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The Riemann-Roch Theorem and Serre Duality

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Resumo

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Neste texto estudamos a cohomologia de feixes em esquemas, culminando no teorema da dualidade de Serre. Começamos estudando cohomologia de feixes em espaços gerais, a relacionando com a cohomologia de Čech. Após introduzir algumas ferramentas e técnicas, nós nos especializamos para o caso de esquemas, começando com esquemas afins e passando para esquemas projetivos sobre anéis. Após isso, estudamos um dos mais impotantes teoremas em geometria algébrica, a dualidade de Serre. Para isto, introduzimos os funtores Ext e os relacionamos com a cohomologia de feixes. No último capítulo, discutimos uma aplicação da teoria na forma do teorema de Riemann-Roch, um teorema poderoso sobre curvas algébricas. O caminho geral do texto segue essencialmente como em [Har77], porém se utilizando do maquinário de categorias derivadas e sequências espectrais. Certas provas e resultados também foram tirados de [SD17], [Ill04], [MO15] e [GW10].

Palavras-chave: Cohomologia, Dualidade de Serre, Esquemas, Geometria Algébrica, Teorema de Riemann-Roch.

Abstract

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In this survey we study sheaf cohomology of schemes, culminating in the Serre duality theorem. We start by studying sheaf cohomology in general spaces, relating it to Čech cohomology. After introducing some tools and techniques we specialize to the case of schemes, starting with affine schemes and eventually passing to projective schemes over rings. Afterward we study one of the most important theorems in algebraic geometry, Serre duality. For that we introduce Ext functors and relate them to sheaf cohomology. In the last chapter we also give an application of the theory in the form of the Riemann-Roch theorem, a powerful theorem about algebraic curves. The general path followed is roughly the same as the one seen in [Har77], but we use the language of derived categories and spectral sequences. Certain proofs and results where also taken from [SD17], [Ill04], [MO15] and [GW10].

Keywords: cohomology, Serre Duality, Schemes, Algebraic Geometry, Riemann-Roch Theorem.

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Conventions, Notations and Prerequisites

Conventions and Notations

Unless otherwise stated, every ring A will be assumed to be commutative and unital. In this case, we denote the category of A-modules by Mod_A. Given an A-module M we always denote its dual module $\mathrm{Hom}_A(M, A)$ by $M^\vee$. Given a topological space X we denote the category of sheaves over X by $\mathsf{Sh}(X)$ and the category of abelian sheaves (i.e., sheaves of abelian groups) over X by $\mathsf{Ab}(X)$. If X is endowed with a sheaf of commutative rings $\mathcal{O}_X$ (i.e. $(X, \mathcal{O}_X)$ is a ringed space), we denote by Mod_X its category of $\mathcal{O}_X$-modules, by $\mathsf{QCoh}(X)$ its category of quasi-coherent sheaves and by $\mathsf{Coh}(X)$ its category of coherent sheaves.

Given an abelian category $\mathcal{A}$, we denote by $\mathsf{D}^*(\mathcal{A})$ ($* = \varnothing, +, -, \flat$) its derived category (and its bounded variants). In the case of $\mathcal{A} = \mathsf{Mod}_A$ or $\mathcal{A} = \mathsf{Mod}_X$ we also denote the derived category by $\mathsf{D}^*(A)$ or $\mathsf{D}^*(X)$ respectively. Given an object $A \in \mathcal{A}$ we also denote by A the complex concentrated in degree 0, $\cdots \to 0 \to A \to 0 \to \dots$ inside $\mathsf{D}(\mathcal{A})$.

During this survey we'll commonly use an equal sign (=) for canonical isomorphisms.

Prerequisites

During the writing of this survey I was faced with the hard task of determining exactly how much background I should assume from a possible reader. At first I decided to assume as little as I could and try to develop most of the theory needed. I soon realized that if I was to do that, I would end up writing a book hundreds of pages long. With that in mind, I scrapped what I had done so far and started again, this time assuming some prerequisites. The following is a short list containing most of the knowledge I decided to assume other than standard undergrad courses.

- Firstly, we assume the reader is well acquainted with category theory. More specifically, we assume prior knowledge in limits, adjunctions, abelian categories, etc. All of these can be found in the first two volumes of Borceux's "Handbook of Categorical Algebra", [Bor94a; Bor94b].

- Basic Knowledge in commutative algebra. Roughly equivalent to what is in [AM16].

- Some modern algebraic geometry. More specifically, we assume prior knowledge of schemes and morphisms of schemes, along with some of its properties. The classical reference is [Har77], which essentially covers the prerequisites in the first two chapters.

- We also assume a good knowledge in homological algebra. This is probably the topic from which we assume the most. We assume essentially what can be found in the first five chapters (minus chapter 4) of [Wei94] and also, a great deal of knowledge in derived categories, which can be found for example in [GM99].

Although we assume all of this, we'll recall some definitions and notions whenever we feel the need.

Chapter 1

Cohomology of Sheaves

We begin this survey by defining cohomology groups of sheaves on topological spaces before focusing on the cases that will be of interest for us, namely, schemes.

1.1 $\mathbf{R}\Gamma$ and $\mathbf{R}f_*$

Let $(X, \mathcal{O}_X)$ be a ringed space. One can show that Mod_X has enough injectives. Let $\Gamma = \Gamma(X, \_)\colon \mathsf{Mod}_X \to \mathsf{Ab}$ be the global sections functor (i.e. $\mathcal{F} \mapsto \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$). Then Γ is additive and left exact, but is not right exact in general. By general abstract nonsense its derived functor $\mathbf{R}\Gamma\colon \mathsf{D}(X) \to \mathsf{D}(\mathsf{Ab})$ exists[1] and so, we define the *cohomology groups* of a complex $\mathcal{F}^\bullet \in \mathsf{D}(X)$ by

$$H^n(X, \mathcal{F}^\bullet) := H^n(\mathbf{R}\Gamma(X, \mathcal{F}^\bullet)).$$

Calculating derived functors in general is quite difficult and in fact the existence of the derived global sections functor (among other important derived functors, such as $\mathbf{R}f_*$, $\mathbf{L}f^*$, etc) on the unbounded derived category was an open problem for a long time, being finally solved by N. Spaltenstein in 1988 in [Spa88]. Luckily enough we will only deal with derived functors in the case where the complex $\mathcal{F}^\bullet$ is bounded below, and so, we have the "classic" method of computing derived functors. Given $\mathcal{F}^\bullet$ bounded below, we find a quasi-isomorphism[2] $\mathcal{F}^\bullet \xrightarrow{\sim} I^\bullet$, where $I^\bullet$ is a bounded below complex of injective sheaves (e.g. an injective resolution if $\mathcal{F}^\bullet$ is a sheaf) and then we have that

$$\mathbf{R}\Gamma(X, \mathcal{F}^\bullet) \cong \Gamma(X, I^\bullet).$$

It is then easy to see that if $\mathcal{F}^i = 0$ for all $i < a$ then $\mathbf{R}\Gamma(X, \mathcal{F}^\bullet)^i = 0$ for all $i < a$. Moreover, as usual with derived functors, we can also use Γ-acyclic sheaves[3] to compute $\mathbf{R}\Gamma$, i.e. change $I^\bullet$ in the above discussion by a complex whose components are all Γ-acyclic sheaves.

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[1] A classical reference is [GM99] but it only deals with derived functors in bounded derived categories. For more complete and modern references dealing with the unbounded case see [KS06, Chapters 13 and 14] or [Lip09, Chapter 2].

[2] i.e., a morphism which induces isomorphisms in cohomology.

[3] Given a left exact additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories, suppose we can define the right derived functor of F. An object $A \in \mathcal{A}$ is said to be F-acyclic if $\mathbf{R}^n F(A) := H^n(\mathbf{R}F(A)) = 0$ for all $n > 0$.

Remark 1.1.1. The definition of the global sections functor we gave can be quite ambiguous, in the sense that for a $\mathcal{O}_X$-module $\mathcal{F}$ we can think of $\Gamma(X, \mathcal{F})$ in two ways:

- As the composition of the forgetful functor $\mathsf{Mod}_X \to \mathsf{Ab}(X)$ with the "abelian" global sections functor $\Gamma_{\mathsf{Ab}} \colon \mathsf{Ab}(X) \to \mathsf{Ab}$.
- As the composition of the global sections functor $\Gamma_X \colon \mathsf{Mod}_X \to \mathsf{Mod}_{\mathcal{O}_X(X)}$ with the forgetful functor $\mathsf{Mod}_{\mathcal{O}_X(X)} \to \mathsf{Ab}$.

Obviously these two functors coincide, both sends an $\mathcal{O}_X$-module $\mathcal{F}$ to the underlying abelian group of the $\mathcal{O}_X(X)$-module $\mathcal{F}(X)$ and similarly for morphisms, but the calculation of their derived functors gives us more ways to compute $\mathbf{R}\Gamma$. In the first case, it's easy to see that the forgetful functor $\mathsf{Mod}_X \to \mathsf{Ab}(X)$ is exact and takes injective $\mathcal{O}_X$-modules to flasque abelian sheaves[4] which are Γ-acyclic, so, by more abstract nonsense, to compute $\mathbf{R}\Gamma$ we can think of a bounded below complex of $\mathcal{O}_X$-modules $\mathcal{F}^\bullet$ as a complex of abelian sheaves, find a quasi-isomorphism $\mathcal{F}^\bullet \xrightarrow{\sim} I^\bullet$ to a complex of abelian injective sheaves and then apply the global sections functor. The second case gives us the computation we have already been using, find a quasi-isomorphism $\mathcal{F}^\bullet \xrightarrow{\sim} I^\bullet$ to a complex of injective $\mathcal{O}_X$-modules and then apply the global sections functor.

For any open set $U \subseteq X$ we can also define the right derived functor $\mathbf{R}\Gamma_U$ of the left exact functor

$$\Gamma_U \colon \mathsf{Mod}_X \to \mathsf{Ab},$$

given by $\mathcal{F} \to \Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ and then define

$$H^n(U, \mathcal{F}^\bullet) = H^n(\mathbf{R}\Gamma_U(\mathcal{F}^\bullet)).$$

All remarks made about the calculation of $\mathbf{R}\Gamma$ also apply here and moreover, once again we have some kind of ambiguity, we can think of Γ_U in an alternative way. Γ_U can be seen as the composition of the restriction functor $\mathsf{Mod}_X \to \mathsf{Mod}_U$, which is exact, with the global sections functor $\Gamma \colon \mathsf{Mod}_U \to \mathsf{Ab}$. Once again all of these interpretations gives us the same result with different methods of computing it.

Now we turn the discussion to a different but related functor, the *derived pushfoward functor*. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces and $f \colon X \to Y$ a morphism of ringed spaces[5]. Recall that we have an additive left exact functor, the pushfoward (or direct image) functor $f_* \colon \mathsf{Mod}_X \to \mathsf{Mod}_Y$, where $f_*\mathcal{F}$ is given by

$$V \mapsto \mathcal{F}(f^{-1}(V)).$$

[4]Recall that a sheaf $\mathcal{F}$ is flasque if for all open sets $V \subseteq U \subseteq X$ the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. We'll say a little bit more about flasque sheaves in the next section.

[5]i.e. a continuous map $f \colon X \to Y$ and a morphism of sheaves of commutative rings $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Where $(f_*\mathcal{O}_X)(V) = \mathcal{O}_X(f^{-1}(V))$.

For all $V \subseteq Y$ open. Similarly to the global sections functor, its derived functor $\mathbf{R}f_*\colon \mathsf{D}(X) \to \mathsf{D}(Y)$ exists and can be computed in the same way as $\mathbf{R}\Gamma$[6]. Once again we have that flasque sheaves are f_*-acyclic and moreover, by definition, f_* takes flasque sheaves to flasque sheaves. This discussion and the fact that

$$\Gamma(X, \_) = \Gamma(Y, \_) \circ f_*$$

implies that

$$\mathbf{R}\Gamma(X, \_) = \mathbf{R}\Gamma(Y, \_) \circ \mathbf{R}f_* \, .$$

Similarly, if we have morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of ringed spaces, then

$$\mathbf{R}(g \circ f)_* = \mathbf{R}g_* \circ \mathbf{R}f_* \, .$$

We also have the following alternative characterization of the higher direct image functors.

Proposition 1.1.2. Given a morphism $f\colon X \to Y$ of ringed spaces, and a $\mathcal{O}_X$-module $\mathcal{F}$, we have that $\mathbf{R}^i f_*(\mathcal{F})$ is canonically isomorphic to the sheafification of the presheaf given by

$$U \mapsto H^i(f^{-1}(U), \mathcal{F}).$$

Proof. See [SD17, Lemma 11.2]. ■

1.2 Flasque Sheaves

As we said before, a sheaf $\mathcal{F}$ is *flasque* if for all open sets $V \subseteq U \subseteq X$ the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. It's clear from the definition that if $\mathcal{F}$ is flasque then $f_*\mathcal{F}$ and $\mathcal{F}|_U$ are also flasque. Now lets prove a proposition that essentially justify all of the assertions about flasque sheaves made on the preceding section.

Proposition 1.2.1. Let X be a ringed space.

1. Suppose
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
is an exact sequence in Mod_X with $\mathcal{F}'$ a flasque sheaf. Then
$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$$
is also exact.

2. Under the same assumptions of 1., if $\mathcal{F}$ is also flasque, then $\mathcal{F}''$ is flasque.

3. An injective sheaf is flasque.

Proof. 1. The proof of 1. is quite tiresome and so we relegate the task to the reader. Alternatively, check out [Ill04, Proposition 6.2].

[6] In fact, Γ can be seen as a special case of f_* where $Y = *$ and $\mathcal{O}_Y = \mathbb{Z}$.

2. Given open sets $V \subseteq U \subseteq X$, since the restriction functor $\mathsf{Mod}_X \to \mathsf{Mod}_U$ is exact and sends flasque sheaves to flasque sheaves, we have by 1. the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}'(U) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}''(U) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}'(V) \arrow[r] & \mathcal{F}(V) \arrow[r] & \mathcal{F}''(V) \arrow[r] & 0\,. \end{tikzcd}$$

By hypothesis the two left vertical arrows are surjective, and so, we conclude that the last arrow is also surjective.

3. This proof needs a little more machinery, so let me use the opportunity to introduce it now. If $U \overset{i_U}{\hookrightarrow} X$ is the inclusion of an open set, one can show that i_U^*[7] is simply given by the restriction functor $\mathcal{F} \mapsto \mathcal{F}|_U$ and moreover has a left adjoint $i_{U!} \colon \mathsf{Mod}_U \to \mathsf{Mod}_X$ which sends a sheaf $\mathcal{F}$ in Mod_U to the sheafification of the presheaf $\mathcal{F}^0$ given by

$$\mathcal{F}^0(V) = \begin{cases} \mathcal{F}(V), & \text{if } V \subseteq U. \\ 0, & \text{otherwise.} \end{cases}$$

$i_{U!}$ is known as the *extension by zero functor* because one can show that $(i_{U!}\mathcal{F})_x = \mathcal{F}_x$ if $x \in U$ and zero otherwise. In particular, we have that for any $\mathcal{O}_X$-module $\mathcal{F}$

$$\mathcal{F}(U) = \mathrm{Hom}_U(\mathcal{O}_U, \mathcal{F}|_U) = \mathrm{Hom}_X(i_{U!}\mathcal{O}_U, \mathcal{F}).$$

Given open sets $V \subseteq U \subseteq X$ one can show by hand that we have a canonical map $i_{V!}\mathcal{O}_V \to i_{U!}\mathcal{O}_U$ which is given at the level of presheaves by the map $\iota \colon \mathcal{O}_V^0 \to \mathcal{O}_U^0$, which is the identity in open sets $W \subseteq V$ and 0 otherwise. It´s clear that this map is a monomorphism. Now, given an injective sheaf I, we obtain a surjection

$$I(U) = \mathrm{Hom}_X(i_{U!}\mathcal{O}_U, I) \twoheadrightarrow \mathrm{Hom}_X(i_{V!}\mathcal{O}_V, I) = I(V).$$

By the universal property of the sheafification, we have that $\mathrm{Hom}_X(i_{U!}\mathcal{O}_U, I) = \mathrm{Hom}_X(\mathcal{O}_U^0, I)$. So we have

$$I(U) = \mathrm{Hom}_X(\mathcal{O}_U^0, I) \twoheadrightarrow \mathrm{Hom}_X(\mathcal{O}_V^0, I) = I(V)\ ,$$

given by

$$\alpha \mapsto \phi = \begin{cases} 1 \mapsto \alpha|_W\,, & \text{if } W \subseteq U. \\ 0, & \text{otherwise.} \end{cases} \mapsto \phi \circ \iota \mapsto (\phi \circ \iota)_V(1) = \alpha|_V\,.$$

[7] Recall that given a morphism of ringed spaces $f \colon X \to Y$ the *pullback* functor $f^* \colon \mathsf{Mod}_Y \to \mathsf{Mod}_X$ is the left adjoint to f_{U*}.

