,\phi}(\eta_0(b^*)\eta_0(b)) = \omega_{\beta,\mu,\phi}(\eta_0(b^*))\omega_{\beta,\mu,\phi}(\eta_0(b)) = \rho_0, \quad \phi \in [0, 2\pi), \tag{2.72}$$

and

$$\omega_{\beta,\mu} = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \omega_{\beta,\mu,\phi}. \tag{2.73}$$

On the Weyl algebra, the limit that defines $\omega_{\beta,\mu,\phi}$, $\phi \in [0, 2\pi)$, *exists along the nets in the variables* (λ, V). *The corresponding states are ergodic and coincide with the states obtained in Proposition* A.1.

Conversely, if the $(\mathrm{CSB})_{\text{q-a}}$ *occurs in the sense that* (2.70) *and* (2.71) *hold with* $\rho_0 \neq 0$, *then we have* $(\mathrm{ODLRO})_{\text{q-a}}$ *and* $(\mathrm{BEC})_{\text{q-a}}$.

Proof. We prove only the first statement. The converse follows from the Schwarz inequality for the states $\omega_{\beta,\mu,\phi}$ together with formula (2.80) presented below.

We show that $(\mathrm{ODLRO})_{\text{q-a}} \implies (\mathrm{GSB})_{\text{q-a}}$. We first assume that some state $\omega_{\beta,\mu,\phi_0}$, $\phi_0 \in [0, 2\pi)$, satisfies $(\mathrm{ODLRO})_{\text{q-a}}$. Then by (2.66),

$$\lim_{\lambda\to+0}\lim_{V\to\infty} \omega_{\beta,\mu,\Lambda,\lambda}(\eta_{\Lambda,0}(b^*)\eta_{\Lambda,0}(b)) = \lim_{\lambda\to+0} |\zeta_{\max}(\lambda)|^2 =: \rho_0 > 0. \tag{2.74}$$

The limit exists by the convexity of $p(\beta, \mu, \lambda)$ in λ and (2.38). Hence,

$$\lim_{\lambda\to+0} \frac{\partial p(\beta, \mu, \lambda)}{\partial \lambda} \neq 0. \tag{2.75}$$

At the same time, (2.65) shows that all states $\omega_{\beta,\mu,\phi}$ satisfy (2.74). Therefore, $(\mathrm{GSB})_{\text{q-a}}$ does not occur in the states $\omega_{\beta,\mu,\phi}$, $\phi \in [0, 2\pi)$. We have thus proved that assumption (2.45) implies that we have $(\mathrm{ODLRO})_{\text{q-a}}$ in all states $\omega_{\beta,\mu,\phi}$, $\phi \in [0, 2\pi)$.

The gauge invariance of $\omega_{\beta,\mu,\Lambda}$ (or, equivalently, Hamiltonian $H_{\Lambda,\mu}$) with (2.26) and (2.47) taken into account yields

$$\omega_{\beta,\mu,\Lambda,\lambda}(\eta_{\Lambda,0}(b^*)\eta_{\Lambda,0}(b)) = \omega_{\beta,\mu,\Lambda,-\lambda}(\eta_{\Lambda,0}(b^*)\eta_{\Lambda,0}(b)). \tag{2.76}$$

Again using (2.26) and (2.36) and the gauge invariance of $H_{\Lambda,\mu}$, we obtain

$$\lim_{\lambda\to-0} \frac{\partial p(\beta, \mu, \lambda)}{\partial \lambda} = -\lim_{\lambda\to+0} \frac{\partial p(\beta, \mu, \lambda)}{\partial \lambda}.$$

Because the derivative $\partial p(\beta, \mu, \lambda)/\partial\lambda$ is convex and monotonically increasing,

$$\lim_{\lambda\to+0} \frac{\partial p(\beta, \mu, \lambda)}{\partial \lambda} = \lim_{\lambda\to+0} \zeta_{\max}(\lambda) = \sqrt{\rho_0}, \tag{2.77}$$

$$\lim_{\lambda\to-0} \frac{\partial p(\beta, \mu, \lambda)}{\partial \lambda} = -\lim_{\lambda\to+0} \zeta_{\max}(\lambda) = -\sqrt{\rho_0}. \tag{2.78}$$

Again using (2.76), we obtain

$$\lim_{\lambda\to-0}\lim_{V\to\infty} \omega_{\beta,\mu,\Lambda,\lambda}(\eta_{\Lambda,0}(b^*)\eta_{\Lambda,0}(b)) = \lim_{\lambda\to+0}\lim_{V\to\infty} \omega_{\beta,\mu,\Lambda,\lambda}(\eta_{\Lambda,0}(b^*)\eta_{\Lambda,0}(b)). \tag{2.79}$$

According to [6], the weight $\mathcal{W}_{\mu,\lambda}$ for $\lambda = 0$ is supported on a disc with the radius equal to right-derivative (2.75). The convexity of the pressure as a function of λ implies that

$$\frac{\partial p(\beta, \mu, \lambda_0^-)}{\partial \lambda_0^-} \le \lim_{\lambda\to-0} \frac{\partial p(\beta, \mu, \lambda)}{\partial \lambda} \le \lim_{\lambda\to+0} \frac{\partial p(\beta, \mu, \lambda)}{\partial \lambda} \le \frac{\partial p(\beta, \mu, \lambda_0^+)}{\partial \lambda_0^+}$$

for any $\lambda_0^- < 0 < \lambda_0^+$. Therefore, by the Griffiths lemma (see, e.g., [22], [6]), we obtain

$$\lim_{\lambda\to -0}\lim_{V\to\infty}\omega_{\beta,\mu,\Lambda,\lambda}(\eta_{\Lambda,0}(b^*)\eta_{\Lambda,0}(b)) \le \lim_{V\to\infty}\omega_{\beta,\mu,\Lambda}\left(\frac{b_0^* b_0}{V}\right) \le$$
$$\le \lim_{\lambda\to +0}\lim_{V\to\infty}\omega_{\beta,\mu,\Lambda,\lambda}(\eta_{\Lambda,0}(b^*)\eta_{\Lambda,0}(b)). \tag{2.80}$$

Then (2.79) and (2.80) yield

$$\lim_{V\to\infty}\omega_{\beta,\mu,\Lambda}\left(\frac{b_0^* b_0}{V}\right) = \lim_{\lambda\to +0}\lim_{V\to\infty}\omega_{\beta,\mu,\Lambda,\lambda}(\eta_{\Lambda,0}(b^*)\eta_{\Lambda,0}(b)), \quad \phi\in[0,2\pi). \tag{2.81}$$

This proves that all $\omega_{\beta,\mu,\phi}$ for all $\phi\in[0,2\pi)$ satisfy $(\text{ODLRO})_{\text{q-a}}$, as asserted.

From (2.65) and (2.77), we obtain (2.70) and (2.71). Then (2.73) is a consequence of the gauge invariance of $\omega_{\beta,\mu}$. The ergodicity of the states $\omega_{\beta,\mu,\phi}$, $\phi\in[0,2\pi)$, follows from (2.70), (2.71), and (2.81) (see the second point in Definition 2.2).

An analogous construction can be presented using the Weyl algebra instead of the polynomial algebra (see Sec. 4.3.2 in [12] and references therein for Proposition A.1). The limit along a subnet in the variables (λ, V) exists by the $*$-weak compactness and by the asymptotic Abelianness of the Weyl algebra under space translations (see, e.g., Example 5.2.19 in [15]); ergodic decomposition (2.73), which is also a central decomposition, is unique. Therefore, the $\omega_{\beta,\mu,\phi}$, $\phi\in[0,2\pi)$, coincide with the states constructed in Proposition A.1.

Remark 2.6. Remark 2.3 and Theorem 2.1 elucidate a problem discussed in [6], where the authors defined a generalized GSB via the quasiaverage $(\text{GSB})_{\text{q-a}}$, i.e., as the condition

$$\lim_{\lambda\to +0}\lim_{V\to\infty}\omega_{\beta,\mu,\Lambda,\lambda}(\eta_{\Lambda,0}(b)) \ne 0.$$

If the condition involves other than the gauge group, then we call it $(\text{SSB})_{\text{q-a}}$. Similarly, in that paper, the definition of one-mode condensation (2.74) (denoted by $(\text{BEC})_{\text{q-a}}$) was modified, and the authors established the equivalence $(\text{GSB})_{\text{q-a}} \iff (\text{BEC})_{\text{q-a}}$. They also asked whether $(\text{BEC})_{\text{q-a}} \iff$ BEC.

In Theorem 2.1, we showed that $(\text{GSB})_{\text{q-a}}$ (2.70) implies $(\text{ODLRO})_{\text{q-a}}$ or $(\text{BEC})_{\text{q-a}}$. We note that for the zero-mode BEC (see Definition 2.4), an analogous theorem shows that the answer to the posed question is affirmative. This follows from the crucial fact that the state $\omega_{\beta,\mu}$ is gauge invariant, which is consistent with decomposition (2.73) and leads to inequalities (2.80).

On the other hand, for other types of condensation (but nevertheless equally important, as confirmed in (2.17)), the comparison (from the standpoint of implication $\Longrightarrow$ or equivalence $\Longleftrightarrow$) of values calculated in the sense of quasiaverages or outside this method can turn out to be false. For example, for the PBG, the value of $(\text{BEC})_{\text{q-a}}$ is strictly greater than the zero-mode BEC for an anisotropy $\alpha_1 \ge 1/2$, and $(\text{BEC})_{\text{q-a}} \not\Longrightarrow$ BEC for $\alpha_1 > 1/2$ (see Sec. 2.1). The same can also be seen for interacting Bose gas (2.17), although in both cases (PBG and model (2.17)), we obtain $(\text{BEC})_{\text{q-a}} \Longrightarrow$ gBEC (see Sec. 2.2). Therefore, the answer to the question is negative in the general case. We note that as established in Theorem 2.1, the quasiaverages lead to ergodic states, and this fact clarifies an important conceptual aspect of the method of quasiaverages.

