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Entire functions on banach spaces with the U-property

Humberto D. Carrión V.

Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brasil

Correspondence
Humberto D. Carrión V., Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo Caixa Postal 66281 - CEP : 05311-970 - São Paulo, Brasil.
Email: leinad@ime.usp.br

Abstract

Let E be a Banach space without a copy of l_1 and with the U-property. We show that every entire function on E which is weakly continuous on bounded sets is bounded on bounded sets of E. We answer this way, in the affirmative, to a problem raised by Aron, Hervés, and Valdivia in 1983, for these spaces. In particular, this is true for every Banach space which is an M-ideal in its bidual.

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1 | INTRODUCTION

Our aim in this paper is to give a positive solution for the denominated "the ℓ_1-problem" (see [6, p. 138]) in certain Banach spaces. Indeed, in 1977 Valdivia [15] showed that a complex Banach space E is reflexive if and only if every weakly continuous function on bounded sets is weakly uniformly continuous on E. Motivated by this result Aron et al. [1] showed that for every Banach space E, every polynomial which is weakly continuous on bounded sets of E is uniformly continuous. So, the reflexivity is not a necessary condition for the polynomial case. A holomorphic version of this problem was proposed by these authors and it is considered an open problem.

Problem 1.1. If $f : E \to \mathbb{C}$ is a holomorphic function which is weakly continuous on bounded sets is f weakly uniformly continuous?

Dineen in [7] showed that this question has a positive answer if $E = c_0$ (the spaces of all null sequences). A careful analysis of the techniques given by Dineen allows us to conclude that the results follows true for all Banach space with unconditional and shrinking basis. So the reflexivity of E is not a necessary condition for giving a positive answer to Problem 1.1. In [2] there is a version of this result generalized to functions defined in balanced open sets of a Banach space E, with unconditionally and shrinking basis. The name given to Problem 1.1 must be awarded to Aron et al. [1], who proved that a positive answer to Problem 1.1 for ℓ_1, the spaces of summable sequences, implies a positive answer for all Banach spaces E.

A partial answer to Problem 1.1 was given, by the author of this paper, in [3] for separable dual Banach space. However, some years later Valdivia found some mistakes in the technique used.

In this paper we extend the result of Dineen given in [7], and we show that Problem 1.1 has a positive answer when the Banach E has the U-property and does not have a copy of ℓ_1. We finished the paper giving some examples.

2 | NOTATION AND BASIC DEFINITIONS

Now we give some notation. Let E be a complex Banach space with unity sphere $S(E)$ and with dual space E'. We denote by $P(^nE, F)$ the space of all continuous n-homogenous polynomials and by $H(E, F)$ the space of all entire functions from E into F. We denote by $H_{bk}(E, F)$ the space of all functions in $H(E, F)$ which are bounded on weakly compact subsets of E and by $H_b(E, F)$ all functions in $H(E, F)$ which are bounded on bounded subset of E. If $F = \mathbb{C}$, as usual we denote $P(^nE) := P(^nE, F)$ and $H(E) := H(E, \mathbb{C})$. For each $P \in P(^nE)$ we denote by $\overset{\vee}{P}$ the unique continuous n-linear symmetric mapping associated to P. If $f : E \to \mathbb{C}$ is a function and Ω is a compact subset of E, we denote $\|f\|_\Omega = \sup_{x\in\Omega} |f(x)|$, the uniform norm of f on Ω.

For background material about polynomials and holomorphic functions in Banach spaces, we refer the reader to [12] or [6].

Let (y_i) be a sequence in E. Consider the formal series $y = \sum_{i=1}^{\infty} y_i$ and for each $m, n \in \mathbb{N}$, $0 < m < n$, define

$$q^n y := \sum_{i=1}^{n} y_i \text{ and } q_m^n y := \sum_{i=m+1}^{n} y_i.$$

Let $(P_n) \subset P(^nE, F)$ be a sequence of n-homogeneous polynomials where, $n = 1, 2, \ldots$ and $(m_i) \subset \mathbb{N}$ a strictly increasing sequence ($m_0 := 0$). Then, if $n > m_1$, we obtain by Leibniz‘s formula [12, Theorem 1.8] the following inequality:

$$\begin{aligned}\|P_n(q^n(y))\| &= \left\| \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \overset{\vee}{P}_n \left((q^{m_1}y)^j \left(q_{m_1}^n y \right)^{n-j} \right) \right\| \\ &\leqslant \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \left\| \overset{\vee}{P}_n \left((q^{m_1}y)^j \left(q_{m_1}^n y \right)^{n-j} \right) \right\|.\end{aligned}$$

Thus for some $j_{n,m_1} \in \mathbb{N}$ with $0 \leqslant j_{n,m_1} \leqslant n$ we have

$$\|P_n(q^n(y))\| \leqslant (n+1) \frac{n!}{j_{n,m_1}!\left(n - j_{n,m_1}\right)!} \left\| \overset{\vee}{P}_n \left((q^{m_1}y)^{j_{n,m_1}} \left(q_{m_1}^n y \right)^{n-j_{n,m_1}} \right) \right\|.$$

If $n \geqslant m_2$ take

$$Q(y) = \overset{\vee}{P}_n\left((q^{m_1}y)^{j_{n,m_1}}\left(q^n_{m_1}y\right)^{n-j_{n,m_1}}\right),$$

then using again the Leibniz‘s formula, we obtain

$$Q(y) = \sum_{j=0}^{n-j_{n,m_1}} \frac{\left(n-j_{n,m_1}\right)!}{j!\left(n-j_{n,m_1}-j\right)!}\overset{\vee}{P}_n\left((q^{m_1}y)^{j_{n,m_1}}\left(q^{m_2}_{m_1}y\right)^{j}\left(q^n_{m_2}y\right)^{n-j_{n,m_1}-j}\right).$$

Now for some $j_{n,m_2} \in \mathbb{N}$ with $0 \leqslant j_{n,m_2} \leqslant n - j_{n,m_1}$ we have that

$$\begin{aligned}\|Q(y)\| \leqslant{}& \left(n-j_{n,m_1}+1\right)\frac{\left(n-j_{n,m_1}\right)!}{j_{n,m_2}!\left(n-j_{n,m_1}-j_{n,m_2}\right)!}\times\\ &\times\left\|\overset{\vee}{P}_n\left((q^{m_1}y)^{j_{n,m_1}}\left(q^{m_2}_{m_1}y\right)^{j_{n,m_2}}\left(q^n_{m_2}y\right)^{n-j_{n,m_1}-j_{n,m_2}}\right)\right\|.\end{aligned}$$

Set $d_{n,0} := n$, $d_{n,1} := n - j_{n,m_1}$, $d_{n,2} := n - j_{n,m_1} - j_{n,m_2}$.

