

RT-MAT 2001-11

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Setembro 2001

Esta é uma publicação preliminar (“preprint”).

Obstruction theory and minimal number of coincidences for maps from a complex into a manifold

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September 21, 2001

Abstract The Nielsen coincidence theory is well understood for a pair of maps between n -dimensional compact manifolds for n greater than or equal to three. We consider coincidence theory of a pair $(f, g) : K \rightarrow N^n$, where K is a finite simplicial complex of the same dimension as the manifold N^n . We construct an algorithm to find the minimal number of coincidences in the homotopy class of the pair based on the obstruction to deform the pair to coincidence free. Some particular cases are analyzed including the one where the target is simply connected.

Key words: Coincidence Nielsen numbers, Index, Obstruction, Simplicial Complexes

1991 AMS classification: Primary 55M20; secondary 54H25.

1 Introduction

Let K be a finite simplicial complex of dimension n and let $f, g: K \rightarrow N^n$ be maps, where N^n is a n -dimensional manifold. The purpose of this work is to define a sharper invariant than the coincidence Nielsen number to study the minimal number of coincidences in the homotopy class of the pair (f, g) . This invariant is based on algebraic and geometric features of the pair (f, g) and of the complex K . The case where K is a manifold has been treated by H. Shirmer, in [Sc2]. In [BG], we study the case where K is the union of two subcomplexes K_1, K_2 each being a closed manifold and $K_i - K_1 \cap K_2$ being by-passing in K_i . On trying to drop the by-passing condition in this

particular case, it became clear that the geometry of K was very relevant, and the difficulties to find a Nielsen type number to describe $MC[f, g]$, the minimal number of coincidences in the homotopy class of the pair (f, g) , were basically the same as if we consider a general complex K . So we treat in this work the general problem. The invariants defined here are homotopy invariants with respect to the pair (f, g) but are not invariants with respect to the homotopy type of K as one can see in examples in [G2, section 4].

The article is divided into 4 sections, besides this one. In section 2 we show that it suffices to work with complexes that are homogeneous and with no $(n - 1)$ -simplices facing only one n -simplex. Section 3 is devoted to the definition of a homotopy invariant, in terms of the obstruction cocycles representing the obstruction class to deform the pair (f, g) to coincidence free. In section 4 we show that this invariant coincides with the minimum number of coincidences, under mild conditions. Finally, in section 5 we analyze some special cases. One of the cases is when the target N^n is simply connected and the other is when K is finite union of manifolds without boundary. In these examples we estimate the difference between the minimal number of coincidence points and the usual Nielsen number.

2 The Geometry of the complex K and the minimal number of coincidences

In this section we show that to solve the coincidence problem it suffices to consider n -homogeneous simplicial complexes K with the property that every $(n - 1)$ - simplex faces at least two n -simplices. We denote by $MC[f, g]$ the minimum number of coincidences in the homotopy class of the pair (f, g) .

Given $f, g : K \rightarrow N^n$ by [G2, Proposition 2.7] we know that $MC[f, g] = MC[f', g']$, where f', g' are the restrictions of f, g to the subcomplex $\hat{K} \subset K$ where \hat{K} is the smallest subcomplex which contains all n -simplices of K . So we assume that K is n -homogeneous. Given a n -homogeneous complex consider the n -simplices containing a $(n-1)$ -face which is not contained in another n -simplex. Define $\bar{K} \subset K$ the subcomplex of K obtained from K by removing these n -simplices as well as its $(n - 1)$ -faces which are not contained in another n -simplex.

Lemma 2.1 Given $f, g : K \rightarrow N^n$ then $MC[f, g] = MC[\bar{f}, \bar{g}]$, where \bar{f}, \bar{g} are the restrictions of f, g , respectively, to \bar{K} .