Remark 2.7. The states $\omega_{\beta,\mu,\phi}$ in Theorem 2.1 have the second property in Proposition A.1: if $\phi_1 \ne \phi_2$, then $\omega_{\beta,\mu,\phi_1} \ne \omega_{\beta,\mu,\phi_2}$. According to the Kadison theorem [23], two factor states are either disjoint or quasiequivalent, and the states $\omega_{\beta,\mu,\phi}$ for different ϕ are therefore mutually disjoint. This phenomenon also occurs for spontaneous magnetization in quantum spin systems. The word "degeneracy" must be understood in this sense (cf. the discussion in [5]).

3. Bogoliubov quasiaverages and critical quantum fluctuations

The aim in this section is to show that scaled symmetry-breaking external sources can have a nontrivial impact on critical quantum fluctuations. This demonstrates that quasiaverages are useful not only in studying phase transitions via $(\mathrm{SSB})_{\text{q-a}}$ but also in analyzing the corresponding critical, commutative and noncommutative, quantum fluctuations. For this, as an illustration, we use the example of a concrete model manifesting a quantum phase transition with a discrete $(\mathrm{SSB})_{\text{q-a}}$ [24].

3.1. Algebra of fluctuation operators. We first consider a general setup to recall the concept of quantum fluctuations based on the noncommutative central limit theorem and the corresponding CCR.

To describe any quantum statistical model (on a lattice $\mathbb{Z}^d$), we must first define a microscopic dynamical system, which is a triplet $(\mathcal{A}, \omega, \alpha_t)$, where

a. $\mathcal{A} = \cup_\Lambda \mathcal{A}_\Lambda$ is the quasilocal algebra of observables (here, the Λ are a bounded subset of $\mathbb{Z}^d$, and $[\mathcal{A}_{\Lambda'}, \mathcal{A}_{\Lambda''}] = 0$ if $\Lambda' \cap \Lambda'' = \varnothing$),

b. ω is a state on $\mathcal{A}$ (if τ_x is space translation automorphism of translations over the distance $x \in \mathbb{Z}^d$, i.e., the map $\tau_x\colon \mathcal{A}_\Lambda \ni A \to \tau_x(A) \in \mathcal{A}_{\Lambda+x}$, then the state ω is translation invariant if $\omega \circ \tau_x(A) \equiv \omega(\tau_x(A)) = \omega(A)$ and space-clustering if $\lim_{|x|\to\infty} \omega(A\tau_x(B)) = \omega(A)\omega(B)$ for $A, B \in \mathcal{A}$), and

c. α_t is the dynamics described by a family of local Hamiltonians $\{H_\Lambda\}_{\Lambda\subset\mathbb{Z}^d}$ (α_t is usually defined as a norm limit of the local dynamics: $\alpha_t(A) := \lim_\Lambda \exp(itH_\Lambda)A\exp(-itH_\Lambda)$, i.e., $\alpha_t\colon \mathcal{A} \to \overline{\mathcal{A}}$ is the norm closure of $\mathcal{A}$; for equilibrium states, we assume time invariance, $\omega \circ \alpha_t = \omega$).

We usually also assume that space and time translations commute: $\tau_x(\alpha_t(A)) = \alpha_t(\tau_x(A))$, where $A \in \mathcal{A}_\Lambda$ and $\Lambda \subset \mathbb{Z}^d$.

Making the transition from the microsystem $(\mathcal{A}, \omega, \alpha_t)$ to the macrosystem of physical observables, we must distinguish two essentially different classes of macrosystems.

The first class (macrosystem I) corresponds to the weak law of large numbers and is well suited for describing order parameters in the system. This class of observables is formally defined as follows. For any $A \in \mathcal{A}$, we define the local spatial mean by the map

$$m_\Lambda\colon A \to m_\Lambda(A) := |\Lambda|^{-1} \sum_{x\in\Lambda} \tau_x(A).$$

The limit map $m\colon A \to \mathcal{C}$, defined as

$$m(A) = w - \lim_\Lambda m_\Lambda(A), \quad A \in \mathcal{A}, \tag{3.1}$$

then exists in the ω-weak topology induced by the ergodic state ω (see point b in the definition).

Let $m(\mathcal{A}) = \{m(A)\colon A \in \mathcal{A}\}$. Macrosystem I then has the properties that

Ia. $m(\mathcal{A})$ is a set of observables at infinity because $[m(\mathcal{A}), \mathcal{A}] = 0$,

Ib. $m(\mathcal{A})$ is an Abelian algebra, $m(A) = \omega(A)\cdot\mathbb{1}$, and the states on $m(\mathcal{A})$ are hence probability measures,

Ic. the map $m\colon \mathcal{A} \to m(\mathcal{A})$ is not injective, because $m(\tau_a(A)) = m(A)$ (this is a mathematical expression of coarse graining under the weak law of large numbers), and

Id. the macrodynamics $\tilde{\alpha}_t(m(A)) := m(\alpha_t(A))$ induced by the microdynamics α_t on $m(\mathcal{A})$ is trivial because $m(\alpha_t(A)) = \omega(\alpha_t(A)) \cdot \mathbb{1} = \omega(A) \cdot \mathbb{1} = m(A)$.

The second class of macroscopic observables (macrosystem II) corresponds to the quantum central limit theorem and is well suited for describing quantum fluctuations and, in particular, collective and elementary excitations (phonons, plasmons, excitons, etc.) in many-body quantum systems [12].

The construction of macrosystem II must be more precise. Let $A \in \mathcal{A}_{\mathrm{sa}} := \{B \in \mathcal{A}\colon B = B^*\}$ be self-adjoint operators in a Hilbert space $\mathfrak{H}$. We can then define the local map $F^{\delta_A}_{k,\Lambda}\colon A \to F^{\delta_A}_{k,\Lambda}(A)$, where

$$F^{\delta_A}_{k,\Lambda}(A) := \frac{1}{|\Lambda|^{1/2+\delta_A}} \sum_{x\in\Lambda} \big(\tau_x(A) - \omega(A)\big) e^{ikx}, \quad k, \delta_A \in \mathbb{R}. \tag{3.2}$$

This is just the local fluctuation operator A for the mode k. If $\delta_A = 0$, then this fluctuation operator is said to be normal.

The next important concept is due to [25]–[27] and a further development in [28].

Quantum Central Limit Theorem. *Let*

$$\gamma_\omega(r) := \sup_{\Lambda,\Lambda'} \sup_{\substack{A\in\mathcal{A}_\Lambda,\\ B\in\mathcal{A}_{\Lambda'}}} \left\{ \frac{\omega(AB) - \omega(A)\omega(B)}{\|A\|\,\|B\|}\colon\ r \le \operatorname{dist}(\Lambda,\Lambda') \right\}, \qquad \sum_{x\in\mathbb{Z}^d} \gamma_\omega(|x|) < \infty.$$

Then for any $A \in \mathcal{A}_{\mathrm{sa}}$, the corresponding limit characteristic function exists for the normal fluctuation operator ($\delta_A = 0$) for the zero mode $k = 0$:

$$\lim_\Lambda \omega(e^{iuF_\Lambda(A)}) = e^{-u^2 S_\omega(A,A)/2}, \quad u \in \mathbb{R}, \tag{3.3}$$

where $S_\omega(A,B)$ is sesquilinear form

$$S_\omega(A,B) := \operatorname{Re} \sum_{x\in\mathbb{Z}^d} \omega((A - \omega(A))\, \tau_x(B - \omega(B))), \quad A, B \in \mathcal{A}_{\mathrm{sa}}.$$

We list the properties of macrosystem II.

IIa. Result (3.3) explains the meaning of the quantum central limit for normal fluctuation operators. If (3.3) exists for $\delta_{A,B} \ne 0$ with the modified sesquilinear form

$$S_{\omega,\delta_{A,B}}(A,B) = \lim_\Lambda \operatorname{Re} \frac{1}{|\Lambda|^{\delta_A+\delta_B}} \sum_{x\in\mathbb{Z}^d} \omega((A - \omega(A))\, \tau_x(B - \omega(B))), \tag{3.4}$$

then we say that the quantum central limit exists for the zero-mode abnormal fluctuations:

$$\lim_\Lambda F^{\delta_A}_\Lambda(A) = F^{\delta_A}(A). \tag{3.5}$$

The fluctuation operators $F^{\delta_A}(A)$, $A \in \mathcal{A}_{\mathrm{sa}}$, act in a Hilbert space $\mathcal{H}$, which is defined by the reconstruction theorem corresponding to (3.3) and (3.4).

IIb. We regard $\mathcal{A}_{\mathrm{sa}}$ as a vector space with the symplectic form $\sigma_\omega(\,\bullet\,,\,\bullet\,)$, which is well defined in the case $\delta_A + \delta_B = 0$ by the weak law of large numbers:

$$i\sigma_\omega(A,B)\cdot \mathbb{1} = \lim_\Lambda [F^{\delta_A}_\Lambda(A), F^{\delta_B}_\Lambda(B)] = 2i\operatorname{Im} \sum_{y\in\mathbb{Z}^d} \big(\omega(A\tau_y(B)) - \omega(A)\omega(B)\big). \tag{3.6}$$

Let $W(\mathcal{A}_{\rm sa}, \sigma_\omega)$ be the Weyl algebra, i.e., the family of Weyl operators $W\colon \mathcal{A}_{\rm sa} \ni A \mapsto W(A)$ such that

$$W(A)W(B) = W(A+B)e^{-i\sigma_\omega(A,B)/2}, \tag{3.7}$$

where the operators $A, B \in \mathcal{A}_{\rm sa}$ act in the Hilbert space $\mathfrak{H}$.