Therefore

$$\begin{aligned}\left\|P_n(q^n(y))\right\| \leqslant{}& (n+1)\frac{n!}{j_{n,m_1}!\left(n-j_{n,m_1}\right)!}.\left(n-j_{n,m_1}+1\right)\\ &\times\frac{\left(n-j_{n,m_1}\right)!}{j_{n,m_2}!\left(n-j_{n,m_1}-j_{n,m_2}\right)!}\\ &\times\left\|\overset{\vee}{P}_n\left((q^{m_1}y)^{j_{n,m_1}}\left(q^{m_2}_{m_1}y\right)^{j_{n,m_2}}\left(q^n_{m_2}y\right)^{n-j_{n,m_1}-j_{n,m_2}}\right)\right\|\\ ={}& (n+1)\left(n-j_{n,m_1}+1\right)\frac{n!}{j_{n,m_1}!j_{n,m_2}!\left(n-j_{n,m_1}-j_{n,m_2}\right)!}\\ &\times\left\|\overset{\vee}{P}_n\left((q^{m_1}y)^{j_{n,m_1}}\left(q^{m_2}_{m_1}y\right)^{j_{n,m_2}}\left(q^n_{m_2}y\right)^{n-j_{n,m_1}-j_{n,m_2}}\right)\right\|\\ ={}& \left(d_{n,0}+1\right)\left(d_{n,1}+1\right)\frac{n!}{j_{n,m_1}!j_{n,m_2}!d_{n,2}!}\\ &\times\left\|\overset{\vee}{P}_n\left((q^{m_1}y)^{j_{n,m_1}}\left(q^{m_2}_{m_1}y\right)^{j_{n,m_2}}\left(q^n_{m_2}y\right)^{d_{n,2}}\right)\right\|.\end{aligned}$$

By induction we obtain for $n \geqslant m_k$

$$\|P_n(q^n(y))\| \leqslant \Pi_{s=0}^{k-1}\left(d_{n,s}+1\right)\frac{n!}{\Pi_{s=1}^{k} j_{n,m_s}.d_{n,k}!} \times \left\|\overset{\vee}{P}_n\left(\Pi_{s=1}^{k}\left(q_{m_{s-1}}^{m_s}(y)\right)^{j_{n,m_s}}.\left(q_{m_k}^{n}(y)\right)^{d_{n,k}}\right)\right\|, \tag{2.1}$$

where for all r with $0 \leqslant r \leqslant k$, we have that $0 \leqslant \sum_{s=1}^{r} j_{n,m_s} \leqslant n$, and

$$d_{n,r} = n - \sum_{k=0}^{r} j_{n,m_k},$$

$(j_{n,m_0} := 0)$. Now, by the Cauchy‘s inequality combined with [12, Corollary 7.10] we have

$$\begin{aligned}\|P_n(q^n(y))\| &\leqslant \Pi_{s=0}^{k-1}\left(d_{n,s}+1\right)\frac{n!}{\Pi_{s=1}^{k} j_{n,m_s}!d_{n,k}!} \\ &\quad\times \left\|\overset{\vee}{P}_n\left(\Pi_{s=1}^{k}\left(q_{m_{s-1}}^{m_s}y\right)^{j_{n,m_s}}\left(q_{m_k}^{n}y\right)^{d_{n,k}}\right)\right\| \\ &\leqslant \Pi_{s=0}^{k-1}\left(d_{n,s}+1\right)\sup_{|\theta_s|=1}\left\|P_n\left(\sum_{s=1}^{k}\theta_s q_{m_{s-1}}^{m_s}(y)+\theta_{k+1}q_{m_k}^{n}(y)\right)\right\|.\end{aligned}$$

Theses inequalities will be used extensively throughout this article.

The following result is well known, see [6], we include it here for the sake of the reader.

Lemma 2.1. *Let $(P_n)_{n\geqslant 0}$ be a sequence of polynomials such that for all $n \in \mathbb{N}$, we have $P_n \in P(^nE,F)$. Then*

(1) $f = \sum_n P_n \in H(E,F)$ *if and only if* $\limsup_n \|P_n\|_K^{1/n} = 0$ *for all compact subsets K of E;*
(2) $f = \sum_n P_n \in H_{bk}(E,F)$ *if and only if* $\limsup_n \|P_n\|_W^{1/n} = 0$ *for all relatively compact subsets W of E;*
(3) $f = \sum_n P_n \in H_b(E,F)$ *if and only if* $\limsup \|P_n\|_{S(E)}^{1/n} = 0$.

Lemma 2.2. *Let $(d_n) \subset \mathbb{N}$ be a strictly increasing sequence with $d_n \leqslant n$, for every n. For each n let $P_n \in P(^{d_n}E)$ be a polynomial of degree d_n. Suppose that*

(1) $(j_{n,1})_n, (j_{n,2})_n, \dots, (j_{n,i})_n \subset \mathbb{N}$ *are sequences such that* $0 \leqslant \sum_{k=1}^{i} j_{n,k} \leqslant n$, *for every* $n \in \mathbb{N}$,
(2) $(y_{n,1})_n, (y_{n,2})_n, \dots, (y_{n,i})_n \subset E$ *are weakly convergent sequences to* $y_k \in E$, $k = 1, 2, \dots, i$,
(3) (Q_n) *is a sequence of homogeneous polynomials with degree* $\deg(Q_n) := d_{n,i} := n - \sum_{k=0}^{i} j_{n,k}$ $(j_{n,0} := 0)$, *defined by:*

$$Q_n(x) = \Pi_{k=0}^{i-1}\left(d_{n,k}+1\right)\frac{d_n!}{\Pi_{k=1}^{i} j_{n,k}!d_{n,i}!}\overset{\vee}{P}_n\left(\Pi_{k=1}^{i}\left(y_{n,k}-y_k\right)^{j_{n,k}}x^{d_{n,i}}\right).$$

Suposse in addition that for all weakly compact subsets W of E, (P_n) satisfies that

$$\limsup_n \|P_n\|_W^{\frac{1}{n}} = 0.$$

Then for all weakly compact subsets W of E we have

$$\limsup \|Q_n\|_W^{\frac{1}{n}} = 0.$$

Proof. Let W be a weakly compact subset of E. By the Eberlein–Smulian theorem, it is not difficult to verify that the set

$$\Theta \otimes W = \left\{ \sum_{k=1}^{i} \theta_k \left(y_{n,k} - y_k\right) + \theta_{i+1} x \,:\, n \geqslant 1, \left|\theta_k\right| = 1, k = 1, 2, \ldots, i+1, x \in W \right\}$$

is a relatively weakly compact set of E. Since

$$\Pi_{k=0}^{i-1} \left(d_{n,k} + 1\right)^{\frac{1}{n}} \leqslant (d_n + 1)^{\frac{i}{n}} \leqslant (n+1)^{\frac{i}{n}},$$

then by the the Cauchy's inequality we have

$$\begin{aligned} \left|Q_n(x)\right|^{1/n} &= \Pi_{k=0}^{i-1} \left(d_{n,k} + 1\right)^{\frac{1}{n}} \\ &\quad \times \left| \frac{d_n!}{\Pi_{k=1}^{i} j_{n,k}! d_{n,i}!} \overset{\vee}{P}_n \left(\Pi_{k=1}^{i} \left(y_{n,k} - y_k\right)^{j_{n,k}} x^{d_{n,i}} \right) \right|^{\frac{1}{n}} \\ &\leqslant (d_n + 1)^{\frac{i}{n}} \sup_{|\theta_k|=1} \left| P_n \left(\sum_{k=1}^{i} \theta_k \left(y_{n,k} - y_k\right) + \theta_{i+1} x \right) \right|^{\frac{1}{n}} \\ &\leqslant (d_n + 1)^{\frac{i}{n}} \|P_n\|_{\Theta \otimes W}^{\frac{1}{n}} \\ &\leqslant (n+1)^{\frac{i}{n}} \|P_n\|_{\overline{\Theta \otimes W}}^{\frac{1}{n}}, \end{aligned}$$

where $\overline{\Theta \otimes W}$ is the closure in the weak topology of $\Theta \otimes W$ in E. Since $x \in W$ is arbitrary, we have

$$\|Q_n\|_W^{\frac{1}{n}} \leqslant (n+1)^{\frac{i}{n}} \|P_n\|_{\overline{\Theta \otimes W}}^{\frac{1}{n}}.$$

