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Toward a Comprehensive Theory for Stability Regions of a Class of Nonlinear Discrete Dynamical Systems

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***Abstract*—A comprehensive theory for the stability boundaries and the stability regions of a general class of nonlinear discrete dynamical systems is developed in this paper. This general class of systems is modeled by diffeomorphisms and admits as limits sets only fixed points and periodic orbits. Topological and dynamical characterizations of stability boundaries are developed. Necessary and sufficient conditions for fixed points and periodic orbits to lie on the stability boundary are derived. Numerical examples, including applications to associative neural-networks, illustrating the theoretical developments are presented.**

***Index Terms*—Nonlinear Discrete Dynamical Systems, Stability Region, Stability Boundary, Periodic Orbits.**

I. Introduction

DISCRETE dynamical systems may exhibit very complex behavior, such as periodic orbits and chaos. Deriving a characterization of the stability region (region of attraction) and stability boundary (the boundary of stability region) of these dynamical systems is a very difficult task. Up to now, only a characterization of stability boundary of a particular class of systems admitting only hyperbolic fixed points on the stability boundary was developed [1]. In this paper, we give a step further in the process of understanding the stability boundary of these systems by extending this characterization to systems that admit both fixed and periodic hyperbolic points on the stability boundary.

The knowledge of stability region is important in many practical discrete dynamical systems, including ecosystems [2] and economic models [3], power systems [4] and sample-data systems [5], [6]. Many practical discrete systems exhibit periodic orbits, including recurrent iterated-map neural networks [7], [8] and dynamics defined on complex number spaces [9]. The concept of stability region is also important in applications of control of discrete systems [10], including studies of networked control systems [11], [12] and control of engines with fuel injection [13]. Hence, determining or estimating stability regions of nonlinear discrete dynamical systems is relevant in many applications.

There has been significant work on analyzing the stability and asymptotic behavior of discrete-time dynamical systems, providing estimates of the stability region. An invariance principle for discrete-time systems was proven by LaSalle in [14], [15] and an extension of this principle was independently derived in [16] and [17]. A survey on the theory of positively invariant sets in the analysis and control of discrete-time nonlinear dynamical systems was provided in [18]. Recent advances on the stability theory of these systems can be found in [19]–[21]. In spite of the enormous amount of work done in the analysis of asymptotic behavior of solutions of discrete-time nonlinear dynamical systems, the problem of characterizing and estimating the stability region of these systems is still an open problem.

Significant progress has been made in the development of theory and estimation of stability regions of nonlinear continuous dynamical systems [22]–[24]. In addition, methodologies for optimally estimating stability regions of nonlinear continuous dynamical systems were derived in [25]. These developments led to several advances and practical methods to estimate stability regions of large-scale nonlinear dynamical systems on the order of 40,000 dimensions [26]. These developments have also attracted great interest from power industries [27] and nonlinear optimization technologies [1].

Compared with nonlinear continuous dynamical systems, few analytical results on the characterization of stability regions of nonlinear discrete dynamical systems exist.

The main contribution of this paper is extending the theory of stability regions for a general class of nonlinear discrete dynamical systems. This theory parallels the ones developed for continuous system in [22] and extends the results of stability regions developed in [1] for a larger class of discrete dynamical systems by admitting not only fixed points but also periodic orbits on the stability boundary. This extension is achieved by studying the characterization of hyperbolic periodic orbits on the boundary of stability regions.

Specifically, this paper develops, for the class of nonlinear discrete dynamical systems modeled by diffeomorphisms:

- necessary and sufficient conditions for hyperbolic p-periodic orbits belonging to the boundary of the stability region.

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- a complete characterization of the boundary of the stability region admitting, in the limit set, only hyperbolic fixed points and hyperbolic p-periodic orbits on the stability boundary.

These characterizations are useful to understand the composition of the boundary of the stability region and to develop algorithms for optimally estimating it.

II. Discrete Dynamical Systems

Consider the autonomous nonlinear time-discrete dynamical systems:

$$x_{k+1} = f(x_k) \tag{1}$$

where $k \in \mathbb{Z}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, i.e. f is continuos, differentiable, invertible and its inverse is also continuous and differentiable. A consequence of f being a dipheomorphism is that solutions are well defined for forward and backward times.

The solution of (1), starting from $x_0 \in \mathbb{R}^n$ at $k = 0$, denoted by $\phi(\cdot, x_0) : \mathbb{Z} \to \mathbb{R}^n$, is called an orbit (or trajectory) of (1), i.e., $x_k = \phi(k, x_0) = f^k(x_0)$, where function f^k stands for the k-fold composition of f.

A set M is said positively invariant with respect to the discrete system (1) if $f(M) \subset M$, which implies that every orbit x_k starting in M remains in M for all $k \geq 0$. A set M is negatively invariant if $f^{-1}(M) \subset M$. A set M is said invariant if $f(M) = M$, see [10].