Proof: Clearly $MC[f, g] \geq MC[\bar{f}, \bar{g}]$. To show the converse let K' be a subcomplex obtained from K by removing one n -simplex, Δ^n , together with one of its $(n - 1)$ faces, Δ^{n-1} , which faces only this n -simplex. We will show that $MC[f', g'] \geq MC[f, g]$ where f', g' are the restrictions of f and g , respectively, to K' . Since $MC[f', g']$ is finite let us consider a pair of maps (f', g') such that $coin(f', g')$ is finite. We will construct a pair of maps (f_1, g_1) in the homotopy class of the pair (f, g) such that its restriction to K' is homotopic to (f', g') and such that $coin(f_1, g_1) = coin(f', g')$.

We can assume that in the boundary of the n -simplex there is only one coincidence point. Then this point is either in the interior of $\partial(\Delta^n - \Delta^{n-1})$ or in the boundary of Δ^{n-1} . We will look in details the first situation. Call x_0 this coincidence point. We consider a small $(n - 1)$ closed disk around this point, as well as half balls, having this disk as base, each lying in a n -simplex, except Δ^n , having the $(n - 1)$ -simplex that contains x_0 as a face, and such that the images of the maps f' and g' restricted to the closed half balls lie inside of an Euclidean neighborhood of $f'(x_0) = g'(x_0) = y_0$. After composing the maps with a chart, we take $f' - g'$, and deform it, relative to the boundary of the half balls and to the point $\{x_0\}$, to a map h' which is linear in the interior of the half balls, and has only one zero at x_0 . Now use h' to define a map g'_1 , homotopic to g' so that $coin(f', g'_1) = coin(f', g')$. Observe that h' restricted to the boundary of the $(n - 1)$ -disk is homotopic to the constant map in $R^n - \{0\}$. Therefore we can extend h' , first to the boundary of the upper hemisphere of a half n -ball in Δ^n having the disk as base, and then extend, by linearity, to the interior of the hemisphere without introducing any other coincidences, see Figure II. Call this extension h_1 and consider any close extension f_1 of f' to the closure of the upper hemisphere. Construct an extension g_1 of g'_1 , so that $h_1 = f_1 - g_1$. Observe that we succeed extending the pair (f', g') to (f_1, g_1) without introducing any other coincidences. Finally, making use of the retraction of Δ^n onto $(\partial\Delta^n - Int(\Delta^{n-1}) \cup H^+$, where H^+ is the closure of the upper hemisphere, we get the desired extension.

The case where the coincidence point is in the boundary of Δ^{n-1} is left to the reader.

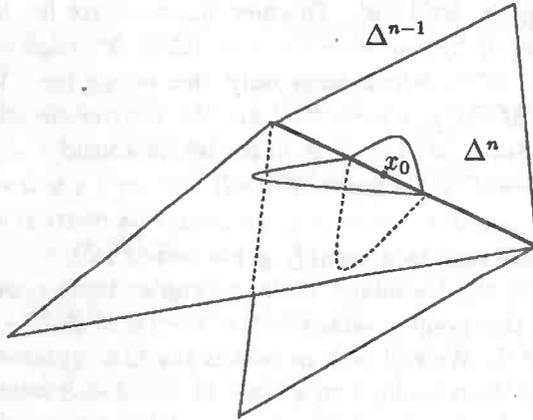


Figure I

Given a n -complex K we can iterate the following two operations: the first is to consider the homogeneous subcomplex $K < n > \subset K$ (see [G2]) and the second is the operation defined in the beginning of the section which consists to eliminate the n -simplices containing faces that do not face any other n -simplex, together with these faces. This process of iterating the two operations will stop after a finite number of steps, and we make the following

Definition 2.2 Given a n -complex K we define the soul of K , denoted by $s(K)$, to be the subcomplex obtained at the end of the process indicated above.

Proposition 2.3 Given $f, g : K \rightarrow N^n$ then $MC[f, g] = MC[f', g']$ where f' and g' are the restriction of the f and g , respectively, to $s(K)$.