By hypothesis we have that $\limsup_n \|P_n\|_{\overline{\Theta \otimes W}}^{1/n} = 0$. So the claim follows. ☐

3 | RESULTS

In 1983 Dineen [7] showed that if $E = c_0$, the space of null sequences, then

$$H_{bk}(c_0) = H_b(c_0).$$

Here we extend this result to Banach spaces E with the U-property and without a copy of ℓ_1. Now, we recall that a series $\sum_{i=1}^{\infty} y_i$ in E is weakly unconditionally Cauchy (wuC) if for all $\rho \in E'$, we have that $\sum_{i=1}^{\infty} |\rho(y_i)| < \infty$. If the series $\sum_{i=1}^{\infty} y_i$ is wuC and $(n_k), (m_k)$ are sequences of strictly increasing positive integers with $m_k < n_k$ for all k, the sequence $(\sum_{i=m_k}^{n_k} y_i)_k$ weakly converges to zero. Also if (α_i) is a scalar sequence with $\lim \alpha_k = 0$, then $\sum_{k=1}^{\infty} \alpha_k y_k$ converges unconditionally in norm, see [11] or [4].

Definition 3.1. A Banach space E is said to have the U-property if for every weakly Cauchy sequence $(x_n) \subset E$, there exists a wuC-series $\sum y_i$ such that the sequence $(x_n - \sum_{i=1}^{n} y_i)_{n\geqslant 1}$ converges to zero in the weak topology.

This property was defined by Pelczynski in [14].

Lemma 3.2. *Let E be a Banach space, $\sum_{i\geqslant 1} y_i$, a weakly unconditionally Cauchy series and $r \in \mathbb{N}$. Suposse that (d_{n_i}), (n_i) are strictly increasing sequences of positive integers and (P_{n_i}) is a sequence of polynomials such that:*

(1) *for every $i \in \mathbb{N}$,*

$$d_{n_i} \leqslant n_i;$$

(2) *for every $i \in \mathbb{N}$, $P_{n_i} \in P(^{d_{n_i}}E)$ and*

$$\left| P_{n_i}\left(\sum_{s=1}^{n_i} y_s\right)\right| > 1;$$

(3) *for all weakly compact subsets W of E*

$$\limsup_i \left\|P_{n_i}\right\|_W^{\frac{1}{n_i}} = 0.$$

Then, there are a positive integer m_1 and sequences $(i(t))_t$, $(n_{i(t)})_t$, $(j_{n_{i(t)},m_1})_t$ of positive integers such that:

(1)
   (a) *The sequences $(i(t))_t$, $(n_{i(t)})_t$, and $(d_{n_{i(t)}} - j_{n_{i(t)},m_1})_t$ are strictly increasing with*

$$\left(n_{i(t)}\right)_t \subset (n_i),$$

$$i(1) > r,$$

$$n_{i(1)} > m_1.$$

(b) *For every* $t \in \mathbb{N}$,

$$0 \leqslant j_{n_{i(t)},m_1} \leqslant d_{n_i(t)}.$$

(2) *The limit*

$$\delta := \lim_t \frac{j_{n_{i(t)},m_1}}{n_{i(t)}}$$

exists and $0 < \delta < 1$.

(3) *For every* $t \in \mathbb{N}$,

(a)

$$\frac{j_{n_{i(t)},m_1}}{n_{i(t)}} \leqslant 2\delta,$$

and

(b)

$$1 \leqslant \left(d_{n_{i(t)}} + 1\right)\binom{d_{n_{i(t)}}}{j_{n_{i(t)},m_1}} \times \left|\overset{\vee}{P}_{n_{i(t)}}\left(\left(\sum_{s=1}^{m_1} y_s\right)^{j_{n_{i(t)},m_1}}\left(\sum_{s=m_1+1}^{n_{i(t)}} y_s\right)^{d_{n_{i(t)}}-j_{n_{i(t)},m_1}}\right)\right|.$$

Proof. For each $m \in \mathbb{N}$, with $m \leqslant n_i$, we have

$$1 \leqslant \left|P_{n_i}\left(\sum_{s=1}^{n_i} y_s\right)\right| \leqslant \left(d_{n_i} + 1\right)\frac{d_{n_i}!}{j_{n_i,m}!\left(d_{n_i} - j_{n_i,m}\right)} \times \left|\overset{\vee}{P}_{n_i}\left(\left(\sum_{s=1}^{m} y_s\right)^{j_{n_i,m}}\left(\sum_{s=m+1}^{n_i} y_s\right)^{d_{n_i}-j_{n_i,m}}\right)\right| \tag{3.1}$$

for some $j_{n_i,m} \in \mathbb{N}$ with $j_{n_i,m} \leqslant d_{n_i}$.
We claim that there exists a m such that

$$0 < \liminf_i \frac{j_{n_i,m}}{n_i}.$$

Indeed, if $\liminf_i \frac{j_{n_i,m}}{n_i} = 0$ for all m, then given m there exists $n_{i_m} > m$ such that

$$\frac{j_{n_{i_m},m}}{n_{i_m}} < \frac{1}{m}.$$

Now, we get

$$\alpha_m = \begin{cases} e^{-\sqrt{\frac{n_{i_m}}{j_{n_{i_m},m}}}} & if \quad j_{n_{i_m},m} \neq 0 \\ 1/m & if \quad j_{n_{i_m},m} = 0, \end{cases}$$

so

$$\lim \alpha_m = 0$$

$$\lim_m (\alpha_m)^{\frac{j_{n_{i_m},m}}{n_{i_m}}} = \lim_m e^{-\sqrt{\frac{j_{n_{i_m},m}}{n_{i_m}}}} = 1.$$

Since the sequence $(\sum_{s=m+1}^{n_{i_m}} y_s)_m$ weakly converges to zero, $\lim_m \|\alpha_m(\sum_{i=1}^m y_i)\| = 0$, and for all weakly compact subsets W of $E \limsup_n \|P_n\|_W^{1/n} = 0$; by Lemma 2.2 we have

$$0 = \limsup_m \left(d_{n_{i_m}} + 1\right)^{\frac{1}{n_{i_m}}} \left(\frac{d_{n_{i_m}}!}{j_{n_{i_m},m}!\left(d_{n_{i_m}} - j_{n_{i_m},m}\right)}\right)^{\frac{1}{n_{i_m}}} \times \left\|\overset{\vee}{P}_{n_{i_m}}\left(\left(\alpha_m \sum_{s=1}^m y_s\right)^{j_{n_{i_m},m}} \left(\sum_{s=m+1}^{n_{i_m}} y_s\right)^{d_{n_{i_m}} - j_{n_{i_m},m}}\right)\right\|^{\frac{1}{n_{i_m}}}. \tag{3.2}$$