A point z is said to be in the ω-limit set (or α-limit set) of x_0 if there is a sequence $k_i \in \mathbb{Z}$ with $k_i \to +\infty$ (or $k_i \to -\infty$) as $i \to \infty$ such that $z = lim_{i\to\infty} x_{k_i}$ [28]. The ω-limit set $\omega(x)$ is closed and invariant. If, in addition, the forward orbit $\{f^k(x_0), k \geq 0\}$ is bounded, then the ω-limit set is nonempty, compact, invariant, and invariantly connected[1] [1]. Moreover, the solution approaches the limit set as $k \to \infty$, i.e.,

$$x_k = f^k(x) \to \omega(x_0) \text{ as } k \to \infty.$$

A point $x^* \in \mathbb{R}^n$ is a *periodic point* of period p of system (1) if $f^p(x^*) = x^*$ and $f^k(x^*) \neq x^*$ for all $k = 1, 2, \ldots, p-1$. Let x^* be a periodic point of period p of system (1). The sequence $\gamma = \left\{x^*, f(x^*), \ldots, f^{p-1}(x^*)\right\}$ is a *periodic orbit of period* p of system (1). In the above definition, if $p = 1$, i.e., $f(x^*) = x^*$, then x^* is a *fixed point* of system (1). We denote by P the set of all periodic-points and fixed points of (1).

We say a fixed point x^* is *hyperbolic* if the Jacobian of the function f at x^*, $Df(x^*)$, has no eigenvalues with modulus equal to 1. Moreover, if all eigenvalues of $Df(x^*)$ have modulus less than 1, then x^* is an asymptotically stable fixed point of system (1). Similarly, x_0 is a *source or repelling fixed point* if all eingenvalues of $Df(x^*)$ have modulus greater than 1.

If f is a $\mathcal{C}^r$-diffeomorphism with $r \geq 1$, and x^s is a hyperbolic fixed point, there are unique manifolds $W^s(x^*)$ and $W^u(x^*)$, respectively called *stable* and *unstable* manifolds, of class C^r, that are invariant with respect to (1) [29], [30]. Every orbit x_k starting in $W^s(x^*)$ tends to x^* as $k \to +\infty$, whereas every orbit starting in $W^u(x^*)$ tends to x^* as $k \to -\infty$.

Let q be a point in $\mathbb{R}^n$ and let M and N be differentiable manifolds in $\mathbb{R}^n$; then M and N satisfy the *transversality condition* at q if: (1) $q \notin M \cap N$ or (2) if $q \in M \cap N$, then $T_qM + T_qN = \mathbb{R}^n$ [31]. If M and N satisfy the transversality condition at every point $q \in \mathbb{R}^n$, then M and N satisfy the transversality condition. [2]

III. Periodic Orbits and the p-iterated System

To investigate p-periodic orbits on the stability boundary of an asymptotically stable fixed point of a discrete system (1), we will explore the relationship between the discrete system (1) and an auxiliary associated p-iterated system. One advantage of this approach is the fact that if p is a multiple of the period of all periodic orbits of system (1), then all of the periodic points of system (1) become fixed points of system (2). The characterization of stability regions for systems that admit only fixed points on the stability boundary has already been developed in the literature [1] and can be applied to system (2).

For an integer p greater than or equal to 1, the p-iterated system associated with system (1) is given by:

$$x_{m+1} = f^p(x_m). \tag{2}$$

.

Fixed points and periodic points of the discrete dynamical system (1) have a close correspondence with fixed points of the p-iterated system (2).

If $x^s \in \mathbb{R}^n$ is a (hyperbolic) fixed point of (1), then x^s is also a (hyperbolic) fixed point of (2) [32]. In particular, if $x^s \in \mathbb{R}^n$ is an asymptotically stable fixed point of (1), then x^s is an asymptotically stable fixed point of (2). If x^* is a periodic point of period p of system (1), then x^* is a fixed point of system (2). Indeed, if x^* is a fixed point of system (2), then either x^* is a fixed point or a periodic point of system (1).

A subindex p will be used to differentiate the manifolds of fixed points with respect to the p-iterated system (2) from those with respect to the system (1). For a hyperbolic fixed point x^*, for example, the stable and unstable manifolds with respect to the p-iterated system will be respectively denoted $W^s_p(x^*)$ and $W^u_p(x^*)$.

A periodic orbit γ with period p of system (1) is *hyperbolic* if and only if every point of the periodic orbit γ is a hyperbolic fixed point for system (2) [32]. For a hyperbolic periodic orbit $\gamma = \left\{x^*, f(x^*), \ldots, f^{p-1}(x^*)\right\}$, where x^* is a periodic point with period p of system (1), we define the *stable manifold* and the *unstable manifold* of γ, respectively, as

$$W^s(\gamma) = \{x \in \mathbb{R}^n; \omega(x) \subset \gamma\}$$

$$W^u(\gamma) = \{x \in \mathbb{R}^n; \alpha(x) \subset \gamma\}.$$

The next lemma establishes a relationship between the invariant manifolds of a periodic orbit γ of period p of the

[1] A closed and invariant set M is invariantly connected if it is not the union of two non-empty disjoint closed invariant sets [15].

[2] T_qM and T_qN denote the tangent spaces of M and N at point q.

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discrete dynamical system (1) with the invariant manifolds of the points belonging to the periodic orbit with respect to the p-iterated system (2).

