Units in group rings of free products of prime cyclic groups

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Abstract

Let G be a free product of cyclic groups of prime order. The structure of the unit group $\mathcal{U}(\mathbf{Q}G)$ of the rational group ring $\mathbf{Q}G$ is given in terms of free products and amalgamated free products of groups. As an application, all finite subgroups of $\mathcal{U}(\mathbf{Q}G)$, up to conjugacy, are described and the Zassenhaus Conjecture for finite subgroups in $\mathbf{Z}G$ is proved. A strong version of the Tits Alternative for $\mathcal{U}(\mathbf{Q}G)$ is obtained as a corollary of the structural result.

1   Introduction

Let $\mathcal{U}(\mathbf{Z}G)$ denote the unit group of the integral group ring $\mathbf{Z}G$ of a group G and let $\mathcal{U}_1(\mathbf{Z}G)$ be the group of units of augmentation 1 in $\mathbf{Z}G$. Similar notation shall be used for the rational group algebra $\mathbf{Q}G$. The Conjecture of Zassenhaus, denoted (ZC3) [14], states that if G is finite and H is a finite subgroup of $\mathcal{U}_1(\mathbf{Z}G)$ then H is conjugate in $\mathcal{U}(\mathbf{Q}G)$ to a subgroup of G. A restricted version of this conjecture, denoted (ZC1) [14], says that every torsion unit of $\mathcal{U}_1(\mathbf{Z}G)$ is conjugate in $\mathcal{U}(\mathbf{Q}G)$ to an element of G. It

is known that **(ZC3)** holds for finite nilpotent groups [16], [17], finite split metacyclic groups [12], [15] and some particular groups . However, **(ZC3)** is false in general and the counterexamples show that it does not hold for finite metabelian groups [7] and [13]. The Zassenhaus Conjecture restricted to finite p-subgroups of $\mathcal{U}_1(\mathbb{Z}G)$ has been established for finite nilpotent-by-nilpotent groups G [4], for finite solvable groups G whose Sylow p-subgoups are either abelian or generalized quaternion [4] and for Frobenius groups G which cannot be mapped homomorphically onto S_5 [5]. More information on the Zassenhaus Conjecture and its various versions can be found in [3],[13], [14]. It is interesting to know which infinite groups satisfy **(ZC3)**.In [11] an infinite nilpotent group is constructed which does not satisfy **(ZC1)** (compare with [2]). Problem 39 of [14] asks whether **(ZC1)** holds for a free product of finite cyclic groups.

Torsion units in integral group rings $\mathbb{Z}G$ where G is a free product of abelian groups were studied by A.I. Lichtman and S.K. Sehgal [10]. They proved that if $u \in \mathcal{U}_1(\mathbb{Z}G)$ has order $m < \infty$ then one of the free factors of G contains an element h of order m. Moreover, if G is a free product of a finite number of finite abelian groups then u is conjugate to h in a large overring of $\mathbb{Q}G$ (Theorem 1 of [10]). In a particular case when G is the infinite dihedral group the conjugating element can be taken even in $\mathbb{Z}[\frac{1}{2}]G$ [9].

In this paper we study the free product $G = *G_\alpha$ $(\alpha \in I)$ of cyclic groups of prime order $|G_\alpha| = p_\alpha$ (the $p_\alpha's$ are not necessarily distinct and I may be infinite). In Section 2 by applying Gerasimov's Theorem [6] we prove that

$$\mathcal{U}(\mathbb{Q}G) = *_{\mathcal{U}(\mathbf{Q})}((A_\alpha * B_\alpha) \rtimes \mathcal{U}(\mathbb{Q}G_\alpha)),$$

where $*_{\mathcal{U}(\mathbf{Q})}$ denotes the amalgamated free product over the multiplicative group $\mathcal{U}(\mathbb{Q})$ of $\mathbb{Q}$ and A_α, B_α are abelian groups isomorphic to the additive groups of some infinite dimensional vector spaces over $\mathbb{Q}$ (Theorem 2.3). As a consequence we prove that every nonabelian subgroup of $\mathcal{U}(\mathbb{Q}G)$ either contains a free noncyclic subgroup or is metabelian (Corollary 2.4). In Section 3 we use Theorem 2.3 to prove that every finite subgroup of $\mathcal{U}_1(\mathbb{Q}G)$ is conjugate in $\mathcal{U}(\mathbb{Q}G)$ to a subgroup of $\mathcal{U}_1(\mathbb{Q}G_\alpha)$ for some $\alpha \in I$ (Theorem 3.4). As a corollary the Zassenhaus Conjecture **(ZC3)** is proved for G (Corollary 3.5).

2   The structure of the rational unit group

Let K be an associative ring with identity and $G = *_H G_\alpha (\alpha \in I)$ be the free product of groups G_α with amalgamated subgroup H. It is easy to verify that KG is isomorphic to the coproduct $\coprod_{KH} KG_\alpha, (\alpha \in I)$ of rings KG_α over KH. In particular, if $G = *G_\alpha (\alpha \in I)$ is the free product of groups G_α, then $KG \cong \coprod_K KG_\alpha (\alpha \in I)$. Thus, Gerasimov's Theorem on units in coproducts of rings [6] can be used in the study of $\mathcal{U}(KG)$.

