ON THE MODULE CATEGORIES WITH INFINITE RADICAL CUBE ZERO

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1. Introduction

This is a report of a joint work with E. Marcos, H. Merklen and A. Skowroński.

Let A be an artin algebra over a commutative ring R, that is, A is an R-algebra which is finitely generated as an R-module. All algebras in this note are basic, connected and indecomposable. By an A-module it is meant a finitely generated left A-module. Let rad(mod$A$) denote the Jacobson radical of mod$A$, that is, the ideal of mod$A$ generated by all non-invertible morphisms and by $\text{rad}^\infty(\text{mod}A)$ the intersection of all powers $\text{rad}^i(\text{mod}A)$ of rad(mod$A$). The study of $\text{rad}^\infty(\text{mod}A)$ gives important informations on the category mod$A$, in particular, in the components of the Auslander-Reiten quiver Γ_A of A (see definition below). We are particularly interested in the case when $\text{rad}^\infty(\text{mod}A)$ is nilpotent. We say that an algebra A is *representation-finite* if mod$A$ has only finitely many non-isomorphic indecomposable modules. Otherwise, A is called *representation-infinite*. The following result has been proven in [4].

Theorem 1.1. *If $(\text{rad}^\infty(\text{mod}A))^2 = 0$, then A is representation-finite.*

We now consider algebras A such that $(\text{rad}^\infty(\text{mod}A))^3 = 0$. We first observe that there are representation-infinite algebras with this property. In order to introduce such examples, we will first recall some notions.

2. Auslander-Reiten quivers

For a given artin algebra A, its Auslander-Reiten quiver Γ_A is defined as follows. The vertices of Γ_A is in a one-to-one correspondence with the isomorphism class of the indecomposable modules in mod$A$. For the definition of the arrows in Γ_A we recall the notion of irreducible morphisms: if X and Y are modules in mod$A$ then a morphism $f: X \longrightarrow Y$ is *irreducible* if (i) f is not a split morphism; and (ii) whenever $f = gh$, then either g is a split epimorphism or h is a split monomorphism. Suppose $[X]$ and $[Y]$ are two vertices in Γ_A corresponding, respectively, to indecomposable

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modules X and Y. Now, by definition, there is an arrow from $[X]$ to $[Y]$ if and only if there is an irreducible morphism between X and Y.

We would like to stress some facts on the quiver defined above. First, Γ_A is locally finite, that is, for each vertex $[X]$, there are at most finitely many arrows with $[X]$ as a start or an end point. Moreover, there are no arrows from a vertex to itself. Finally, Γ_A is a so-called translation quiver, that is, there exists a bijection $\tau\colon \Gamma' \longrightarrow \Gamma''$, where Γ' (respectively, Γ'') is the set of vertices not corresponding to projective (respectively, injective) modules, such that for each $x \in \Gamma'$, there exists an arrow $y \longrightarrow x$ if and only if there exists an arrow $\tau x \longrightarrow y$. The quiver Γ_A is, in general, not connected. For details on the above construction we refer the reader to, for instance, [1, 6]. We also recall the following result due to Auslander-Reiten [2](1.7).

Proposition 2.1. *If $f \in rad(X,Y)$, then $f = \Sigma g_i + h$, where $h \in rad^\infty(modA)$ and, for each i, g_i is a composite of irreducible morphisms.*

Corollary 2.2. *Any morphism between modules belonging to distinct components of Γ_A belongs to $rad^\infty(modA)$.*

3. Tame concealed algebras

Let H be a hereditary algebra. It is known that in this case R is a field and H is in fact a finite dimensional algebra over R. Moreover, there exists a bilinear form on the Grothendieck group $K_0(H)$ of H given by

$$< M, N > = \dim_R \mathrm{Hom}_H(M,N) - \dim_R \mathrm{Ext}^1_H(M,N)$$

which induces a quadratic form q_H on $K_0(H) \otimes_{\mathbb{Z}} \mathbb{Q}$. It is well-known that H is representation-finite if and only if q_H is positive definite. The algebra H is said to be of *tame* type if it is not representation-finite and q_H is positive semidefinite.