Proof: The proof follows from Lemma 2.1 and [G2, Proposition 2.7].

The figure below shows a complex and its soul.

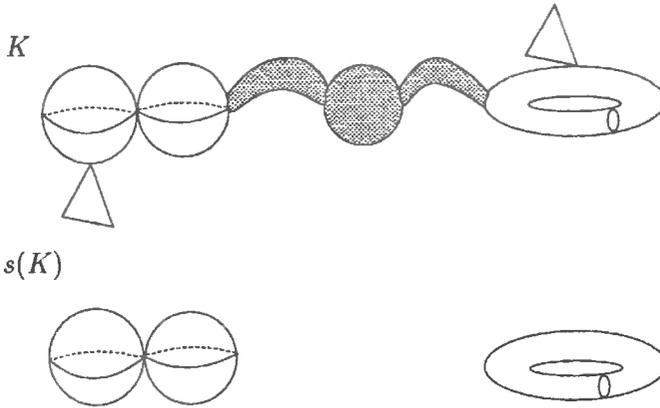


Figure II

Remark 2.4 *A typical example of a n -complex which coincides with its soul is the union of n -manifolds without boundary. The converse is not true, though. To see this consider three disjoint spheres joint to a 2-simplex so that each of its faces belongs to one of the spheres.*

3 Local coincidence index and the number $NO(f, g; K)$

In this section we define a homotopy invariant which will coincide, under mild conditions, with the minimal number of coincidences in the homotopy class of the pair $f, g : K \rightarrow N^n$. This invariant is constructed in terms of the primary obstruction to deform a pair of maps to coincidence free as well as in terms of the geometry of the complex K . We will start by reviewing the notion of local index as formulated by E. Fadell and S. Husseini in [FH] where we adapted the terminology to the coincidence case.

Let U be an open set of K and $(f, g) : U \rightarrow N^n$ be a pair of maps where the set of coincidence points are compact.

As in [FH], we consider the diagonal Δ in $N^n \times N^n$ and replace the inclusion $N^n \times N^n - \Delta \hookrightarrow N^n \times N^n$ by a fiber map $p: E \rightarrow N^n \times N^n$, where $E = \{(\alpha, \beta) : \alpha(0) \neq \beta(0)\}$, and $p(\alpha, \beta) = (\alpha(1), \beta(1))$. For $b = (x, y)$ in $N^n \times N^n$ and $F_b = p^{-1}(b)$, $\pi_{m-1}(F_b)$ is a local system of coefficients on $N^n \times N^n$. There is an isomorphism of local systems on $N^n \times N^n$

$$\zeta: \pi_{m-1}(F_b, b) \rightarrow \mathcal{Z}[\pi],$$

where $\pi = \pi_1(N^n, x)$ and the action of $\pi \times \pi$ on $\mathcal{Z}[\pi]$ is given by

$$\alpha \cdot (\sigma, \tau) = \text{sgn} \sigma \sigma^{-1} \cdot \alpha \cdot \tau$$

We will refer to this system as \mathcal{B} .

Let the local system on U be the one induced from \mathcal{B} by $f \times g: U \rightarrow N^n \times N^n$ and denote it by $\mathcal{B}(f \times g)$. Consider the fiber space $E(f, g)$ obtained by pulling back $p: E \rightarrow N^n \times N^n$ over U by $f \times g$.

The obstruction to deform the pair (f, g) to a coincidence free pair is related to the obstruction to extend sections of the fiber map $E(f, g) \rightarrow U$.