On the other hand, the inequality (3.1) implies that

$$\begin{aligned} &\left(d_{n_{i_m}} + 1\right)^{\frac{1}{n_{i_m}}} \left(\frac{d_{n_{i_m}}!}{j_{n_{i_m},m}!\left(d_{n_{i_m}} - j_{n_{i_m},m}\right)}\right)^{\frac{1}{n_{i_m}}} \\ &\quad \times \left\|\overset{\vee}{P}_{n_{i_m}}\left(\left(\alpha_m \sum_{s=1}^m y_s\right)^{j_{n_{i_m},m}} \left(\sum_{s=m+1}^{n_{i_m}} y_s\right)^{d_{n_{i_m}} - j_{n_{i_m},m}}\right)\right\|^{\frac{1}{n_{i_m}}} \\ &= (\alpha_m)^{\frac{j_{n_{i_m},m}}{n_{i_m}}} \left(d_{n_{i_m}} + 1\right)^{\frac{1}{n_{i_m}}} \left(\frac{d_{n_{i_m}}!}{j_{n_{i_m},m}!\left(d_{n_{i_m}} - j_{n_{i_m},m}\right)}\right)^{\frac{1}{n_{i_m}}} \\ &\quad \times \left\|\overset{\vee}{P}_{n_{i_m}}\left(\left(\sum_{s=1}^m y_s\right)^{j_{n_{i_m},m}} \left(\sum_{s=m+1}^{n_{i_m}} y_s\right)^{d_{n_{i_m}} - j_{n_{i_m},m}}\right)\right\|^{\frac{1}{n_{i_m}}} \\ &\geqslant (\alpha_m)^{\frac{j_{n_{i_m},m}}{n_{i_m}}} \end{aligned}$$

and $\limsup_m (\alpha_m)^{\frac{j_{n_{i_m},m}}{n_{i_m}}} = 1$. This is a contradiction to (3.2).

We fix m_1, satisfying the above condition. Then

$$0 < \delta := \liminf_i \frac{j_{n_i,m_1}}{n_i}.$$

We choose a strictly increasing sequence $(i(t))$ such that $(n_{i(t)})$ is strictly increasing

$$i(1) > r,\ n_{i(1)} > m_1$$

$$\lim_t \frac{j_{n_{i(t)},m_1}}{n_{i(t)}} = \liminf_i \frac{j_{n_i,m_1}}{n_i}$$

$$\frac{j_{n_{i(t)},m_1}}{n_{i(t)}} \leqslant 2\delta, \quad for\ every\ t = 1,2,\ldots$$

As $(n_{i(t)}) \subset (n_i)$ then for every t we have

$$1 \leqslant \left(d_{n_{i(t)}} + 1\right)\binom{d_{n_{i(t)}}}{j_{n_{i(t)},m_1}} \times \left| \overset{\vee}{P}_{n_{i(t)}}\left(\left(\sum_{s=1}^{m_1} y_s\right)^{j_{n_{i(t)},m_1}} \left(\sum_{s=m_1+1}^{n_{i(t)}} y_s\right)^{d_{n_{i(t)}}-j_{n_{i(t)},m_1}}\right)\right\|.$$

Now we show that $\lim_t \frac{j_{n_{i(t)},m_1}}{n_{i(t)}} < 1$. In fact, if $\lim_t \frac{j_{n_{i(t)},m_1}}{n_{i(t)}} = 1$, then

$$\lim_t \frac{n_{i(t)} - j_{n_{i(t)},m_1}}{n_{i(t)}} = 0.$$

As $d_{n_i(t)} \leqslant n_i(t)$ then we have

$$\lim_t \frac{d_{n_{i(t)}} - j_{n_{i(t)},m_1}}{n_{i(t)}} = 0. \tag{3.3}$$

Equation (3.3) leads to a contradiction. In fact, consider

$$\alpha_t := \begin{cases} e^{-\sqrt{\frac{n_{i(t)}}{d_{n_{i(t)}}-j_{n_{i(t)},m_1}}}} & if \quad d_{n_{i(t)}} - j_{n_{i(t)},m_1} \neq 0 \\ 1/t & if \quad d_{n_{i(t)}} - j_{n_{i(t)},m_1} = 0, \end{cases}$$

then $\lim_t \alpha_t = 0$ and $\lim_t \alpha_t^{\frac{d_{n_{i(t)}}-j_{n_{i(t)},m_1}}{n_{i(t)}}} = 1$. Now, since $\lim_t \|\alpha_t(\sum_{s=m_1+1}^{n_{i(t)}} y_s)\| = 0$ and $\sum_{s=1}^{m_1} y_s$ is a fix vector of E, Lemma 2.2 implies that

$$
\begin{aligned}
0 =\ & \limsup_t \left(d_{n_{i(t)}} + 1\right)^{\frac{1}{n_{i(t)}}} \left(\frac{d_{n_{i(t)}}!}{j_{n_{i(t)},m_1}!\left(d_{n_{i(t)}} - j_{n_{i(t)},m_1}\right)!} \right)^{\frac{1}{n_{i(t)}}} \\
& \times \left| \overset{\vee}{P}_{n_{i(t)}} \left(\left(\sum_{s=1}^{m_1} y_s \right)^{j_{n_{i(t)},m_1}} \left(\alpha_t \sum_{s=m+1}^{n_{i(t)}} y_s \right)^{d_{n_{i(t)}} - j_{n_{i(t)},m_1}} \right) \right|^{\frac{1}{n_{i(t)}}} \\
=\ & \limsup_t \alpha_t^{\frac{d_{n_{i(t)}} - j_{n_{i(t)},m_1}}{n_{i(t)}}} \left(d_{n_{i(t)}} + 1\right)^{\frac{1}{n_{i(t)}}} \left(\frac{d_{n_{i(t)}}!}{j_{n_{i(t)},m_1}!\left(d_{n_{i(t)}} - j_{n_{i(t)},m_1}\right)!} \right)^{\frac{1}{n_{i(t)}}} \\
& \times \left| \overset{\vee}{P}_{n_{i(t)}} \left(\left(\sum_{s=1}^{m_1} y_i \right)^{j_{n_{i(t)},m_1}} \left(\sum_{s=m_1+1}^{n_{i(t)}} y_s \right)^{d_{n_{i(t)}} - j_{n_{i(t)},m_1}} \right) \right|^{\frac{1}{n_{i(t)}}} \\
\geqslant\ & \limsup_t \alpha_t^{\frac{d_{n_{i(t)}} - j_{n_{i(t)},m_1}}{n_{i(t)}}} = 1,
\end{aligned}
$$

which is a contradiction. Thus

$$
0 < \delta := \lim \frac{j_{n_{i(t)},m_1}}{n_{i(t)}} < 1.
$$

Finally, the sequence $(d_{n_{i(t)}} - j_{n_{i(t)},m_1})_t$ is unbounded. Otherwise

$$
\lim_t \left(\frac{d_{n_{i(t)}} - j_{n_{i(t)},m_1}}{n_{i(t)}} \right) = 0
$$

and we have already shown that this equation leads to a contradiction. □

Lemma 3.3. *For each $n \in \mathbb{N}$, let $P_n \in P(^nE)$ be a polynomial. If for all weakly compact subsets W of E, $\limsup \|P_n\|_W^{\frac{1}{n}} = 0$, then for all weakly unconditionally Cauchy series, $\sum y_i$, we have*

$$
\limsup_n \left| P_n\left(\sum_{s=1}^{n} y_s \right) \right|^{\frac{1}{n}} = 0.
$$