Lemma 1. *Let $\gamma = \{x^*, f(x^*), \ldots, f^{p-1}(x^*)\}$ be a hyperbolic periodic orbit of period p of system (1) where function f is a diffeomorphism. Then,*

$$W^s(\gamma) = W_p^s(x^*) \cup W_p^s(f(x^*)) \cup \ldots \cup W_p^s(f^{p-1}(x^*))$$
$$W^u(\gamma) = W_p^u(x^*) \cup W_p^u(f(x^*)) \cup \ldots \cup W_p^u(f^{p-1}(x^*)).$$

The validity of the above Lemma 1 can be verified by means of the Proposition 9.1 demonstrated in [32] and exploring the fact that the stable and unstable manifolds of the Definition 6.1 of [32] coincide with the stable and unstable manifolds of the fixed points $f^i(x^*)$, $i = 0, ..., p-1$ of the p-iterated system (2), respectively.

IV. Stability Regions

Suppose that x^s is an asymptotically stable fixed point of system (1). The *Stability Region* of x^s is the set of initial conditions whose trajectories tend to the fixed point x^s, i.e.,

$$A(x^s) = \left\{ x \in \mathbb{R}^n; \lim_{k\to\infty} f^k(x) = x^s \right\}.$$

Stability Region is also called *Region of Attraction* and its topological boundary, called *Stability Boundary* , will be denoted by $\partial A(x^s)$.

Several topological characterizations of stability regions of nonlinear discrete dynamical systems were studied in [1]. The stability region $A(x^s)$ is: (i) open, (ii) positively and negatively invariant, (iii) invariant, and (iv) path connected; and the stability boundary $\partial A(x^s)$ is: (i) closed and (ii) invariant. See [1], [33].

An asymptotically stable fixed point x^s of the discrete dynamical system (1) is also an asymptotically stable fixed point of the p-iterated system (2). Consequently, the stability region of x^s with respect to the p-iterated system can be defined as:

$$A_p(x^s) = \left\{ x \in \mathbb{R}^n; \lim_{k\to\infty} f^{pk}(x) = x^s \right\}$$

and its topological boundary is denoted as $\partial A_p(x^s)$.

Theorem 1 establishes a relationship between the stability region of system (1) and the stability region of the associated p-iterated system (2).

Theorem 1. *If x^s is an asymptotically stable fixed point of system (1), then $A(x^s) = A_p(x^s)$ for all $p \in \mathbb{Z}_+$.*

Proof. First, suppose that $x \in A(x^s)$, i.e. $f^k(x) \to x^s$ as $k \to \infty$. In particular, $f^{kp}(x) \to x^s$ as $k \to \infty$, thus $x \in A_p(x^s)$ and $A(x^s) \subset A_p(x^s)$.

Suppose now that $x \in A_p(x^s)$ for some p, i.e. $f^{kp}(x) \to x^s$ as $k \to \infty$. We must prove that $f^n(x) \to x^s$ as $n \to \infty$. More precisely, we must prove that, for a given $\epsilon > 0$, there exists an integer N such that $||f^n(x) - x^s|| < \epsilon$ for all $n > N$. The fixed point x^s is an asymptotically stable fixed point of (1). Then, for a given $\epsilon > 0$, there exists $\delta > 0$ such that $||x - x^s|| < \delta \Rightarrow ||f^n(x) - x^s|| < \epsilon$ for all $n > 0$ and $f^n(x) \to x^s$ as $n \to \infty$. Since x belongs to the stability region of the p-iterated system (2), there is a number k^* such that $||f^{k^*p}(x) - x^s|| < \delta$. Then $||f^n(x) - x^s|| < \epsilon$ for all $n > k^*p$ and $f^n(x) \to x^s$ as $n \to \infty$. Thus $A_p(x^s) \subset A(x^s)$. This concludes the proof.

□

Theorem 1 shows that the stability region of any asymptotically stable fixed point of the discrete dynamical system (1) is equal to the stability region of this same point in the associated p-iterated system (2). Consequently, we can study the stability region of the original system by studying the stability region of the associated p-iterated system.

Since $A(x^s) = A_p(x^s)$ and $\partial A(x^s) = \partial A_p(x^s)$ for all $p = 1, 2, \ldots$, then all of the topological properties of stability regions and stability boundaries discussed in section IV also hold for the p-iterated system (2).

V. Characterization of Stability Boundary

A characterization of stability boundaries of a class of discrete nonlinear dynamical systems will be developed in this section. We first derive a local characterization for a fixed point and a periodic orbit to lie on the stability boundary $\partial A_p(x^s)$ of the p-iterated system. Next, additional conditions are imposed on the discrete dynamical system and the results are further sharpened. Finally, we develop a characterization of the stability boundary as the union of the invariant stable manifolds of critical elements on the stability boundary.

Theorem 2 offers necessary and sufficient conditions for a periodic orbit lying on the stability boundary $\partial A(x^s)$ of system (1).