Following the steps in [FH] and making the usual adaptations to the coincidence case, we end up with:

Definition 3.1 *The coincidence index of $(f, g): U \rightarrow N^n$ is the cohomology class $i(f, g)$ in $H_c^n(U; \mathcal{B}(f \times g))$ with the property that (f, g) can be deformed by a compact homotopy to a coincidence free pair if and only if $i(f, g)$ vanishes.*

Consider now F an isolated set of coincidences of (f, g) and let V be an open set of U such that $F = V \cap \text{coin}(f, g)$. Consider the diagram

$$H^n(V, V - F; \mathcal{B}(f \times g)) \xrightarrow{j^{*-1}} H^n(U, U - F; \mathcal{B}(f \times g)) \xrightarrow{k^*} H^n(U, U - \text{coin}(f, g); \mathcal{B}(f \times g))$$

where the first arrow is the inverse of the excision isomorphism and the second is induced by the inclusion. Recall that $H_c^n(U; \mathcal{B}(f \times g))$ is the inverse limit of $H^n(U, U - C; \mathcal{B}(f \times g))$, where the limit is taken over all compact subsets C of U .

Definition 3.2 *The local coincidence index of F , denoted by $i(f, g; F)$, is the element in $H_c^n(U; \mathcal{B}(f \times g))$ given by $k^*(j^*)^{-1}(\alpha)$, where α in $H^n(V, V - F; \mathcal{B}(f \times g))$ corresponds to the coincidence index of $(f, g): V \rightarrow N^n$.*

Let us consider the group $H^n(K, A)$, the n -th simplicial cohomology group of K with local coefficients, where A is a free abelian group and identified with the direct sum of Z 's indexed by some set J . We call a cochain $c_n \in C^n(K, A)$ elementary if c_n is nonzero in only one n -simplex, called its support, and has value in one summand Z of A indexed by $j \in J$. So we can associate to each elementary cochain a pair (Δ^n, j) , where Δ^n is its support and j is the index of the summand $Z \subset A$ where the cochain assumes its value. Two elementary cochains are disjoint if the pairs $(\Delta^n, j), (\Delta^n, j')$ are not equal. Given an arbitrary cocycle (or cochain) $c_n \in C^n(K, A)$ we define an integer, $\ell(c_n)$, as follows:

The cocycle c_n can be uniquely written as a sum of disjoint elementary cocycles i.e. $c_n = c_{n,1} + c_{n,2} + \dots + c_{n,r}$, where each $c_{n,i}$ is elementary.

Definition 3.3 A cocycle is essential if it represents a nonzero cohomology class.

Definition 3.4 A partial sum $c_{n,i_1} + \dots + c_{n,i_s}$ of the decomposition of c_n is said to be combinable if the intersection of the supports of all elementary summands is nonempty and they have values in the same summand Z of A . Define $\ell(c_n)$ to be the minimal number of combinable partial summands among all decompositions of c_n .

Now let $f, g : K \rightarrow N^n$ be maps.

Definition 3.5 The number $NO(f, g; K)$ is defined as the minimum of the numbers $\ell(c_n)$, where c_n runs over the set of all cocycles representing the obstruction $O^n(f, g) \in H^n(K, Z[\pi])$ to deform (f, g) to coincidence free.

Theorem 3.6 $NO(f, g; K)$ is a homotopy invariant.

Proof: The result follows from the fact that $O^n(f, g) = O^n(f_1, g_1)$, for (f_1, g_1) homotopic to (f, g) ■

4 The minimal number of coincidences and the realization of the number $NO(f, g; K)$

We will now prove that the number $NO(f, g; K)$ coincides with the minimal number of coincidences in the homotopy class of the pair (f, g) . The techniques applied are based on works by H. Schirmer [Sc2], X.Zhao [Z], L.D. Borsari and D.L. Gonçalves [BG] and D. Gonçalves [G2]. From what we have seen before, we may assume that K coincides with its soul.