Proof. We will show the existence of subsequences $(P_{n_{i_k(t)}})_t$ $(k = 1, 2, \ldots)$ of the sequence of polynomials (P_n), which satisfy some properties. Indeed, using the diagonal process we choose the sequence $(P_{n_{i_k(k)}})$, and we obtain a contradiction.

Suppose that $\limsup |P_n(\sum_{i=1}^n y_i)|^{1/n} = \rho > 0$. So, taking a subsequence of (P_n), if necessary, and the series $\rho^{-1}\sum_j y_j$, we can suppose that

$$\left|P_n\left(\sum_{s=1}^{n} y_s\right)\right| \geqslant 1$$

for all n.

Now, we use a procedure analogous to that given in [7, Theorem 7]: As $\lim_n n/(\ln(n+1)) = +\infty$, given $i \in \mathbb{N}$ there exists $n_i \geqslant i$ such that $n_i/(\ln(n_i+1)) \geqslant i$. So we obtain $1/(n_i+1) \geqslant 1/e^{n_i/i}$ and therefore

$$\frac{1}{(n_i+1)^{1/n_i}} \geqslant \frac{1}{e^{\frac{1}{i}}} > \frac{1}{e}, \quad for\ every\ i = 1, 2, \ldots \tag{3.4}$$

and

$$\left|P_{n_i}\left(\sum_{s=1}^{n_i} y_s\right)\right| \geqslant 1$$

for every i.

By Lemma 3.2, there exist a positive integer m_1 and strictly increasing sequences $(i_1(t))$, $(n_{i_1(t)})$, $(n_{i_1(t)} - j_{n_{i_1(t)},m_1})$ with $i_1(1) > 1$, $n_{i_1}(1) > m_1$, and

$$0 \leqslant j_{n_{i_1(t)},m_1} \leqslant n_{i_1(t)}, \quad for\ every\ t = 1, 2, \ldots$$

$$0 < \lim_t \frac{j_{n_{i_1(t)},m_1}}{n_{i_1(t)}} := \delta_1 < 1,\ with\ \frac{j_{n_{i_1(t)},m_1}}{n_{i_1(t)}} < 2\delta_1, \quad for\ every\ t = 1, 2, \ldots$$

$$and$$

$$1 \leqslant \left(n_{i_1(t)} + 1\right)\frac{n_{i_1(t)}!}{j_{n_{i_1(t)},m_1}!\left(n_{i_1(t)} - j_{n_{i_1(t)},m_1}\right)!}$$

$$\times\left|\overset{\vee}{P}_{n_{i_1(t)}}\left(\left(\sum_{s=1}^{m_1} y_s\right)^{j_{n_{i_1(t)},m_1}}\left(\sum_{s=m_1+1}^{n_i} y_s\right)^{n_{i_1(t)}-j_{n_{i_1(t)},m_1}}\right)\right|.$$

Now we define $d_{n_{i_1(t)},m_0} := n_{i_1(t)}$ and $d_{n_{i_1(t)},m_1} := d_{n_{i_1(t)},m_0} - j_{n_{i_1(t)},m_1}$. Then we have that

$$1 \leqslant \left(d_{n_{i_1(t)},m_0} + 1\right)\frac{n_{i_1(t)}!}{j_{n_{i_1(t)},m_1}!\,d_{n_{i_1(t)},m_1}!}$$

$$\times\left|\overset{\vee}{P}_{n_{i_1(t)}}\left((q^{m_1}y)^{j_{n_{i_1(t)},m_1}}\left(q_{m_1}^{n_{i_1(t)}}y\right)^{d_{n_{i_1(t)},m_1}}\right),\right|$$

where $q^{m_1}y = \sum_{s=1}^{m_1} y_s$ and $q_{m_1}^{n_{i_1(t)}}(y) = \sum_{s=m_1+1}^{n_{i_1(t)}} y_s$.

For t with $n_{i_1(t)} > m_1$, consider the polynomial $Q : E \to \mathbb{C}$ defined by

$$Q_t(x) = \left(d_{n_{i_1(t)},m_0} + 1\right) \frac{n_{i_1(t)}!}{j_{n_{i_1(t)},m_1}! d_{n_{i_1(t)},m_1}!} \overset{\vee}{P}_{n_{i_1(t)}} \left((q^{m_1} y)^{j_{n_{i_1(t)},m_1}} (x)^{d_{n_{i_1(t)},m_1}} \right).$$

Then $\deg Q_t(x) = d_{n_{i_1(t)},m_1} \leqslant n_{i_1(t)}$ and

$$\left| Q_t\left(q_{m_1}^{n_{i_1(t)}} y\right)\right| = \left| Q_t\left(\sum_{s=m_1+1}^{n_{i_1(t)}} y_s \right)\right| \geqslant 1.$$

As $q^{m_1} y$ is a fixed vector and, by hypothesis, for every weakly compact subsets $W \subset E$, we have $\limsup_n \|P_n\|_W^{1/n} = 0$, by Lemma 2.2, we have that for every weakly compact subsets $W \subset E$

$$\limsup_t \|Q_t\|_W^{\frac{1}{n_{i_1(t)}}} = 0.$$

Since the sequence $(d_{n_{i_1(t)},m_1})_t$ is strictly increasing, by Lemma 3.2, there exists a positive integer $m_2 > m_1$, and strictly increasing sequences $(i_2(t))$, $(n_{i_2(t)})$, $(d_{i_2(t)} - j_{n_{i_2(t)},m_2})$ with

$$(i_2(t)) \subset (i_1(t)),\ i_2(1) > 2,\ n_{i_2}(1) > m_2,$$

$$0 \leqslant j_{n_{i_2(t)},m_2} \leqslant d_{n_{i_2(t)},m_1} \leqslant n_{i_2(t)},\ \textit{for every}\ t = 1, 2, \ldots$$

$$0 < \lim \frac{j_{n_{i_2(t)},m_2}}{n_{i_2(t)}} = \delta_2 < 1$$

$$\frac{j_{n_{i_2(t)},m_2}}{n_{i_2(t)}} \leqslant 2\delta_2,\quad \textit{for every}\ t = 1, 2, \ldots$$