Theorem 2 (Characterization of periodic orbits on the stability boundary). *Let $A(x^s)$ be the stability region of an asymptotically stable fixed point x^s of system (1). Let $\gamma = \{x^*, f(x^*), \ldots, f^{p-1}(x^*)\}$ be a hyperbolic periodic orbit with period p of system (1) and suppose f is a diffeomorphism. Then,*

(i) *if $\gamma \subseteq \partial A(x^s)$, then $\{W_p^u(\hat{x}) - \{\hat{x}\}\} \cap \overline{A_p(x^s)} \neq \emptyset$ for all $\hat{x} \in \{x^*, f(x^*), \ldots, f^{p-1}(x^*)\}$;*

(ii) *if $\hat{x} \in \{x^*, f(x^*), \ldots, f^{p-1}(x^*)\}$ and $\{W_p^u(\hat{x}) - \{\hat{x}\})\} \cap \overline{A_p(x^s)} \neq \emptyset$, then $\gamma \subset \partial A(x^s)$;*

(iii) *if $\gamma \subseteq \partial A(x^s)$ and $\hat{x}$ is not a source of system (2) for all $\hat{x} \in \{x^*, f(x^*), \ldots, f^{p-1}(x^*)\}$, then $\{W_p^s(\hat{x}) - \{\hat{x}\}\} \cap \partial A_p(x^s) \neq \emptyset$ for all $\hat{x} \in \{x^*, f(x^*), \ldots, f^{p-1}(x^*)\}$;*

(iv) *if $\hat{x} \in \{x^*, f(x^*), \ldots, f^{p-1}(x^*)\}$ is not a source of system (2) and $\{W_p^s(\hat{x}) - \{\hat{x}\}\} \cap \partial A_p(x^s) \neq \emptyset$, then $\gamma \subseteq \partial A(x^s)$;*

Theorem 2 was proved in [1] for $p = 1$. The proof for $p > 1$ is very similar to the ones of theorems 9-8 and 9-9 in [1] and will be omitted. Theorem 2 indicates that we can check, from a computational point of view, whether a periodic orbit γ lies on the stability boundary $\partial A(x^s)$ of the original system (1) by checking that only one of its periodic points

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lies on the stability boundary $\partial A_p(x^s)$ of the associated p-iterated system (2). This verification can be numerically made by checking the existence of a point on the unstable manifold $W_p^u(x^*)$ of the periodic point x^* with a forward trajectory of system (2) converging to the asymptotically stable fixed point x^s.

Theorem 2 can be sharpened by imposing the following conditions on the map of system (1):

- **(B1)** All the fixed points and periodic orbits on $\partial A(x^s)$ are hyperbolic;
- **(B2)** The stable and unstable manifolds of fixed points and periodic orbits on $\partial A(x^s)$ satisfy the transversality condition;
- **(B3)** Every trajectory x_k on $\partial A(x^s)$ approaches one of the fixed points or periodic orbits as $k \to \infty$.

The next lemma establishes a relation between the assumptions (B1), (B2) and, (B3) of the map of system (1) with the following assumptions (A1), (A2), and (A3) of the map of the p-iterated system (2):

- **(A1)** All the fixed points on $\partial A_p(x^s)$ are hyperbolic;
- **(A2)** The stable and unstable manifolds of fixed points on $\partial A_p(x^s)$ satisfy the transversality condition;
- **(A3)** Every orbit x_k on $\partial A_p(x^s)$ approaches one of the fixed points as $k \to \infty$.

Particularly, it shows that assumptions (B1), (B2), and (B3) on system (1) imply assumptions (A1), (A2), and (A3) on system (2).

Lemma 2. *If **(B1)**, **(B2)**, and **(B3)** are valid for the discrete dynamical system (1) and $p \in \mathbb{N}$ is a multiple of all periods of the periodic orbits on $\partial A(x^s)$, then assumptions **(A1)**, **(A2)**, and **(A3)** hold for the p-iterated system (2).*

Proof. **(B1)** $\Rightarrow$ **(A1)**

Let $x^* \in \partial A_p(x^s)$ be a fixed point of system (2). Then either x^* is a fixed point or a periodic point of system (1). If x^* is a fixed point of system (1), then it is hyperbolic due to assumption **(B1)**. Consequently, x^* is also a hyperbolic fixed point of system (2) [32]. If x^* is a periodic point of system (1), the periodic orbit containing x^* is hyperbolic due to **(B1)**. Then x^* is a hyperbolic fixed point of (2). Thus **(A1)** holds.

(B2) $\Rightarrow$ **(A2)**

Let x^* and y^* be fixed points on $\partial A_p(x^s)$ of system (2). Then, they are periodic points of system (1), observing that we are not excluding the case they are trivial periodic orbits, i.e. fixed points of system (1). Let $\gamma = \{x^*, f(x^*), \ldots, f^{p-1}(x^*)\}$ and $\beta = \{y^*, f(y^*), \ldots, f^{p-1}(y^*)\}$ be the periodic orbits with period p on $\partial A(x^s)$ of system (1) containing the points x^* and y^*. Since **(B1)** and **(B2)** are satisfied, then γ and β are hyperbolic and their manifolds satisfy the transversality condition.

From Lemma 1, one concludes that $W_p^s(f^i(x^*))$ and $W_p^u(f^j(y^*))$ satisfy the transversality condition for all $i, j \in \{0, 1, \ldots, p-1\}$. Therefore, the invariant manifolds of fixed points of system (2) satisfy the transversality condition (A2).

(B3) $\Rightarrow$ **(A3)**

For any $y \in \partial A_p(x^s)$, we want to show that $f^{np}(y)$ approaches a fixed point on $\partial A_p(x^s)$ as $n \to \infty$. Since $\partial A_p(x_s) = \partial A(x^s)$, $y \in \partial A(x^s)$ and, by assumption **(B3)**, trajectory $f^k(y)$ of system (1) approaches either a fixed point or a periodic orbit as $k \to \infty$. Suppose that $f^k(y)$ approaches a fixed point x^* of system (1) as $k \to \infty$, then, in particular, $f^{kp}(y) \to x^*$ as $k \to \infty$ and the proof is complete.