We will define a decomposition of K in terms of a simplicial structure of K , although it can be shown that this decomposition does not depend on the particular simplicial structure. For each maximal simplex Δ^n let $C(\Delta^n)$ be the smallest subcomplex which contains all n -simplices Δ^n such that there is a sequence of n -simplices starting at Δ^n and ending at Δ^n so that the intersection of two consecutive ones is a $(n - 1)$ -simplex which faces only these two n -simplices. This defines a covering of K by homogeneous simplicial subcomplexes which we denote by $\{K_1, \dots, K_r\}$. These subcomplexes happen to be, in many situations, manifolds but not necessarily. Take for example, K to be the n -sphere with its poles identified. Associated to this covering we have the subcomplex $K_0 = \bigcup_{i \neq j} K_i \cap K_j$. Observe that the points of K_0 are characterized by the property that they are not locally Euclidean in K .

Theorem 4.1 *Let $(f, g): K \rightarrow N^n$ be a pair of maps where K and N^n have dimension bigger than or equal to three. Assume every component of K_0 is of non-zero dimension. Then the minimum number of coincidences in the homotopy class of the pair (f, g) is given by $NO(f, g; K)$.*

Proof: The process of deforming the pair (f, g) to (f_1, g_1) having all coincidences lying in the interior of n -simplices is based on [Sc2] and it guarantees that the cocycle associated to this new pair, $c_n = c_n(f_1, g_1)$, satisfies $\ell(c_n) \leq \text{Coin}(f, g)$. Therefore $NO(f, g; K) \leq \text{Coin}(f, g)$, and being a homotopy invariant, it becomes a lower bound for the minimum number of coincidences in the homotopy class of the pair (f, g) . It remains to prove that $NO(f, g; K)$ can be realized and this is done in what follows.

Let c_n be a n -cocycle representing the obstruction to deform (f, g) to coincidence free and such that $\ell(c_n) = NO(f, g; K)$. Consider (f', g') a pair homotopic to (f, g) so that $c_n(f', g') = c_n$. This means that (f', g') has coincidences appearing in the simplices that are support for each elementary

cocycle in the decomposition of c_n . Each combinable partial sum of c_n will correspond to a set of n -simplices having non-empty intersection each of them containing one coincidence.

Moreover, since the values of these elementary cocycles lie in the same summand Z of $\mathcal{Z}[\pi]$, there can be found paths α_i starting in a coincidence point and ending in a common point a in the intersection of all these simplices so that $f(\alpha_1\alpha_i^{-1}) \cong g(\alpha_1\alpha_i^{-1})$, for all i . These conditions, together with the techniques developed in [BG], allow us to deform the pair (f', g') so that all these coincidences coalesce to a . Repeating this procedure to all others combinable sets we end up with $\ell(c_n) = NO(f, g; K)$ coincidence points. ■

Remark 4.2 *In the case where some, if not all, components of K_0 have zero dimension, it could happen that two or more combinable partial sums have the intersection of their supports being only one point. In this case, only one set of coincidences, arising from the combinable partial sums, would be joint to this point. Therefore, we would have to add to the number $\ell(c_n)$ the number of elements of all, except the biggest, combinable partial sums for which the intersection of supports is the same single point. Then, the minimum of these numbers, as c_n runs through all possible cocycles representing the obstruction class, will give us the minimum number of coincidences in the homotopy class of the pair (f, g) .*

As an application of the above result, let $K' \subset K$ be any subcomplex such that the homomorphism $i^* : H^n(K, Z[\pi]) \rightarrow H^n(K', Z[\pi])$, induced by the inclusion map, is a cohomology isomorphism with local coefficients where $\pi = \pi_1(N^n)$. Observe that if two subcomplexes have this property then their intersection does too. Hence, we may always consider the minimal one, namely, the intersection of all subcomplexes satisfying the above condition.