Reconstruction Theorem. *Let $\tilde{\omega}$ be a quasifree state on the Weyl algebra $W(\mathcal{A}_{\rm sa}, \sigma_\omega)$ defined by the sesquilinear form $S_\omega(\,\bullet\,,\,\bullet\,)$:*

$$\tilde{\omega}(W(A)) := e^{-S_\omega(A,A)/2}. \tag{3.8}$$

It follows from (3.7) *that $W(A) := e^{i\Phi(A)}$, where $\Phi\colon A \mapsto \Phi(A)$ are boson field operators acting in the Hilbert space $\mathcal{H}_{\tilde{\omega}}$ of representations of the CCR corresponding to the state $\tilde{\omega}$. Relations* (3.2)–(3.8) *therefore yield identifications of the spaces $\mathcal{H} = \mathcal{H}_{\tilde{\omega}}$ and of the operators*

$$\lim_{\Lambda} F_\Lambda^{\delta_A}(A) =: F^{\delta_A}(A) = \Phi(A). \tag{3.9}$$

IIc. The reconstruction theorem gives a transition from the microsystem $(\mathcal{A}_{\rm sa}, \omega)$ to the macrosystem of fluctuation operators $(F(\mathcal{A}_{\rm sa}, \sigma_\omega), \tilde{\omega})$. We note that $F(\mathcal{A}_{\rm sa}, \sigma_\omega) = \{F^{\delta_A}(A)\}_{A\in\mathcal{H}_{\rm sa}}$ is the CCR algebra on the symplectic space $(\mathcal{A}_{\rm sa}, \sigma_\omega)$ (see (3.7)–(3.9)).

IId. The map $F\colon \mathcal{A}_{\rm sa} \to F(\mathcal{A}_{\rm sa}, \sigma_\omega)$ is not injective (the zero-mode coarse graining). For example, $\tilde{\tau}_x(F(A)) := F(\tau_x(A)) = F(A)$, but it has a nontrivial macrodynamics $\tilde{\alpha}_t(F(A)) := F(\alpha_t(A))$. Therefore, macrosystem II defined by the algebra of fluctuation operators is the triplet $(F(\mathcal{A}_{\rm sa}, \sigma_\omega), \tilde{\omega}, \tilde{\alpha}_t)$.

The definition of the algebra of fluctuation operators $F(\mathcal{A}_{\rm sa}, \sigma_\omega)$ for a given microsystem $((\mathcal{A}, \omega, \alpha_t))$ with the CCR algebra of the boson field operators allows describing the so-called collective excitations (phonons, plasmons, excitons, etc.) in the pure state ω mathematically.

Deviating from the theme of this paper, we note that this approach also allows mathematically justifying one more physical idea, linear response theory [26]. In this case, it is understandable that the fluctuation algebra is more sensitive to "gentle" perturbations of the microscopic Hamiltonian by external sources than, for example, the algebra at infinity $m(\mathcal{A})$. This property becomes even more significant if the (pure) state ω belongs to the critical domain [24]. Perturbations of the microscopic Hamiltonian that do not change the equilibrium state ω ("gentle" pertubations) can produce different fluctuation algebras independent of quantum or classical nature of the microsystem.

As we understood in Sec. 2.2, the idea that perturbation of the Hamiltonian produces equilibrium states goes back to the Bogoliubov quasiaverages. This method was generalized to mixed states [29]. We recall that it can be formulated as follows.

Let $\{B_l = \tau_l\,(B)\}_{l\in\mathbb{Z}}$ be operators breaking the symmetry of the original system and

$$H_\Lambda(h) := H_\Lambda - \sum_{l\in\Lambda} h_l B_l, \quad h_l \in \mathbb{R}^1.$$

Then the limit states for $h_l = h$,

$$\langle\,\bullet\,\rangle = \lim_{h\to 0}\lim_{\Lambda}\langle\,\bullet\,\rangle_{\Lambda,h}, \tag{3.10}$$

select pure states in the sense of the decomposition corresponding to the symmetry (Bogoliubov quasiaverages).

If the external field $h = \hat{h}/|\Lambda|^\alpha$, then the obvious generalization of (3.10) gives either pure states (for $\alpha < \alpha_{\rm c}$) or a family of mixed states indexed by $\hat{h}$ and $\alpha \geq \alpha_{\rm c}$ (see [29]).

It was shown in [24] that the algebra of fluctuations for a quantum model of a ferroelectric (structural phase transitions) depends on the parameter α in the critical domain (below the critical line) even for pure states, i.e., for $\alpha < \alpha_c = 1$, we obtain $\delta_Q = \alpha/2$ for correlation critical exponents (3.2), while $\delta_P = 0$ (for $T \neq 0$, where T is the temperature). Here, $A := Q$ and $B := P$ are the respective atomic displacement and momentum operators at the site $l = 0$ of the lattice $\mathbb{Z}$. The second observation in [24] concerns the quantum nature of the critical fluctuations $F^{\delta}(\bullet)$, i.e., fluctuations in the pure state ω, which belongs to the critical line. It was shown that expected Abelian properties of critical fluctuations can change into non-Abelian commutations between $F^{\delta_Q}(Q)$ and $F^{\delta_P}(P)$ with $\delta_Q = -\delta_P > 0$ at the quantum critical point $T = 0$, $\lambda = \lambda_c$. Here, $\lambda := \hbar/\sqrt{m}$ is the quantum parameter of the model, where m is the mass of atoms in the sites of the lattice $\mathbb{Z}$.

Because we usually have long-range correlations on the critical line, we can expect that the critical fluctuations are sensitive to the "gentle" perturbations $h = \hat{h}/|\Lambda|^{\alpha}$ mentioned above. On the other hand, they must also be sensitive to decay of a direct interaction between the particles. In our model, the decay of the harmonic force matrix elements is given by

$$\phi_{l,l'} \sim |l - l'|^{-(d+\sigma)} \quad \text{as } |l - l'| \to \infty. \tag{3.11}$$

If $\sigma \geq 2$, then we assume that interaction (3.11) is short-range, and if $0 < \sigma < 2$, then we assume that it is long range because the corresponding discrete Fourier transform has the two types of asymptotic forms as $k \to 0$:

$$\tilde{\phi}(k) \sim \begin{cases} a^{\sigma} k^{\sigma} + o(k^{\sigma}), & 0 < \sigma < 2, \\ a^2 k^2 + o(k^2), & \sigma \geq 2. \end{cases} \tag{3.12}$$

Therefore, our goal is to find exponents δ_A as a function of α and σ for a quantum ferroelectric model with local interaction (3.11). We note that $\delta_Q = \delta_Q(\alpha, \sigma)$ is directly related to the critical exponent η describing decay of the two-point correlation function for displacements on the critical line: $\eta = 2 - 2d\delta_A$ [30].

3.2. Quantum phase transition, fluctuations, and quasiaverages. Let $\mathbb{Z}$ a d-dimensional square lattice. with each lattice site l occupied by a particle of mass m, we associate the position operator $Q_l \in \mathbb{R}^1$ and the momentum operator $P_l = \frac{\hbar}{i}\frac{\partial}{\partial Q_l}$ in the Hilbert space $\mathcal{H}_l = L^2(\mathbb{R}^1, dx)$. Let Λ be a finite cubic subset of $\mathbb{Z}$, $V = |\Lambda|$, and the set Λ^* be dual to Λ with respect to periodic boundary conditions. The local Hamiltonian H_Λ of the model is a self-adjoint operator on the domain $\mathrm{dom}(H_\Lambda) \subset \mathcal{H}_\Lambda$ given by

$$H_\Lambda = \sum_{l\in\Lambda} \frac{P_l^2}{2m} + \frac{1}{4}\sum_{l,l'\in\Lambda} \phi_{l,l'}(Q_l - Q_{l'})^2 + \sum_{l\in\Lambda} U(Q_l) - h\sum_{l\in\Lambda} Q_l. \tag{3.13}$$

Here, the local Hilbert space is $\mathcal{H}_\Lambda := \otimes_{l\in\Lambda}\mathcal{H}_l$. We note that the second term in (3.13) represents the harmonic interaction between particles, the last term represents the action of an external field, and the third term is the anharmonic on-site potential acting in each $l \in \mathbb{Z}$. We recall that potential U must have a double-well form to describe a displacing structural phase transition attributed to the one-component ferroelectric [30]. For example,

$$U(x) = \frac{a}{2}Q_l^2 + W(Q_l^2), \qquad W(x) = \frac{1}{2}bx^2, \quad a < 0, \quad b > 0.$$

Another example is a nonpolynomial U such that $a > 0$ and $W(x) = be^{-\eta x}/2$, $\eta > 0$ for $b > 0$. Then (3.13) becomes

$$H_\Lambda = \sum_{l\in\Lambda} \frac{P_l^2}{2m} + \frac{1}{4}\sum_{l,l'\in\Lambda} \phi_{l,l'}(Q_l - Q_{l'})^2 + \frac{a}{2}\sum_{l\in\Lambda} Q_l^2 + \sum_{l\in\Lambda} W(Q_l^2) - h\sum_{l\in\Lambda} Q_l. \tag{3.14}$$

We recall that model (2.2) manifests a structural phase transition breaking the Z_2 symmetry $\{Q_l \to -Q_l\}_{l\in\mathbb{Z}}$ at low temperatures if the quantum parameter $\lambda < \lambda_c$, [31], [32].