In conclusion, we find that the surjection described above is the restriction map, proving the claim.

■

The preceding proposition is essential to prove that flasque sheaves are f_*-acyclic. Indeed, by Proposition 1.1.2 it is enough to prove that every flasque sheaf is Γ-acyclic.

Proposition 1.2.2. Every flasque sheaf $\mathcal{F}$ is Γ-acyclic.

Proof. Let $\mathcal{F} \hookrightarrow I$ be a monomorphism into an injective sheaf and $\mathcal{G} = I/\mathcal{F}$. By Proposition 1.2.1 2. we have that $\mathcal{G}$ is flasque and by 1., that $H^1(X, \mathcal{F}) = 0$. Moreover, since $H^i(X, I) = 0$ for all $i > 0$, by the long exact sequence in cohomology we see that $H^i(X, \mathcal{G}) = H^{i+1}(X, \mathcal{F})$. The proposition now follows by induction on i since $\mathcal{G}$ is also flasque. ■

1.3 Čech Cohomology

Given that cohomology will play an important role in future sections, it's important to have a concrete and effective way to compute it. Certainly the abstract definitions given up until now won't help us with computations in general. In this section we introduce *Čech Cohomology* groups, which are more friendly and more concrete groups that (in most cases) coincide with the usual derived functor cohomology.

Definition 1.3.1. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of a topological space X and $\mathcal{F} \in \mathsf{Ab}(X)$. For $n \geqslant 0$ we define groups $\check{C}^n(\mathcal{U}, \mathcal{F})$ by

$$\check{C}^n(\mathcal{U}, \mathcal{F}) := \prod_{(i_0, \dots, i_n) \in I^{n+1}} \mathcal{F}(U_{i_0 \dots i_n}) \,,$$

where $U_{i_0 \dots i_n} = \bigcap_{j=0}^{n} U_{i_j}$. Given an element $a \in \check{C}^n(\mathcal{U}, \mathcal{F})$ we denote its $(i_0, \dots, i_n)$ component by $a_{i_0 \dots i_n} \in \mathcal{F}(U_{i_0 \dots i_n})$. We also define a homomorphism of groups $d \colon \check{C}^n(\mathcal{U}, \mathcal{F}) \to \check{C}^{n+1}(\mathcal{U}, \mathcal{F})$

$$(da)_{i_0 \dots i_{n+1}} := \sum_{j=0}^{n+1} (-1)^j \, a_{i_0 \dots \widehat{i_j} \dots i_{n+1}} \Big|_{U_{i_0 \cdots i_{n+1}}} \in \mathcal{F}(U_{i_0 \dots i_{n+1}}).$$

It's not hard to verify that this is a homomorphism of groups and moreover, that $d^2 = 0$. In fact, direct computation give us

$$\begin{aligned} (d^2 a)_{i_0 \dots i_{n+2}} = & \sum_{0 \leqslant j < k \leqslant n+2} (-1)^{j+k} \, a_{i_0 \dots \widehat{i_j} \dots \widehat{i_k} \dots i_{n+2}} \Big|_{U_{i_0 \cdots i_{n+2}}} \\ & - \sum_{0 \leqslant k < j \leqslant n+2} (-1)^{j+k} \, a_{i_0 \dots \widehat{i_k} \dots \widehat{i_j} \dots i_{n+2}} \Big|_{U_{i_0 \cdots i_{n+2}}} \\ = & \; 0. \end{aligned}$$

We then get a complex $\check{C}^\bullet(\mathcal{U},\mathcal{F}) \in \mathsf{D}^+(\mathsf{Ab})$, the *Čech complex*. The cohomology groups of this complex are called the *Čech cohomology groups* and are denoted by $\check{H}^i(\mathcal{U},\mathcal{F})$.

It's clear by definition that $\check{H}^0(\mathcal{U},\mathcal{F}) = \Gamma(X,\mathcal{F})$ for any open cover $\mathcal{U}$ of X and for any sheaf $\mathcal{F}$.

Remark 1.3.2. We can also define a related complex, the *alternated Čech complex* $\check{C}^\bullet_{alt}(\mathcal{U},\mathcal{F}) \in \mathsf{D}^+(\mathsf{Ab})$. Let $\mathcal{U}$, X and $\mathcal{F}$ be as in 1.3.1. Choose a total order $\leqslant$ on the set I. The alternated Čech complex is defined by

$$\check{C}^m_{alt}(\mathcal{U},\mathcal{F}) := \prod_{i_0<\cdots<i_n\in I} \mathcal{F}(U_{i_0\dots i_n}) \ ,$$

with differentials as defined in the usual Čech complex. Obviously, we have a monomorphism $\check{C}^\bullet_{alt}(\mathcal{U},\mathcal{F}) \hookrightarrow \check{C}^\bullet(\mathcal{U},\mathcal{F})$ which is in fact a quasi-isomorphism, i.e. an isomorphism in $\mathsf{D}(\mathsf{Ab})$. Moreover, different choices of total orders on I gives quasi-isomorphic complexes. With this in mind, we'll denote both of these complexes by $\check{C}^\bullet(\mathcal{U},\mathcal{F})$ and let the precise meaning be specified by the context.

Now we define a "sheafified" version of the Čech complex.

Definition 1.3.3. Let $\mathcal{U}, X$ and $\mathcal{F}$ be as in 1.3.1. Given an open set $V \subseteq X$, define $\mathcal{U}\cap V := \{U_i \cap V\}_{i\in I}$, which is an open cover of V. For each integer $n \geqslant 0$ we define a presheaf $\check{\mathcal{C}}^n(\mathcal{U},\mathcal{F})$ by

$$\check{\mathcal{C}}^n(\mathcal{U},\mathcal{F})(V) := \check{C}^n(\mathcal{U}\cap V, \mathcal{F}|_V) \ ,$$

with obvious restriction morphisms. The differentials of the Čech complex assemble into a differential $\check{\mathcal{C}}^n(\mathcal{U},\mathcal{F}) \to \check{\mathcal{C}}^{n+1}(\mathcal{U},\mathcal{F})$ and so we get a complex of presheaves $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$.

Remark 1.3.4. By definition it's clear that

$$\check{\mathcal{C}}^n(\mathcal{U},\mathcal{F}) = \prod i_{U_{i_0\dots i_n}*}\, \mathcal{F}|_{U_{i_0\dots i_n}} \ ,$$

where $i_{U_{i_0\dots i_n}} \colon U_{i_0\dots i_n} \hookrightarrow X$ is the inclusion the open set $U_{i_0\dots i_n}$. We can easily see that $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$ is in fact a complex of sheaves. Moreover, if $\mathcal{F}$ is a $\mathcal{O}_X$-module then $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}) \in \mathsf{D}(X)$.

The natural morphisms $\varepsilon_V \colon \mathcal{F}(V) \to \check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})(V)$, $a \mapsto (a|_{U_i\cap V})_{i\in I}$ induces a morphism of complexes of sheaves

$$\varepsilon \colon \mathcal{F} \to \check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}).$$

In fact, this is a quasi-isomorphism.

Proposition 1.3.5. Let $\mathcal{F}$ be a sheaf on X and $\mathcal{U}$ an open cover of X. Then $\varepsilon \colon \mathcal{F} \to \check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$ is a quasi-isomorphism.

Proof. To avoid confusion, given a complex of sheaves $\mathcal{F}^\bullet$ we sometimes use the notation $\mathcal{H}^i(\mathcal{F}^\bullet)$ for the chain complex cohomology of $\mathcal{F}^\bullet$, i.e.

$$\mathcal{H}^i(\mathcal{F}^\bullet) = \frac{\ker(\mathcal{F}^i \to \mathcal{F}^{i+1})}{\operatorname{im}(\mathcal{F}^{i-1} \to \mathcal{F}^i)} \,.$$

Now, to prove that ε is a quasi-isomorphism, we have to check that the morphisms of sheaves

$$\mathcal{H}^i(\varepsilon) \colon \mathcal{H}^i(\mathcal{F}) \to \mathcal{H}^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$$

is an isomorphism. To show this, it's enough to see that for each $x \in X$ the induced morphism on the stalk

$$\mathcal{H}^i(\varepsilon)_x \colon \mathcal{H}^i(\mathcal{F})_x \to \mathcal{H}^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))_x$$

is an isomorphism. As cohomology commutes with the stalk[8], this is the same as the morphism

$$\mathcal{H}^i(\varepsilon_x) \colon \mathcal{H}^i(\mathcal{F}_x) \to \mathcal{H}^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})_x).$$

We then conclude that it's enough to prove that for each $x \in X$ the morphism ε_x is a quasi-isomorphism. By the definition of the stalk, to prove that ε_x is a quasi-isomorphism it's enough to prove that there exists an open neighborhood U of x such that for every neighborhood $V \subseteq U$ of x we have that ε_V is a quasi-isomorphism. To show this, we take an open set U_i of the cover $\mathcal{U}$ such that $x \in U_i$. Then, for every open set $V \subseteq U_i$, we apply the following lemma with $X, \mathcal{U}, \mathcal{F}$ replaced respectively by V,$\mathcal{U} \cap V$, $F|_V$. ■

Lemma 1.3.6. Let $\mathcal{U}$, X and $\mathcal{F}$ be as in 1.3.1. Suppose that there exists $i \in I$ such that $U_i = X$. Then $\varepsilon_X \colon \mathcal{F}(X) \to \check{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a homotopy equivalence.

Proof. Define the morphism $\alpha \colon \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \mathcal{F}(X)$ by $\alpha^j = 0$ for $j \neq 0$ and

$$\alpha^0((a_j)_{j \in I}) = a_i.$$

It's clear that $\alpha \circ \varepsilon = id_{\mathcal{F}(X)}$. On the other hand, consider the homotopy $h^n \colon \check{C}^n(\mathcal{U}, \mathcal{F}) \to \check{C}^{n-1}(\mathcal{U}, \mathcal{F})$ given by $(h^n a)_{i_0 \ldots i_{n-1}} = a_{i i_0 \ldots i_{n-1}}$ (this makes sense because $U_{i_0 \ldots i_{n-1}} = U_{i i_0 \ldots i_{n-1}}$). Now we have to check that

$$id_{\check{C}^\bullet(\mathcal{U}, \mathcal{F})} - \varepsilon \circ \alpha = hd + dh.$$

In degree 0 we have,

$$(h^1 d^0 a)_{i_0} = (d^0 a)_{i i_0} = a_{i_0}|_{U_{i i_0}} - a_i|_{U_{i i_0}} = ((id_{\check{C}^0(\mathcal{U}, \mathcal{F})} - \varepsilon^0 \alpha^0)a)_{i_0}.$$

[8] This follows from the fact that the stalk functor is exact.

In degree $n > 0$, we have

$$\begin{aligned}(h^{n+1}d^n a)_{i_0\dots i_n} &= (d^n a)_{ii_0\dots i_n}\\ &= a_{i_0\dots i_n} + \sum_{j=0}^{n}(-1)^{j+1}\, a_{ii_0\dots\widehat{i_j}\dots i_n}\Big|_{U_{i_0\dots i_n}}\\ &((id_{\check{C}^n(\mathcal{U},\mathcal{F})} - d^{n-1}h^n)a)_{i_0\dots i_n}.\end{aligned}$$

This concludes the proof. ∎

Recall that by the theory of derived functors, given a complex $\mathcal{F}^\bullet \in \mathsf{D}(X)$ we always have a natural morphism $\beta\colon \Gamma(X,\mathcal{F}^\bullet) \to \mathbf{R}\Gamma(X,\mathcal{F}^\bullet)$ in $\mathsf{D}(\mathsf{Ab})$[9]. In particular, since $\Gamma(X,\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})) = \check{C}^\bullet(\mathcal{U},\mathcal{F})$ we get a morphism

$$\beta\colon \check{C}^\bullet(\mathcal{U},\mathcal{F}) \to \mathbf{R}\Gamma(X,\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})).$$

Using ε as in 1.3.5, we get an isomorphism (in the derived category)

$$\mathbf{R}\Gamma(X,\varepsilon)\colon \mathbf{R}\Gamma(X,\mathcal{F}) \xrightarrow{\sim} \mathbf{R}\Gamma(X,\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})).$$

Composing β with $\mathbf{R}\Gamma(X,\varepsilon)^{-1}$ we get a morphism (once again in the derived category)

$$\gamma\colon \check{C}^\bullet(\mathcal{U},\mathcal{F}) \to \mathbf{R}\Gamma(X,\mathcal{F})\ ,$$

which in turns induces morphisms

$$\check{H}^i(\mathcal{U},\mathcal{F}) \to H^i(X,\mathcal{F})$$

for all i. Finally, we have the following theorem.

Theorem 1.3.7. Let X be a topological space and $\mathcal{F}$ an abelian sheaf. Let $\mathcal{B}$ be a basis of X such that

- $\mathcal{B}$ is closed under pairwise intersection.
- For every $U \in \mathcal{B}$ and every $i > 0$, $H^i(U,\mathcal{F}) = 0$.

Then for any open cover $\mathcal{U} \subseteq \mathcal{B}$ of X, γ defined above is an isomorphism.

Proof. Clearly, it's enough to prove that the natural map

$$\beta\colon \check{C}^\bullet(\mathcal{U},\mathcal{F}) = \Gamma(X,\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})) \to \mathbf{R}\Gamma(X,\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}))$$

is an isomorphism. It's a known fact in homological algebra that the above morphism will be an isomorphism if each object in the complex $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$ is Γ-acyclic. Indeed, if every object is Γ-acyclic we can use $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$ itself to compute $\mathbf{R}\Gamma(X,\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}))$ and the map β will be the identity (at least up to a natural isomorphism) by construction.

[9] This morphism is essentially part of the definition of derived functors, for specific details see [Lip09, Chapter 2] or [GM99, Section III.6].