Suppose now that $f^k(y)$ approaches a periodic orbit β, i.e., $f^k(y) \to \beta$ as $k \to \infty$, then $f^{kp}(y) \to \beta$ as $k \to \infty$. Since the periodic orbit $\beta = \{y^*, f(y^*), \ldots, f^{p-1}(y^*)\}$ is a finite set of isolated points, then there exists a sub-sequence k_j such that $f^{k_j p}(y) \to f^i(y^*)$, for some $i \in \{0, 1, \ldots, p-1\}$, as $k_j \to \infty$.

Without loosing generality, assume $i = 0$ to simplify our notation. Then, $f^{k_j p}(y) \to y^*$ as $k_j \to \infty$. From the continuity of f and the p-periodicity of y^*, given $\delta > 0$, there exists $\varepsilon > 0$, with $\varepsilon < \delta$ such that $\|x - y^*\| < \varepsilon$ implies $\|f^p(x) - y^*\| < \delta$. Notice that each $f^i(y^*)$ is a hyperbolic isolated fixed point of the p-iterated system. Then, δ can be chosen sufficiently small such that $\min_{1 \le i \le p-1} dist(f^i(y^*), B_\delta(y^*)) > \delta$, where $B_\delta(y^*)$ is the ball with radius δ centered at y^*.

Now, for the chosen number ε, there exist natural numbers N and M, with $pN > M$, such that $dist(f^k(y), \beta) < \varepsilon$ for all $k > M$ and $dist(f^{k_j p}(y), y^*) < \varepsilon$ for all $k_j > N$. Consequently, for any $k_j > N$, one has that $\|f^{(k_j+1)p}(y) - y^*\| < \delta$. But $k_j + 1 > N$ implies $(k_j + 1)p > M$, which implies that $dist(f^{(k_j+1)p}(y), \beta) < \varepsilon < \delta$. Thus, necessarily $\|f^{(k_j+1)p}(y) - y^*\| < \varepsilon$ and then $\|f^{(k_j+r)p}(y) - y^*\| < \varepsilon$ for any $k_j > N$ and any $r \geq 0$. This implies that $f^{kp}(y) \to y^*$ as $k \to \infty$ and this concludes the proof. □

Under assumptions **(B1)**, **(B2)**, and **(B3)**, the next theorem offers a local characterization of a periodic orbit on the stability boundary of dynamical system (1), which is sharper than the one derived in Theorem 2.

Theorem 3 (**Periodic Orbits on the Stability boundary**). *Let $A(x^s)$ be the stability region of an asymptotically stable fixed point x^s and $\gamma = \{x^*, f(x^*), \ldots, f^{p-1}(x^*)\}$ be a hyperbolic periodic orbit of period p of system (1) where function f is a diffeomorphism. Suppose that assumptions **(B1)**, **(B2)**, and **(B3)** are satisfied for system (1) and $p \in \mathbb{N}$ is a multiple of all periods of the periodic orbits on $\partial A(x^s)$. Then, the following characterizations hold:*

- (i) $\gamma \subseteq \partial A(x^s)$ *if and only if* $W^u(\gamma) \cap A(x^s) \neq \emptyset$;
- (ii) $\gamma \subseteq \partial A(x^s)$ *if and only if* $W^s(\gamma) \subseteq \partial A(x^s)$.

The proof of this theorem is similar to the one of theorem 9-10 in [1] and will be omitted.

From Theorem 2, the existence of a trajectory of system (2) starting in $W_p^u(\hat{x})$, for some $\hat{x} \in \gamma$, converging to the stable fixed point x^s is a sufficient condition to ensure $\gamma \subset \partial A(x^s)$. Under assumptions **(B1)**, **(B2)** and **(B3)**, this condition is also necessary. Consequently, the characterization of periodic orbits on the stability boundary derived in Theorem 3 is more practical, from a computational viewpoint, than Theorem 2 to check if a periodic orbit lies on the stability boundary of system (1).

Letting $p = 1$ in Theorem 3, one obtains Corollary 1, which offers a local characterization of hyperbolic fixed points on the stability boundary. This Corollary is a generalization of the result proven in [1]. Corollary 1 is proven under the more general assumptions **(B1)**, **(B2)**, and **(B3)**, while the equivalent result in [1] is proven under assumptions **(A1)**, **(A2)**, and **(A3)**.

Corollary 1. *Let $A(x^s)$ be the stability region of an asymptotically stable fixed point x^s and let $\hat{x} \neq x^s$ be a hyperbolic fixed point of system (1), where function f is a diffeomorphism. Suppose that assumptions (**B1**), (**B2**) and (**B3**) are satisfied for system (1) and $p \in \mathbb{N}$ is a multiple of all periods of the periodic orbits on $\partial A(x^s)$. Then, the following characterization holds:*

1) *$\hat{x} \in \partial A(x^s)$ if and only if $W^u(\hat{x}) \cap A(x^s) \neq \emptyset$;*
2) *if $\hat{x}$ is not a source, then $\hat{x} \in \partial A(x^s)$ if and only if $W^s(\hat{x}) \subseteq \partial A(x^s)$.*

Now we are in a position to develop a characterization of the stability boundary of a class of nonlinear discrete systems that admit fixed points and periodic orbits on the stability boundary.