We note that a modified model (3.14) can be solved exactly in the approximation

$$\sum_{l\in\Lambda} W(Q_l^2) \to VW\biggl(\frac{1}{V}\sum_{l\in\Lambda} Q_l^2\biggr),$$

known as the model of self-consistent phonons (SCP) [30]. In this case, we obtain a model with the Hamiltonian

$$H_\Lambda^{\text{SCP}} = \sum_{l\in\Lambda}\frac{P_l^2}{2m} + \frac{1}{4}\sum_{l,l'\in\Lambda}\phi_{l,l'}(Q_l - Q_{l'})^2 + \frac{a}{2}\sum_{l\in\Lambda} Q_l^2 + VW\biggl(\frac{1}{V}\sum_{l\in\Lambda} Q_l^2\biggr) - h\sum_{l\in\Lambda} Q_l, \tag{3.15}$$

which can be solved by the approximating Hamiltonian method [33] (also see [34] and [24]). The free-energy density for the Hamiltonian $H_\Lambda(c)$ approximating (3.15) is the equal to

$$f_\Lambda[H_\Lambda(c)] := -\frac{1}{\beta V}\log \operatorname{Tr}_{\mathcal{H}_\Lambda} e^{-\beta H_\Lambda(c)}, \qquad \beta := \frac{1}{k_{\text{B}} T}. \tag{3.16}$$

According to the approximating Hamiltonian method,

$$\begin{aligned} H_\Lambda(c) := \sum_{l\in\Lambda}\frac{P_l^2}{2m} + \frac{1}{4}\sum_{l,l'\in\Lambda}\phi_{l,l'}(Q_l - Q_{l'})^2 + \frac{a}{2}\sum_{l\in\Lambda} Q_l^2 + \\ + V\biggl[W(c) + W'(c)\biggl(\frac{1}{V}\sum_{l\in\Lambda} Q_l^2 - c\biggr)\biggr] - h\sum_{l\in\Lambda} Q_l, \end{aligned} \tag{3.17}$$

and we can therefore write free-energy density (3.16) in the explicit form

$$\begin{aligned} f_\Lambda[H_\Lambda(c_{\Lambda,h}(T,\lambda))] = \frac{1}{\beta V}\sum_{q\in\Lambda^*}\log\biggl[2\sinh\frac{\beta\lambda\Omega_q(c_{\Lambda,h}(T,\lambda))}{2}\biggr] - \frac{1}{2}\frac{h^2}{\Delta(c_{\Lambda,h}(T_c,\lambda))} + \\ + [W(c_{\Lambda,h}(T,\lambda)) - c_{\Lambda,h}(T,\lambda)W'(c_{\Lambda,h}(T,\lambda))]. \end{aligned}$$

Here, $c = c_{\Lambda,h}(T,\lambda)$ is a solution of the self-consistency equation

$$c = \frac{h^2}{\Delta^2(c)} + \frac{1}{V}\sum_{q\in\Lambda^*}\frac{\lambda}{2\Omega_q(c)}\coth\frac{\beta\lambda}{2}\Omega_q(c). \tag{3.18}$$

The spectrum $\Omega_q(c_{\Lambda,h}(T,\lambda))$, $q\in\Lambda^*$, of $H_\Lambda(c_{\Lambda,h}(T,\lambda))$ is defined by the harmonic spectrum ω_q and by the gap $\Delta(c_{\Lambda,h}(T,\lambda))$:

$$\Omega_q^2(c_{\Lambda,h}(T,\lambda)) := \Delta(c_{\Lambda,h}(T,\lambda)) + \omega_q^2,$$

where $\Delta(c_{\Lambda,h}(T,\lambda)) := a + 2W'(c_{\Lambda,h}(T,\lambda))$ and

$$\omega_q^2 =: \tilde\phi(0) - \tilde\phi(q), \qquad \tilde\phi(q) := \sum_{l\in\Lambda}\phi_{l,0}e^{-iql}.$$

The approximating Hamiltonian method in the thermodynamic limit $\Lambda\to\mathbb{Z}$ gives the stability condition for $H_\Lambda(c_{\Lambda,h}(T,\lambda)) \ge 0$:

$$\Delta(c_h(T,\lambda)) = \lim_\Lambda \Delta(c_{\Lambda,h}(T,\lambda)) \ge 0, \qquad c_h(T,\lambda) := \lim_\Lambda c_{\Lambda,h}(T,\lambda). \tag{3.19}$$

Let $a > 0$ and $W\colon \mathbb{R}_+^1 \to \mathbb{R}_+^1$ be a monotonically decreasing function with $W''(c) \ge w > 0$. Then by the definition of the spectral gap $\Delta(c_{\Lambda,h}(T,\lambda))$ and with (3.19) taken into account, we obtain the stability domain: $D = [c^*,\infty)$, where $c^* = \inf\{c\ge 0\colon \Delta(c)\ge 0\}$ and $\Delta(c^*) = a + 2W'(c^*) = 0$.

Theorem 3.1. *We have the equality*

$$\lim_{\Lambda} f_{\Lambda}[H_{\Lambda}^{\text{SCP}}] = \lim_{\Lambda} \sup_{c \geq c^*} f_{\Lambda}[H_{\Lambda}(c)] =: f(\beta, h). \tag{3.20}$$

By the main theorem of the approximating method [24], the thermodynamics of the systems H_{Λ}^{SCP} and $H_{\Lambda}(c)$ are equivalent for $c = c_{\Lambda,h}(T,\lambda)$ given by (3.18). Therefore, to study the phase diagram of model (3.15), we must consider Eq. (3.18) in the thermodynamic limit $\Lambda \to \mathbb{Z}$:

$$c_h(T,\lambda) = \rho(T,\lambda,h) + I_d(c_h(T,\lambda),T,\lambda). \tag{3.21}$$

Here, we split the thermodynamic limit of integral sum (3.18) into the zero-mode term plus h-term and the rest:

$$\rho(T,\lambda,h) = \lim_{\Lambda} \rho_{\Lambda}(T,\lambda,h) :=$$

$$:= \lim_{\Lambda}\left\{\frac{h^2}{\Delta^2(c_{\Lambda,h}(T,\lambda))} + \frac{1}{V}\frac{\lambda}{2\sqrt{\Delta(c_{\Lambda,h}(T,\lambda))}}\coth\frac{\beta\lambda}{2}\sqrt{\Delta(c_{\Lambda,h}(T,\lambda))}\right\}, \tag{3.22}$$

$$I_d(c_h(T,\lambda),T,\lambda) := \frac{\lambda}{(2\pi)^d}\int_{q\in B_d} d^d q\,\frac{1}{2\Omega_q(c_h(T,\lambda))}\coth\frac{\beta\lambda}{2}\Omega_q(c_h(T,\lambda)),$$

Here, $B_d = \{q \in \mathbb{R}^d : |q| \leq \pi\}$ is the first Brillouin zone.

To analyze the solution of (3.21), we consider two cases $h = 0$ and $h \neq 0$ below.

Case $h = 0$. From (3.21) and (3.22), we easily find that for $T = 0$, there exists a λ_{c} such that $c^* \leq I_d(c^*,0,\lambda)$ for $\lambda \geq \lambda_{\text{c}}$ and $c^* = I_d(c^*,0,\lambda_{\text{c}})$ defines the critical value of the quantum parameter λ. The line $(\lambda, T_{\text{c}}(\lambda))$ of critical temperatures, which separates the phase diagram (λ, T) into two domains (A)-(B), then satisfies the identity

$$c^* = I_d(c^*, T_{\text{c}}(\lambda), \lambda) \quad \text{for } \lambda \leq \lambda_{\text{c}}, \quad T_{\text{c}}(\lambda_{\text{c}}) = 0. \tag{3.23}$$

Taking (3.21) and (3.22) into account, we can express conditions (3.23) as the critical-line equation

$$\rho_{c^*}(T_{\text{c}}(\lambda),\lambda) := \rho(T,\lambda,h)\big|_{c_h(T,\lambda)=c^*} = c^* - I_d(c^*, T_{\text{c}}(\lambda),\lambda) = 0. \tag{3.24}$$

We thus obtain two solutions of (3.21) distinguished by the value of gap (3.19):

$$\rho(T,\lambda,0) = 0, \quad c_0(T,\lambda) > c^* \quad \text{or} \quad \Delta(c_0(T,\lambda)) > 0\colon\ T > T_{\text{c}}(\lambda) \vee \lambda > \lambda_{\text{c}}, \tag{A}$$

$$\rho(T,\lambda,0) \geq 0, \quad c_0(T,\lambda) = c^* \quad \text{or} \quad \Delta(c_0(T,\lambda)) = 0\colon\ 0 \leq T_{\text{c}}(\lambda) \wedge \lambda \leq \lambda_{\text{c}}. \tag{B}$$

For a fixed $\lambda < \lambda_{\text{c}}$, looking along the vertical line ($\lambda = \text{const}$), we observe the well-known temperature-driven phase transition at $T_{\text{c}}(\lambda) > 0$ with an order parameter that can be defined in terms of ρ. On the other hand, for a fixed $T < T_{\text{c}}(0)$, looking along the horizontal line ($T = \text{const}$), we observe a phase transition at λ, $T_{\text{c}}(\lambda) = T$, which is driven by the quantum parameter $\lambda = \hbar/\sqrt{m}$.

We note that for $\lambda > \lambda_{\text{c}}$, i.e., for light atoms, the temperature-driven phase transition is suppressed by quantum tunneling or quantum fluctuations. The decreasing $T_{\text{c}}(\lambda)$ for light atoms is well known as an isotopic effect in ferroelectrics [30]. Because the thermodynamics of model (3.15) and the approximating Hamiltonian $H_{\Lambda}(c_{\Lambda,h}(T,\lambda))$ are equivalent by Theorem 3.1, the proof that we have the same effect in model (3.15) including the existence of λ_{c} therefore follows from the solution of (3.15) and the monotonicity of $\lambda \mapsto I_d(c^*,0,\lambda)$. The proof of the isotopic effect for model (3.13) was obtained in [35] (also see [31], [32]).