An element of KG of the form $1 + x\nu y$ where $x, y \in KG_\alpha, yx = 0, \nu \in KG$ is called a KG_α-*transvection*. Let $\Gamma(KG_\alpha)$ be the subgroup of $\mathcal{U}(KG)$ generated by $\mathcal{U}(KG_\alpha)$ and all the KG_α-transvections of KG. A ring R with the identity element 1 is called 1-*commutative* if $xy = 1$ implies $yx = 1$ ($x, y \in R$). The following statement is an immediate consequence of Gerasimov's Theorem.

Statment 2.1. Let $G = *G_\alpha (\alpha \in I)$ and K be a division ring. If each KG_α is 1-commutative then

$$\mathcal{U}(KG) \cong *_{\mathcal{U}(K)} \Gamma(KG_\alpha), \quad (\alpha \in I),$$

where $\mathcal{U}(K)$ denotes the multiplicative group of K.

It is easy to see that the subgroup $T(KG_\alpha)$ generated by all the KG_α-transvections of KG is normal in $\Gamma(KG_\alpha)$.

Suppose now that $K = \mathbf{Q}$ and that each $|G_\alpha| = p_\alpha$ is a prime $(\alpha \in I)$. The $p_\alpha's$ are not necessarily distinct and I may be infinite. Let S be the disjoint union of the $G_\alpha \setminus \{1\}, (\alpha \in I)$. We say that the product $g = g_1 \cdots g_n$, $(g_i \in S)$ is *reduced* if either $n = 1$ or $n \geq 2$ and no adjacent factors belong to the same G_α. In this case n is called the *length* of g and shall be denoted by $\ell(g)$.

Let β be a fixed index and $G_\beta = <c>$. Take any ordering on each $G_\alpha \setminus \{1\}, (\alpha \neq \beta)$. Set $c^i < c^j$ if and only if $i < j, (0 < i, j \leq p - 1, p = p_\beta)$.

Now take an ordering on I such that $\beta < \alpha$ for every $\alpha \in I, (\alpha \neq \beta)$ and assume that the identity element $1 \in G$ has length 0. This determines an ordering in S.

Suppose now that every element of G is given as a reduced product and order them first by their length and then lexicografically from left to right.

For a $\nu \in KG$ the leading term, *lead*(ν), of ν is the maximum of

$\{g : g \in supp(\nu)\}$, that is $lead(\nu) \geq g$ for every g of the support of ν.

Let C_β be the **Q**-subspace of **Q**G generated by all reduced products $c^\imath g_1 \cdots g_n$, $(g_\jmath \in S, n \geq 1, g_n \notin G_\beta, 0 \leq \imath \leq p-2)$ and let D_β be the **Q**-subspace of **Q**G generated by all reduced products $g_1 \cdots g_n c^\imath$, $(g_\jmath \in S, n \geq 1, g_1 \notin G_\beta, 0 \leq \imath \leq p-2)$.

Set $\hat{c} = 1 + c + \cdots + c^{p-1}$ and consider the following maps:
$\varphi : C_\beta \longrightarrow \mathcal{U}(\mathbf{Q}G)$ and $\psi : D_\beta \longrightarrow \mathcal{U}(\mathbf{Q}G)$ defined by $\varphi(\nu) = 1 + (1-c)\nu\hat{c}$, and $\psi(\nu) = 1 + \hat{c}\nu(1-c)$. It is easily seen that φ and ψ are homomorphisms from the additive groups C_β and D_β respectively into $T(\mathbf{Q}G_\beta)$.

Lemma 2.2. Set $A_\beta = Im\varphi$ and $B_\beta = Im\psi$. Then $T(\mathbf{Q}G_\beta) = <A_\beta, B_\beta>$ and $\varphi : C_\beta \longrightarrow A_\beta$, $\psi : D_\beta \longrightarrow B_\beta$ are isomorphisms.

Proof. It is easily seen that if $xy = 0$ for some $x, y \in \mathbf{Q}G_\beta$ then one of these elements belongs to $(1-c)\mathbf{Q}G_\beta$ and the other to $\mathbf{Q}\hat{c}$. Hence $T(\mathbf{Q}G_\beta)$ is generated by all elements of the form $1 + (1-c)\nu\hat{c}$, $1 + \hat{c}\nu(1-c)$, $\nu \in \mathbf{Q}G$. Then it follows from the equality

$$(1-c)c^{p-1} = -(1-c)(1 + c + \cdots + c^{p-2})$$

that $T(\mathbf{Q}G_\beta)$ is generated by $Im\varphi$ and $Im\psi$. This proves the first statement. It remains to be shown that $Ker\varphi = Ker\psi = \{0\}$.

Let $0 \neq \nu \in C_\beta$ and $lead(\nu) = c^\imath g_1 \cdots g_n$, $(n \geq 1, 0 \leq \imath \leq p-2)$ be written as a reduced product.

Let $c^\jmath h_1 \cdots h_k \neq lead(\nu)$ be a reduced product from the support of ν. Observe that since $\beta < \alpha$ for every $\alpha \in I$, $(\alpha \neq \beta)$, we have that $k \leq n$. (Note that this observation will be used in (8)). Then either $k < n$ or $k = n$ and $\jmath < \imath$ or $k = n, \jmath = \imath$ and $h_1 \cdots h_n < g_1 \cdots g_n$.