Let now H be a representation-infinite hereditary algebra and let n denote the rank of $K_0(H)$. Let T be a multiplicity-free preprojective tilting H-module, that is, $\mathrm{Ext}^1_H(T,T) = 0$, $\mathrm{rad}^\infty(-,T) = 0$ and T is a direct sum of n pairwise non-isomorphic indecomposable H-modules. The algebra $B = \mathrm{End}_H(T)$ is called a *concealed algebra* and if H is tame hereditary then B is called *tame concealed.*

Let now A denote a tame concealed algebra (which can be, in particular a hereditary algebra). The Auslander-Reiten quiver of A defined as above has

-a component consisting of modules X such that $\mathrm{rad}^\infty(-,X) = 0$ and containing all indecomposable projectives, called *preprojective component*;

-a component consisting of modules X such that $\mathrm{rad}^\infty(X,-) = 0$ and containing all indecomposable injectives, called *preinjective component*; and

-an infinite family of generalized standard pairwise orthogonal stable tubes $(\mathcal{T}_\rho)_{\rho\in\Omega}$, that is, for each $\lambda \in \Omega$, $\mathcal{T}_\rho$ is a quiver of the form $\mathbb{Z}A_\infty/(\tau^m)$, for some m, and $\mathrm{rad}^\infty(X,Y) = 0$ for all X and Y belonging to components in this family.

With this description of the components of the Auslander-Reiten quiver of A, it is not difficult to see that $(\text{rad}^\infty(\text{mod}A))^3 = 0$ and since A is not representation-finite, we infer from (1.1) that $(\text{rad}^\infty(\text{mod}A))^2 \neq 0$. For details on the results discussed above we refer to [6].

4. Main result

Let A be an artin algebra and C be a component of Γ_A. The component C is called *regular* if it does not contain neither projective nor injective modules and it is called *faithful* if it contains all indecomposable summands of a faithful module. Recall that a module Z is *faithful* if $\text{ann}Z = 0$.

¿From now on, we assume that $(\text{rad}^\infty(\text{mod}A))^3 = 0$. The main result in this note is the following.

Theorem 4.1. *Let A be an artin algebra such that $(\text{rad}^\infty(\text{mod}A))^3 = 0$. If Γ_A contains a faithful regular component, then A is tame concealed.*

For a proof of this result, we refer the reader to [5]. However, we shall discuss quickly some intermediate steps in order to show the techniques used. By hypothesis, Γ_A contains a regular component. Let then $(\mathcal{T}_\rho)_{\rho\in\Omega}$ be the family of all regular components. We shall first show that $(\mathcal{T}_\rho)_{\rho\in\Omega}$ is a family of generalized standard pairwise orthogonal components. Indeed, suppose there exists a non-zero morphism $f \in \text{rad}^\infty(X,Y)$ with X and Y belonging to the family $(\mathcal{T}_\rho)_{\rho\in\Omega}$ and consider the projective cover $\pi\colon P_A(X) \longrightarrow X$ of X and the injective envelope $\iota\colon Y \longrightarrow I_A(Y)$ of Y. By Corollary 2.2 both π and ι belong to $\text{rad}^\infty(\text{mod}A)$. Therefore, $\iota f \pi$ is a non-zero morphim in $(\text{rad}^\infty(\text{mod}A))^3$, a contradiction. Using results from *tilting theory* we can also conclude that $\mathcal{T}_\lambda$ is a stable tube

Let now Γ be a faithful regular component of Γ_A. Then there exists a module $Z \in \text{add}\Gamma$ and a monomorphism $q\colon A \longrightarrow Z$. Consider now the injective envelope $\iota\colon Z \longrightarrow I_A(Z)$ of Z. Again by Corollary 2.2 both q and ι belong to $\text{rad}^\infty(\text{mod}A)$. Since ιq is a monomorphism, we conclude that $\text{rad}^\infty(-,A) = 0$. Similarly, we have also that $\text{rad}^\infty(DA,-) = 0$. By [3](3.4) or [7](3.3) we infer that A is concealed. Moreover, A is in fact tame concealed because Γ_A contains stable tubes (see [6]).

For finite dimensional algebras over algebraically closed fields we have the following consequence. Recall that an algebra A is said to be *minimal representation-infinite* if it is representation-infinite but for each ideal I, the algebra A/I is representation-finite.

Corollary 4.2. *Let A be a connected finite dimensional algebra over an algebraically closed field. Then A is tame concealed if and only if A is minimal representation-infinite and $(\text{rad}^\infty(\text{mod}A))^3 = 0$.*

Proof. The necessity is clear. To prove the sufficiency, it is enough to observe that if A is a representation-infinite algebra as in the statement, then there exists a regular component Γ. Since A is minimal representation-infinite, it follows that Γ is faithful and then, by Theorem 4.1. A is tame concealed. □

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