Further, we introduce canonical Gibbs states for Hamiltonians (3.14), (3.15), and (3.17):

$$\omega_{\beta,\Lambda,*}(\,\bullet\,) = \frac{\mathrm{Tr}_{\mathcal{H}_\Lambda}[\exp(-\beta H_{\Lambda,*})(\,\bullet\,)]}{\mathrm{Tr}_{\mathcal{H}_\Lambda}\exp(-\beta H_{\Lambda,*})}, \qquad H_{\Lambda,*} = H_\Lambda, H_\Lambda^{\mathrm{SCP}}, H_\Lambda(c). \tag{3.25}$$

We note that these states for $h = 0$ inherit the $\mathbb{Z}^2$ symmetry of Hamiltonians (3.14), (3.15), and (3.17): $Q_l \to -Q_l$:

$$\omega_{\beta,\Lambda,*}(Q_l) = \lim_\Lambda \omega_{\beta,\Lambda,*}(-Q_l) = 0. \tag{3.26}$$

Case $h \neq 0$. In this case, we obtain

$$\omega_{\beta,c_h}(Q_l) = \frac{h}{\Delta(c_h(T,\lambda))}. \tag{3.27}$$

For disordered phase (A), we have $\lim_{h\to 0} c_h(T,\lambda) = c(T,\lambda) > c^*$. Hence, $\Delta(c) > 0$ and

$$\lim_{h\to 0} \omega_{\beta,c_h}(Q_l) = 0. \tag{3.28}$$

For ordered phase (B), we have $\lim_{h\to 0} c_h(T,\lambda) = c^*$, and then by (3.22),

$$\rho_{c^*}(T,\lambda) = c^* - I_d(c^*,T,\lambda) = \lim_{h\to 0} \frac{h^2}{\Delta^2(c_h)} > 0. \tag{3.29}$$

Finally, (3.27) and (3.29) yield the values of the physical order parameters

$$\omega_{\beta,\pm}(Q_l) := \lim_{h\to\pm 0} \omega_{\beta,c_h}(Q_l) = \pm\sqrt{\rho_{c^*}(T,\lambda)} \neq 0. \tag{3.30}$$

Therefore, using Bogoliubov quasiaverage (3.30) and the results in Sec. 2.2, we obtain two extremal translation-invariant equilibrium states $\omega_{\beta,+}$ and $\omega_{\beta,-}$ such that

$$\omega_{\beta,+}(Q_l) = -\omega_{\beta,-}(Q_l) = [\rho_{c^*}(T,\lambda)]^{1/2} \neq 0, \quad l \in \mathbb{Z}. \tag{3.31}$$

In this case, it is easily verified that position and momentum fluctuations are normal, $\delta_Q = \delta_P = 0$ [24]. We return to this observation below in the framework of a more general approach of scaled Bogoliubov quasiaverages.

Definition 3.1. We say that external sources in (3.14), (3.15), and (3.17) correspond to the *scaled Bogoliubov quasiaverage* $h \to 0$ if it is coupled to the thermodynamic limit $\Lambda \uparrow \mathbb{Z}$ by the relation

$$h_\alpha := \frac{\hat{h}}{V^\alpha}, \quad \alpha > 0. \tag{3.32}$$

This choice of the quasiaverage is sufficiently flexible to scan between weak and strong external sources as a function of $\alpha > 0$. This gives a basis for the following proposition [24].

Proposition 3.1. *If* $\alpha < 1$, *then the limit equilibrium states remain pure,*

$$\lim_\Lambda \omega_{\beta,c_h}(Q_l) = \operatorname{sgn}\hat{h}\,[\rho_{c^*}(T,\lambda)]^{1/2},$$

which is similar to the limit $h \to \pm 0$ *of standard* (*nonscaled*) *Bogoliubov quasiaverage* (3.31).

If $\alpha \geq 1$, then the limit state $\omega_{\beta,\hat{h}}(Q_l)$ becomes a mixture of pure states:

$$\omega_{\beta,\hat{h}}(Q_l) = a\omega_{\beta,+}(Q_l) + (1-a)\omega_{\beta,-}(Q_l),$$

where $a := a(\hat{h}, \alpha, \rho_{c^*}(T,\lambda)) \in [0,1]$ is given by

$$a(\hat{h}, \alpha, \rho) = \begin{cases} \frac{1}{2}\left(1 + \frac{\hat{h}}{\xi\sqrt{\rho}}\right), & \alpha = 1, \\ \frac{1}{2}, & \alpha > 1. \end{cases} \tag{3.33}$$

Here, $\xi := \lim_\Lambda[\Delta(c_{\Lambda,h}(T,\lambda))V] = (2\beta\rho)^{-1} + \sqrt{(2\beta\rho)^{-2} + \hat{h}^2/\rho}$.

Our next step is to study the influence of the scaled quasiaverage sources on the quantum fluctuation operators. We consider the zero-mode ($k = 0$ in (3.2)) position and momentum fluctuation operators:

$$F_{\delta_Q}(Q) = \lim_\Lambda \frac{1}{V^{1/2+\delta_Q}} \sum_{i\in\Lambda} \big(Q_i - \omega_{\beta,\Lambda,c_h}(Q_i)\big), \tag{3.34}$$

$$F_{\delta_P}(P) = \lim_\Lambda \frac{1}{V^{1/2+\delta_P}} \sum_{i\in\Lambda} \big(P_i - \omega_{\beta,\Lambda,c_h}(P_i)\big). \tag{3.35}$$

Because the approximating Hamiltonian is quadratic operator form (3.17), we can calculate the variances of fluctuation operators (3.34) and (3.35) explicitly:

$$\begin{aligned} &\lim_\Lambda \omega_{\beta,\Lambda,c_h}\left(\left\{\frac{1}{V^{1/2+\delta_Q}} \sum_{i\in\Lambda}\big(Q_i - \omega_{\beta,\Lambda,c_h}(Q_i)\big)\right\}^2\right) = \\ &\qquad = \lim_\Lambda \frac{1}{V^{2\delta_Q}} \frac{\lambda}{2\sqrt{\Delta(c_{\Lambda,h}(T,\lambda))}} \coth\frac{\beta\lambda}{2}\sqrt{\Delta(c_{\Lambda,h}(T,\lambda))}, \end{aligned} \tag{3.36}$$

$$\begin{aligned} &\lim_\Lambda \omega_{\beta,\Lambda,c_h}\left(\left\{\frac{1}{V^{1/2+\delta_P}} \sum_{i\in\Lambda}\big(P_i - \omega_{\beta,\Lambda,c_h}(P_i)\big)\right\}^2\right) = \\ &\qquad = \lim_\Lambda \frac{1}{V^{2\delta_P}} \frac{\lambda m\sqrt{\Delta(c_{\Lambda,h}(T,\lambda))}}{2} \coth\frac{\beta\lambda}{2}\sqrt{\Delta(c_{\Lambda,h}(T,\lambda))}. \end{aligned} \tag{3.37}$$

Here, $c_h := c_{\Lambda,h}(T,\lambda)$ is a solution of self-consistent equation (3.18) and by the $\mathbb{Z}^2$ symmetry $P_l \to -P_l$ of Hamiltonian (3.15), we have $\omega_{\beta,\Lambda,c_h}(P_l) = 0$ in (3.37) for all $l \in \Lambda$ and for any values of β and h.

We note that existence of nontrivial variances (3.36) and (3.37) suffices to prove the existence of characteristic function (3.3) with the sesquilinear form $S_\omega(\bullet\,,\bullet)$. The next ingredient is the symplectic form $\sigma_\omega(\bullet\,,\bullet)$ corresponding to the fluctuation operator algebra. For this, we must calculate the limit of commutator (3.6). From (3.34) and (3.35), we obtain

$$\lim_\Lambda [F_\Lambda^{\delta_P}(P), F_\Lambda^{\delta_Q}(Q)] = \lim_\Lambda \frac{1}{V^{1+\delta_P+\delta_Q}} \sum_{l,l'\in\Lambda} [P_l, Q_{l'}] = \lim_\Lambda \frac{1}{V^{\delta_P+\delta_Q}} \frac{\hbar}{i}. \tag{3.38}$$

We conclude this subsection with a list of remarks and comments.

1(A). Let $h = 0$ and $[0, \lambda_c] \ni \lambda \mapsto T_c(\lambda)$. If the point (λ, T) on the phase diagram is above the critical line, $T > T_c(\lambda)$, or if $\lambda > \lambda_c$ (see (3.23)), then we have case (A), where $\Delta(c_h(T,\lambda)) > 0$ for

$h = 0$. Consequently, nontrivial variances (3.36) and (3.37) are possible only for $\delta_Q = \delta_P = 0$. Hence, momentum and displacement fluctuation operators (3.34) and (3.35) satisfy the central limit theorem. They are called normal or noncritical fluctuation operators. In this case, commutator (3.38) is nonzero because the operators $F_0(P)$ and $F_0(Q)$ are generators of the non-Abelian algebra of normal fluctuations. Because in this domain of the phase diagram, the order parameter $\rho(T, \lambda, h) = 0$ for $h = 0$ (see (3.22)), we say that this pure phase is disordered. We note that $\omega_{\beta,\Lambda,c_h}(Q_l) = 0$ even without reliance on Z_2 symmetry (3.22).

1(B). Let $h = 0$. If the point (λ, T) on the phase diagram is below the critical line, $T < T_c(\lambda)$ and $\lambda < \lambda_c$, then $\lim_\Lambda c_{\Lambda,h=0}(T, \lambda) = c^*$, i.e., for (3.19), we have $\lim_\Lambda \Delta(c_{\Lambda,h}(T, \lambda))|_{h=0} = 0$, and the order parameter $\rho(T, \lambda, h)|_{h=0} > 0$ (see (3.22)). Consequently, we obtain $\delta_Q = 1/2$ using (3.36) and $\delta_P = 0$ using (3.37), which ensures a nonzero central limit. Hence, the displacement fluctuation operator $F_{1/2}(Q)$ is abnormal, while the momentum fluctuation operator $F_0(P)$ is normal. Based on (3.38), we conclude that the operators $F_{1/2}(Q)$ and $F_0(P)$ given by (3.34) and (3.35) commute, i.e., they generate an Abelian algebra of fluctuations. We note that $\rho(T, \lambda, h)|_{h=0} > 0$. The presence of Z_2 symmetry (3.22) implies that displacement order parameter $\omega_{\beta,c^*}(Q_l) = 0$. Bogoliubov quasiaverage (3.30) gives a nonzero value for this order parameter. This means that ω_{β,c^*} is the one-half mixture of the pure states $\omega_{\beta,\pm}$ given by (3.31) and explains the abnormal displacement fluctuation.