It is easy to see that in all cases $c^{\imath+1} g_1 \cdots g_n > c^{\jmath+1} h_1 \cdots h_k$ and, consequently,

$$lead((1-c)\nu) = c^{\imath+1} g_1 \cdots g_n,$$
$$lead(\varphi(\nu)) = c^{\imath+1} g_1 \cdots g_n c^{p-1} = c(lead(\nu))c^{p-1}. \tag{1}$$

Thus, $\varphi(\nu) \neq 1$ and $Ker\varphi = \{0\}$.

Let $0 \neq \nu \in D_\beta$ and for a reduced product $g = h_1 \cdots h_k c^\jmath$ from the support of ν, set $\omega(g) = h_1 \cdots h_k$. Let

$$g_1 \cdots g_n = max\{\omega(g) : g \in supp(\nu)\}$$

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and

$$g_1 \cdots g_n c^i = max\{g \in supp(\nu) : \omega(g) = g_1 \cdots g_n\}.$$

If $h_1 \cdots h_k c^j$ is any other reduced product from $supp(\nu)$, then either $h_1 \cdots h_k < g_1 \cdots g_n$ or $k = n$, $h_1 \cdots h_k = g_1 \cdots g_n$ and $j < \imath$. In both cases we have that $h_1 \cdots h_k c^{j+1} < g_1 \cdots g_n c^{\imath+1}$, therefore,

$$lead(\nu(1-c)) = g_1 \cdots g_n c^{\imath+1},$$
$$lead\psi(\nu) = lead(\hat{c}\nu(1-c)) \quad = c^{p-1} g_1 \cdots g_n c^{\imath+1}. \qquad (2)$$

Thus $Ker\psi = \{0\}$. $\square$

Now we shall prove the main result of this Section.

Theorem 2.3. Let $G = *G_\alpha$, $(\alpha \in I)$ where $\mid G_\alpha \mid = p_\alpha$ is a prime. Then

$$\mathcal{U}(\mathbf{Q}G) = *_{\mathcal{U}(\bullet)}((A_\alpha * B_\alpha) \rtimes \mathcal{U}(\mathbf{Q}G_\alpha)), \quad (\alpha \in I)$$

where $A_\alpha * B_\alpha = T(\mathbf{Q}G_\alpha)$ is the group generated by all $\mathbf{Q}G_\alpha$−transvections of $\mathbf{Q}G$, A_α and B_α are abelian groups isomorphic to the additive groups of some infinite dimensional vector spaces over $\mathbf{Q}$ (see Lemma 2.2).

Proof. Fix $\beta \in I$. We shall use the notation and the ordering introduced above. By Statment 2.1 and Lemma 2.2 it suffices to prove that $T(\mathbf{Q}G_\beta) \cap \mathcal{U}(\mathbf{Q}G_\beta) = \{1\}$ and $T(\mathbf{Q}G_\beta) = A_\beta * B_\beta$. We shall do this by calculating the leading term of an arbitrary element of $T(\mathbf{Q}G_\beta)$.

We shall say that two $\mathbf{Q}G_\beta$−transvections t_1 and t_2 have *the same type* if $t_1, t_2 \in A_\beta$ or $t_1, t_2 \in B_\beta$. A product of $\mathbf{Q}G_\beta$−transvections $u = t_1 \cdots t_n$ shall be called *reduced* if no adjacent factors have the same type. It is easy to see that an arbitrary reduced product u of transvectios is a sum of the identity and elements of the form

$$0 \neq w = [(1-c)\nu_0 \hat{c}]^{\varepsilon_1} \hat{c}\nu_1 (1-c)^2 \nu_2 \hat{c} \cdots$$
$$\hat{c}\nu_{2n-1}(1-c)^2 \nu_{2n} \hat{c}[\hat{c}\nu_{2n+1}(1-c)]^{\varepsilon_2}, \qquad (3)$$

where $\nu_0, \nu_2, \cdots, \nu_{2n} \in C_\beta$, $\nu_1, \nu_3, \cdots, \nu_{2n+1} \in D_\beta$, and $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$.

We shall proceed by finding the leading term of $\nu_i(1-c)^2 \nu_j$ where $i < j$, $i < 2n+1$, i is odd and j is even. Write an arbitrary element $g \in G$

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as $g = g_1 \omega(g) g_2$ where $g_1, g_2 \in\, <c>$ and $\omega(g)$ does not begin or end in a nonidentity element of $<c>$. Set $t_k = max\{\omega(g) : g \in supp(\nu_k)\}$, $0 \leq k \leq 2n+1$. If k is odd write $\nu_k = t_k x_k + r_k$, where $x_k \in \mathbf{Q} <c>$ and for every $g \in supp(r_k)$, $\omega(g) < t_k$. For an even k, $0 \leq k \leq 2n$ write

$$\nu_k = x_k^{(1)} t_k^{(1)} + \sum_{s=2}^{m} x_k^{(s)} t_k^{(s)} + r_k', \tag{4}$$

where $x_k^{(1)}, \cdots, x_k^{(m)} \in \mathbf{Q} <c>$, $t_k^{(1)} = t_k$, $\ell(t_k^{(s)}) = \ell(t_k)$, $(2 \leq s \leq m)$ and $\ell(\omega(g)) < \ell(t_k)$ for every $g \in supp(r_k')$.