and

$$1 \leqslant \left(d_{n_{i_2(t)},m_1} + 1\right) \frac{d_{n_{i_2(t)},m_1}!}{j_{n_{i_2(t)},m_2}! \left(d_{n_{i_2(t)},m_1} - j_{n_{i_2(t)},m_2}\right)!} \times \left| \overset{\vee}{Q}_t \left(q_{m_1}^{m_2} y\right)^{j_{n_{i_2(t)},m_2}} \left(q_{m_2}^{n_{i_2(t)}} y\right)^{d_{n_{i_2(t)},m_1} - j_{n_{i_2(t)},m_2}} \right|$$

or

$$1 \leqslant \left(d_{n_{i_2(t)},m_1} + 1\right) \frac{d_{n_{i_2(t)},m_1}!}{j_{n_{i_2(t)},m_2}! \left(d_{n_{i_2(t)},m_1} - j_{n_{i_2(t)},m_2}\right)!}$$
$$\times \left(d_{n_{i_2(t)},m_0} + 1\right) \frac{n_{i_2(t)}!}{j_{n_{i_2(t)},m_1}! d_{n_{i_2(t)},m_1}!}$$
$$\times \left| \overset{\vee}{P}_{n_{i_2(t)}} \left((q^{m_1} y)^{j_{n_{i_2(t)},m_1}} \left(q_{m_1}^{m_2} y\right)^{j_{n_{i_2(t)},m_2}} \left(q_{m_2}^{n_{i_1(t)}} y\right)^{d_{n_{i_2(t)},m_1} - j_{n_{i_2(t)},m_2}} \right) \right|,$$

this is,

$$
\begin{aligned}
1 \leqslant & \left(d_{n_{i_2(t)},m_0}+1\right)\left(d_{n_{i_2(t)},m_1}+1\right) \\
& \times \frac{n_{i_2(t)}!}{j_{n_{i_2(t)},m_1}!\,j_{n_{i_2(t)},m_2}!\left(d_{n_{i_2(t)},m_1}-j_{n_{i_2(t)},m_2}\right)!} \\
& \times \left|\overset{\vee}{P}_{n_{i_2(t)}}\left(\left(q^{m_1}y\right)^{j_{n_{i_2(t)},m_1}}\left(q^{m_2}_{m_1}y\right)^{j_{n_{i_2(t)},m_2}}\left(q^{n_{i_2(t)}}_{m_2}y\right)^{d_{n_{i_2(t)},m_2}}\right)\right|,
\end{aligned}
\tag{3.5}
$$

where $d_{n_{i_2(t)},m_2}=d_{n_{i_2(t)},m_1}-j_{n_{i_2(t)},m_2}$. Observe that $i_2(1)>2$ implies $i_2(2)>i_2(1)>2$.

Proceeding inductively we find

(i) a strictly increasing sequence $(m_i)_{i\geqslant 1}$ of positive integer,
(ii) strictly increasing sequences $(i_k(t))_t$, $(k=1,2,\ldots)$ of positive integers such that every k

$$
\begin{gathered}
\left(i_{k+1}(t)\right)\subset(i_k(t)),\\
i_k(k)>k,\\
n_{i_k}(1)>m_k,
\end{gathered}
$$

(iii) sequences $(j_{n_{i_k(t)},m_k})_t$ $(k=1,2,\ldots)$ of positive integers with

$$0<\lim_t \frac{j_{n_{i_k(t)},m_k}}{n_{i_k(t)}}=\delta_k<1,\ \ for\ every\ k=1,2,\ldots$$

$$\frac{j_{n_{i_k(t)},m_k}}{n_{i_k(t)}}<2\delta_k,\ \ for\ every\ t=1,2,\ldots.$$

and

$$
\begin{aligned}
1 \leqslant & \prod_{s=1}^{k}\left(d_{n_{i_k(t)},m_{s-1}}+1\right).\frac{n_{i_k(t)}!}{d_{n_{i_k(t)},m_k}!\prod_{s=1}^{k}j_{n_{i_k(t)},m_s}!} \\
& \times\left|\overset{\vee}{P}_{n_{i_k(t)}}\left(\prod_{s=1}^{k}\left(q^{m_s}_{m_{s-1}}y\right)^{j_{n_{i_k(t)},m_s}}\left(q^{n_{i_k(t)}}_{m_k}y\right)^{d_{n_{i_k(t)},m_k}}\right)\right|,
\end{aligned}
\tag{3.6}
$$

where $d_{n_{i_k(t)},m_k}:=d_{n_{i_k(t)},m_{k-1}}-j_{n_{i_k(t)},m_k}\leqslant d_{n_{i_k(t)},m_{k-1}}\leqslant n_{i_k(t)}$ and $(d_{n_{i_k(t)},m_k})_t$ are strictly creasing, for $k=1,2,\ldots$

We observe that $(i_k(t))\subset(i_s(t))$ for $k\geqslant s$ and therefore

$$\lim_t \frac{j_{n_{i_k(t)},m_s}}{n_{i_k(t)}}=\delta_s,\ \ for\ every\ k=s,s+1,\ldots$$

$$\frac{j_{n_{i_k(t)},m_s}}{n_{i_k(t)}} < 2\delta_s, \quad for\ every\ k = s, s+1, \ldots\ and\ t = 1, 2, \ldots$$

Now, we show that the sequence (δ_k) is summable. Indeed, as

$$0 \leqslant \sum_{s=1}^{k} j_{n_{i_k(t)},m_s} \leqslant n_{i_k(t)},$$

it is

$$0 \leqslant \sum_{s=1}^{k} \frac{j_{n_{i_k(t)},m_s}}{n_{i_k(t)}} \leqslant 1$$

for all $t = 1, 2, \ldots$ and $k = 1, 2, \ldots$. Then we have

$$\begin{aligned}\sum_{s=1}^{k} \delta_s &= \lim_t \frac{j_{n_{i_k(t)},m_1}}{n_{i_k(t)}} + \lim_t \frac{j_{n_{i_k(t)},m_2}}{n_{i_k(t)}} + \cdots + \lim_t \frac{j_{n_{i_k(t)},m_k}}{n_{i_k(t)}} \\ &= \lim_t \left(\frac{j_{n_{i_k(t)},m_1}}{n_{i_k(t)}} + \frac{j_{n_{i_k(t)},m_2}}{n_{i_k(t)}} + \cdots + \frac{j_{n_{i_k(t)},m_k}}{n_{i_k(t)}} \right) \leqslant 1.\end{aligned}$$

Since k is arbitrary, we obtain that $\sum_{k=1}^{\infty} \delta_k \leqslant 1$.