2. We now assume that $h \neq 0$ and consider the standard Bogoliubov quasiaverages.

2(A). Because $\Delta(c_h(T, \lambda)) > 0$, using (3.36) and (3.37), we obtain the finite quasiaverages

$$\lim_{h\to 0} \lim_{\Lambda} \omega_{\beta,\Lambda,c_h}\left(\left\{\frac{1}{V^{1/2}}\sum_{i\in\Lambda}\bigl(Q_i - \omega_{\beta,\Lambda,c_h}(Q_i)\bigr)\right\}^2\right),$$

$$\lim_{h\to 0} \lim_{\Lambda} \omega_{\beta,\Lambda,c_h}\left(\left\{\frac{1}{V^{1/2}}\sum_{i\in\Lambda}\bigl(P_i - \omega_{\beta,\Lambda,c_h}(P_i)\bigr)\right\}^2\right).$$

This leads to normal fluctuations as in case 1(A).

2(B). Because $h \neq 0$, the difference from case 1(B) arises because of $\lim_\Lambda \Delta(c_{\Lambda,h}(T, \lambda)) > 0$ in (3.19) and inequality (3.29), which holds in the ordered phase. The quasiaverage used to calculate displacement variance (3.36),

$$\lim_{h\to 0} \lim_{\Lambda} \frac{1}{V^{2\delta_Q}} \frac{\lambda}{2\sqrt{\Delta(c_{\Lambda,h}(T, \lambda))}} \coth \frac{\beta\lambda}{2}\sqrt{\Delta(c_{\Lambda,h}(T, \lambda))}, \tag{3.39}$$

then has no nontrivial value for any δ_Q. Moreover, the quasiaverage for momentum variance (3.37) is nontrivial only with $\delta_P = 0$.

2^*(B). The difficulty in analyzing case 2(A) is one reason to consider the scaled Bogoliubov quasiaverage with the parameter h_α given by (3.32) instead of (3.39).

1^*(B). We conclude our remarks with the case where $h = 0$ and the point (λ, T) belongs to the critical line: $(\lambda, T_c(\lambda))$ and $\lambda \leq \lambda_c$. Therefore, for $h = 0$, the gap $\lim_\Lambda \Delta(c_{\Lambda,h}(T_c(\lambda), \lambda)) = 0$, and the order parameter $\rho(T_c(\lambda), \lambda, h)|_{h=0} = 0$.

If $\lambda < \lambda_c$, then $T_c(\lambda) > 0$. Hence, with (3.37) taken into, the momentum fluctuation operator is normal, $\delta_P = 0$, while the displacement fluctuation operator is abnormal with the degree $\delta_Q > 0$, which depends on the asymptotic order $O(V^{-\gamma})$, $\gamma > 0$, of the gap $\Delta(c_{\Lambda,h}(T_c(\lambda), \lambda))$ for $h = 0$ in thermodynamic limit. We note that in the scaled limit $\lim_\Lambda \Delta(c_{\Lambda,h_\alpha}(T_c(\lambda), \lambda)) = 0$, the asymptotic order $O(V^{-\gamma})$ and hence $\delta_Q > 0$ can be modified by varying the degree α, although it leaves $\delta_P = 0$ unchanged. We study this phenomenon in the next section. Based on (3.38), we conclude that the corresponding algebra of fluctuations is Abelian.

If $\lambda = \lambda_c$, then $T_c(\lambda_c) = 0$ in (3.22), and we observe a zero-temperature quantum phase transition at the critical point $(0, \lambda_c)$ by varying the quantum parameter λ. In this case, variances (3.36) and (3.37) become

$$\lim_{\Lambda} \frac{1}{V^{2\delta_Q}} \frac{\lambda_{\mathrm{c}}}{2\sqrt{\Delta(c_{\Lambda,h}(0,\lambda_{\mathrm{c}}))}}, \tag{3.40}$$

$$\lim_{\Lambda} \frac{1}{V^{2\delta_P}} \frac{\lambda_{\mathrm{c}} m \sqrt{\Delta(c_{\Lambda,h}(0,\lambda_{\mathrm{c}}))}}{2}. \tag{3.41}$$

Because $\Delta(c_{\Lambda,h_\alpha}(0,\lambda_{\mathrm{c}})) = O(V^{-\gamma})$, expression (3.40) shows that the displacement fluctuation operator is abnormal with the degree $\delta_Q = \gamma/4 > 0$, which can be modified by varying the degree α. The momentum fluctuation operator is also abnormal, but it satisfies the strict condition $\delta_P = -\gamma/4 < 0$, which follows from (3.41). We note that $\delta_Q + \delta_P = 0$ lead to a nontrivial commutator (3.38). Therefore, the algebra of abnormal fluctuations generated by $F_{\delta_Q}(Q)$ and $F_{\delta_P}(P)$ is non-Abelian and possibly depends on α.

In the next section, we elucidate a relation between the definition of quantum fluctuation operators and the scaled Bogoliubov quasiaverages considered in the remarks above.

4. Quasiaverages for critical quantum fluctuations

4.1. Quantum fluctuations below the critical line. We now consider the (difficult) case 2^*(B). We show that the scaled Bogoliubov quasiaverage with the parameter h_α given by (3.32) is well suited for analyzing fluctuations below the critical line.

Proposition 4.1. *Let $0 \le T < T_{\mathrm{c}}(\lambda)$ and $\lambda < \lambda_{\mathrm{c}}$. Then the momentum fluctuation operator is normal, $\delta_P = 0$, while the displacement fluctuation operator is abnormal with a degree $0 < \delta_Q \le 1/2$, which depends on the scaled Bogoliubov quasiaverage parameter α (see (3.32)). The fluctuation algebra is Abelian.*

Proof. Let $0 < \alpha < 1$. Using (3.22), (3.29), and (3.32), we obtain

$$\omega_{\beta,\operatorname{sgn}\hat{h}}(Q_l) = \lim_{\Lambda} \omega_{\beta,\Lambda,c_h}(Q_l) = \lim_{\Lambda} \frac{\hat{h}}{V^\alpha \Delta(c_{\Lambda,h}(T,\lambda))} = \operatorname{sgn}\hat{h}\,\sqrt{\rho_{c^*}(T,\lambda)}. \tag{4.1}$$

This indicates that the limit corresponding to the scaled quasiaverage gives pure state (3.31) and that the variance of the displacement fluctuation operator with (3.36) taken into account is finite only if $\delta_Q = \alpha/2$. In this case,

$$0 < \lim_{\Lambda} \omega_{\beta,\Lambda,c_h}\left(\left\{\frac{1}{V^{1/2+\delta_Q}} \sum_{i\in\Lambda} \bigl(Q_i - \omega_{\beta,\Lambda,c_h}(Q_i)\bigr)\right\}^2\right) = \lim_{\Lambda} \frac{1}{V^{2\delta_Q-\alpha}} \frac{\sqrt{\rho_{c^*}(T,\lambda)}}{2\beta|\hat{h}|} < \infty. \tag{4.2}$$

On the other hand, finite limit (3.37) implies that $\delta_P = 0$, i.e., the momentum fluctuation operator is normal. Because $\alpha > 0$, using (3.38), we conclude that the fluctuation algebra is Abelian.

Now let $\alpha = 1$. Taking (3.21), (3.22), and (3.32) into account, we then obtain

$$\rho_{c^*}(T,\lambda) = \lim_{\Lambda} \rho_\Lambda(T,\lambda,h) = \lim_{\Lambda}\left\{\frac{\hat{h}^2}{[V\Delta(c_{\Lambda,h}(T,\lambda))]^2} + \frac{1}{V\Delta(c_{\Lambda,h}(T,\lambda))}\right\} =$$

$$= c^* - I_d(c^*,T,\lambda) > 0, \tag{4.3}$$

and this implies that $w_{\hat{h}}(T,\lambda) := \lim_\Lambda [V\Delta(c_{\Lambda,h}(T,\lambda))] > 0$ is bounded for $h = \hat{h}/V$. The displacement order parameter then satisfies the relations

$$-\sqrt{\rho_{c^*}(T,\lambda)} < \lim_{\Lambda} \omega_{\beta,\Lambda,c_h}(Q_l) = \lim_{\Lambda} \frac{\hat{h}}{V\Delta(c_{\Lambda,h}(T,\lambda))} = \frac{\hat{h}}{w_{\hat{h}}((T,\lambda)))} < \sqrt{\rho_{c^*}(T,\lambda)}.$$

This means that the equilibrium Gibbs state

$$\omega_{\beta,\hat{h}}(\,\bullet\,) = \xi\omega_{\beta,+}(\,\bullet\,) + (1-\xi)\omega_{\beta,-}(\,\bullet\,), \quad \xi = \frac{1}{2}\left[1 + \frac{\hat{h}}{(w_{\hat{h}}(T,\lambda)\sqrt{\rho_{c^*}(T,\lambda)}\,)}\right] \in (0,1), \tag{4.4}$$

is a convex combination of pure states (4.1). We note that (3.36) and the boundedness of $w_{\hat{h}}(T,\lambda)$ imply that $\delta_Q = 1/2$, while (3.37) gives $\delta_P = 0$. Hence, in the mixed state $\omega_{\beta,\hat{h}}(\,\bullet\,)$, the displacement fluctuations are abnormal, but the momentum fluctuation operator remains normal, and the fluctuation algebra is Abelian, as in the preceding case.