Fix an odd i, $1 \leq i < 2n+1$, and an even j, $j \leq 2n$ such that $i < j$. Let

$$c^l = max\{g : g \in \cup_{s=1}^{m} supp(x_i(1-c)^2 x_j^{(s)})\}$$

and

$$f_j = max\{t_j^{(s)} : c^l \in supp(x_i(1-c)^2 x_j^{(s)})\}$$

where $t_j^{(s)}$ is defined in (4). We claim that $c^l \neq 1$ and that

$$lead[\nu_i(1-c)^2 \nu_j] = t_i c^l f_j. \tag{5}$$

In particular,

$$\ell(lead[\nu_i(1-c)^2 \nu_j]) = \ell(t_i) + 1 + \ell(t_j) \geq 3. \tag{6}$$

Let ζ be a primitive $p = p_\beta - th$ root of unity and $\pi : \mathbf{Q} <c> \to \mathbf{Q}(\zeta)$ be the map determined by $\pi(c) = \zeta$. It follows from the definitions of C_β and D_β that c^{p-1} does not belong to the supports of x_i and $x_j^{(s)}$, $(1 \leq s \leq m)$, hence $\pi(x_i)\pi(x_j^{(s)}) \neq 0$, $(1 \leq s \leq m)$ and therefore $\pi(x_i(1-c)^2 x_j^{(s)}) \neq 0$ $(1 \leq s \leq m)$. Consequently $x_i(1-c)^2 x_j^{(s)} \neq 0$, $(1 \leq s \leq m)$ and as $(1-c)^2$ is not a unit in $\mathbf{Q} <c>$ we see that $x_i(1-c)^2 x_j^{(s)} \notin \mathbf{Q}$. Thus $supp(x_i(1-c)^2 x_j^{(s)})$ contains a nonidentity element of $<c>$ for every s, $(1 \leq s \leq m)$. In particular, $c^l \neq 1$.

Let $g_1 c^a g_2$ be an arbitrary element from $supp(\nu_i(1-c)^2 \nu_j)$, where $g_1 = \omega(h_1)$, $g_2 = \omega(h_2)$ for some $h_1 \in supp(\nu_i)$ and $h_2 \in supp(\nu_j)$. It follows from the definitions of t_i and f_j that $g_1 \leq t_i$ and $\ell(g_2) \leq \ell(f_j)$.

If $g_1 < t_i$ then clearly $g_1 c^a g_2 < t_i c^l f_j$. So let $g_1 = t_i$. If $\ell(g_2) < \ell(f_j)$ then

$$\ell(g_1 c^a g_2) \leq \ell(g_1) + 1 + \ell(g_2) < \ell(t_i) + 1 + \ell(f_j) = \ell(t_i c^l f_j)$$

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and therefore again

$$g_1 c^a g_2 = t_i c^a g_2 < t_i c^l f_j.$$

Thus we may suppose that $\ell(g_2) = \ell(f_j)$. Then $g_2 = t_j^{(s)}$ for some s, $(1 \leq s \leq m)$ and consequently $c^a \in supp(x_i(1-c)^2 x_j^{(s)})$. Thus $c^l \geq c^a$ and since $c^l > c^a$ implies

$$g_1 c^a g_2 = t_i c^a t_j^{(s)} < t_i c^l f_j$$

we may suppose that $a = l$. But then $c^l \in supp(x_i(1-c)^2 x_j^{(s)})$ and by the definition of f_j we get that $f_j \geq g_2$. Finally, as $g_2 < f_j$ implies

$$g_1 c^a g_2 = t_i c^l g_2 < t_i c^l f_j,$$

we conclude that $t_i c^l f_j$ is indeed the leading term of $\nu_i (1-c)^2 \nu_j$, proving our claim.

Now we obtain from (3) that

$$lead(w) = (lead[(1-c)\nu_0])^{\epsilon_1}[c^{p-1}lead[\nu_1(1-c)^2\nu_2]c^{p-1}]\cdots$$
$$[c^{p-1}lead[\nu_{2n-1}(1-c)^2\nu_{2n}]c^{p-1}](lead[\nu_{2n+1}(1-c)])^{\epsilon_2}. \qquad (7)$$

Clearly this product is reduced if all the leading terms are given as reduced products. In particular, $lead(w) \notin G_\beta$ and consequently, $T(\mathbb{Q}G_\beta) \cap \mathcal{U}(\mathbb{Q}G_\beta) = \{1\}$.

Applying (1) and (2) to ν_i and ν_j respectively, and keeping in mind the observation made in the proof of Lemma 2.2, we obtain

$$\begin{aligned} \ell(lead[(1-c)\nu_j]) &= 1 + \ell(\omega(lead\nu_j)) = 1 + \ell(t_j) \geq 2, \\ \ell(lead[\nu_i(1-c)]) &= 1 + \ell(t_i) \geq 2. \end{aligned} \qquad (8)$$

Comparing (5) and (6) we see that

$$\ell(lead[\nu_i(1-c)^2\nu_j]) \geq max\{\ell(lead[(1-c)\nu_j]), \ell(lead[\nu_i(1-c)])\}. \qquad (9)$$

Note that (8) holds for arbitrary even j, $(0 \leq j \leq 2n)$ and for arbitrary odd i, $(1 \leq i \leq 2n+1)$. Observe that (7),(8) and (5) imply that

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$$\ell(lead(w)) \geq 3 \qquad (10)$$

for all w as in (3).