Consider $(\alpha_i) \in c_0$ such that

$$\Pi_{i=1}^{\infty} \alpha_i^{\delta_i} = 2. \tag{3.7}$$

Since $\frac{j_{n_{i_k(t)},m_s}}{2n_{i_k(t)}} < \delta_s < 1$, for every $k \geqslant s$, $t = 1, 2, \ldots$ and $\lim_s \alpha_s = 0$ we can suppose that

$$\Pi_{s=1}^{k} \alpha_s^{\frac{j_{n_{i_k(k)},m_s}}{n_{i_k(k)}}} > \Pi_{s=1}^{k} \alpha_s^{2\delta_s}.$$

Now, by the inequality (3.6) we obtain that

$$\begin{aligned}G_k(y) :&= \left(\frac{n_{i_k(k)}!}{d_{n_{i_k(k)},m_k}! \Pi_{s=1}^{k} j_{n_{i_k(k)},m_s}!} \right)^{\frac{1}{n_{i_k(k)}}} \\ &\times \left| \overset{\vee}{P}_{n_{i_k(k)}} \left(\Pi_{s=1}^{k} \left(\alpha_s q_{m_{s-1}}^{m_s} y \right)^{j_{n_{i_k(k)},m_s}} \left(q_{m_k}^{n_{i_k(k)}} y \right)^{d_{n_{i_k(k)},m_k}} \right) \right|^{\frac{1}{n_{i_k(k)}}} \\ &= \Pi_{s=1}^{k} \alpha_s^{\frac{j_{n_{i_k(k)},ms}}{n_{i_k(k)}}} \left(\frac{n_{i_k(k)}!}{d_{n_{i_k(k)},k}! \Pi_{s=1}^{k} j_{n_{i_k(k)},m_s}!} \right)^{\frac{1}{n_{i_k(k)}}}\end{aligned}$$

$$\times \left| \overset{\vee}{P}_{n_{i_k(k)}} \left(\Pi_{s=1}^{k} \left(q_{m_{s-1}}^{m_s} y \right)^{j_{n_{i_k(k)},m_s}} \left(q_{m_k}^{n_{i_k(k)}} y \right)^{d_{n_{i_k(k)},m_k}} \right) \right|^{\frac{1}{n_{i_k(k)}}}$$

$$\geqslant \Pi_{s=1}^{k} \alpha_s^{2\delta_s} \frac{1}{\Pi_{s=1}^{k} \left(d_{n_{i_k(k)},m_{s-1}} + 1 \right)^{1/n_{i_k(k)}}}.$$

Now by 3.4, as $i_k(k) \geqslant k$, for every $k = 1, 2, \ldots$, we have

$$\frac{1}{\Pi_{s=1}^{k} \left(d_{n_{i_k(k)},m_{s-1}} + 1 \right)^{1/n_{i_k(k)}}} \geqslant \frac{1}{\Pi_{s=1}^{k} \left(n_{i_k(k)} + 1 \right)^{1/n_{i_k(k)}}} \geqslant \frac{1}{e^{\frac{k}{i_k(k)}}} \geqslant \frac{1}{e}.$$

Thus,

$$G_k(y) \geqslant \frac{1}{e} \Pi_{s=1}^{k} \alpha_s^{2\delta_s},$$

and by (3.7), we have

$$\limsup_k G_k(y) \geqslant \limsup_k \frac{1}{e} \Pi_{s=1}^{k} \alpha_s^{2\delta_s} = 4/e. \tag{3.8}$$

On the other hand, as $\sum_{j=1} y_j$ is a weakly unconditionally Cauchy series, then the series $\sum_{i=1}^{\infty} \sum_{j=m_{i-1}+1}^{m_i} y_i$ is also weakly unconditionally Cauchy and therefore the series

$$\alpha_1 \left(\sum_{k=1}^{m_1} y_i \right) + \alpha_2 \left(\sum_{k=m_1+1}^{m_2} y_i \right) + \cdots + \alpha_k \left(\sum_{k=m_{k-1}+1}^{m_k} y_i \right) + \cdots$$

is unconditionally convergent in norm and the sequence

$$\left(q_{m_k}^{n_{i_k(k)}}(y) \right)_k = \left(\sum_{s=m_k+1}^{n_{i_k(k)}} y_s \right)_k$$

weakly converges to zero. Then the set

$$W = \left\{ \sum_{s=1}^{k} \alpha_s \theta_s q_{m_{s-1}}^{m_s}(y) + \theta_{k+1} . q_{m_k}^{n_{i_k(k)}}(y) \,:\, |\theta_s| = 1,\ k \geqslant 1,\ m_0 := 0 \right\}$$

is relatively weakly compact. So by [12, Corollary 7.10],

$$G_k(y) \leqslant \sup_{|\theta_j|=1} \left\{ \left| P_{n_{i_k(k)}} \left(\sum_{s=1}^{k} \alpha_s \theta_s q_{m_{s-1}}^{m_s} y + \theta_{k+1} . q_{m_k}^{n_{i_k(k)}} y \right) \right|^{\frac{1}{n_{i_k(k)}}} \right\}$$

$$\leqslant \left\| P_{n_{i_k(k)}} \right\|_{\overline{W}}^{\frac{1}{n_{i_k(k)}}},$$

where $\overline{W}$ is the weak closure of W. Now, by assumption, we have

$$\limsup_k G_k(y) \leqslant \limsup_k \left\|P_{n_{i_k(k)}}\right\|_{\overline{W}}^{\frac{1}{n_{i_k(k)}}} = 0.$$

It is a contradiction to inequality (3.8). □

The following theorem generalizes a result obtained by Dineen in [7] for the space c_0.

Theorem 3.4. *Let E be a Banach space with the U-property and without a copy of ℓ_1. Then*

$$H_{bk}(E) = H_b(E).$$

Proof. It is clear that $H_b(E) \subset H_{bk}(E)$. Consider $f = \sum_{n=0}^{\infty} P_n \in H_{bk}(E)$ and suppose that $f \notin H_b(E)$, so $\limsup \|P_n\|^{1/n} = \rho > 0$. We can choose a subsequence, if necessary, and we suppose that $\|P_n\|^{1/n} \geqslant \rho$ for all $n \geqslant 1$. Then, by continuity, there exist a sequence $(x_n) \subset E$ with $\|x_n\| = 1$ for all n, such that

$$\left|P_n(x_n)\right|^{\frac{1}{n}} > \rho/2$$

or $|P_n(\frac{2}{\rho}x_n)| > 1$ for all $n \geqslant 1$. Since E does not have a copy of ℓ_1 and the sequence $(z_n) := (\frac{2}{\rho}x_n)$ is bounded, then by Rosenthal's Theorem, there exists a weakly Cauchy subsequence $(z_{n_i}) := (\frac{2}{\rho}x_{n_i}) \subset (z_n)$. As E has the U-property, there is a weakly unconditionally Cauchy series (wuC) $\sum_{k=1}^{\infty} y_k$, such that the sequence $(z_{n_i} - \sum_{k=1}^{n_i} y_k)_i$ converges to zero in the weak topology. Now, for each n_i we have, by the Leibniz formula,

$$P_{n_i}\left(z_{n_i}\right) = \sum_{j=0}^{n_i} \frac{n_i!}{j!(n_i-j)!} \overset{\vee}{P}_{n_i}\left(\left(z_{n_i} - \sum_{k=1}^{n_i} y_k\right)^j \left(\sum_{k=1}^{n_i} y_k\right)^{n_i-j}\right).$$

So given n_i there exist j_{n_i} with $0 \leqslant j_{n_i} \leqslant n_i$ for each i such that

$$1 \leqslant \left|P_{n_i}\left(z_{n_i}\right)\right| \leqslant (n_i+1)\frac{n_i!}{j_{n_i}!\left(n_i - j_{n_i}\right)!} \tag{3.9}$$
$$\times \left|\overset{\vee}{P}_{n_i}\left(\left(z_{n_i} - \sum_{k=1}^{n_i} y_k\right)^{j_{n_i}} \left(\sum_{k=1}^{n_i} y_k\right)^{n_i - j_{n_i}}\right)\right|.$$