Let $\alpha > 1$. Then again by (3.21), (3.22), and (3.32), we obtain

$$\rho_{c^*}(T,\lambda) = \lim_{\Lambda} \frac{1}{V\Delta(c_{\Lambda,h}(T,\lambda))} = c^* - I_d(c^*,T,\lambda) > 0, \tag{4.5}$$

whence using (3.27), we obtain the order parameter for the displacement operator:

$$\lim_{\Lambda} \omega_{\beta,\Lambda,c_h}(Q_l) = \lim_{\Lambda} \frac{\hat{h}}{V^{\alpha}\Delta(c_{\Lambda,h}(T,\lambda))} = 0. \tag{4.6}$$

We note that scaled quasiaverages (4.5) and (4.6) in ordered phase (B) differ essentially from standard quasiaverages (3.29) and (3.30). Using (3.36) and (3.37), we obtain $\delta_Q = 1/2$ and $\delta_P = 0$, which coincide with the case $\alpha = 1$, including the Abelian fluctuation algebra. The case $\alpha > 1$ is formally equivalent to the case $\alpha = 1$ for $\hat{h} \to 0$, which implies $\xi \to 1/2$ (see (4.4)). This can also be deduced from (4.6).

4.2. Abelian algebra of fluctuations on the critical line. Here, we find the exponents δ_Q and δ_P on the critical line as functions of the parameters d, σ, and α (if it depends on α. For this, we proceed as follows. The critical line is defined by (3.24). Hence, $\rho_{c^*}(T_{\mathrm{c}}(\lambda),\lambda) = 0$, and expression (3.22) in the limit $\lim_{\Lambda} \Delta(c_{\Lambda,h}(T,\lambda)) = c^*$ then becomes

$$\lim_{\Lambda}\left\{\frac{1}{V}\frac{\lambda}{2\sqrt{\Delta(c_{\Lambda,h}(T,\lambda))}}\coth\frac{\beta_{\mathrm{c}}\lambda}{2}\sqrt{\Delta(c_{\Lambda,h}(T,\lambda))} + \frac{\hat{h}^2}{V^{2\alpha}\Delta^2(c_{\Lambda,h}(T,\lambda))}\right\} = 0, \tag{4.7}$$

where $\beta_{\mathrm{c}} := 1/k_{\mathrm{B}}T_{\mathrm{c}}(\lambda)$.

Because we choose $h = \hat{h}/V^{\alpha}$ for scaled quasiaverage (3.32), from (3.22) and (3.24), we find that $\lim_{\Lambda} c_{\Lambda,h}(T_{\mathrm{c}}(\lambda),\lambda) = c^*$. Hence, $\lim_{\Lambda} \Delta(c_{\Lambda,h}(T,\lambda)) = 0$. We must now consider two cases:

a. if $T_{\mathrm{c}}(\lambda) > 0$, then (4.7) is equivalent to

$$\lim_{\Lambda}\left\{\frac{1}{V\Delta(c_{\Lambda,h}(T,\lambda))\beta_{\mathrm{c}}} + \frac{\hat{h}^2}{V^{2\alpha}\Delta^2(c_{\Lambda,h}(T,\lambda))}\right\} = 0, \tag{4.8}$$

b. if $T_{\mathrm{c}}(\lambda_{\mathrm{c}}) = 0$, then (4.7) is equivalent to

$$\lim_{\Lambda}\left\{\frac{\lambda}{2V\sqrt{\Delta(c_{\Lambda,h}(0,\lambda))}} + \frac{\hat{h}^2}{V^{2\alpha}\Delta^2(c_{\Lambda,h}(0,\lambda))}\right\} = 0. \tag{4.9}$$

In both cases, the gap Δ in (4.7) behaves asymptotically as $V \to \infty$ as $\Delta \simeq V^{-\gamma}$ either with $0 < \gamma < 1$ and $0 < \gamma < \alpha$ in (4.8) or with $0 < \gamma < 2$ and $0 < \gamma < \alpha$ in (4.9). We note that Eq. (3.18) is the key for calculating this behavior. To make this obvious, we rewrite (3.18) in the equivalent form

$$(c_{\Lambda} - c^*) + [c^* - I_d(c_{\Lambda},T_{\mathrm{c}}(\lambda),\lambda)] + \\ + \left[I_d(c_{\Lambda},T_{\mathrm{c}}(\lambda),\lambda) - \frac{1}{V}\sum_{\substack{q\in\Lambda^*,\\ q\neq 0}}\frac{\lambda}{2\Omega_q(c_{\Lambda})}\coth\frac{\beta_{\mathrm{c}}\lambda\Omega_q(c_{\Lambda})}{2}\right] = \left(\frac{\hat{h}}{V^{\alpha}\Delta}\right)^2 + \frac{1}{V}\frac{\lambda}{2\sqrt{\Delta}}\coth\frac{\beta_{\mathrm{c}}\lambda\sqrt{\Delta}}{2}, \tag{4.10}$$

where we set $c_\Lambda := c_{\Lambda,h}(T_c(\lambda),\lambda)$ and $\Delta := \Delta(c_{\Lambda,h}(T_c(\lambda),\lambda)$. The asymptotic behavior of the left-hand side of (4.10) results from the assumption that $h = \hat{h}/V^\alpha$ and from the rate of convergence of the Darboux–Riemann sum to the integral $I_d(c_\Lambda, T_c(\lambda), \lambda)$. This together with the asymptotic form of the right-hand side gives the exponent γ.

Proposition 4.2. *If (T,λ) belongs to the critical line $(T_c(\lambda),\lambda)$ with $T_c(\lambda) > 0$, then the spectral gap $\Delta(c_{\Lambda,h}(T,\lambda))$ behaves asymptotically with respect to the volume as*

$$\gamma = \begin{cases} \dfrac{2\alpha}{3}, & \alpha < \alpha_c, \\ \dfrac{1}{2}, & \alpha \geq \alpha_c, \end{cases} \qquad \alpha_c = \frac{3}{4},$$

if $d > 2\sigma$, as

$$\gamma = \begin{cases} \dfrac{2\alpha}{3} + 0, & \alpha < \alpha_c, \\ \dfrac{1}{2} + 0, & \alpha \geq \alpha_c, \end{cases} \qquad \alpha_c = \frac{3}{4},$$

if $d = 2\sigma$, and as

$$\gamma = \begin{cases} 2\alpha \dfrac{\sigma}{d+\sigma}, & \alpha < \alpha_c, \\ \dfrac{1}{2}, & \alpha \geq \alpha_c, \end{cases} \qquad \alpha_c = \frac{1}{2} + \frac{\sigma}{2d},$$

if $\sigma < d < 2\sigma$.

Because $T_c(\lambda) > 0$, the right-hand side of (4.10) has asymptotic behavior (4.8) or

$$O[(V\Delta)^{-1} + (V^\alpha \Delta)^{-2}]. \tag{4.11}$$

We choose α_c such that $O[(V\Delta)^{-1}] = O[(V^{\alpha_c}\Delta)^{-2}]$. We then obviously obtain $O[(V\Delta)^{-1} + (V^{\alpha_c}\Delta)^{-2}] = O[(V\Delta)^{-1}]$ for asymptotic behavior (4.11), i.e., for $\alpha = \alpha_c$, the gap Δ has the same asymptotic behavior as for $\hat{h} = 0$. The three regimes of the potential decreasing in dependence on σ presented in Proposition 4.2 were considered in detail in [36].

Theorem 4.1. *If (T,λ) belongs to the critical line $(T_c(\lambda),\lambda)$ with $T_c(\lambda) > 0$, then the algebra of fluctuation operators is Abelian. The momentum fluctuation operator $F_{\delta_P}(P)$ is normal ($\delta_P = 0$), while the position fluctuation operator $F_{\delta_Q}(Q)$ is abnormal with the critical exponent given by*

$$\delta_Q = \begin{cases} \dfrac{\alpha}{3}, & \alpha < \alpha_c, \\ \dfrac{1}{4}, & \alpha \geq \alpha_c, \end{cases} \qquad \alpha_c = \frac{3}{4},$$

if $d > 2\sigma$, by

$$\delta_Q = \begin{cases} \dfrac{\alpha}{3} + 0, & \alpha < \alpha_c, \\ \dfrac{1}{4} + 0, & \alpha \geq \alpha_c, \end{cases} \qquad \alpha_c = \frac{3}{4},$$

if $d = 2\sigma$, and by

$$\delta_Q = \begin{cases} \alpha \dfrac{\sigma}{d+\sigma}, & \alpha < \alpha_c, \\ \dfrac{\sigma}{2d}, & \alpha \geq \alpha_c, \end{cases}, \qquad \alpha_c = \frac{1}{2} + \frac{\sigma}{2d},$$

if $\sigma < d < 2\sigma$.

Proof. To verify that the algebra of fluctuation operators generated by F^{δ_Q} and F^{δ_P} is Abelian, it suffices to note that the limit of the commutator satisfies the equality

$$\lim_{\Lambda}[F_{\Lambda}^{\delta_P}, F_{\Lambda}^{\delta_Q}] = \lim_{\Lambda} \frac{1}{|\Lambda|^{1+\delta_P+\delta_Q}} \sum_{l,l'\in\Lambda} [P_l, Q_{l'}] = 0.$$

The second part of the theorem follows from (3.36) and (3.37), which on the critical line for $h = \hat{h}/V^{\alpha}$ become

$$\lim_{\Lambda} \omega_{\beta,\Lambda,c_h}\left(\left\{\frac{1}{V^{1/2+\delta_Q}}\sum_{i\in\Lambda}\big(Q_i - \omega_{\beta,\Lambda,c_h}(Q_i)\big)\right\}^2\right) = \lim_{\Lambda}\frac{1}{V^{2\delta_Q}}\frac{k_{\rm B}T_{\rm c}(\lambda)}{\Delta(c_{\Lambda,h}(T_{\rm c}(\lambda)),\lambda)}, \tag{4.12}$$

$$\lim_{\Lambda} \omega_{\beta,\Lambda,c_h}\left(\left\{\frac{1}{V^{1/2+\delta_P}}\sum_{i\in\Lambda}\big(P_i - \omega_{\beta,\Lambda,c_h}(P_i)\big)\right\}^2\right) = \lim_{\Lambda}\frac{1}{V^{2\delta_P}} m k_{\rm B}T_{\rm c}(\lambda). \tag{4.13}$$

Therefore, variance (4.12) is nontrivial if and only if $\delta_Q = \gamma/2$, and (4.13) is nontrivial if and only if $\delta_P = 0$. Here, the value of δ_Q is taken from Proposition 4.2.