Now suppose that $1, 0 \neq w'$ is obtained from w by dropping some consecutive factors $\hat{c}\nu_i(1-c), (1-c)\nu_j\hat{c}$. Then we can write $w = w_1 w_2 w_3$, $w' = w_1 w_3$ where w_2 has the form (3) with less $\nu_k's$ involved. We shall prove that

$$\ell(lead(w)) > \ell(lead(w')). \qquad (11)$$

Suppose first that one of w_1 or w_3 is 1. It is enough to treat the case $w_1 = 1$, since the other one is similar. So let $w_1 = 1$; then $w = w_2 w_3$, $w' = w_3$. If w_2 ends in $\hat{c}$ then w_3 begins with $\hat{c}$ and by (7)

$$\ell(lead(w)) = \ell(lead(w_2)) + \ell(lead(w_3)) - 1.$$

It follows from (10) that $\ell(lead(w_2)) \geq 3$ and therefore $\ell(lead(w)) > \ell(lead(w_3))$. Let w_2 be ending in $1 - c$. Then w_3 begins with $1 - c$ and we can write $w_2 = w_2'\hat{c}\nu_k(1 - c)$ and $w_3 = (1 - c)\nu_{k+1}\hat{c}w_3'$. Call

$$\lambda_j = \begin{cases} \ell(lead(w_j')) & \text{if } w_j' \neq 1, \\ 1 & \text{otherwise.} \end{cases}$$

It follows from (7) and (6) that

$$\ell(lead\, w) = \lambda_2 + \ell(t_k) + \ell(t_{k+1}) + 1 + \lambda_3.$$

By (7) and (8) we have

$$\ell(lead\, w') = \ell(t_{k+1}) + 1 + \lambda_3$$

Consequently, $\ell(lead\, w) > \ell(lead\, w')$.

Now suppose that $w_1 \neq 1$ and $w_3 \neq 1$. If w_2 begins with $\hat{c}$ then w_1 ends in $\hat{c}$, and therefore w_3 begins with $\hat{c}$ and w_2 ends in $\hat{c}$. By (7) we get

$$\ell(lead(w)) = \ell(lead(w_1)) - 1 + \ell(lead(w_2)) - 1 + \ell(lead(w_3))$$

$$\ell(lead(w')) = \ell(lead(w_1)) + \ell(lead(w_3)) - 1,$$

hence

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$$\ell(lead(w)) > \ell(lead(w')).$$

If w_2 begins with $1 - c$ then w_1 ends in $1 - c$, w_3 begins with $1 - c$ and w_2 ends in $1 - c$. Write $w_1 = w_1' \hat{c} \nu_k (1 - c)$, $w_2 = (1 - c) \nu_{k+1} \hat{c} w_2' \hat{c} \nu_s (1 - c)$, $w_3 = (1 - c) \nu_{s+1} \hat{c} w_3'$. Applying (7) and (6) we obtain

$$\ell(lead\, w) = \lambda_1 + \ell(t_k) + 1 + \ell(t_{k+1}) + \lambda_2 + \ell(t_s) + 1 + \ell(t_{s+1}) + \lambda_3,$$

$$\ell(lead\, w') = \lambda_1 + \ell(t_k) + \ell(t_{s+1}) + 1 + \lambda_3,$$

and clearly $\ell(lead(w)) > \ell(lead(w'))$ which completes the proof of (11).

Now let

$$u = (1 + (1 - c)\nu_0 \hat{c})^{\varepsilon_1} \prod_{i=1}^{n} [(1 + \hat{c}\nu_{2i-1}(1-c))(1 + (1-c)\nu_{2i}\hat{c})](1 + \hat{c}\nu_{2n+1}(1-c))^{\varepsilon_2}$$

be an arbitrary reduced product of transvections. Assume that ε_i, ν_i are as in (3). Then $u = w + \sum_{w' \in J} w' + 1$ where each $w' \in J$ is obtained from w by dropping some factors $\hat{c}\nu_i(1 - c)$, $(1 - c)\nu_j \hat{c}$.

Fix a $w' \in J$. Then there exists a sequence of elements $w' = w_1', \cdots, w_s' = w$ such that each w_k', $(1 \le k \le s - 1)$ is obtained from w_{k+1}' by dropping some consecutive factors $\hat{c}\nu_i(1 - c)$, $(1 - c)\nu_j \hat{c}$.

It follows from (11) that $\ell(lead(w)) > \ell(lead(w_{s-1}')) > \cdots > \ell(lead(w'))$. Thus, $lead(u) = lead(w)$ and since $\ell(lead(w)) \ge 3$, $u \ne 1$. We conclude that $T(\mathbb{Q}G_\beta)$ is the free product of A_α and B_β and as $\beta \in I$ is arbitrary, the theorem is proved. $\square$

As a corollary we obtain a strong version of the Tits Alternative for $\mathcal{U}(\mathbb{Q}G)$.