First of all, recall that if a sheaf vanishes on a base then it vanishes everywhere. Consider for each $U_{i_0}, \dots, U_{i_n} \in \mathcal{U} = \{U_i\}_{i \in I}$ the inclusion $j_{i_0 \dots i_n} \colon U_{i_0 \dots i_n} \to X$. Using the assumptions on $\mathcal{B}$ and our observation on the vanishing of sections of a sheaf, by Proposition 1.1.2 we can conclude that $\mathbf{R}^n j_{i_0 \dots i_n, *} \mathcal{F}|_{U_{i_0 \cdots i_n}} = 0$ for all $n > 0$. Thus, we have canonical isomorphisms $j_{i_0 \dots i_n, *} \mathcal{F}|_{U_{i_0 \cdots i_n}} \cong \mathbf{R} j_{i_0 \dots i_n, *} \mathcal{F}|_{U_{i_0 \cdots i_n}}$. Now, for each $n \geqslant 0$, we have the following.

$$
\begin{aligned}
\mathbf{R}\Gamma(X, \check{\mathcal{C}}^n(\mathcal{U}, \mathcal{F})) &= \prod_{i_0 < \cdots < i_n} \mathbf{R}\Gamma(X, j_{i_0 \dots i_n, *} \mathcal{F}|_{U_{i_0 \cdots i_n}}) \\
&= \prod_{i_0 < \cdots < i_n} \mathbf{R}\Gamma(X, \mathbf{R} j_{i_0 \dots i_n, *} \mathcal{F}|_{U_{i_0 \cdots i_n}}) \\
\text{(by composition of derived functors)} \quad &= \prod_{i_0 < \cdots < i_n} \mathbf{R}\Gamma(U_{i_0 \dots i_n}, \mathcal{F}|_{U_{i_0 \cdots i_n}}) \\
\text{(by our hypothesis on } \mathcal{B}\text{)} \quad &= \prod_{i_0 < \cdots < i_n} \Gamma(U_{i_0 \dots i_n}, \mathcal{F}|_{U_{i_0 \cdots i_n}}) \\
&= \check{C}^n(\mathcal{U}, \mathcal{F}).
\end{aligned}
$$

And so, we conclude that each $\check{\mathcal{C}}^n(\mathcal{U}, \mathcal{F})$ is Γ-acyclic, thus concluding the proof. ■

In the next chapter we'll use this last theorem to compare sheaf and Čech cohomology of quasi-coherent sheaves on schemes.

Chapter 2

Cohomology of Quasi-Coherent Sheaves on Schemes

2.1 Cohomology on Affine Schemes

During this section X will denote a fixed affine scheme $\operatorname{Spec} A$ unless stated otherwise. Recall[1] that for X, the category $\mathsf{QCoh}(X)$ of quasi-coherent $\mathcal{O}_X$-modules over X is equivalent (as an abelian category) to Mod_A, the category of A-modules. This equivalence is given by the pair of functors

$$\Gamma\colon \mathsf{QCoh}(X) \to \mathsf{Mod}_A$$
$$\widetilde{(\_)}\colon \mathsf{Mod}_A \to \mathsf{QCoh}(X).$$

Where Γ is the usual global sections functor and $\widetilde{(\_)}$ is essentially characterized by the property that $\widetilde{M}(D(f)) \cong M_f$ (as A_f-modules) for all $f \in A$ and restrictions between such open sets are identified under these isomorphisms with the localization maps.[2] For a general scheme X, an $\mathcal{O}_X$-module $\mathcal{F}$ over is quasi-coherent if and only if for an affine open cover $\mathcal{U}$ of X the restriction $\mathcal{F}|_U$ is quasi-coherent for all $U \in \mathcal{U}$ (equivalently, for every affine open set $U \subseteq X$, $\mathcal{F}|_U$ is quasi-coherent).

We'll prove that for a quasi-coherent sheaf over X all of the higher cohomology vanishes. This will allow us to prove in the next section that, under mild assumptions, the morphism $\gamma\colon \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \mathbf{R}\Gamma(X, \mathcal{F})$ defined last chapter is an isomorphism when working with well behaved schemes and open covers. With that in mind, lets start with some important lemmas.

Lemma 2.1.1. Let $\mathcal{U} = \{D(f_i)\}_{i=0}^n$ be a finite open covering of an open set $D(g) \subseteq X$ by principal open sets. Let $\mathcal{F} \in \mathsf{QCoh}(X)$. Then

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = 0$$

for all $i > 0$.

[1] See [Har77, Section III.5].

[2] Recall that given $f \in A$, we have an open set $D(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$ which is canonically isomorphic as a scheme to $\operatorname{Spec} A_f$. Moreover, the collection of all such open sets forms a base for the Zariski topology on $\operatorname{Spec} A$ which is closed under intersections, in fact, $D(f) \cap D(g) = D(fg)$.

Proof. Since $D(g)$ is affine and $D(f_i) \cap D(g) = D(f_i g)$ it is clearly enough to consider the case $D(g) = X$. We know that $\mathcal{F} \cong \widetilde{M}$ for some A-module M. Then, $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ is isomorphic to $\check{C}^\bullet(\mathcal{U}, \widetilde{M})$, which is given by

$$\bigoplus_{0 \leqslant i \leqslant n} M_{f_i} \to \bigoplus_{0 \leqslant i < j \leqslant n} M_{f_i f_j} \to \cdots \to M_{f_1 \dots f_n}$$

with the usual alternating sums as morphisms. To prove the claim, it's enough to prove that the following sequence of A-modules

$$0 \to M \to \bigoplus_{0 \leqslant i \leqslant n} M_{f_i} \to \bigoplus_{0 \leqslant i < j \leqslant n} M_{f_i f_j} \to \cdots \to M_{f_1 \dots f_n} \to 0$$

where the map $M \to \bigoplus_{0 \leqslant i \leqslant n} M_{f_i}$ is the sum of the canonical localization maps. To prove this, we use a baby case of the powerful theorem of faithfully flat descent[3]. Consider the canonical morphism of rings $A \to B := \prod_{0 \leqslant i \leqslant n} A_{f_i}$ given by the localization maps. Since localization is exact, its clear that this morphism is flat[4] and moreover, the map induced on affine schemes

$$\coprod_{0 \leqslant i \leqslant n} D(f_i) \to X$$

is the product of the inclusions $D(f_i) \hookrightarrow X$. Since $\mathcal{U}$ covers X, the map above is surjective and we conclude that $A \to B$ is faithfully flat. That is to say that a sequence in Mod_A is exact if an only if the sequence tensored with B over A is an exact sequence in Mod_B. So, it suffices to prove that

$$0 \to \prod_{0 \leqslant k \leqslant n} M_{f_k} \to \prod_{0 \leqslant k \leqslant n} \bigoplus_{0 \leqslant i \leqslant n} M_{f_k f_i} \to \prod_{0 \leqslant k \leqslant n} \bigoplus_{0 \leqslant i < j \leqslant n} M_{f_k f_i f_j} \to \cdots \to \prod_{0 \leqslant k \leqslant n} M_{f_k f_1 \dots f_n} \to 0$$

is an exact sequence of B-modules. Since B is a finite product of rings, we are done if we prove that for each $0 \leqslant k \leqslant n$ fixed, the sequence

$$0 \to M_{f_k} \to \bigoplus_{0 \leqslant i \leqslant n} M_{f_k f_i} \to \bigoplus_{0 \leqslant i < j \leqslant n} M_{f_k f_i f_j} \to \cdots \to M_{f_k f_1 \dots f_n} \to 0$$

is an exact sequence of A_{f_k}-modules. But this sequence is simply the canonical sequence

$$0 \to \widetilde{M_{f_k}}(D(f_k)) \to \check{C}^\bullet(\mathcal{U} \cap D(f_k), \widetilde{M_{f_k}})$$

of 1.3.6 for $X = D(f_k) \cong \operatorname{Spec} A_{f_k}, \mathcal{U} = \{D(f_i) \cap D(f_k)\}_{0 \leqslant i \leqslant n}$ and $\mathcal{F} = \widetilde{M_{f_k}}$. The exactness of such sequence is then simply a restatement of Lemma 1.3.6. ■

Lemma 2.1.2. Let

$$0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0$$

[3] All of the results used about faithfully flat morphisms are in [Stacks, Tag 00H9]. For a more advanced introduction to the theory of descent on can look at [Vis05].

[4] I.e., the morphism gives B the structure of a flat A-module.

be a short exact sequence of $\mathcal{O}_X$-modules. If $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ for all finite open covers of an open set $D(g) \subseteq X$ by principal open sets, then the map $\mathcal{G}(D(g)) \to \mathcal{H}(D(g))$ is surjective. In particular, we conclude that $H^1(D(g), \mathcal{F}) = 0$ for all $g \in A$.

Proof. Let $U = D(g)$.Take an element $s \in \mathcal{H}(U)$. Choose an open covering $\mathcal{U} = \{D(f_i)\}_{i=1}^n$ of U such that the following holds

- $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$.
- $s|_{D(f_i)}$ is the image of a section $s_i \in \mathcal{G}\left(D(f_i)\right)$.

Such a cover always exists. In fact, since ψ is surjective and U is compact, we can find a finite covering such that the second condition holds. Since principal open sets forms a base for the topology of X, the existence of such a cover follows from Lemma 2.1.1. Now, consider the sections

$$\widetilde{s}_{ij} = s_i|_{D(f_i f_j)} - s_j|_{D(f_i f_j)} \in \mathcal{G}(D(f_i f_j)).$$

Since s_i maps to $s|_{D(f_i)}$ we see that s_{ij} is mapped to zero in $\mathcal{H}$. But then, there exists $\widetilde{s}_{ij} \in \mathcal{F}\left(D(f_i f_j)\right)$ with image s_{ij} in $\mathcal{G}$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}\left(D(f_i)\right)$ such that

$$s_{ij} = t_i|_{D(f_i f_j)} - t_j|_{D(f_i f_j)}.$$

It's then easy to see that the sections $s_i - \varphi(t_i) \in \mathcal{G}(D(f_i))$ glue to a section of $\mathcal{G}$ over U which maps to s. ■

With all of these lemmas in hand, we can prove the following.

Theorem 2.1.3. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module an suppose that for each $g \in A$

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$$

for all $p > 0$ and any finite open covering $\mathcal{U} = \{D(f_i)\}_{i=0}^n$ of $D(g)$. Then $H^p(X, \mathcal{F}) = 0$ for all $p > 0$.

Proof. Choose a monomorphism $\mathcal{F} \hookrightarrow I$ with I injective. We then have an exact sequence

$$0 \to \mathcal{F} \to I \to \mathcal{Q} \to 0$$

with $\mathcal{Q} = I/\mathcal{F}$. By Lemma 2.1.2 we conclude that $I(D(g)) \to \mathcal{Q}(D(g))$ is surjective for any $g \in A$. Taking products, we conclude that for any finite open cover $\mathcal{U} = \{D(f_i)\}_{i=0}^n$ of X the sequence

$$0 \to \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{C}^\bullet(\mathcal{U}, I) \to \check{C}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

is exact. Since both $\check{H}^p(\mathcal{U}, \mathcal{F})$ and $\check{H}^p(\mathcal{U}, I)$ vanishes for $p > 0$, we conclude by taking the long exact sequence in cohomology of the above short exact sequence that the same holds for $\mathcal{Q}$. In fact, we conclude that $\mathcal{Q}$ also satisfies the hypothesis of the theorem.

Now, by the taking cohomology of the exact sequence $0 \to \mathcal{F} \to I \to \mathcal{Q} \to 0$, we see that

- $H^1(X, \mathcal{F}) = 0$.
- $H^p(X, \mathcal{F}) = H^{p-1}(X, \mathcal{Q})$ for all $p > 1$.

Since $\mathcal{Q}$ satisfies the hypothesis of the theorem, we conclude the proof by induction on p. ■

We can now prove the following.

Theorem 2.1.4. Let X be an affine scheme and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Proof. This follows directly from Lemmas 2.1.1 and 2.1.3. ■

Finally, if we have a separated scheme[5], using Theorem 2.1.5, we get the following.

Theorem 2.1.5. Let X be a separated scheme and let $\mathcal{F} \in \mathsf{QCoh}(X)$. Then, for any open cover $\mathcal{U}$ of X be affine open sets we have that

$$\gamma\colon \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \mathbf{R}\Gamma(X, \mathcal{F})$$

is an isomorphism.

This essentially justifies the use of Čech cohomology to calculate the cohomology of sheaves on well behaved schemes. In particular, any projective scheme is separated. We now go into a revision of properties of such schemes.

2.2 Some Reminders on Projective Schemes

For this section, the principal reference was [MO15]. Let $R = \bigoplus_{i \geqslant 0} R_i$ be any graded ring (i.e., $R_i \cdot R_j \subset R_{i+j}$), and let $R_+ = \bigoplus_{i \geqslant 1} R_i$ be the ideal of elements of positive degree. We define a scheme $\operatorname{Proj} R$ as follows:

1. As a point set:

$$\operatorname{Proj} R = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \text{ is homogeneous and } R_+ \not\subseteq \mathfrak{p}\}.$$

2. As a topological space:

   For all subsets $S \subset R$, let $V(S) = \{\mathfrak{p} \in \operatorname{Proj} R \mid S \subset \mathfrak{p}\}$. If a is the homogeneous ideal generated by the homogeneous parts of all $f \in S$, then

$$V(S) = V(\mathfrak{a}).$$

[5] This essentially means that a finite intersections of affine open sets is affine.

We define the $V(S)$ to be the closed sets of $\operatorname{Proj} R$. It´s then easy to see that the "distinguished open subsets"

$$D(f) = \{\mathfrak{p} \in \operatorname{Proj} R \mid f \notin \mathfrak{p}\} ,$$

where $f \in R_k$ for some $k \geqslant 1$, forms a basis of open sets.

3. The structure sheaf:

The structure sheaf of $X = \operatorname{Proj} R$ is essentially characterized by

$$\mathcal{O}_X(D(f))) \cong (R_f)_0$$

for all $f \in R_k$, $k \geqslant 1$, where $(R_f)_0$ is the degree 0 component of the localization R_f.

Let $f \in R_k, k \geqslant 1$. Then there is a canonical isomorphism:

$$\left(D(f), \mathcal{O}_X|_{D(f)}\right) \cong \left(\operatorname{Spec}(R_f)_0, \mathcal{O}_{\operatorname{Spec}((R_f)_0)}\right).$$

Moreover, just as with Spec, the construction of the structure sheaf can be turned into a functor of modules too. In this case, for every graded R-module M, we can define a quasi-coherent $\mathcal{O}_X$-module $\widetilde{M}$ by the requirement:

$$\widetilde{M}(D(f)) \cong (M_f)_0 .$$

Remark 2.2.1. We give next a list of fairly straightforward properties of the operations Proj and of $\widetilde{(\_)}$:

a) The homomorphisms $R_0 \to (R_f)_0$ for all $f \in R_k$, all $k \geqslant 1$ induce a morphism

$$\operatorname{Proj} R \longrightarrow \operatorname{Spec} R_0 ,$$

which is of finite type[6] if R is a finitely generated R_0-algebra.

b) If S_0 is an R_0-algebra, then

$$\operatorname{Proj}(R \otimes_{R_0} S_0) \cong \operatorname{Proj}(R) \times_{\operatorname{Spec} R_0} \operatorname{Spec} S_0.$$

c) If $d \geqslant 1$ and $R\langle d\rangle = \bigoplus_{k=0}^{\infty} R_{dk}$, then $\operatorname{Proj} R \cong \operatorname{Proj} R\langle d\rangle$.

d) If $I \subset R$ is a homogeneous ideal, then there is a canonical closed immersion[7]:

$$\operatorname{Proj} R/I \hookrightarrow \operatorname{Proj} R.$$

[6] See [Stacks, Tag 01T0] for a definition and some properties.

[7] i.e., a morphism of schemes $i\colon Z \to X$ inducing a homeomorphism of Z onto a closed subset of X and such that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective.

Moreover, if R is finitely generated over R_0, every closed subscheme Z of $\operatorname{Proj} R$ is isomorphic to $\operatorname{Proj}\ R/I$ a for some homogeneous ideal I.

e) Proj is not a functor but it does satisfy a partial functoriality property. Let R and R' be two graded rings and let $\phi : R \longrightarrow R'$ be a ring homomorphism such that for some $d > 0$
$$\varphi(R_n) \subset R'_{nd}$$
for all $n \geqslant 0$ (i.e., a morphism of graded rings $\varphi\colon R \to R'\langle d\rangle$). Let $I = \phi(R_+)R'$. Then φ induces a natural map by taking the inverse image of primes:
$$f : \operatorname{Proj} R' \setminus V(I) \longrightarrow \operatorname{Proj} R.$$
In fact,
$$\operatorname{Proj} R' \setminus V(I) = \bigcup_{n\geqslant 1} \bigcup_{a\in R_n} D(\varphi(a)).$$
The restriction of f to $D(\varphi(a))$ is the morphism from $D(\varphi(a))$ to $D(a)$ induced by
$$\begin{aligned} \varphi : (R_a)_0 &\longrightarrow \left(R'_{\varphi(a)}\right)_0 \\ \varphi\left(\tfrac{b}{a^l}\right) &= \tfrac{\varphi(b)}{\varphi(a)^l}, \quad b \in R_{nl}. \end{aligned}$$

f) If R and R' are two graded rings such that $R_0 = R'_0$, then
$$\operatorname{Proj}\ R \times_{\operatorname{Spec}}\ R_0 \operatorname{Proj}\ R' = \operatorname{Proj} R''\ ,$$
where
$$R'' = \bigoplus_{n=0}^{\infty} R_n \otimes_{R_0} R'_n.$$

g) $M \longmapsto \widetilde{M}$ is an exact functor.

h) There is a natural map $M_0 \longrightarrow \Gamma(\operatorname{Proj} R, \widetilde{M})$ given by gluing the sections $m/1 \in (M_f)_0 = \widetilde{M}(D(f))$.