If we set $\sigma = 2$ in the theorem, then the statement corresponds to short-range interactions with $\sigma \geq 2$ (see (3.12)). This observation coincides with the result in [24] if we set $\alpha = \infty$, i.e., in the case with no quasiaverage sources.

4.3. Non-Abelian algebra of fluctuations on the critical line. We have the following statement.

Proposition 4.3. *If (T, λ) coincides with the critical point $(0, \lambda_{\rm c})$, then the asymptotic volume behavior of the gap $\Delta(c_{\Lambda,h}(0,\lambda), 0)$ is given by*

$$\gamma = \begin{cases} \dfrac{2\alpha}{3}, & \alpha < \alpha_{\rm c}, \\[2mm] \dfrac{2}{3}, & \alpha \geq \alpha_{\rm c}, \end{cases} \qquad \alpha_{\rm c} = 1,$$

if $d > 3\sigma/2$, by

$$\gamma = \begin{cases} \dfrac{2\alpha}{3} + 0, & \alpha < \alpha_{\rm c}, \\[2mm] \dfrac{1}{2} + 0, & \alpha \geq \alpha_{\rm c}, \end{cases} \qquad \alpha_{\rm c} = 1,$$

if $d = 3\sigma/2$, and by

$$\gamma = \begin{cases} 2\alpha\dfrac{\sigma}{2d+3\sigma}, & \alpha < \alpha_{\rm c}, \\[2mm] \dfrac{\sigma}{d}, & \alpha \geq \alpha_{\rm c}, \end{cases} \qquad \alpha_{\rm c} = \frac{1}{2} + \frac{3\sigma}{4d},$$

if $\sigma/2\sigma < d < 3\sigma/2$.

At the point $(0, \lambda_{\rm c})$ on the critical line, we obtain limit (4.9), i.e., the gap has the asymptotic value $\Delta \simeq V^{-\gamma}$. The right-hand side of (4.10) then has the asymptotic form $O[(V^{\alpha}\Delta)^{-2} + (V\Delta^{1/2})^{-1}]$. Similarly to Proposition 4.2, we define $\alpha = \alpha_{\rm c}$ such that $O[(V^{\alpha}\Delta)^{-2}] = O[(V\Delta^{1/2})^{-1}]$. We must again consider three regimes for the value of σ in Proposition 4.3 [36].

Theorem 4.2. *If (T, λ) coincides with the critical point $(0, \lambda_c)$, then the algebra of fluctuation operators is non-Abelian because the position fluctuation operator $F_{\delta_Q}(Q)$ is abnormal ($\delta_Q > 0$), while the momentum fluctuation operator $F_{\delta_P}(P)$ is supernormal (squeezed) with $\delta_P = -\delta_Q$, and the exponent δ_Q is given by*

$$\delta_Q = \begin{cases} \dfrac{\alpha}{6}, & \alpha < \alpha_c, \\ \dfrac{1}{6}, & \alpha \geq \alpha_c, \end{cases} \qquad \alpha_c = 1,$$

if $d > 3\sigma/2$, by

$$\delta_Q = \begin{cases} \dfrac{\alpha}{6} + 0, & \alpha < \alpha_c, \\ \dfrac{1}{8} + 0, & \alpha \geq \alpha_c, \end{cases} \qquad \alpha_c = 1,$$

if $d = 3\sigma/2$, and by

$$\delta_Q = \begin{cases} \alpha \dfrac{\sigma}{2d + 3\sigma}, & \alpha < \alpha_c, \\ \dfrac{\sigma}{4d}, & \alpha \geq \alpha_c, \end{cases} \qquad \alpha_c = \frac{1}{2} + \frac{3\sigma}{4d},$$

if $\sigma/2\sigma < d < 3\sigma/2$.

Proof. Taking (3.23) into account in $\lim_{\lambda \to \lambda_c - 0}(T_c(\lambda), \lambda) = (0, \lambda_c)$, we obtain $\beta_c = (k_B T_c(\lambda))^{-1} \to \infty$. Variances (3.36) and (3.37) then become

$$\lim_{\Lambda} \omega_{\infty,\Lambda,c_h}\left(\left\{\frac{1}{V^{1/2+\delta_Q}} \sum_{i \in \Lambda} \big(Q_i - \omega_{\beta,\Lambda,c_h}(Q_i)\big)\right\}^2\right) = \lim_{\Lambda} \frac{1}{V^{2\delta_Q}} \frac{\lambda}{\sqrt{\Delta(c_{\Lambda,h}(0, \lambda_c))}},$$

$$\lim_{\Lambda} \omega_{\infty,\Lambda,c_h}\left(\left\{\frac{1}{V^{1/2+\delta_P}} \sum_{i \in \Lambda} \big(P_i - \omega_{\beta,\Lambda,c_h}(P_i)\big)\right\}^2\right) = \lim_{\Lambda} \frac{1}{V^{2\delta_P}} \frac{\lambda m}{2} \sqrt{\Delta(c_{\Lambda,h}(0, \lambda_c))}.$$

Because $\Delta \simeq V^{-\gamma}$, it suffices to apply Proposition 4.3 with $\delta_Q = \gamma/4 = -\delta_P$ to obtain the possible values of δ_Q for nontrivial variances. The non-Abelian nature of the algebra of fluctuation operators follows from commutator (3.38).

We note that the same remark about $\sigma = 2$ at the end of Sec. 4.2 also holds for the quantum fluctuations at the point $(0, \lambda_c)$.

5. Concluding remarks

We have analyzed the Bogoliubov method of quasiaverages for quantum systems in which phase transitions with an order parameter occur.

First, we verified the possibility to use this method in analyzing phase transitions with SSB. For this, we considered examples of the BEC in continuous perfect and interacting systems. The existence of different types of condensation led to the conclusion that only quantities defined in terms of Bogoliubov quasiaverages are physically meaningful (see Secs. 2.2 and 2.3).

The innovation in the second part of the paper is that we presented arguments supporting the use of the Bogoluibov method of the scaled quasiaverages. Taking the structural phase transition as a basic example, we investigated the relation between SSB and critical quantum fluctuations. Our analysis in Sec. 3 showed that the scaled quasiaverages are again a tool suitable for describing the algebra of quantum fluctuation operators. The subtlety of quantum fluctuations already becomes noticeable on the level of the question of the existence of the order parameter, which can be destroyed by quantum fluctuations even at a zero temperature (see Sec. 3.2). We note that the standard Bogoluibov method suffices for this analysis.

A relevance of the scaled Bogoluibov quasiaverages becomes evident for mesoscopic quantum fluctuation operators defined by the quantum central limit because this limit is sensitive to the quantity α determining the behavior of the scaling parameter. In contrast to the non-Abelian algebra of normal fluctuation operators in the disordered phase, the critical quantum fluctuations in the ordered phase and on the critical line depend on α (see Sec. 4). This concerns both abnormal and supernormal (squeezed) quantum fluctuations. They form various Abelian and non-Abelian algebras of fluctuation operators, which all depend on α (see Secs. 4.1 and 4.2).

Several points connected with this paper were developed in [37], [38].

Appendix A

For the reader's convenience, we present the statement of the basic Fannes–Pulé–Verbeure theorem [39] (also see the extension to nonzero momenta in [40] and [12]). Unfortunately, neither [39] nor [40] show that the states $\omega_{\beta,\mu,\phi}, \phi \in [0, 2\pi)$ in the theorem below are ergodic. The simple, but instructive, proof of this was given in [12].

Proposition A.1. *Let $\omega_{\beta,\mu}$ be an analytic gauge-invariant equilibrium state. If ODLRO* (2.45) *holds for $\omega_{\beta,\mu}$, then there exist ergodic states $\omega_{\beta,\mu,\phi}$, $\phi \in [0, 2\pi)$, that are not gauge invariant and satisfy the following conditions*:

1. $\omega_{\beta,\mu,\phi} \neq \omega_{\beta,\mu,\theta}$ *for all $\theta, \phi \in [0, 2\pi)$ such that $\theta \neq \phi$,*

2. *the state $\omega_{\beta,\mu}$ has the decomposition*

$$\omega_{\beta,\mu} = \frac{1}{2\pi} \int_0^{2\pi} d\phi\, \omega_{\beta,\mu,\phi},$$

3. *for each polynomial Q in the operators $\eta(b_0)$ and $\eta(b_0^*)$ and for each $\phi \in [0, 2\pi)$,*

$$\omega_{\beta,\mu,\phi}(Q(\eta(b_0^*), \eta(b_0)X) = \omega_{\beta,\mu,\phi}(Q(\sqrt{\rho_0}e^{-i\phi}, \sqrt{\rho_0}e^{i\phi})X), \quad X \in \mathcal{A}.$$

Following [12], we note that the proof of this proposition is constructive. An important point is that the state $\omega_{\beta,\mu}$ is separating (or faithful), i.e., if $\omega_{\beta,\mu}(A) = 0$, then $A = 0$. This property, which depends on the extension of the states $\omega_{\beta,\mu}$ to the von Neumann algebra $\pi_\omega(\mathcal{A})''$ [15], [41] applies to thermal states but not to ground states, even without their extension. In fact, a ground state (or vacuum) is unfaithful on $\mathcal{A}$ (see Proposition 3 in [42]). We therefore see that thermal states and ground states can differ with regard to the ergodic decomposition in point 2 of Proposition A.1 (also cf. our discussion in Sec. 5).

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