Theorem 4 (**Stability Boundary Characterization**). *Let $A(x^s)$ be the stability region of an asymptotically stable fixed point x^s of the discrete system (1). Suppose that function f is a diffeomorphism and satisfies assumptions **(B1)** and **(B3)**. Let x_i, $i = 1, 2, ...$, be the unstable fixed points and γ_j, $j = 1, 2, ...$, be the periodic orbits on the stability boundary $\partial A(x^s)$. Suppose that $p \in \mathbb{N}$ is a multiple of all periods of the periodic orbits. Then*

$$\partial A(x^s) \subseteq \bigcup_i W^s(x_i) \bigcup_j W^s(\gamma_j).$$

*Moreover, if assumption **(B2)** is also satisfied, then*

$$\partial A(x^s) = \bigcup_i W^s(x_i) \bigcup_j W^s(\gamma_j).$$

Proof. Suppose $y \in \partial A(x^s)$. From assumptions **(B1)** and **(B3)**, it follows that either $y \in W^s(x_i)$ for some $i = 1, 2, \ldots$ or $y \in W^s(\gamma_j)$ for some $j = 1, 2, \ldots$. Thus, $\partial A(x^s) \subseteq \bigcup_i W^s(x_i) \bigcup_j W^s(\gamma_j)$.

Corollary 1 implies that $\bigcup_i W^s(x_i) \subseteq \partial A(x^s)$ and Theorem 3 implies that $\bigcup_j W^s(\gamma_j) \subseteq \partial A(x^s)$. Thus, $\bigcup_i W^s(x_i) \bigcup_j W^s(\gamma_j) \subseteq \partial A(x^s)$. This completes the proof. □

As expected, the stability boundary characterization of Theorem 4 is similar to the characterization of the stability boundary for continuous dynamical systems with periodic orbits on the stability boundary [22]. In other words, stability boundary is composed of the union of the stable manifolds of critical elements on the stability boundary.

Assumptions **(B1)**, **(B2)** and **(B3)** are sufficient conditions for the characterization of stability boundary in Theorem 4. Examples showing that the characterization of Theorem 4 fails when **(B2)** or **(B3)** do not hold are given in [1]. Assumption **(B1)** can be relaxed to accommodate some types of nonhyperbolic fixed points on the stability boundary. This relaxation has been already studied for continuous system [24], [34]–[36].

The next example illustrates these analytical results.

Example 1. *Consider the following two-dimensional nonlinear discrete system:*

$$\begin{aligned} x_{k+1} &= -ax_k^3 + bx_k \\ y_{k+1} &= cy_k^3 - dy_k \end{aligned} \tag{3}$$

where $a = c = 1$, $b = \frac{1}{10}$, and $d = \frac{1}{8}$. The function $f(x_k, y_k)$ is a diffeomorphism and system (3) possesses 3 fixed points: (0,0), an asymptotically stable fixed point, and (0,1.0607), (0,-1.0607), which are unstable fixed points. The system also possesses 3 periodic orbits of period 2: {(-1.0488, 1.0607), (1.0488, 1.0607)}, {(-1.0488, -1.0607), (1.0488, -1.0607)} and {(1.0488,

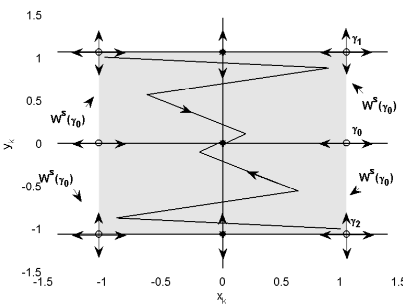

Fig. 1. The stability region of (0,0) and the invariant manifolds of the fixed points of system (3)

Consider the corresponding 2-iterated system:

$$\begin{aligned} x_{m+1} &= a^4x_m^9 - 3a^3bx_m^7 + 3a^2b^2x_m^5 \\ &\quad -ab^3x_m^3 - abx_m^3 + b^2x_m \\ y_{m+1} &= c^4y_m^9 + 3c^3dy_m^7 + 3c^2d^2y_m^5 \\ &\quad +cd^3y_m^3 - dcy_m^3 + d^2y_m \end{aligned} \tag{4}$$

Function $f^2(x_k, y_k)$ is a diffeomorphism and system (4) possesses 9 fixed points: (0,0), an asymptotically stable fixed point, (0,1.0607), (0,-1.0607),(1.0488, 0), (-1.0488, 0), which are unstable type-one fixed points; and (-1.0488, 1.0607), (1.0488, 1.0607), (-1.0488, -1.0607), (1.0488, -1.0607), which are type-2

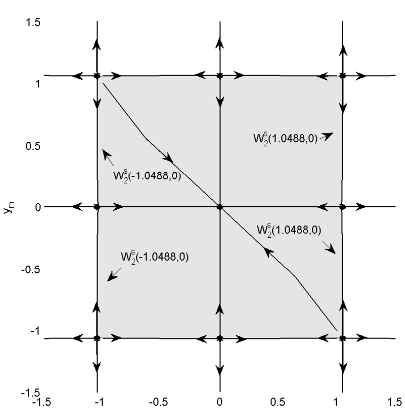

Fig. 2. Stability region of system (4). Its stability boundary is composed of the union of the stable manifolds of the 8 unstable fixed points lying on the stability boundary.