The most important example of Proj for us is $\mathbb{P}^n_A = \operatorname{Proj} A\,[x_0, \ldots, x_n]$. Since $x_0, \ldots, x_n$ generate the ideal of elements of positive degree, $\mathbb{P}^n_A$ is covered by the distinguished open sets $D(x_i)$, i.e., by the $n+1$ copies of $\mathbb{A}^n_A$:
$$U_i = \operatorname{Spec} A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]\ ,$$
glued in the usual way. Moreover if $R = \oplus_{i=0}^{\infty} R_i$ is any graded ring generated over R_0 by elements of R_1 and with R_1 finitely generated as R_0-module, then R is a quotient of $R_0\,[x_0, \ldots, x_n]$ for some n. Therefore, $\operatorname{Proj} R$ is a closed subscheme of $\mathbb{P}^n_{R_0}$.

With this example in mind, we give the following definition.

Definition 2.2.2. Let X be a scheme over a ring A. We say X is *projective* if it is isomorphic to a closed subscheme of $\mathbb{P}^n_A$ for some $n \geqslant 0$.

Now, we give a brief overview of invertible sheaves (or line bundles) on Proj schemes.

Definition 2.2.3. Let X be a scheme. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called locally free of rank n (or a vector bundle of rank n) if each point of X has an open neighborhood U such that

$$\mathcal{F}|_U \cong \mathcal{O}_X^n|_U \ ,$$

or equivalently, there exists an open covering $\{U_\alpha\}$ of X such that for each α,

$$\mathcal{F}|_{U_\alpha} \cong \mathcal{O}_X^n\big|_{U_\alpha} .$$

If $\mathcal{L}$ is locally free of rank one (a line bundle), we say that $\mathcal{L}$ is *invertible.* The reason why invertible sheaves are called invertible is that their isomorphism classes form a group under the tensor product over $\mathcal{O}_X$ for multiplication. In fact, the following properties holds

a) If $\mathcal{L}, \mathcal{L}'$ are invertible, so is $\mathcal{L} \otimes \mathcal{L}'$.

b) It is clear that $\mathcal{L} \otimes \mathcal{O}_X \cong \mathcal{L} \cong \mathcal{O}_X \otimes \mathcal{L}$, so $\mathcal{O}_X$ is a unit element for the multiplication, up to isomorphism.

c) Let $\mathcal{L}^\vee = \mathcal{H}om\,(\mathcal{L}, \mathcal{O}_X)$, the dual sheaf of $\mathcal{L}$. Then $\mathcal{L}^\vee$ is invertible and the natural map
$$\mathcal{L} \otimes \mathcal{L}^\vee \longrightarrow \mathcal{O}_X$$
is an isomorphism.

d) Let $f : X \to Y$ be a morphism and $\mathcal{L}$ an invertible sheaf on Y. Then $f^*\mathcal{L}$ is an invertible sheal on X.

This proves that isomorphism classes of invertible sheaves over $\mathcal{O}_X$ form a group, denoted by $\mathrm{Pic}(X)$, the *Picard group* of X. Moreover, this construction is functorial.

Let $\mathcal{L}$ be an invertible sheaf on a scheme X. Let $f \in \Gamma(X, \mathcal{L})$ be a section. We define

$$X_f := \{x \in X \mid f(x) \neq 0\}.$$

We recall that $f(x)$ is the value of f in $\mathcal{L}_x/\mathfrak{m}_x\mathcal{L}_x$, as distinguished from $f_x \in \mathcal{L}_x$.

During the rest of this section, we assume that every graded ring R is generated by R_1 over R_0 as an algebra, that is

$$R = R_0\,[R_1]\,.$$

Example 2.2.4. Let A be any commutative ring, and let.

$$R = A\,[T_0, \ldots, T_r]$$

be the polynomial ring in $r+1$ variables. Then $R_0 = A$, and R_n consists of the homogeneous polynomials of degree n with coefficients in A. Furthermore R_1 is the free module over A, with basis $T_0, \ldots, T_r$.

For simplicity, denote

$$X = \operatorname{Proj} R.$$

Given any graded module M. For all integers $d \in \mathbb{Z}$ we may define the d-twist $M(d)$ of M, which is the module M but with the new grading

$$M(d)_n = M_{d+n}.$$

Then we define

$$\mathcal{O}_X(1) = \widetilde{R(1)}.$$

We have the following properties.

a) $\mathcal{O}_X(1)$ is invertible on X. In fact: Given $f \in R_1$, multiplication by f

$$\cdot f : R \longrightarrow R(1)$$

is a graded homomorphism of degree 0 , whose induced sheaf homomorphism

$$\cdot f : \mathcal{O}_X \to \mathcal{O}_X(1)$$

restricts to an isomorphism on $D(f)$. Moreover, let $\varphi_f = \cdot f$. For $f, g \in R_1$, the sheaf map $\varphi_f^{-1} \circ \varphi_g$ is multiplication by g/f on $D(f) \cap D(g)$.

b) Let M be a graded R-module. Then the isomorphism

$$M \otimes_R R(1) \longrightarrow M(1)$$

induces an isomorphism

$$\widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \longrightarrow \widetilde{M(1)}.$$

c) If for every integer d we define

$$\mathcal{O}_X(d) = \widetilde{R(d)}$$

and for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we define

$$\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d).$$

Then for $d, m \in \mathbb{Z}$ we have $\mathcal{F}(d+m) \cong \mathcal{F}(d) \otimes \mathcal{O}_X(m)$.

d) For d positive,

$$\mathcal{O}_X(d) \cong \mathcal{O}_X(1) \otimes \cdots \otimes \mathcal{O}_X(1) \quad (\text{ product taken } d \text{ times }).$$

e) For $d \in \mathbb{Z}$ the natural pairing

$$\mathcal{O}_X(d) \otimes \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_X$$

identifies $\mathcal{O}_X(-d)$ with the dual sheaf $\mathcal{O}_X(d)^\vee$.

f) For a graded module M, we have $\widetilde{M(d)} \cong \widetilde{M}(d)$.

At last, we look at the functoriality of twists with respect to graded ring homomorphisms. Let R' be a graded ring which we now assume generated by R'_1 over R^t_0. Let

$$\varphi : R \longrightarrow R'$$

be a graded homomorphism of degree 0. Let V be the subset (which is actually open) of $\operatorname{Proj} R'$ consisting of those primes p' such that $p' \notin D(\varphi(R_1))$. Then the inverse image map on prime ideals

$$f : V \longrightarrow \operatorname{Proj} R = X$$

defines a morphism of schemes. We then have the following

Proposition 2.2.5. Let $X' = \operatorname{Proj} R'$. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{G}$

$$f^*(\mathcal{F}(d)) = (f^*\mathcal{F})(d) = (f^*\mathcal{F}) \otimes_{\mathcal{O}_V} \mathcal{O}_{X'}(d)|_V \quad \text{and} \quad f_*\left(\mathcal{G}(d)|_V\right) = \left(f_*\,\mathcal{G}|_V\right)(d).$$

2.3 Cohomology of Projective Space

In this section we perform some explicit calculations of the cohomology of the line bundles $\mathcal{O}_{\mathbb{P}^n_A}(d)$ in projective space over a ring A. This will prove to be an important step for Serre duality.

Theorem 2.3.1. Let A be a ring and $S = A[x_0, \ldots, x_n]$. Let $X = \operatorname{Proj} S = \mathbb{P}^n_A$ with $n > 0$. Denote $\mathcal{O}(d) = \mathcal{O}_{\mathbb{P}^n_A}(d)$. Then

a) $H^i(X, \mathcal{G}) = 0$ for all $i > n$ and for any $\mathcal{G} \in \mathsf{QCoh}(X)$.

b) $H^0(X, \mathcal{O}(d)) = S_d$.

c) $H^n(X, \mathcal{O}(d)) = A[x_0^{\alpha_0} \ldots x_n^{\alpha_n} \mid \alpha_i < 0 \text{ for all } i \text{ and } \sum \alpha_i = d]$.

d) $H^i(X, \mathcal{O}(d)) = 0$ for all $0 < i < n$ and for all $d \in \mathbb{Z}$.

Proof. For this proof, we use the fact that projective space is a separated scheme and moreover, comes with a canonical affine open cover given by $\mathcal{U} = \{D(x_i)\}_{0 \leqslant i \leqslant n}$. By Theorem 2.1.5, we can compute quasi-coherent sheaf cohomology using Čech cohomology. Define the graded sheaf $\mathcal{F}$ as $\mathcal{F} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}(d)$.

a) This follows directly by using the (alternated) Čech complex with the covering $\mathcal{U}$.

b) Using the cover $\mathcal{U}$ to compute Čech cohomology, we see that

$$\bigoplus_{d\in\mathbb{Z}} H^0(X,\mathcal{O}(d)) = H^0(X,\mathcal{F}) = \ker\left(\prod_{0\leqslant i\leqslant n}\mathcal{F}(D(x_i)) \xrightarrow{d} \prod_{0\leqslant i<j\leqslant n}\mathcal{F}(D(x_ix_j))\right)$$
$$= \ker\left(\prod_{0\leqslant i\leqslant n} S_{x_i} \xrightarrow{\mathrm{inc}^i_{ij}-\mathrm{inc}^j_{ij}} \prod_{0\leqslant i<j\leqslant n} S_{x_ix_j}\right),$$

where $\mathrm{inc}^i_{ij}\colon S_{x_i}\to S_{x_ix_j}$ is simply the inclusion of these rings seen as subrings of $S_{x_0\ldots x_n}$. In conclusion, we have that

$$\bigoplus_{d\in\mathbb{Z}} H^0(X,\mathcal{O}(d)) = H^0(X,\mathcal{F}) = \left\{(f_i)\in\prod S_{x_i} \mid f_i=f_j \text{ in } S_{x_ix_j}\right\} = S.$$

Now, it´s easy enough to see that this isomorphism preserve the grading. And so, in fact we have

$$H^0(X,\mathcal{O}(d)) = S_d.$$

c) Using $\mathcal{U}$ to compute Čech cohomology once again, we have that

$$H^n(X,\mathcal{F}) = \operatorname{coker}\left(\prod_{0\leqslant i\leqslant n}\mathcal{F}(D(x_0\ldots\widehat{x_i}\ldots x_n)) \xrightarrow{d} \mathcal{F}(D(x_0\ldots x_n))\right)$$
$$= \operatorname{coker}\left(\prod_{0\leqslant i\leqslant n} S_{x_0\ldots\widehat{x_i}\ldots x_n} \xrightarrow{\mathrm{inc_i}} S_{x_0\ldots x_n}\right),$$

where inc_i is the inclusion $S_{x_0\ldots\widehat{x_i}\ldots x_n}\hookrightarrow S_{x_0\ldots x_n}$. It's easy to see that for each i, the image of inc_i is the subring

$$A[x_0^{\alpha_0}\ldots x_n^{\alpha_n} \mid \alpha_i\geqslant 0]\subseteq S_{x_0\ldots x_n}.$$

Therefore, the entire image is given by

$$A[x_0^{\alpha_0}\ldots x_n^{\alpha_n} \mid \alpha_i\geqslant 0 \text{ for some } i]\subseteq S_{x_0\ldots x_n}.$$

Since all of these are free A-modules, we have that

$$H^n(X,\mathcal{F}) = A[x_0^{\alpha_0}\ldots x_n^{\alpha_n} \mid \alpha_i<0 \text{ for all } i].$$

Once again, it's easy to see that this preserves the graduation. In conclusion, we get

$$H^n(X,\mathcal{O}(d)) = A[x_0^{\alpha_0}\ldots x_n^{\alpha_n} \mid \alpha_i<0 \text{ for all } i \text{ and } \sum\alpha_i=d].$$

d) This last item is proved by induction on n. The base case is $n=1$, which leaves us nothing to prove. Now, note that if we localize the Čech complex $C^\bullet=\check{C}^\bullet(\mathcal{U},\mathcal{F})$ by x_n, we get the complex $C^\bullet_{x_n}=\check{C}^\bullet(\mathcal{U}\cap D(x_n),\mathcal{F})$. Since $D(x_n)$ is affine and the complex $C^\bullet_{x_n}$ computes sheaf cohomology of F in $D(x_n)$, we conclude that it has trivial higher cohomology. On the other hand, since localization is exact, the cohomology of this complex is the localization of the cohomology of $\mathcal{F}$. i.e., we conclude that

$$H^i(X, \mathcal{F})_{x_n} = 0$$

for all $i > 0$. In particular, every element of $H^i(X, \mathcal{F})$ is annihilated by a power of x_n. We'll now prove that for $0 < i < n$, multiplication by x_n gives an injective endomorphism of $H^i(X, \mathcal{F})$. This will conclude the proof.

Consider the exact sequence of graded S-modules

$$0 \to S(-1) \xrightarrow{\cdot x_n} S \to S/(x_n) \to 0$$

Since the twisted modules $S(d)$ are all free, we get for each $d \in \mathbb{Z}$ an exact sequence

$$0 \to S(-1)(d) \xrightarrow{\cdot x_n} S(d) \to (S/(x_n))(d) \to 0 \tag{$*$}$$

Now, sheafifying and summing over d, we get the exact sequence

$$0 \to \mathcal{F}(-1) \xrightarrow{\cdot x_n} \mathcal{F} \to \bigoplus_{d \in \mathbb{Z}} (j_* \mathcal{O}_H)(d) \to 0$$

where $j \colon H = \operatorname{Proj}(S/(x_n)) \hookrightarrow X$ is the closed immersion induced by the quotient map $S \twoheadrightarrow S/(x_n)$. Since $\mathcal{F} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}(d)$, we have that $\mathcal{F} = \mathcal{F}(-1)$. Now, since j is an affine morphism (the inverse image of an affine open set is affine), we easily get by Proposition 1.1.2 that $\mathbf{R}j_* = j$ and since $\mathbf{R}\Gamma_H = \mathbf{R}\Gamma_X \circ \mathbf{R}j_*$, we conclude that

$$H^i(X, \bigoplus_{d \in \mathbb{Z}} (j_* \mathcal{O}_H)(d)) = H^i(H, \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_H(d)).$$

Since $H \cong \mathbb{P}^{n-1}_A$, by induction hypothesis we have that,

$$H^i(X, \bigoplus_{d \in \mathbb{Z}} (j_* \mathcal{O}_H)(d)) = H^i(H, \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_H(d)) = 0$$

for all $0 < i < n - 1$. In particular, by the long exact sequence in cohomology, we conclude that

$$H^i(X, \mathcal{F}) \xrightarrow{\cdot x_n} H^i(X, \mathcal{F})$$

in an isomorphism for all $1 < i < n - 1$ and an injection for $i = n - 1$. Moreover, since ($*$) is exact, we see that the connecting morphism $(S/(x_n)) = H^0(X, \bigoplus_{d \in \mathbb{Z}} (j_* \mathcal{O}_H)(d)) \to H^1(X, \mathcal{F})$ is zero, concluding that

$$H^1(X, \mathcal{F}) \xrightarrow{\cdot x_n} H^1(X, \mathcal{F})$$

is also an isomorphism, finishing the proof.