Corollary 2.4. Let G be as in Theorem 2.3. Then every subgroup of $\mathcal{U}(\mathbb{Q}G)$ either contains a free noncyclic subgroup or is solvable of derived length at most 2.

Proof. Let H be a subgroup of $\mathcal{U}(\mathbb{Q}G)$ which does not contain a noncyclic free subgroup. As

$$\mathcal{U}(\mathbb{Q}G) = *_{\mathcal{U}(\bullet)}(T(\mathbb{Q}G_\alpha) \rtimes \mathcal{U}(\mathbb{Q}G_\alpha)), \quad (\alpha \in I)$$

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and $\mathcal{U}(\mathbf{Q})$ is central in $\mathcal{U}(\mathbf{Q}G)$, applying the Kurosh Subgroup Theorem [8, p.17] to the factor group $\mathcal{U}(\mathbf{Q}G)/\mathcal{U}(\mathbf{Q})$ we conclude that, modulo $\mathcal{U}(\mathbf{Q})$, H is either infinite cyclic, or a free product of two cyclic groups of order 2, or is conjugate to a subgroup of $T(\mathbf{Q}G_\alpha) \rtimes \mathcal{U}(\mathbf{Q}G_\alpha)$ for some α. In the first case H is obviously abelian, and in the second it is metabelian. In the third case we may suppose that H is a subgroup of $T(\mathbf{Q}G_\alpha) \rtimes \mathcal{U}(\mathbf{Q}G_\alpha)$.

Now $T(\mathbf{Q}G_\alpha) = A_\alpha * B_\alpha$ where A_α and B_α are torsion-free abelian groups. Since

$$H \cap T(\mathbf{Q}G_\alpha) \subseteq A_\alpha * B_\alpha$$

applying again the Kurosh Subgroup Theorem we see that $H \cap T(\mathbf{Q}G_\alpha)$ is abelian. But $H/(H \cap T(\mathbf{Q}G_\alpha))$ is isomorphic to a subgroup of $\mathcal{U}(\mathbf{Q}G_\alpha)$ and therefore is abelian. Hence H is either abelian or metabelian. $\square$

3 The Zassenhaus Conjecture

Let G be a group, $G(i) = \{g \in G : o(g) = i\}$, and C_g be the conjugacy class of $g \in G$. For $u = \sum_{g \in G} u(g)g \in \mathbf{Q}G$ set $T^{(i)}(u) = \sum_{g \in G(i)} u(g)$ and $\tilde{u}(g) = \sum_{h \in C_g} u(h)$. We recall a result on generalized traces $T^{(i)}$:

Lemma 3.1. (see [1, Lemma 2.4]) Let G be a group and p a prime. If $u \in \mathcal{U}_1(\mathbf{Z}G)$ is a torsion unit of order p^n then $T^{(p^n)}(u) \equiv 1 (mod\ p)$ and $T^{(p^i)}(u) \equiv 0 (mod\ p)$ for all $i < n$.

Lemma 3.2. Suppose that $u, w \in \mathbf{Q}G$ and $x^{-1}ux = w$ where $x \in \mathcal{U}(\mathbf{Q}G)$. Then $\tilde{u}(g) = \tilde{w}(g)$ for all $g \in G$.

Proof. Let $[\mathbf{Q}G, \mathbf{Q}G]$ be the $\mathbf{Q}$-submodule of $\mathbf{Q}G$, generated by all $gh - hg\ (g, h, \in G)$. Then $y = x^{-1}ux - u = x^{-1}(ux) - (ux)x^{-1} \in [\mathbf{Q}G, \mathbf{Q}G]$, and therefore $\tilde{y}(g) = 0$ for all $g \in G$. The result follows. $\square$

The next result is an adaptation of [14, Lemma 37.13] to the case of an infinite group G.

Lemma 3.3. Let G be a group, $t = 1 + x\nu y$ where $x, \nu, y \in \mathbf{Q}G$, $yx = 0$ and

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let tw $(w \in \mathcal{U}(\mathbf{Q}G))$ be a torsion unit of order n such that

$$(1 + x\mathbf{Q}Gy) \cap <w> = \{1\}.$$

If w commutes with x and y, then the element

$$z = 1 + t + tt^w + t t^w t^{w^2} + \cdots + tt^w \cdots t^{w^{n-2}},$$

where $t^{w^i} = w^i t w^{-i}$, is invertible in $\mathbf{Q}G$, and $z^{-1}twz = w$.

Proof. Since w commutes with x and y we see that $t^{w^j} \in 1 + x\mathbf{Q}Gy$, for all j. Therefore we get from $(tw)^n = 1$ that

$$w^n = tt^w t^{w^2} \cdots t^{w^{n-1}} = 1. \tag{12}$$

We have that

$$
\begin{aligned}
z = 1 + (1 + x\nu y) + (1 + x\nu y)(1 + x\nu^w y) + \cdots \\
+ (1 + x\nu y)(1 + x\nu^w y)\cdots(1 + x\nu^{w^{n-2}} y) = n + x\bar{\nu}y
\end{aligned}
$$

for some $\bar{\nu} \in \mathbf{Q}G$. Thus, $z^{-1} = \frac{1}{n}(1 - \frac{1}{n}x\bar{\nu}y) \in \mathbf{Q}G$.