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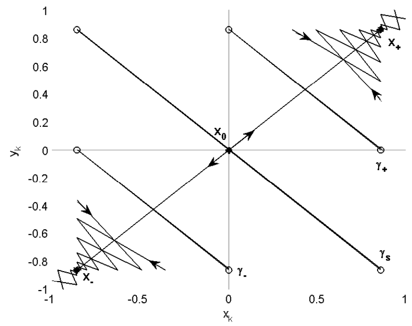

Fig. 3. Phase portrait of the 2-neuron network (5) with the activation function (6) for $a = 1.5$. The origin is a repellor. The fixed points x_+ and x_- are asymptotically stable. The stable periodic orbit γ_s is formed by the points in $\{(-0.8586, 0.8586), (0.8586, -0.8586)\}$ while the 2-periodic orbits γ_+ and γ_- are formed by the points $\{(0, 0.8586), (0.8586, 0)\}$ and $\{(-0.8586, 0), (0, -0.8586)\}$.

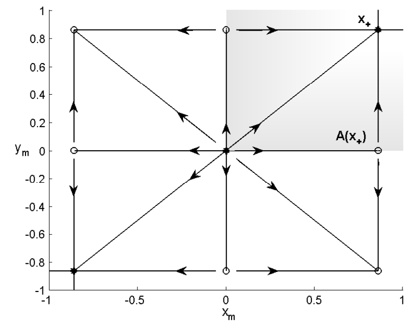

Fig. 4. Phase portrait of the 2-iterated system associated with the 2-neuron network (5) with the activation function (6) for $a = 1.5$. The origin is a repellor. The fixed points x_+ and x_- are asymptotically stable. The 2-periodic points of system (5) are now fixed points for the 2-iterated system.The gray area represents the stability region of the fixed point x_+.

The stability regions of the asymptotically fixed point $(0, 0)$ of system (3), depicted in Fig. 1, and of system (4), depicted in Fig. 2, are equal. The stability boundary $\partial A_p(0, 0)$ of the p-iterated system (4), depicted in Fig. 2 is composed of the stable manifolds of all unstable fixed points on the stability boundary of system (4). The stability boundary $\partial A(0, 0)$ of system (3), depicted in Fig. 1, is composed of the union of the stable manifolds of the unstable fixed points and periodic orbits on the stability boundary of system (3).

Example 2. *Consider the following set of difference equations:*

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = F\left(W \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right), \tag{5}$$

where $W = [\omega_{ij}]_{2\times 2}$ is a symmetric weight matrix, and which models a class of 2-neuron symmetric time discrete analog networks [8]. The nonlinear function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u) \\ f(v) \end{pmatrix}$, where $f : \mathbb{R} \to \mathbb{R}$ is the following activation function:

$$f(z) = \tanh(az), \tag{6}$$

It was demonstrated in [8], for $w_{11} = w_{22} = 0$ and $w_{12} = w_{21} = 1$, that the limit sets of the neural network (5) are only fixed points and 2-periodic orbits.

For $a > 1$, system (5) has three fixed points and six 2-periodic points. The origin is a repellor fixed point. For the particular value $a = 1.5$, we can see, from Fig. 3, the other two fixed points of system (5), namely, $x_+ = (0.8586, 0.8586)$ and $x_- = (-0.8586, -0.8586)$. Theses points are asymptotically stable fixed points of system (5). Furthermore, for $a > 1$, system (5) exhibits one asymptotically stable 2-periodic orbit and two unstable 2-periodic orbits. See Fig. 3 for $a = 1.5$.

Fig. 4 illustrates the dynamics of the 2-iterated system associated with the 2-neuron network model (5). All of the six 2-periodic points of system (5) are now fixed points of the 2-iterated system. The 2-iterated system has 4 asymptotically stable equilibrium points and 5 unstable fixed points.

The stability region of the fixed point x_+ of the 2-neuron network is depicted at Fig. 4. Observe that the origin and the periodic orbit γ_+ are on the stability boundary $A(x_+)$ of the fixed point x_+ of the 2-neuron network (5) and the stability boundary is composed of the union of their stable manifold as indicated in the figure.

VI. Sufficient Conditions for Assumption (B3)

Assumptions **(B1)** and **(B2)** are generic conditions of diffeomorphisms, which means that they are satisfied for almost all systems in the form of (1). On the contrary, assumption **(B3)** is not generic, thus it is crucial to verify this assumption in the application of the results of section V on the stability boundary characterization. In this section, sufficient conditions for assumption **(B3)** are derived.