∎

Corollary 2.3.2. The natural A-bilinear map

$$\operatorname{Hom}_X(\mathcal{O}(d), \mathcal{O}(-n-1)) \times H^n(X, \mathcal{O}(d)) \to H^n(X, \mathcal{O}(-n-1)) \cong A$$

is a perfect pairing of finitely generated free A-modules, for each $d \in \mathbb{Z}$. (i.e., it induces isomorphisms of each side into the dual module of the other side.)

Proof. It´s a classic fact that we have canonical isomorphisms

$$\operatorname{Hom}_X(\mathcal{O}(d), \mathcal{O}(-n-1)) \cong \operatorname{Hom}_X(\mathcal{O}, \mathcal{O}(-d-n-1)) \cong H^0(X, \mathcal{O}(-d-n-1)).$$

Under these isomorphisms, it's easy to see that the pairing above becomes a pairing

$$H^0(X, \mathcal{O}(-d-n-1)) \times H^n(X, \mathcal{O}(d)) \to H^n(X, \mathcal{O}(-n-1)) \cong A\,,$$

which is given (under the calculations of Theorem 2.3.1) by multiplying elements $f \in H^0(X, \mathcal{O}(-d-n-1)) = S_{-d-n-1}$ and $g \in H^n(X, \mathcal{O}(d)) = A[x_0^{\alpha_0} \dots x_n^{\alpha_n} \mid \alpha_i < 0$ for all i and $\sum \alpha_i = d]$ and then taking the coefficient multiplying the monomial $x_0^{-1} \dots x_n^{-1}$ in $fg \in S_{x_0 \dots x_n}$[8]. Since all of the modules involved are finitely generated and free, to prove that this pairing is perfect, it's enough to prove that it is non degenerate. Now, this is clear, since R_{-d-n-1} has a basis given by monomials $x_0^{\alpha_0} \dots x_n^{\alpha_n}$ with $\sum \alpha_i = -d-n-1$ and $\alpha_i \geqslant 0$ and we can give a explicit dual to each of this monomials, namely, $x_0^{-\alpha_0-1} \dots x_n^{-\alpha_n-1} \in H^n(X, \mathcal{O}(d))$. ■

Remark 2.3.3. Clearly, since cohomology commutes with finite sums, we can also conclude the preceding corollary for finite sums of sheaves of the form $\mathcal{O}(d)$.

2.4 Finiteness and Vanishing Theorems in Cohomology

First of all, recall that for a general scheme X a quasi-coherent sheaf $\mathcal{F}$ is *finitely generated* if and only if for every affine open set $U = \operatorname{Spec} A$, $\mathcal{F}|_U$ is isomorphic to $\widetilde{M}$ for some finitely generated A-module M (as usual this is equivalent to the same condition restricted to an affine open cover). If X is locally noetherian[9] a quasi-coherent sheaf is coherent if and only if it's finitely generated. In this section we prove some results about the cohomology of finitely generated quasi-coherent sheaves over projective schemes. For that we need some lemmas.

Lemma 2.4.1. Let $\mathcal{L}$ be an invertible sheaf on a quasi-compact scheme X and $f \in \Gamma(X, \mathcal{L})$. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then the following holds:

a) Let $s \in \Gamma(X, \mathcal{F})$ be a section whose restriction to X_f is 0. Then for some $n > 0$ we have $f^n s = 0$, where $f^n s \in \Gamma(X, \mathcal{L}^n \otimes \mathcal{F})$[10] is the image of $f^{\otimes n} \otimes s \in \mathcal{L}(X)^{\otimes n} \otimes \mathcal{F}(X)$ in $\Gamma(X, \mathcal{L}^n \otimes \mathcal{F})$.

[8]This makes sense since f has degree $-d-n-1$ and g has degree d in $S_{x_0 \dots x_n}$.
[9]I.e, every affine open set is the spectrum of a noetherian ring
[10]$\mathcal{L}^n = \mathcal{L}^{\otimes n}$.

b) Let $t \in \Gamma(X_f, \mathcal{F})$ be a section over X_f. Then there exists $n > 0$ such that the section $f^n t \in \Gamma(X_f, \mathcal{L}^n \otimes \mathcal{F})$ extends to a global section of $\mathcal{L}^n \otimes \mathcal{F}$ over X.

Proof. a) Since X is quasi-compact, we can find a finite covering of X by affine open sets on which $\mathcal{L}$ is free of rank 1. Since a global section is zero if and only if it restricts to zero on an open cover of X, it's enough to prove that if $U = \mathrm{Spec}(A)$ is affine open set that trivializes $\mathcal{L}$, then there is some $n > 0$ such that $f^n s = 0$ on U. The lemma then follows by taking the maximum of these n. Since $\mathcal{F}$ is quasi-coherent, we have that $\mathcal{F}|_U = \widetilde{M}$ for some A-module M. So, we can view s as an element of M, and f as an element of A. By definition of localization, the fact that $s = 0$ in X_f means that $s/1 = 0$ in M_f, and so there is some $n \geqslant 0$ such that $f^n s = 0$.

b) Once again, we can cover X by a finite number of affine opens $U_i = \mathrm{Spec}(A_i)$ such that $\mathcal{L}|_{U_i}$ is free. On each U_i there is an A_i-module M_i such that $\mathcal{F}|_{U_i} = \widetilde{M_i}$. The restriction of t to $X_f \cap U_i = (U_i)_f$ can be seen as an element of $(M_i)_{f_i}$, where $f_i = f|_{U_i}$ can be viewed as an element of A_i since $\mathcal{L}|_{U_i}$ is free of rank 1. By definition of localization, for each i there is an integer $n_i \geqslant 0$ and a section $t_i \in \Gamma(U_i, \mathcal{F}) = M_i$ such that the restriction of t_i to $(U_i)_{f_i}$ is equal to $f^{n_i} t$ over $(U_i)_{f_i}$. Changing all n_i by their maximum, we can work with the same n for all i. On $U_i \cap U_j$ the two sections t_i and t_j are defined, and are equal to $f^n t$ when restricted to $X_f \cap U_i \cap U_j$. By a), there is an integer $m \geqslant 0$ such that $f^m(t_i - t_j) = 0$ on $U_i \cap U_j$ for all i, j, again using the fact that there is only a finite number of pairs (i, j). But then, the sections $f^m t_i \in \Gamma(U_i, \mathcal{L}^m \otimes \mathcal{F})$ agree on intersections an so, define a global section of $\mathcal{L}^m \otimes \mathcal{F}$, whose restriction to X_f is $f^{n+m} t$.

■

For the rest of this section let R a graded ring, finitely generated by R_1 over R_0. We also let $X = \mathrm{Proj}\, R$[11].

Lemma 2.4.2. Let $\mathcal{F}$ be a finitely generated quasi-coherent sheaf on X. Then there is some n_0 such that for all $n \geqslant n_0$, the sheaf $\mathcal{F}(n)$ is generated by a finite number of global sections (i.e., every stalk is generated by the image of one of these global sections. Equivalently, there is a epimorphism $\mathcal{O}_X^m \twoheadrightarrow \mathcal{F}(n)$.).

Proof. Let $f_0, \ldots, f_r$ be generators of R_1 over R_0, and let $X_i = D(f_i)$. Our assumptions of R guarantees that the X_i covers X. For each i there is a finitely generated module M_i over $\mathcal{O}(X_i)$ such that $\mathcal{F}|_{X_i} = \widetilde{M_i}$. For each i, let s_{ij} be a finite number of generators of M_i. By Lemma 2.4.1 there is an integer n such that for all i, j the sections $f_i^n s_{ij}$ extend to global sections of $\mathcal{O}(n) \otimes_{\mathcal{O}} \mathcal{F} = \mathcal{F}(n)$. But for fixed i, the global sections $f_i^n s_{ij}$ generate M_i over $\mathcal{O}(X_i)$ since f_i^n is invertible on $\mathcal{O}(X_i)$. Since the open sets X_i cover X, the image of this sections generates all stalks. ■

The main results of the section are the following.

[11] Observe that the conditions on R guarantee that X is quasi-compact.

Theorem 2.4.3. Let X be a proper scheme[12] over a ring A. Suppose $d = \dim(X)$ is finite. Then

$$H^i(X, \mathcal{F}) = 0$$

for all $i > d$.

Proof. The proof is quite tiresome and so we just give a reference which proves an even more general statement. [Sch92, Corollary 4.6]. ■

Theorem 2.4.4. Suppose R_0 is noetherian, so that X is noetherian. For every coherent $\mathcal{O}_X$-module $\mathcal{F}$ the following holds:

a) For any i the cohomology group $H^i(X, \mathcal{F})$ is a finitely generated R_0-module.

b) There exists $m_0 \in \mathbb{Z}$ such that $H^i(X, \mathcal{F}(m)) = 0$ for all $i > 0$ and $m \geqslant m_0$.

Proof. Let's first prove the theorem for $X = \mathbb{P}^n_A$ (i.e., $R = A[x_0 \dots x_n]$). In this setting, we actually prove both items at the same time by descending induction on i. By Theorem 2.4.3 the results are clear for $i > n$. Suppose now that the results are true for some $i + 1$ (if $i = 0$ b) is vacuous). By Lemma 2.4.2 we can find a surjection

$$\mathcal{O}_X^r \twoheadrightarrow \mathcal{F}(d) \ ,$$

or equivalently (since $\mathcal{O}_X(d)$ is invertible),

$$\mathcal{O}_X(-d)^r \twoheadrightarrow \mathcal{F}.$$

Since the kernel of a morphism between coherent sheaves is coherent[13], we have an exact sequence of coherent sheaves

$$0 \to \mathcal{G} \to \mathcal{O}_X(-d)^r \to \mathcal{F} \to 0$$

The long exact sequence in cohomology gives an exact sequence

$$H^i(X, \mathcal{O}_X(-d)^r) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{G}).$$

By induction hypothesis the A-module on the right is finitely generated and by direct computation done earlier the A-module on the left is also finitely generated. This implies the A-module on the middle is also finitely generated, thus proving the induction step for a). For b), same short exact sequence as before tensored with $\mathcal{O}_X(e)$ for any $e \in \mathbb{Z}$.

$$0 \to \mathcal{G}(e) \to \mathcal{O}_X(e-d)^r \to \mathcal{F}(e) \to 0.$$

The resulting exact sequence in cohomology gives

$$H^i(X, \mathcal{O}_X(e-d)^r) \to H^i(X, \mathcal{F}(e)) \to H^{i+1}(X, \mathcal{G}(e)).$$

[12] See [Stacks, Tag 01W0]. In particular, one can see that every projective scheme is proper.

[13] This is in fact true for every locally ringed space, but would be false if we just used finitely generated sheaves instead. Since we have not defined coherent sheaves in general we assume the noetherian hypothesis on X.

By induction hypothesis, the A-module on the right vanishes for e sufficiently large. On the other hand, by Theorem 2.3.1, the A-module on the left also vanishes if $i > 0$ and $i \neq n$. If $i = n$, then as soon as $e - d > -n - 1$ the module vanishes. Thus, for e large enough both the A-modules on the left and the right vanishes, and so, we get that

$$H^i(X, \mathcal{F}(e)) = 0$$

for all e large enough, thus proving the induction step.

For the general case, our hypothesis on R guarantee a surjection of R_0-algebras

$$R_0[t_0, \ldots, t_n] \twoheadrightarrow R\ ,$$

which in turn induces a closed immersion

$$j \colon X \hookrightarrow \mathbb{P}^n_{R_0}.$$

Since every closed immersion is affine (inverse image of affine open is affine), we have that

$$H^i(X, \mathcal{F}) = H^i(\mathbb{P}^n_{R_0}, j_*\mathcal{F}).$$

Since $j_*\mathcal{F}$ is coherent[14] we have the results for $j_*\mathcal{F}$ over $\mathbb{P}^n_{R_0}$, in particular we conclude a). By Proposition 2.2.5, we also conclude b). ■

[14]see [Stacks, Tag 087T].

Chapter 3

Serre Duality

3.1 A Brief Review of δ-Functors

We briefly recall the definition and some properties of δ-functors.

Definition 3.1.1. A δ-functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$ is a sequence of additive functors

$$F^i : \mathcal{A} \to \mathcal{B} \quad (i = 0, 1, \ldots)$$

plus for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ a morphism $\delta^i : F^i(C) \to F^{i+1}(A)$ functorial in the exact sequence, such that

$$0 \to F^0(A) \to F^0(B) \to F^0(C) \xrightarrow{\delta^0} F^1(A) \to F^1(B) \to F^1(C) \xrightarrow{\delta^1} F^2(A) \to \cdots$$

is exact. A δ-functor $\{F^i\}_{i \geqslant 0}$ is universal if given any other δ-functor $\{G^i\}_{i \geqslant 0}$ and a natural transformation $f^0 : F^0 \to G^0$, there is a unique sequence of natural transformations $f^i : F^i \to G^i$ starting with f^0 which commute with the δ^i.

Remark 3.1.2. If $\{F^i\}_{i \geqslant 0}$ and $\{G^i\}_{i \geqslant 0}$ are both universal δ-functors, given an isomorphism $F^0 \xrightarrow{\sim} G^0$, we obviously get a unique natural isomorphism of δ-functors between $\{F^i\}_{i \geqslant 0}$ and $\{G^i\}_{i \geqslant 0}$.

Definition 3.1.3. An additive functor $G : \mathcal{A} \to \mathcal{B}$ is *effaceable* if for each $X \in \mathcal{A}$ there exists a monomorphism $X \to A$ such that $G(A) = 0$. A cohomological δ-functor $\{F^i\}_{i \geqslant 0}$ is effaceable if F^i are effaceable for all $i \geqslant 1$.

Proposition 3.1.4. If $\{F^i\}_{i \geqslant 0}$ is an effaceable δ-functor, then it is universal.

Proof. See [Stacks, Tag 010T]. ■

Example 3.1.5. If $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $\mathcal{A}$ has enough injectives, then given a left exact functor $F \colon \mathcal{A} \to \mathcal{B}$ it's right derived functors $\{\mathbf{R}^i F\}_{i \geqslant 0}$ form a universal δ-functor extending F.

3.2 Ext Groups and Sheaves

Definition 3.2.1. Recall that for an abelian category $\mathcal{A}$ with enough injectives, we define $\mathrm{Ext}^i_{\mathcal{A}}(A, B)$ for $A, B \in \mathcal{A}$ as

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) = H^i(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(A, B)).$$

If $\mathcal{A} = \mathsf{Mod}_X$ for some scheme X, we simply write $\mathrm{Ext}^i_X(\mathcal{F}, \mathcal{G})$ for $\mathrm{Ext}^i_{\mathsf{Mod}_X}(\mathcal{F}, \mathcal{G})$. In this case, we can also define a *sheaf ext* by

$$\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G}) = H^i(\mathbf{R}\mathcal{H}om_X(\mathcal{F}, \mathcal{G})) ,$$

where $\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$ is the sheaf

$$U \mapsto \mathrm{Hom}_U(\mathcal{F}|_U , \mathcal{G}|_U).$$

Remark 3.2.2. There is a natural isomorphism

$$\mathbf{R}\mathrm{Hom}_X (\mathcal{O}_X, \mathcal{F}) \cong \mathbf{R}\Gamma(X, \mathcal{F})$$

because these are the derived functors of the naturally isomorphic functors $\mathrm{Hom}_X (\mathcal{O}_X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$. On the other hand, $\mathcal{H}om_X (\mathcal{O}_X, \mathcal{F})$ is naturally isomorphic to the identity functor, so

$$\mathbf{R}\mathcal{H}om_X (\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}.$$

Moreover, it´s possible to see that for any open subset U of X, there are natural isomorphisms

$$\mathbf{R}\mathcal{H}om_X (\mathcal{F}, \mathcal{G})|_U \cong \mathbf{R}\mathcal{H}om_U(\mathcal{F}|_U , \mathcal{G}|_U).$$

Proposition 3.2.3. Suppose that

$$\cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

is an exact sequence in Mod_X, where each $\mathcal{L}_i$ is locally free of finite rank. (We say the $\mathcal{L}_\bullet$ form a locally free resolution of $\mathcal{F}$.) Then there is a canonical isomorphism

$$\mathbf{R}\mathcal{H}om_X(\mathcal{F}, \mathcal{G}) \cong \mathcal{H}om_X(\mathcal{L}_\bullet, \mathcal{G}) ,$$

which is functorial both in $\mathcal{G}$ and in the resolution of $\mathcal{F}$. The same holds for the derived functor of Hom_X.