Now by (12) we get

$$
\begin{aligned}
twz = tz^w w = t(1 + t^w + t^w t^{w^2} + \cdots + t^w t^{w^2} \cdots t^{w^{n-1}})w = \\
(t + tt^w + tt^w t^{w^2} + \cdots + tt^w \cdots t^{w^{n-1}})w = zw.
\end{aligned}
$$

Hence, $z^{-1}twz = w$ as desired. $\square$

Let G be the free product $G = *G_\alpha$, $(\alpha \in I)$ of cyclic groups of prime order $|G_\alpha| = p_\alpha$ (the p_α's are not necessarily distinct and I may be infinite). For $\alpha \in I$ fix a generator $c = c_\alpha$ of G_α and set

$$w_\alpha = \begin{cases} \frac{2}{p}\hat{c} - c & \text{if } p > 2 \\ c & \text{otherwise}, \end{cases}$$

where $p = p_\alpha$ and $\hat{c} = 1 + c + \cdots + c^{p-1}$. It is easy to see that $w_\alpha^2 = c^2$.

Theorem 3.4. A finite subgroup of $\mathcal{U}_1(\mathbf{Q}G)$ is conjugate in $\mathcal{U}(\mathbf{Q}G)$ to a subgroup of $<w_\alpha>$ for some $\alpha \in I$.

Proof. Let $H \neq \{1\}$ be a finite subgroup of $\mathcal{U}_1(\mathbf{Q}G)$. By Theorem 2.3,

$$\mathcal{U}(\mathbf{Q}G) = *_{\mathcal{U}(\mathbf{Q})} (T(\mathbf{Q}G_\alpha) \rtimes \mathcal{U}(\mathbf{Q}G_\alpha)), \quad (\alpha \in I),$$

and $T(\mathbf{Q}G_\alpha) = A_\alpha * B_\alpha$ where A_α and B_α are the torsion-free abelian groups defined in Section 2.

Applying the Kurosh Subgroup Theorem to the factor group $\mathcal{U}(\mathbf{Q}G)/\mathcal{U}(\mathbf{Q})$ [8] or a subgroup theorem for amalgamated free products we get that H is conjugate in $\mathcal{U}(\mathbf{Q}G)$ (and therefore in $\mathcal{U}_1(\mathbf{Q}G)$) to a subgroup of $T(\mathbf{Q}G_\alpha) \rtimes \mathcal{U}(\mathbf{Q}G_\alpha)$ for some $\alpha \in I$. Thus, replacing H by its conjugate we may assume that

$$H \subseteq T(\mathbf{Q}G_\alpha) \rtimes \mathcal{U}(\mathbf{Q}G_\alpha).$$

Since every element of $T(\mathbf{Q}G_\alpha)$ has augmentation 1, we really have

$$H \subseteq T(\mathbf{Q}G_\alpha) \rtimes \mathcal{U}_1(\mathbf{Q}G_\alpha).$$

Let u be a nonidentity element of H. Then $u = tw$ where $t \in T(\mathbf{Q}G_\alpha)$ and $w \in \mathcal{U}_1(\mathbf{Q}G_\alpha)$. Moreover, since $T(\mathbf{Q}G_\alpha)$ is torsion-free, w is a torsion unit of the same order as u.

Take a $p-th$ primitive root of unity ζ and consider the isomorphism $\phi : \mathbf{Q} < c > \to \mathbf{Q} \oplus \mathbf{Q}(\zeta)$ defined by $\phi(c) = (1, \zeta)$ and linearly extended. Since the torsion units of $\mathbf{Q}(\zeta)$ are of the form: ζ^k or $-\zeta^k$, we see that $\phi(w) = (1, \zeta^k)$ or $\phi(w) = (1, -\zeta^k)$. If $p > 2$ then $\phi^{-1}(1, -\zeta) = \frac{2}{p}\hat{c} - c = w_\alpha$. Thus, in any case $w \in \; < w_\alpha >$.

Let $v \neq 1$ be another element of H. Similarly we can write $v = fw'$ where $f \in T(\mathbf{Q}G_\alpha)$ and $1 \neq w' \in \; < w_\alpha >$. Replacing u or v by an appropriate power of it we may suppose that $v = fw^{-1}$. Then

$$uv = twfw^{-1} \in T(\mathbf{Q}G_\alpha) \cap H.$$

As $T(\mathbf{Q}G_\alpha)$ is torsion-free $T(\mathbf{Q}G_\alpha) \cap H = \{1\}$ and hence $v = u^{-1}$. It follows that H is cyclic whose order divides $2p$, and we may suppose that $H = < u >$.

We shall now show that u is conjugate to w in $\mathcal{U}(\mathbf{Q}G)$ and this will complete the proof of the theorem.