Theorem 5. *Consider the nonlinear discrete system (1) and the set P of fixed points and periodic points of system (1). Suppose the existence of a continuous scalar function $V : \mathbb{R}^n \to \mathbb{R}$ with the following properties:*

- (i) $\Delta V(x) = V(f(x)) - V(x) < 0$ *for all $x \notin P$;*
- (ii) *for each arbitrarily small $\epsilon > 0$, there exists a number $\eta > 0$, depending on ϵ, such that $\Delta V(x) < -\eta$ for all $x \notin \bigcup_{p\in P} B_\epsilon(p)$.*

*Then assumption (**B3**) holds.*

Proof. First note that $V(x^s)$ is a bound from below for V on $\partial A(x^s)$. Suppose $x \in \partial A(x^s)$ and $f^k(x)$ does not approach set P, the fixed and periodic points of (1), as $k \to \infty$. Then, there exists $\epsilon > 0$ such that for every $N \in \mathbb{N}$, there exists $m > N$ such that dist$(f^m(x),P) > \epsilon$. Equivalently, there exists an increasing infinite sequence of natural numbers $\{m_i\}$, with $m_i \to \infty$, such that dist$(f^{m_i}(x), P) > \epsilon$. From assumptions (i) and (ii), we conclude that $V(f^k(x))$ becomes unbounded from below, contradicting the fact that V is bounded from below in $\partial A(x^s)$. Consequently, $f^k(x)$ approaches P for every $x \in \partial A(x^s)$. □

Corollary 2. *Consider the nonlinear discrete system (1) and suppose the existence of a continuous scalar function $V : \mathbb{R}^n \to \mathbb{R}$ with the following properties:*

- (i) $\Delta V(x) = V(f(x)) - V(x) < 0$ *for all $x \notin P$;*
- (ii) *set P has a finite number of isolated points.*

Then assumption (B3) holds.

Proof. From condition (ii) of the Corollary and continuity of V and f, we can easily prove that condition (ii) of Theorem 5 is satisfied. This completes the proof. □

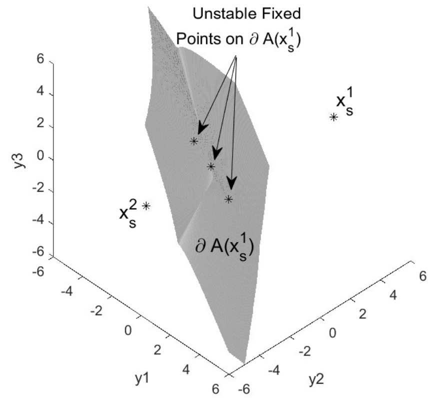

Fig. 5. The stability boundary of the recurrent neural network (7). There are 3 fixed points on the stability boundary. The stability boundary is a 2-dimensional surface composed of the unstable manifolds of these fixed points.

Example 3. *Consider the following model of a recurrent discrete neural network:*

$$y_i(k+1) = \sigma\left(\sum_{j=1}^{3} \mu_i \omega_{ij} y_j(k) + \mu_i s_i\right) \quad i = 1,2,3, \quad (7)$$

where σ is a diffeomorphism called activation function of the neural network. Matrix $W = [\omega_{ij}]_{3\times 3}$ is an invertible matrix called the synaptic weight matrix. The vectors s and μ are respectively the input and activation gain of the network.

If σ is a diffeomorphism and matrix W is invertible, then the map of the discrete model (7) is a diffeomorphism. Consider the scalar function proposed in [7]:

$$V(y) = -\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \omega_{ij} y_i y_j - \sum_{i=1}^{3} s_i y_i + \sum_{i=1}^{3} \frac{1}{\mu_i}\int_0^{y_i} \sigma^{-1}(\tau)d\tau. \quad (8)$$

It can be easily shown that $\Delta V(y) < 0$ for all $y \notin P$. Thus, function v satisfies condition (i) of Theorem 5. Since this system admits a finite number of fixed points on the stability boundary, then all assumptions of Corollary 2 are satisfied and assumption (B3) is true for system (7). Since assumptions (B1) and (B2) are generically satisfied, then the characterization of stability boundary proven in Theorem 4 applies to this system. To illustrate this characterization, consider the activation function $\sigma(z) = z^{\frac{1}{3}} + 0.01z$ and the following set of parameters, $\mu_1 = \mu_2 = \mu_3 = 10$, $s_1 = 0.5$, $s_2 = 0.2$, $s_3 = 0.1$. Consider also the following weight matrix:

$$W = \begin{bmatrix} 0.3 & 0.4 & 0.2 \\ 0.4 & 0.2 & 0.1 \\ 0.2 & 0.1 & 0.4 \end{bmatrix}.$$

For these parameters, the network has 2 stable fixed points. They are $x_1^s = [3.62, 3.22, 3.12]$ and $x_2^s = [-2.96, -2.82, -2.87]$. These two stable fixed points share the same stability boundary, which splits the space phase in 2 parts. The stability boundary of these fixed points is depicted in Fig. 5. Three unstable fixed points lie on this boundary and the boundary, according to Theorem 4, is composed of the stable manifolds of these fixed points.

VII. Conclusion

The theory of stability regions of nonlinear autonomous discrete dynamical systems has been extended in this paper to consider the class of systems modeled by diffeomorphisms admiting fixed and periodic points on the stability boundary. A characterization of stability regions and stability boundaries of this large class of nonlinear discrete dynamical systems was derived. It was shown, for this class, that the stability boundary is composed of the union of the stable manifolds of all fixed points and periodic orbits that lie on the stability boundary. These characterizations were developed exploring the close relationship between the stability regions of the original dynamical systems and the stability regions of its associated p-iterated system. Our further work includes the development of effective schemes to numerically obtain estimates of the stability regions by exploring the characterizations proven in this paper.

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