Proof. This is essentially [Har77, Proposition 6.5]. ■

Note that locally free resolutions are much easier to write down in practice than injective resolutions. For instance, if X is projective over a noetherian ring A, and $\mathcal{F}$ is coherent, then Lemma 2.4.2 gives a surjection $\mathcal{E} \to \mathcal{F}$ where $\mathcal{E}$ is locally free. As usual, repeated application gives a locally free resolution.

Proposition 3.2.4. For any coherent sheaves $\mathcal{F}, \mathcal{G}$ on a projective scheme X over a noetherian ring A, $\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G})$ is again coherent.

Proof. Let $\mathcal{L}_\bullet$ be a free resolution of $\mathcal{F}$. By the previous proposition,

$$\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G}) = H^i(\mathcal{H}om_X(\mathcal{L}_\bullet, \mathcal{G})).$$

Since $\mathcal{G}$ and each $\mathcal{L}_i$ are coherent, we have that each $\mathcal{H}om_X(\mathcal{L}_i, \mathcal{G})$ is also coherent. Since quotients of coherent sheaves are coherent we are done. ■

Lemma 3.2.5. For $\mathcal{F}, \mathcal{G}, \mathcal{L} \in \mathsf{Mod}_X$ with X projective over a noetherian ring A and $\mathcal{L}$ locally free of finite rank, there are canonical isomorphisms

$$\operatorname{Ext}^i_X(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \operatorname{Ext}^i_X(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G})$$

and

$$\mathcal{E}xt^i_X(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathcal{E}xt^i_X(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}) \cong \mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^\vee.$$

Proof. For $i = 0$ these results are classical (See [Har77, Exercise II.5.1].). For general i, we observe that given a locally free resolution $\mathcal{M}_\bullet$ of $\mathcal{F}$, since tensoring by a locally free sheaf is exact, the sequence $\mathcal{M}_\bullet \otimes_{\mathcal{O}_X} \mathcal{L}$ (tensor in each degree) is a locally free resolution of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$. Using Proposition 3.2.3 the results follows from the case $i = 0$ and the observation above. ∎

Proposition 3.2.6. Let X be a locally noetherian scheme, let $\mathcal{F}$ be a coherent sheaf on X, let $\mathcal{G}$ be any $\mathcal{O}_x$-module, and let $x \in X$ be a point. Then we have

$$\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G})_x \cong \operatorname{Ext}^i_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x)$$

for any $i \geqslant 0$, where the right-hand side is Ext over the local ring $\mathcal{O}_x$.

Proof. Since the question is local, we can suppose that X is affine. In this setup, $\mathcal{F}$ has a free resolution $\mathcal{L}_\bullet$ which gives at each stalk a free resolution

$$(\mathcal{L}_\bullet)_x \to \mathcal{F}_x.$$

Computation with both sides of the identity we want to prove and using the fact that the stalk functor is exact, the result follows from $i = 0$ which is true by direct calculation. ∎

Proposition 3.2.7. Let X be a projective scheme over a noetherian ring A. Let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on X. Then there is an integer $n_0 > 0$, depending on $\mathcal{F}, \mathcal{G}$, and i, such that for every $n \geqslant n_0$ we have

$$\operatorname{Ext}^i_X(\mathcal{F}, \mathcal{G}(n)) \cong \Gamma\left(X, \mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G}(n))\right).$$

Proof. It's clear that $\Gamma \circ \mathcal{H}om_X(\mathcal{F}, \_) = \operatorname{Hom}_X(\mathcal{F}, \_)$ and also, it's easy to see that $\mathcal{H}om_X(\mathcal{F}, \_)$ sends injective sheaves to Γ-acyclic sheaves. We then get the identity

$$\mathbf{R}\Gamma \circ \mathbf{R}\mathcal{H}om_X(\mathcal{F}, \_) = \mathbf{R}\operatorname{Hom}_X(\mathcal{F}, \_)\ ,$$

which in turn gives us spectral sequences (Grothendieck's spectral sequence[1])

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q_X(\mathcal{F}, \mathcal{G}(n))) \Rightarrow \operatorname{Ext}^{p+q}_X(\mathcal{F}, \mathcal{G}(n)).$$

[1]See [Wei94, Section 5.8].

By Lemma 3.2.5 the (p, q) term in $E_2^{p,q}$ is the same as $H^p(X, \mathcal{E}xt^q_X(\mathcal{F}, \mathcal{G})(n))$. By Proposition 3.2.4 and Theorem 2.4.4 we see that for each q, if n is large enough then

$$H^p(X, \mathcal{E}xt^q_X(\mathcal{F}, \mathcal{G}(n))) = 0$$

for all $p > 0$. Now, for each $i \geqslant 0$ let N_0 be large enough so that

$$H^p(X, \mathcal{E}xt^q_X(\mathcal{F}, \mathcal{G}(N_0))) = 0.$$

For all $q < i$ and for all $p \geqslant 0$. Thus for this N_0, the E_2 page of the spectral sequence looks like

$$E_2 \qquad \begin{array}{c|ccccc}
q & & & & & \\
i & E_2^{0,i} & E_2^{1,i} & E_2^{2,i} & E_2^{3,i} & E_2^{4,i} \\
i-1 & E_2^{0,i-1} & 0 & 0 & 0 & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & E_2^{0,1} & 0 & 0 & 0 & 0 \\
0 & E_2^{0,2} & 0 & 0 & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4 \quad p
\end{array}$$

In particular, the $(0, i)$-term stabilizes and the convergence of the spectral sequence implies that

$$\mathrm{Ext}^i_X(\mathcal{F}, \mathcal{G}(n)) \cong \Gamma\left(X, \mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G}(n))\right).$$

Since $i \geqslant 0$ was arbitrary we are done. ∎

Remark 3.2.8 (Ext as a δ-functor)**.** As we observed in Example 3.1.5, for a fixed $\mathcal{K}$ $\{\mathrm{Ext}^i_X(\mathcal{K}, \_)\}_{i\geqslant 0}$ is a universal δ-functor in the category Mod_X extending $\mathrm{Hom}_X(\_, \mathcal{K})$. It turns out we can also look at the sequence $\{\mathrm{Ext}^i_X(\_, \mathcal{K})\}_{i\geqslant 0}$ as a δ-functor in the category Mod_X^{op} extending $\mathrm{Hom}_X(\_, \mathcal{K})$. This is not obvious at all since the $\mathrm{Ext}^i_X(\_, \mathcal{K})$ are not exactly the derived functors of $\mathrm{Hom}_X(\_, \mathcal{K})$. This happens because the category Mod_X^{op} doesn't have enough injectives in general. Nonetheless consider an exact sequence

$$0 \to \mathcal{H} \to \mathcal{G} \to \mathcal{F} \to 0$$

in Mod_X^{op}. This is really just an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

in Mod_X. Now, given an injective resolution $I^\bullet$ of $\mathcal{K}$ in Mod_X, to calculate for example $\mathrm{Ext}^i_X(\mathcal{F},\mathcal{K})$ we look at complex

$$\mathrm{Hom}_X(\mathcal{F},\mathcal{I}^0) \xrightarrow{\delta^0_{\mathcal{F}}} \mathrm{Hom}_X(\mathcal{F},\mathcal{I}^1) \xrightarrow{\delta^1_{\mathcal{F}}} \mathrm{Hom}_X(\mathcal{F},\mathcal{I}^2) \xrightarrow{\delta^2_{\mathcal{F}}} \dots$$

Since each I^i is injective, it´s easy to see that we actually have a commutative diagram with exact columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\mathrm{Hom}_X(\mathcal{H},\mathcal{I}^0) \arrow[r, "\delta^0_{\mathcal{H}}"] \arrow[d] & \mathrm{Hom}_X(\mathcal{H},\mathcal{I}^1) \arrow[r, "\delta^1_{\mathcal{H}}"] \arrow[d] & \mathrm{Hom}_X(\mathcal{H},\mathcal{I}^2) \arrow[r, "\delta^2_{\mathcal{H}}"] \arrow[d] & \dots \\
\mathrm{Hom}_X(\mathcal{G},\mathcal{I}^0) \arrow[r, "\delta^0_{\mathcal{G}}"] \arrow[d] & \mathrm{Hom}_X(\mathcal{G},\mathcal{I}^1) \arrow[r, "\delta^1_{\mathcal{G}}"] \arrow[d] & \mathrm{Hom}_X(\mathcal{G},\mathcal{I}^2) \arrow[r, "\delta^2_{\mathcal{G}}"] \arrow[d] & \dots \\
\mathrm{Hom}_X(\mathcal{F},\mathcal{I}^0) \arrow[r, "\delta^0_{\mathcal{F}}"] \arrow[d] & \mathrm{Hom}_X(\mathcal{F},\mathcal{I}^1) \arrow[r, "\delta^1_{\mathcal{F}}"] \arrow[d] & \mathrm{Hom}_X(\mathcal{F},\mathcal{I}^2) \arrow[r, "\delta^2_{\mathcal{F}}"] \arrow[d] & \dots \\
0 & 0 & 0 &
\end{tikzcd}$$

Which induces for each $i \geqslant 0$ a commutative diagram with exact rows

$$\begin{tikzcd}
& \frac{\mathrm{Hom}_X(\mathcal{H},I^i)}{\mathrm{im}\,\delta^{i-1}_{\mathcal{H}}} \arrow[r] \arrow[d] & \frac{\mathrm{Hom}_X(\mathcal{G},I^i)}{\mathrm{im}\,\delta^{i-1}_{\mathcal{G}}} \arrow[r] \arrow[d] & \frac{\mathrm{Hom}_X(\mathcal{F},I^i)}{\mathrm{im}\,\delta^{i-1}_{\mathcal{F}}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\delta^{i+1}_{\mathcal{H}} \arrow[r] & \ker\delta^{i+1}_{\mathcal{G}} \arrow[r] & \ker\delta^{i+1}_{\mathcal{F}} &
\end{tikzcd}$$

Applying the snake lemma we get connecting morphisms

$$\mathrm{Ext}^i_X(\mathcal{F},\mathcal{K}) \to \mathrm{Ext}^{i+1}_X(\mathcal{H},\mathcal{K}).$$

One can then show that these connecting morphisms will give a δ-functor structure to $\{\mathrm{Ext}^i_X(\_,\mathcal{K})\}_{i\geqslant 0}$. The same conclusion remains valid if we change Ext_X by $\mathcal{E}\mathrm{xt}_X$.

3.3 Serre Duality on Projective Space

For the remaining sections of this chapter we´ll always work over a base field k (not necessarily algebraically closed).

Theorem 3.3.1 (Duality on projective space)**.** Put $X = \mathbb{P}^n_k$. Let $\mathcal{F}$ be a coherent sheaf on X. Recall that $H^n(X,\mathcal{O}_X(-n-1)) \cong k$.

a) The map

$$\mathrm{Hom}_X(\mathcal{F},\mathcal{O}_X(-n-1)) \times H^n(X,\mathcal{F}) \to H^n(X,\mathcal{O}_X(-n-1)) \cong k$$

is a perfect pairing of finite dimensional k-vector spaces.

b) The pairing of a) canonically induces perfect pairings of k-vector spaces

$$\operatorname{Ext}^i_X\left(\mathcal{F}, \mathcal{O}_X(-n-1)\right) \times H^{n-i}(X, \mathcal{F}) \to H^n\left(X, \mathcal{O}_X(-n-1)\right) \cong k$$

for each $i > 0$.

Proof. a) First of all, proving this claim is the same as proving that the natural morphism[2]

$$\operatorname{Hom}_X(\mathcal{F}, \mathcal{O}_X(-n-1)) \to H^n(X, \mathcal{F})^\vee$$

induced by the pairing is an isomorphism for every coherent sheaf $\mathcal{F}$. As we've seen in Corollary 2.3.2, a) is valid if $\mathcal{F} = \bigoplus_{1 \leqslant i \leqslant k} \mathcal{O}_X(m_i)$. For a general $\mathcal{F}$, by Lemma 2.4.2 we can find a presentation of $\mathcal{F}$

$$\mathcal{O}_X(m_1)^{r_1} \to \mathcal{O}_X(m_0)^{r_0} \to \mathcal{F} \to 0.$$

Let $\mathcal{E}_i = \mathcal{O}_X(m_i)^{r_i}$, $i = 0, 1$ and $\omega = \mathcal{O}_X(-n-1)$. Since both $\operatorname{Hom}_X(\_, \omega)$ and $H^n(X, \_)$ are left exact as functors from Mod_X^{op}, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \operatorname{Hom}_X(\mathcal{F}, \omega) \arrow[r] \arrow[d] & \operatorname{Hom}_X(\mathcal{E}_0, \omega) \arrow[r] \arrow[d, "\wr"] & \operatorname{Hom}_X(\mathcal{E}_1, \omega) \arrow[d, "\wr"] \\
0 \arrow[r] & 0 \arrow[r] & H^n(X, \mathcal{F})^\vee \arrow[r] & H^n(X, \mathcal{E}_0)^\vee \arrow[r] & H^n(X, \mathcal{E}_1)^\vee
\end{tikzcd}$$

whose rows are exact. By the five lemma we conclude our claim.

b) Instead of directly finding the pairing, we find natural morphisms

$$\operatorname{Ext}^i_X(\mathcal{F}, \omega) \to H^{n-i}(X, \mathcal{F})^\vee.$$

Consider the δ-functors $\{\operatorname{Ext}^i_X(\_, \omega)\}_{i \geqslant 0}$ and $\{H^{n-i}(X, \_)^\vee\}_{i \geqslant 0}$ on the category Mod_X^{op}. The first sequence is indeed a δ-functor by Remark 3.2.8. The second one is easily shown to be a δ-functor from the fact that $\{H^i(X, \_)\}_{i \geqslant 0}$ is a δ-functor in Mod_X. Consider their restrictions to $\mathsf{Coh}(X)^{op}$, the opposite category of the category of coherent $\mathcal{O}_X$-modules[3]. In this setup, we have two δ-functors along with an isomorphism at index 0 by a). By Lemma 2.4.2, for large enough m, we have a monomorphism

$$\mathcal{F} \hookrightarrow \mathcal{O}_X(-m)^r$$

in $\mathsf{Coh}(X)^{op}$. For $i > 0$, we have that

$$\operatorname{Ext}^i_X(\mathcal{O}_X(-m), \omega) = \operatorname{Ext}^i_X(\mathcal{O}_X, \omega(m)) = H^i(X, \omega(m)) \,,$$

[2]Here we really mean that this is a natural transformation from $\operatorname{Hom}_X(\_, \mathcal{O}_X(-n-1))$ to $H^n(X, \_)^\vee$ seen as functors on Mod_X^{op}. This is easy enough to check.