Using the fact that $T(\mathbf{Q}G_\alpha) = A_\alpha * B_\alpha$, write t as a reduced product $t_1 \cdots t_n$ of elements from A_α and B_α. Since the order of $u = t_1 \cdots t_n w$ divides $2p$ we have $(t_1 \cdots t_n w)^{2p} = 1$ which implies that

$$t_1 \cdots t_n (wt_1 \cdots t_n w^{-1})(w^2 t_1 \cdots t_n w^{-2}) \cdots (w^{2p-1} t_1 \cdots t_n w^{-(2p-1)}) = 1.$$

Note that $wA_\alpha w^{-1} \subseteq A_\alpha$ and $wB_\alpha w^{-1} \subseteq B_\alpha$, because w commutes with c. As the product $t_1 \cdots t_n$ is reduced and $T(\mathbb{Q}G_\alpha) = A_\alpha * B_\alpha$, we have that n is odd and that

$$t_n w t_1 w^{-1} = t_{n-2} w t_2 w^{-1} = \cdots = t_{\frac{n+1}{2}+1} w t_{\frac{n+1}{2}-1} w^{-1} = 1.$$

Thus,

$$u = (t_1 t_2 \cdots t_{\frac{n+1}{2}-1}) t_{\frac{n+1}{2}} w (t_{\frac{n+1}{2}-1}^{-1} \cdots t_2^{-1} t_1^{-1}),$$

and u is conjugate by transvections to $t_{(n+1)/2} w$. Since $T(\mathbb{Q}G_\alpha) \cap H = \{1\}$ it follows from Lemma 3.3 that $t_{(n+1)/2} w$ is conjugate in $\mathcal{U}(\mathbb{Q}G)$ to w. $\square$

The theorem implies the Zassenhaus Conjecture (ZC3) for G:

Corollary 3.5. Let G be as in Theorem 3.4. Then every nonidentity finite subgroup of $\mathcal{U}_1(\mathbb{Z}G)$ is conjugate in $\mathcal{U}(\mathbb{Q}G)$ to one of the G_α, $(\alpha \in I)$.

Proof. Let $H \neq \{1\}$ be a finite subgroup of $\mathcal{U}_1(\mathbb{Q}G)$. By Theorem 3.4 $x^{-1}Hx \subseteq <w_\alpha>$, for some $x \in \mathcal{U}(\mathbb{Q}G)$ and some $\alpha \in I$. Thus H is cyclic; its order divides $2p$ if p_α is odd and is equal to 2 otherwise. Obviously, in the last case $x^{-1}Hx = G_\alpha$, so we may assume that $p_\alpha > 2$.

Suppose that H contains an element u of order 2 and set $w = x^{-1}ux$. It follows from Lemma 3.1 that there exists an element $g \in G$ of order 2 such that $\tilde{u}(g) \neq 0$. Hence by Lemma 3.2, $\tilde{w}(g) \neq 0$ which is impossible as $w \in \mathbb{Q}G_\alpha$, where G_α has order $p_\alpha > 2$. We conclude that the order of H is p_α and that $x^{-1}Hx = G_\alpha$. $\square$

4   Acknowledgements

References

[1] Bovdi, A.A., *The Multiplicative Group of an Integral Group Ring*, Uzhgorod, 1987 (210 pp.,Dep. UkrNIINTI, 24.09.87, 2712-Uk87).

[2] Bovdi, A.A., Marciniak, Z., Sehgal, S. K., Torsion Units in Infinite Group Rings, *J. Number Theory* 47 (1994), 284-299.

[3] Dokuchaev, M.A. Finite Subgroups of Units in Integral group Rings, *Resenhas IME-USP* , 2 (3), 1996, 293-302.

[4] Dokuchaev, M.A. and Juriaans, S.O., Finite Subgroups in Integral Group Rings, *Canad. J. Math.* (to appear).

[5] Dokuchaev, M.A., Juriaans, S.O., Polcino Milies, C., Integral Group Rings of Frobenius Groups and the Conjectures of H.J. Zassenhaus, (preprint).

[6] Gerasimov, V.N., The group of units of a free product of rings, *Math. USSR Sbornik*, 62 (1), 1989, 41-63.

[7] Klinger, L., Construction of a counterexample to a conjecture of Zassenhaus, *Commun. Algebra* 19 (1993), 2303-2330.

[8] Kurosh, A.G., *The theory of groups*, II Chelsea Publ. Co., New York, 1956.

[9] Levin, F., Sehgal, S.K., Zassenhaus conjecture for the infinite dihedral group, *Proc A.M.S. Special Session, Series in Algebra* 1, *World Scientific, Singapore* (1993), 57-68.

[10] Lichtman, A.I. and Sehgal, S.K., The elements of finite order in the group of units of group rings of free products of groups, *Commun. Algebra* 17 (1989), 2223-2253.

[11] Marciniak, Z. S., Sehgal, S. K., Zassenhaus Conjecture and Infinite Nilpotent Groups, *J. Algebra* 184, 1996, 207-212.

[12] Polcino Milies, C., Ritter, J. and Sehgal, S. K., On a conjecture of Zassenhaus on torsion units in integral group rings II, *Proc. Amer. Math. Soc.* 97 (2) (1986), 206-210.

[13] Roggenkamp, K.W., Taylor, M. J., *Group Rings and Class Groups*, Birkhäuser Verlag, Basel, 1992.

[14] Sehgal, S.K., *Units of Integral Group Rings*, Longman's, Essex, 1993.

[15] Valenti, A., Torsion Units in Integral Group Rings, *Proc. Amer. Math. Soc.* **120** (1) (1994), 1-4.

[16] Weiss, A., Rigidity of p-adic torsion, *Ann. Math.* **127** (1988), 317-332.

[17] Weiss, A., Torsion units in integral group rings, *J. Reine Angew. Math.* **415** (1991), 175–187.

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