We define the polynomials $Q_{n_i} : E \to \mathbb{C}$ by

$$Q_{n_i}(x) = \frac{(n_i+1)n_i!}{j_{n_i}!\left(n_i - j_{n_i}\right)!} \overset{\vee}{P}_{n_i}\left(\left(z_{n_i} - \sum_{k=1}^{n_i} y_k\right)^{j_{n_i}} (x)^{n_i - j_{n_i}}\right), \quad for\ i = 1, 2, \ldots$$

As $f = \sum_{n=0}^{\infty} P_n \in H_{bk}(E)$ then for all weakly compact subsets W of E we have $\lim \|P_n\|_W^{1/n} = 0$, then by Lemma 2.2, we have

$$0 = \limsup_i \left\|Q_{n_i}\right\|_W^{\frac{1}{n_i}}$$

for all weakly compact subset $W \subset E$. By Lemma 3.3, this implies that

$$\begin{aligned} 0 &= \limsup_i \left|Q_{n_i}\left(\sum_{k=1}^{n_i} y_k\right)\right|^{\frac{1}{n_i}} \\ &= \limsup_i \left|\frac{(n_i+1)n_i!}{j_{n_i}!\left(n_i - j_{n_i}\right)!} \overset{\vee}{P}_{n_i}\left(\left(z_{n_i} - \sum_{k=1}^{n_i} y_k\right)^{j_{n_i}} \left(\sum_{k=1}^{n_i} y_k\right)^{n_i - j_{n_i}}\right)\right|^{\frac{1}{n_i}}, \end{aligned}$$

which is a contradiction by inequality (3.9). □

Corollary 3.5. *Let E be a Banach space with the U-property and without a copy of l_1. Then for every Banach space F we have*

$$H_{bk}(E,F) = H_b(E,F).$$

Proof. It is clear that $H_b(E,F) \subset H_{bk}(E,F)$. Consider $f = \sum_{n=0}^{\infty} P_n \in H_{bk}(E,F)$ and suppose that $f \notin H_b(E,F)$. Then there exist a sequence $(x_n) \in S(E)$ such that $\lim_{n\to\infty} \|f(x_n)\| = \infty$. On the other hand, we have that for every $\phi \in F'$, $\phi \circ f \in H_{bk}(E)$. Thus, by Theorem 3.4, there is $M_\phi \in \mathbb{R}$ such that for every $x \in S(E)$, $|\phi(f(x))| \leqslant M_\phi$. This is, if $y_n = f(x_n) \in F$, then for every $\phi \in F'$, we have that $|\phi(y_n)| \leqslant M_\phi$. By the uniform boundedness principle, we have that $\sup_n \|f(x_n)\| = \sup_n \|y_n\| < \infty$. A contradiction. Thus $f \in H_b(E,F)$ and therefore $H_{bk}(E,F) = H_b(E,F)$. □

We introduce some necessary notation for the next corollary. Let $H_w(E,F)$ denote the space of all entire functions on E into F which are weakly continuous on bounded sets of E. We denote $H_{wu}(E,F)$ the space of all entire functions on E into F which are weakly uniformly continuous on bounded sets of E. In [1] Aron, Herves, and Valdivia posed the following question: Is every weakly continuous function on bounded sets of E also weakly uniformly continuous on bounded sets of E? or is $H_w(E,F) = H_{wu}(E,F)$? for all Banach space E? It is clear that the inclusion $H_{wu}(E,F) \subset H_w(E,F)$ is true. In [1, Example 3.5], they showed that the problem is equivalent to show that the inclusion $H_w(E,F) \subset H_b(E,F)$ holds. Since every weakly continuous function on bounded sets on E is bounded in weakly compact sets on E, Theorem 3.4 implies that when E is a Banach space with the U-property and without a copy of ℓ_1, then $H_{wu}(E,F) \subset H_w(E,F)$. So, for theses spaces the ℓ_1 problem has a positive answer. Also it is true for all subspace $G \subset E$, since the U-property is hereditary.

Let $H_{wsc}(E,F)$ be the space of all entire functions which apply weakly convergent sequences of E into convergent sequences of F. So $H_{wu}(E,F) \subset H_w(E,F) \subset H_{wsc}(E,F)$. In the next corollary we give conditions on E such that we get equality between these spaces.

Corollary 3.6. *Let E be a Banach space with the U-property and without a copy of l_1. Then*

$$H_w(E,F) = H_{wsc}(E,F) = H_{wu}(E,F).$$

Proof. By [1, Proposition 3.3], $H_w(E,F) = H_{wsc}(E,F)$ and by the remark above we have that $H_w(E,F) = H_{wu}(E,F)$. □

We recall that a Banach space has Dunford–Pettis property and does not contain a copy of ℓ_1, if only if E' is a Schur space. This condition permits us to get the next corollary.

Corollary 3.7. *Let E be a Banach space with the U-property such that E' is a Schur space. Then*

$$H_{wu}(E) = H_w(E) = H_b(E) = H_{bk}(E).$$

Proof. Since E' is Schur, then E has the Dunford–Pettis property and by [13, Proposition 4] we have that $H_{bk}(E) = H_w(E)$, once E does not have a copy of ℓ_1. The corollary follows from Theorem 3.4 and Corollary 3.6 because E has the U-property. □

Now, we give some examples of spaces that satisfy the conditions of Corollary 3.7

Example 3.8. Every M-ideal in its bidual is an Asplund space and therefore does not contain a copy of ℓ_1. This space has the U-property, see [9, Theorem 3.8] and [9, Theorem 3.1]). A list of Banach spaces that are M-ideals in their bidual is given in [9, Example 1.4, p. 105].

The spaces $l_2(c_0)$ and $\mathbb{C}\bigoplus_1 c_0$ are examples of Banach spaces without a copy of ℓ_1 with the U-property, which are not M-ideal in their bidual. These are examples of $h - ideals$. See [8, Examples 5, 6 of Section 4] and [8, final remark of Section 8].

There are spaces $C(K)$ which are not M-ideals in their biduals, yet all separable subspaces of $C(K)$ are isomorphic to subspaces of c_0, in particular this is an Asplund space (see [10, Example 2]). For such space we have that the conclusion of Theorem 3.4 is true. In fact, if $f \in H_{bk}(C(K))$ and $f \notin H_b(C(K))$, then there exist a bounded sequence $(x_n) \subset C(K)$ such that $|f(x_n)| > n$ for n. Now, consider $G = \overline{span(x_n)}$ the closed subspace generated by (x_n), F a closed subspace of c_0, $T : F \to G$ the isomorphism, and $(a_n) \subset F$ the bounded sequence in G such that $T(a_n) = x_n$. Let $i : G \to C(K)$ be the inclusion operator. Then we have that $f \circ i \circ T \in H_{bk}(F)$. Since $F \subset c_0$ we obtain that $f \circ i \circ T \in H_b(F)$, that means $(f \circ i \circ T(a_n))_n = (f(x_n))_n$ is bounded. It is a contradiction.

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