[3]Coh(X) is in fact abelian, see [Stacks, Tag 01BY] for more details.

which is zero for large enough m by Theorem 2.4.4. Since $\operatorname{Ext}^i_X(\_, \omega)$ is aditive, we proved that there always exists a monomorphism $\mathcal{F} \hookrightarrow \mathcal{E}$ in $\mathsf{Coh}(X)^{op}$ with $\operatorname{Ext}^i_X(\mathcal{E}, \omega) = 0$ for all $i > 0$. In other words, we proved that $\{\operatorname{Ext}^i_X(\_, \omega)\}_{i \geqslant 0}$ is an effaceable and thus universal δ-functor extending $\operatorname{Hom}_X(\_, \omega)$. This gives us the desired natural morphisms

$$\operatorname{Ext}^i_X(\mathcal{F}, \omega) \to H^{n-i}(X, \mathcal{F})^\vee.$$

By Remark 3.1.2, to prove that it is an isomorphism for every $i > 0$ and every coherent sheaf F, it's enough to prove that $\{H^{n-i}(X, \_)\}_{i \geqslant 0}$ is also an universal δ-functor. By direct calculation done in Theorem 2.3.1, we know that for each $i > 0$,

$$H^{n-i}(X, \mathcal{O}_X(-m)) = 0$$

for any $m > 0$. Using the same monomorphism as before, this proves that $\{H^{n-i}(X, \_)\}_{i \geqslant 0}$ is effaceable and thus also universal. This finishes the proof.

■

Let´s now try to understand what we did in this proof. Suppose now that X is a general projective scheme over a field k with $\dim(X) = d$. Suppose as well that there exists a coherent sheaf $\omega \in \mathsf{Coh(X)}$ along with a morphism

$$\operatorname{tr}\colon H^d(X, \omega) \to k$$

(often called a *trace morphism*) such that for all coherent sheaves $\mathcal{F}$ on X, the natural pairing

$$\operatorname{Hom}_X(\mathcal{F}, \omega) \times H^d(X, \mathcal{F}) \to H^d(X, \omega) \xrightarrow{\operatorname{tr}} k$$

induces an isomorphism

$$\operatorname{Hom}_X(\mathcal{F}, \omega) \xrightarrow{\sim} H^d(X, \mathcal{F})^\vee.$$

Then the same proof we just did would give us that $\{\operatorname{Ext}^i_X(\_, \omega)\}_{i \geqslant 0}$ is a universal δ-functor and moreover, it would give us a natural morphism

$$\{\operatorname{Ext}^i_X(\_, \omega)\}_{i \geqslant 0} \to \{H^{d-i}(X, \_)\}_{i \geqslant 0}$$

of δ-functors. In general there is no reason why this should be an isomorphism, but if we also impose that $\{H^{d-i}(X, \_)\}_{i \geqslant 0}$ is universal then it's an isomorphism. This essentially reduces Serre duality to finding ω.

In the next section, we'll prove that ω always exists and in the case that X is smooth, we have the desired isomorphisms

$$\operatorname{Ext}^i_X(\mathcal{F}, \omega) \xrightarrow{\sim} H^{d-i}(X, \mathcal{F}).$$

3.4 Duality on Projective Schemes

First of all, we fix a projective scheme X over a field k, $\dim(X) = d$, along with a closed immersion $j\colon X \hookrightarrow \mathbb{P}^n_k$. Now, lets give names to the notions discussed at the end of the last section.

Definition 3.4.1. A *dualizing sheaf* for X is a coherent sheaf ω_X on X together with a trace morphism $\mathrm{tr} : H^d(X, \omega_X) \to k$ such that for all coherent sheaves $\mathcal{F}$ on X, the natural pairing

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X) \times H^d(X, \mathcal{F}) \to H^d(X, \omega_X) \xrightarrow{\mathrm{tr}} k$$

induces an isomorphism $\mathrm{Hom}_X(\mathscr{F}, \omega_X) \xrightarrow{\sim} H^d(X, \mathscr{F})^\vee$ of k-vector spaces.

Remark 3.4.2. Dualizing sheaves are unique up to unique isomorphism. More precisely, if (ω_X, tr) and $(\omega'_X, \mathrm{tr}')$ are dualizing sheaves for X along with their trace morphisms, then there is a unique isomorphism $\varphi : \omega_X \xrightarrow{\sim} \omega'_X$ such that $\mathrm{tr} = \mathrm{tr}' \circ H^r(\varphi)$. This happens because the pair (ω_X, tr) represents the functor $H^d(X, \_)^\vee \colon \mathsf{Coh}(X)^{op} \to \mathsf{Vect}_k$.

We now construct dualizing sheaves for projective schemes over fields. For that, we use the fact that we already know a dualizing sheaf for $P = \mathbb{P}^n_k$, namely, $\omega_P = \mathcal{O}_P(-n-1)$. Fixing an isomorphism $H^n(P, \omega_P) \cong k$, for any coherent sheaf $\mathcal{F}$ on X we have

$$H^i(X, \mathcal{F})^\vee = H^i(P, j_*\mathcal{F})^\vee = \mathrm{Ext}^{n-i}_P(j_*\mathcal{F}, \omega_P).$$

For all $i \geqslant 0$. In particular, if a dualizing sheaf for X exists, then

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X) = H^d(X, \mathcal{F})^\vee = \mathrm{Ext}^{n-d}_P(j_*\mathcal{F}, \omega_P).$$

So we have essentially reduced the problem to finding a coherent sheaf ω_X on X for which there is a natural isomorphism

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X) \cong \mathrm{Ext}^{n-d}_P(j_*\mathcal{F}, \omega_P).$$

The trace morphism can then be found by tracing the identity $\mathrm{id}_{\omega_X} \in \mathrm{Hom}_X(\omega_X, \omega_X)$ through the identifications.

One might suspect that this isomorphism comes from an isomorphism of sheaves

$$\mathcal{H}om_X(\mathcal{F}, \omega_X) \overset{?}{\cong} j^*\mathcal{E}xt^{n-d}_P(j_*\mathcal{F}, \omega_P).$$

Taking $\mathcal{F} = \mathcal{O}_X$ in this hypothetical isomorphism would then give us

$$\omega_X = j^*\mathcal{E}xt^{n-d}_P(j_*\mathcal{O}_X, \omega_P).$$

We now carry out the verification that $j^*\mathcal{E}xt^{n-d}_P(j_*\mathcal{O}_X, \omega_P)$ has the required properties of a dualizing sheaf.

Lemma 3.4.3. $j^*\mathcal{E}\mathrm{xt}^i_P(j_*\mathcal{O}_X, \omega_P) = 0$ for all $i < n - d$.

Proof. It's enough to prove that

$$\mathcal{E}\mathrm{xt}^i_P(j_*\mathcal{O}_X, \omega_P) = 0$$

for all $i > n-d$. Since the sheaves $\mathcal{E}\mathrm{xt}^i_P(j_*\mathcal{O}_X, \omega_P)$ are all coherent, by Lemma 2.4.2 we know that for each i, if we choose m large enough then $\mathcal{E}\mathrm{xt}^i_P(j_*\mathcal{O}_X, \omega_P)(m)$ is generated by global sections. On the other hand, by Proposition 3.2.7 we also know that for m large enough

$$\begin{aligned}\Gamma(P, \mathcal{E}\mathrm{xt}^i_P(j_*\mathcal{O}_X, \omega_P)(m)) = \Gamma(P, \mathcal{E}\mathrm{xt}^i_P(j_*\mathcal{O}_X, \omega_P(m))) &= \mathrm{Ext}^i_P(j_*\mathcal{O}_X, \omega_P(m)) \\ &= \mathrm{Ext}^i_P(j_*\mathcal{O}_X(-m), \omega_P) \\ &= H^{n-i}(P, j_*\mathcal{O}_X(-m))^\vee \\ &= H^{n-i}(X, \mathcal{O}_X(-m))^\vee ,\end{aligned}$$

where the first and third equalities follows from Lemma 3.2.5 and the fourth follows from Serre duality for P. Now, if $i < n-d$, then $n-i > d$ and we conclude by Theorem 2.4.3 that

$$\Gamma(P, \mathcal{E}\mathrm{xt}^i_P(j_*\mathcal{O}_X, \omega_P)(m)) = 0.$$

Since $\mathcal{E}\mathrm{xt}^i_P(j_*\mathcal{O}_X, \omega_P)(m)$ is generated by global sections, we find that

$$\mathcal{E}\mathrm{xt}^i_P(j_*\mathcal{O}_X, \omega_P)(m) = 0 ,$$

and since $\mathcal{O}_P(m)$ is invertible, it follows that

$$\mathcal{E}\mathrm{xt}^i_P(j_*\mathcal{O}_X, \omega_P) = 0 ,$$

concluding the proof. ■

We can now prove that $j^*\mathcal{E}\mathrm{xt}^{n-d}_P(j_*\mathcal{O}_X, \omega_P)$ is a dualizing sheaf.

Theorem 3.4.4. $j^*\mathcal{E}\mathrm{xt}^{n-d}_P(j_*\mathcal{O}_X, \omega_P)$ is a dualizing sheaf for X.

Proof. Recall that since $j\colon X \to P$ is a closed immersion then $j_*\colon \mathsf{Mod}_X \to \mathsf{Mod}_P$ is exact and fully faithful with a right adjoint which I'll denote by $j^!$[4]. Moreover, the counit map $j^*j_* \Rightarrow \mathrm{id}_{\mathsf{QCoh}(X)}$ is an isomorphism[5]. One can also prove that given $\mathcal{F} \in \mathsf{Mod}_P$ we have a natural isomorphism

$$j_*j^!(\mathcal{F}) = \mathcal{H}om_P(j_*\mathcal{O}_X, \mathcal{F}) ,$$

and so we conclude that

$$j^!(\_) = j^*\mathcal{H}om_P(j_*\mathcal{O}_X, \_) ,$$

[4] This is a highly non-standard notation.

[5] For these facts see [Stacks, Tag 0A74] and [Stacks, Tag 08KS].

in particular, $j^*\mathcal{H}om_P(j_*\mathcal{O}_X, \_)$ is left exact. Now fix $\mathcal{F} \in \mathsf{Mod}_X$, the composition

$$\mathsf{Mod}_P \xrightarrow{j^*\mathcal{H}om_P(j_*\mathcal{O}_X,\_)} \mathsf{Mod}_X \xrightarrow{\mathrm{Hom}_X(\mathcal{F},\_)} \mathsf{Vect}_k$$

satisfies

$$\mathrm{Hom}_X(\mathcal{F}, j^*\mathcal{H}om_P(j_*\mathcal{O}_X, \mathcal{G})) = \mathrm{Hom}_P(j_*\mathcal{F}, \mathcal{G})$$

for all $\mathcal{G} \in \mathsf{Mod}_P$. I.e., the composition is the functor

$$\mathrm{Hom}_P(j_*\mathcal{F}, \_).$$

Moreover, it's easy to see that $j^*\mathcal{H}om_P(j_*\mathcal{O}_X, \_)$ preserves injectives. Indeed, given $I \in \mathsf{Mod}_P$ injective, we have

$$\mathrm{Hom}_X(\mathcal{F}, j^*\mathcal{H}om_P(j_*\mathcal{O}_X, I)) = \mathrm{Hom}_P(j_*\mathcal{F}, I).$$

The latter expression is exact in $\mathcal{F}$ since j is closed and I is injective. All of this discussion allow us to use Grothendieck's spectral sequence to the composition above to obtain a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_X(\mathcal{F}, j^*\mathcal{E}xt^q_X(j_*\mathcal{O}_X, \omega_P)) \Rightarrow \mathrm{Ext}^{p+q}_P(j_*\mathcal{F}, \omega_P).$$

By Lemma 3.4.3 all of the (p, q)-terms with $q < n - d$ in the second page are zero. Using this, we see that the $(0, n-d)$-term stabilizes in the second page and so, we have by the convergence of the spectral sequence that

$$\mathrm{Hom}_X(\mathcal{F}, j^*\mathcal{E}xt^{n-d}_P(j_*\mathcal{O}_X, \omega_P)) = \mathrm{Ext}^{n-d}_P(j_*\mathcal{F}, \omega_P).$$

Since Grothendieck's spectral sequence is natural in $\mathcal{F}$ we see that the above isomorphism is also natural in $\mathcal{F}$. By the discussion at the start of the section we conclude that $j^*\mathcal{E}xt^{n-d}_P(j_*\mathcal{O}_X, \omega_P)$ is a dualizing sheaf for X. ■

The discussion at the end of last section then gives us the following.

Corollary 3.4.5 (Serre Duality on Projective Schemes)**.** Let X be a projective scheme of dimension d over a field k, and let ω_X be a dualizing sheaf for X. Then for all $0 \leqslant i \leqslant d$ and coherent sheaves $\mathcal{F}$ on X, there exist natural morphisms

$$\mathrm{Ext}^i_X(\mathscr{F}, \omega_X) \to H^{d-i}(X, \mathscr{F})^\vee.$$

If $i = 0$ this is an isomorphism.

Proof. This follows from our previous discussions. ■

If X is smooth then one can show that $j^*\mathcal{E}xt^i_P(j_*\mathcal{O}_X, \omega_P) = 0$ for all $i > n - d$ and $j^*\mathcal{E}xt^{n-d}_P(j_*\mathcal{O}_X, \omega_P) = \Omega^d_{X/k} = \bigwedge^d \Omega^1_{X/k}$, where $\Omega^1_{X/k}$ is the sheaf of Kähler differentials of X over k. Moreover, in this case $\{H^{d-i}(X, \_)^\vee\}_{i \geqslant 0}$ is a universal δ-functor. See [Har77, Chapter 3] for more details.

Remark 3.4.6. Using Chow's lemma [GW10, Theorem 13.100], one can prove that Serre duality is actually valid for proper schemes over k. More precisely, there exists a dualizing sheaf for X and moreover, if X is also smooth then we also have the desired isomorphisms.

To summarize we have the following.

Theorem 3.4.7. Let k be a field and X a proper scheme over k of dimension d. Then there exists a dualizing sheaf ω_X for X and a trace morphism,

$$\mathrm{tr} : H^d(X, \omega_X) \longrightarrow k ,$$

such that for all coherent sheaves $\mathcal{F}$ on X we have induced pairings

$$\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) \times H^{d-i}(X, \mathcal{F}) \longrightarrow H^d(X, \omega_X) \xrightarrow{\mathrm{tr}} k.$$

Moreover, this pairing is always perfect if $i = 0$ and if X is smooth then $\omega_X = \Omega^d_{X/k}$ and all of these pairings are perfect.

We now state a last corollary.

Corollary 3.4.8. Let k be a field and X a proper smooth scheme over k of dimension d. If $\mathcal{F}$ in the theorem above is locally free, we have natural isomorphisms

$$H^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{F}^\vee) \xrightarrow{\sim} H^{d-i}(X, \mathcal{F})$$

for every $0 \leqslant i \leqslant d$.

Proof. This follows directly from last theorem since

$$\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) = \mathrm{Ext}^i_X(\mathcal{O}_X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{F}^\vee) = H^i(X, \omega_X \otimes \mathcal{F}^\vee).$$

■

Chapter 4

The Riemann-Roch Theorem and Applications

In this chapter we fix a not necessarily algebraic closed field k and always work on the category Sch/k of schemes over k.

4.1 Divisors and the Riemann-Roch Theorem

Definition 4.1.1. Let X be a scheme of finite type over k. X is said to be a curve if it is geometrically integral (i.e. $X_K = X \times_k \operatorname{Spec} K$ is integral for every field extension K/k), smooth, proper and of dimension 1.

Remark 4.1.2. This definition is not standard but will be the most useful to us.

Example 4.1.3. Let $p(x,y,z) = y^2z - x^3 + xz^2 - z^3 \in \mathbb{C}[x,y,z]$. The scheme $E = \operatorname{Proj} \mathbb{C}[x,y,z]/p(x,y,z)$ can be easily seen to satisfy the conditions above. Indeed, we can see that $p(x,y,z)$ is irreducible and moreover, that it doesn't have any common zeros with its partial derivatives. Since we are working over an algebraically closed field, this implies that E is geometrically irreducible and smooth. Properness follows from the fact that it's a closed subscheme of projective space and at last, since E is integral and of finite type over a field, the dimension of E is the same as in any open subset of E, where we can easily see that the dimension is 1. This is an example of an *elliptic curve* over $\mathbb{C}$. If we look at the complex points of E,

$$E(\mathbb{C}) = \operatorname{Hom}_{\operatorname{Spec}\mathbb{C}}(\operatorname{Spec}\mathbb{C}, E) = \{(a:b:c) \in \mathbb{P}^2(\mathbb{C})^{1} \mid p(a,b,c) = 0\},$$

it has a canonical structure of a compact Riemann Surface, recovering the usual definition of a complex elliptic curve.

Let X be a curve over k. One can show that the subset of closed points of X, denoted by X_0[2], is exactly the set

$$\{x \in X \mid k(x) \text{ is a finite extension of } k\}.$$

[1]Here we use the notation $\mathbb{P}^2(\mathbb{C})$ to denote the classical complex projective space of lines through the origin in $\mathbb{C}^3$ (equivalently, this is the set of complex points of the scheme $\mathbb{P}^2_{\mathbb{C}}$).

[2]It's easy to show that in the case of a curve X_0 is non-empty and every point is closed except for the generic point of X.

Remark 4.1.4. If k is algebraically closed, then we have that

$$\begin{aligned} X_0 &= \{x \in X \mid k(x) \text{ is a finite extension of } k\} \\ &= \{x \in X \mid k(x) = k\} = \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, k, X) = X(k). \end{aligned}$$

The third equality comes from the fact that a morphism $\mathrm{Spec}\, k \to X$ contains exactly the same information as a choice of a closed point x in X and a k-homomorphism of fields $k(x) \hookrightarrow k$. In general this always identifies $X(k)$ with a subset of X_0, but for a general field extension K/k this identification doesn't quite work. This happens because K might have non-trivial k-automorphisms and so, a closed point might give rise to more than one morphism $\mathrm{Spec}\, K \to X$.

We then define a *divisor* on X as a formal sum of closed points of X. More formally:

Definition 4.1.5. Let X be a curve over k. The divisor group of X, $\mathrm{Div}(X)$, is the free abelian group generated on the closed points of X. Given a divisor $D = \sum n_p[p]$ we define its support $\mathrm{supp}(D)$ as the points in which $n_p \neq 0$. We also define the *degree* of a divisor $D = \sum n_p[p]$ as

$$\deg(D) = \sum [K(p) \colon k] n_p \in \mathbb{Z}.$$

In fact, this defines a group homomorphism

$$\deg \colon \mathrm{Div}(X) \to \mathbb{Z}\ ,$$

with kernel denoted by $\mathrm{Div}^0(X)$. We also say that a divisor $D = \sum n_p[p]$ is *effective* if $n_p \geqslant 0$ for all p.

Since X is integral, we can also look at its function field $k(X) = \mathrm{Frac}(\mathcal{O}_{X,\eta})$, where η is the generic point of X[3]. This should really be thought as the field of meromorphic functions on X. Indeed, observe that in our case given any point $x \in X_0$, we have that $\mathcal{O}_{X,x}$ is a discrete valuation ring with valuation given by

$$v_x(f) = \sup\{n \in \mathbb{N} \mid f \in \mathfrak{m}_x^n \text{ but } f \notin \mathfrak{m}_x^{n+1}\}.$$

Since $k(X) = \mathrm{Frac}(\mathcal{O}_{X,x})$, we can extend this valuation to $k(X)$. We then say that $f \in k(X)$ has a zero of order $n \in \mathbb{N}$ at $x \in X_0$ if $v_x(f) = n$ and similarly, that f has a pole of order $n \in \mathbb{N}$ at x if $v_x(f) = -n$. It's non-trivial to prove that any $f \in k(X)^\times$ has only a finite number of zeros and poles and moreover, that

$$\sum_{x \in X_0} v_x(f) = 0.$$

This gives us a homomorphism of groups

$$\mathrm{div} \colon k(X)^\times \to \mathrm{Div}^0(X)$$

[3]In fact, $k(X)$ is the fraction field of $\mathcal{O}_X(U)$ for any open set $U \subseteq X$.

given by $\mathrm{div}(f) = \sum v_p(f)[p]$. Divisors of this form are called *principal* divisors and form a subgroup of $\mathrm{Div}(X)$. Two divisors $D, E \in \mathrm{Div}(X)$ are said to be *equivalent* if $D-E$ is principal. We denote this equivalence relation by $D \sim E$. The quotient group of $\mathrm{Div}(X)$ (resp. $\mathrm{Div}^0(X)$) by principal divisors is denoted by $\mathrm{Cl}(X)$ (resp. $\mathrm{Cl}^0(X)$).

An amazing fact about divisors is the following.

Theorem 4.1.6. Let X be a curve over k. Given a divisor $D \in \mathrm{Div}(X)$ we define a sheaf $\mathcal{O}_X(D)$ by

$$\begin{aligned}\mathcal{O}_X(D)(U) &= \{f \in k(X)^\times \mid v_p(f) \geqslant -n_p \text{ for all } p \in U \text{ closed}\} \cup \{0\}\\ &= \{f \in k(X)^\times \mid D + \mathrm{div}(f) \text{ is effective.}\} \cup \{0\}\ ,\end{aligned}$$

i.e., f has at most a pole of order n_p for all p in U closed. Then $O_X(D)$ is an invertible sheaf and moreover, this association defines an isomorphism of groups

$$\mathrm{Cl}(X) \xrightarrow[\mathcal{O}_X(\_)]{\sim} \mathrm{Pic}(X).$$

Proof. This can be seen in the chapter about divisors on the online notes [EO20]. ■

With this theorem in hand we define a canonical divisor $K_X \in \mathrm{Div}(X)$ as any divisor such that $\mathcal{O}_X(K_X) = \Omega^1_{X/k} = \omega_X$[4]. We can now state the Riemann-Roch theorem.

Theorem 4.1.7 (Riemann-Roch)**.** Let X be a curve over k. Then for any divisor $D \in \mathrm{Div}(X)$ we have

$$\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \mathcal{O}_X(K_X - D)) = \deg(D) - g + 1\ ,$$

where $g = \dim_k H^0(X, \omega_X) = \dim_k H^1(X, \mathcal{O}_X)$ is the *genus* of X.

Proof. By Corollary 3.4.8, it's enough to prove that

$$\chi(\mathcal{O}_X(D)) := \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^1(X, \mathcal{O}_X(D)) = \deg(D) - g + 1.$$

Now, note that $H^0(X, \mathcal{O}_X) = k$. Indeed, since X is irreducible and of finite type over k, we have that $H^0(X, \mathcal{O}_X)$ is a finite dimensional k-algebra without non-trivial nilpotents and idempotents. Using the structural theorem for commutative artinian rings we conclude that $H^0(X, \mathcal{O}_X)$ is a field finite over k. If k is algebraically closed we're finished. If k is not algebraically closed, the conclusion is still valid but involves more work involving the smoothness of X. See [MO15, Chapter VIII, Theorem 1.1] for more details. Thus, we have reduced Riemann-Roch to proving that

$$\chi(\mathcal{O}_X(D)) = \deg(D) + \chi(\mathcal{O}_X).$$

Given a closed point $p \in X$, we have an inclusion $\mathcal{O}_X(-[p])(U) \subseteq \mathcal{O}_X(U)$ which induces a monomorphism $\mathcal{O}_{\mathcal{X}}(-[p]) \hookrightarrow \mathcal{O}_X$ with kernel given by the skyscraper sheaf with value

[4] As we´ve seen before in 3.4.7, this is the dualizing sheaf of the curve.

$k(p)$ at the pont p. We then have an exact sequence

$$0 \to \mathcal{O}_X(-[p]) \to \mathcal{O}_X \to k(p) \to 0.$$

Since $\mathcal{O}_X(D)$ is invertible for any divisor D, then tensoring we have an exact sequence

$$0 \to \mathcal{O}_X(D - [p]) \to \mathcal{O}_X(D) \to k(p) \to 0$$

for any divisor D. Using the long exact sequence in cohomology we easily find that

$$\chi(\mathcal{O}_X(D - [p])) = \chi(\mathcal{O}_X(D)) - \chi(K(p)) = \chi(\mathcal{O}_X(D)) - 1.$$

With this formula, it's clear that the theorem holds for a divisor D if and only if it holds for $D - [p]$ for any closed point $p \in X$. By adding and subtracting the points in the support of a divisor D we reduce to the case of $D = 0$. In this case the result is obvious. ■

We now state some direct consequences of the Riemann-Roch theorem.

Corollary 4.1.8. Let X be a curve over k. Then

a) $\deg(K_X) = 2g - 2$.

b) If $\deg(D) < 0$ then $H^0(X, \mathcal{O}_X(D)) = 0$.

c) If $\deg(D) > 2g - 2$ then

$$\dim_k H^0(X, \mathcal{O}_X(D)) = \deg D - g + 1.$$

Proof. a) This follows from Riemann-Roch directly by substituting $D = K_X$.

b) If $H^0(X, \mathcal{O}_X(D)) \neq 0$ then there exists $f \in k(X)^\times$ such that $D + \operatorname{div}(f)$ is effective. In particular, $\deg(D) = \deg(D + \operatorname{div}(f)) \geqslant 0$.

c) If $\deg(D) > 2g - 2$ then $\deg(K_X - D) < 0$ and so the claim follows from b) and Riemann-Roch.

■

4.2 Applications of Riemann-Roch to the Theory of Algebraic Curves

Recall that a invertible sheaf $\mathcal{L}$ on a scheme X over k is *very ample* if there exists a closed immersion $i\colon X \hookrightarrow \mathbb{P}^n_k$ such that $\mathcal{L} = i^*\mathcal{O}_X(1)$. An invertible sheaf is *ample* if $\mathcal{L}^{\otimes n}$ is very ample for some $n > 0$. Using the machinery of Riemann-Roch one can prove the following.

Theorem 4.2.1. Let X be a curve over k. Let $\mathcal{L} = \mathcal{O}_X(D)$ be an invertible sheaf.

a) $\mathcal{L} = \mathcal{O}_X(D)$ is ample if and only if $\deg(D) > 0$.

b) $\mathcal{L} = \mathcal{O}_X(D)$ is very ample if $\deg(D) > 2g$.

Proof. See [MO15, Chapter VIII, Proposition 1.7]. ■

With this theorem in hand, we have the following simple consequence.

Corollary 4.2.2. Let X be a curve over k. Then X is projective.

Proof. Given a closed point $p \in X$, consider the divisor $D = [p]$. Then $\deg(nD) = n[K(p)\colon k]$. By last theorem, for n large we find that $\mathcal{O}_X(nD)$ is very ample. ■

For curves of small genus we can do even better. Using Riemann-Roch we can show that any curve of genus $\leqslant 0$ is a smooth cubic and $X \cong \mathbb{P}^1_k$ if and only if $X(k) \neq \varnothing$. We now give a little more focus to curves of genus 1, the so called elliptic curves. For great classical references on elliptic curves see [Sil09] and [Sil94].

Definition 4.2.3. An *elliptic curve* E over k is a curve E of genus 1 with a k-rational point $O \in X(k)$.

Example 4.2.4. Let $p(x,y,z) = y^2z + a_1xyz - a_2x^2z + a_3yz^2 - a_4xz^2 - x^3 - a_6z^3$ such that $E = \operatorname{Proj}\left(\frac{k[x,y,z]}{p(x,y,z)}\right)$ is smooth. Then E is an elliptic curve. If $\operatorname{char} k \neq 2,3$ we can do a variable change to suppose that $p(x,y,z) = y^2 - x^3 - Ax - B$, finding the usual representation of an elliptic curves.

We already know that elliptic curves are projective, but a more precise statement can be proved. Will see that every elliptic curve is of the form given in the above example.

Proposition 4.2.5. Let E be an elliptic curve over k. Then there exists a closed immersion $E \hookrightarrow \mathbb{P}^2_k$ inducing an isomorphism

$$E \cong \operatorname{Proj}\left(\frac{k[x,y,z]}{p(x,y,z)}\right) ,$$

with $p(x,y,z)$ given as in Example 4.2.4.

Proof. Using Theorem 4.2.1 we know that $\mathcal{O}_E(3[O])$ is very ample. By Theorem 4.1.7 we also find that $\dim_k H^0(X, \mathcal{O}_E(3[O]) = 3$. One can then show that this implies that $\mathcal{O}_E(3[O])$ is generated by 3 global sections and so the embedding induced by this very ample sheaf has codomain $\mathbb{P}^2_k$. Moreover, one can prove that every embedded curve in $\mathbb{P}^2_k$ is given by some homogeneous polynomial and if this polynomial has degree d then the genus of the curve is $(d-1)(d-2)/2$. This proves the claim. ■

A more interesting fact about elliptic curves is the following.

Theorem 4.2.6. Let E be an elliptic curve over k. Then for every field extension K/k the set $E(K)$ has a group structure.

First we need some lemmas.

Lemma 4.2.7. Let E be an elliptic curve over k. Given two closed points $p, q \in E_0$ then

$$[p] \sim [q] \iff p = q.$$

Proof. $p = q$ obviously implies $[p] \sim [q]$. If $[p] \sim [q]$, there exists $f \in k(E)^\times$ such that $[p] = [q] + \mathrm{div}(f)$. In particular, we find that $f \in H^0(E, \mathcal{O}_E([q])$. By Theorem 4.1.7 we find that $\dim_k H^0(E, \mathcal{O}_E([q]) = 1$. Since $H^0(E, \mathcal{O}_E([q])$ contain the constant functions we conclude that f is constant and so p=q. ■

Lemma 4.2.8. Let E be an elliptic curve over k.

a) For every $D \in \mathrm{Div}^0(E)$ there exists a unique point $p \in E(k)$ such that

$$D \sim [p] - [O].$$

b) The map

$$\sigma\colon \mathrm{Div}^0(E) \to E(k)$$

given by $\sigma(D) = p$ if $D \sim [p] - [O]$ is surjective and induces a bijection

$$\overline{\sigma}\colon \mathrm{Cl}^0(E) \to E(k).$$

Proof. a) By Theorem 4.1.7 we have that $\dim_k H^0(E, \mathcal{O}_E(D + [O])) = 1$. Let $f \in H^0(E, \mathcal{O}_E(D + [O]))$ be non-zero, so that $D' = D + [O] + \mathrm{div}(f)$ is effective. If $D = \sum n_p[p]$ then

$$\sum_{p \neq O} [k(p) \colon k](n_p + v_p(f)) + (n_O + v_O(f) + 1) = 1.$$

By the effectiveness of D' every term in this sum is positive and so, only one of then is non-zero equal to 1. In any case we conclude that

$$\mathrm{div}(f) = [p] - [O] - D$$

for some $p \in E(k)$ i.e., $D \sim [p] - [O]$. The uniqueness of p is then clear by Lemma 4.2.7.

b) The map is clearly surjective since $\sigma([p] - [O]) = p$. Moreover, by definition we have

$$D_1 - D_2 \sim [\sigma(D_1)] - [\sigma(D_2)].$$

By Lemma 4.2.7 we conclude that

$$D_1 \sim D_2 \iff [\sigma(D_1)] \sim [\sigma(D_2)] \iff \sigma(D_1) = \sigma(D_2).$$

Thus the induced map $\overline{\sigma}\colon \mathrm{Cl}^0(E) \to E(k)$ is a bijection.

■

Proof of Theorem 4.2.6. First of all, we observe that it's enough to prove that $E(k)$ has a group structure. Indeed, given a field extension K/k consider a cartesian diagram

$$\begin{tikzcd} E_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} k. \end{tikzcd}$$

In our case being of finite type, geometrically integral, smooth, proper and of dimension 1 are all properties closed under base change, so E_K is a curve over K. Moreover, by flat base change theorems (See [Har77, Proposition II.9.3].) we have that

$$H^i(E_K, \Omega^1_{E_K/K}) = H^i(E, \Omega^1_{E/k}) \otimes_k K \ ,$$

and so, the genus of E_K is also 1. Using the canonical morphism $O_K \colon \operatorname{Spec} K \to E_K$ induced by $O \in E(k)$ we conclude that E_K is an elliptic curve over K. Since $E_K(K) = E(K)$ it's enough to prove the theorem for $K = k$. In this case we can use Theorem 4.2.8 to transport the group structure of $\mathrm{Cl}^0(E)$ to $E(k)$. ∎

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