Theorem 4.3 *Given $f, g : K \rightarrow N^n$ then $MC[f, g] = MC[f', g']$, where f', g' are the restrictions of f, g , respectively, to K' .*

Proof: Given (f, g) consider its restriction (f', g') to K' , and let (f'', g'') a pair of maps in K' homotopic to (f', g') . Take any cocycle c_n representing the obstruction $O^n(f', g')$. Since K' reflects all cohomology of K , this cocycle also represents the obstruction $O^n(f, g)$. Therefore there exists (f_1, g_1) homotopic to (f, g) such that $c^n(f_1, g_1)$ is the cocycle c_n . Observe that any combinable partial sum of c_n in K' is also combinable in K . Therefore, from Theorem 4.1, it follows that $MC[f, g] \leq MC[f', g']$. The other inequality is clear and the result follows. ■

5 Some special cases

Before analyzing the special cases, let us observe that in the context we are working, we do not expect the minimum number of coincidences to coincide with the Nielsen number. This can be seen in [BG] even in the case where the target is simply connected. The coincidence Nielsen number can be defined as in [B1] or in [G2], since, in our context, the given definitions are equivalent.

In the fixed point case, in dimension two, it is well known that the *Wecken property does not hold*, i.e., the minimal number of fixed points does not coincide with the Nielsen number. It was observed in [Ke] that for maps $f: P \rightarrow P$, where P is the pantalon, the disk with two holes, the difference $MC[f] - N(f)$ can become arbitrarily large as we vary over the homotopy classes of self-maps on P . Inspired on these facts we set

Definition 5.1 *Let $W\{K, N^n\}$ be the maximum of all $MC[f, g] - N(f, g)$, where $[f, g]$ runs over all homotopy classes of pairs of maps from K to N^n .*

It is well known that this number is zero in the fixed point case when the complex K has no local cut points, it is not a surface, and it has dimension greater than or equal to two, see [J]. This is also the case in the coincidence context, where the spaces are manifolds of dimension greater than or equal to three, see [Sc2]. In our context, we will be seeing that $W\{K, N^n\}$ can be either finite or infinity.

The case where N^n is simply connected

Let $f, g: K \rightarrow N^n$ be a pair of maps. Since N^n is simply connected we have only one Nielsen class. We will show that in general the number $MC[f, g]$ is bounded for a fixed K but it goes to infinity as we vary K . Many examples can be construct having $MC[f, g] > 1$, and a upper bound for the maximum of $MC[f, g]$, among all pairs (f, g) , is given.

Let $C = \{K_{i_1}, \dots, K_{i_r}\}$ be the covering of K defined in the previous section, and assume that all components of K_0 have nonzero dimension.

Definition 5.2 *A subset $\{K_{i_1}, \dots, K_{i_r}\}$ of the covering $C = \{K_1, \dots, K_r\}$ is called admissible if the intersection $K_{i_1} \cap \dots \cap K_{i_r} \neq \emptyset$. Let $\ell(C)$ be the minimal number of admissible subsets which cover C . For the purpose of computing $\ell(C)$ we can assume, without loss of generality, that the admissible sets are maximal in the sense that for any $K_j \neq K_{i_t}$, $t = 1, \dots, r$, we have $K_j \cap K_{i_1} \cap \dots \cap K_{i_r} = \emptyset$.*

Proposition 5.3 *Given $f, g : K \rightarrow N^n$ then $MC[f, g] \leq \ell(C)$. In particular $W\{K, N^n\}$ is finite.*

Proof: Let c^n be any cocycle representing the obstruction to deform the pair (f, g) to coincidence free. For each admissible covering of K we obtain a decomposition of c^n into combinable partial sums. The number of elements in this decomposition is less than or equal to the number of elements of the covering. So it follows that $MC[f, g] \leq \ell(C)$. ■

Remark Observe that in this context, where the target is simply connected, we may replace the complex K by any subcomplex K' of K so that the inclusion $i : K' \rightarrow K$ induces an isomorphism of the n -th cohomology group with coefficient in Z , not twisted. The minimal of these complexes has been considered in [G2]. Therefore $W\{K, N^n\}$ can be computed as $W\{K', N^n\}$, where K' varies over a larger family of subcomplexes of K , than the family considered in section 4.

Examples

Consider the simplicial complex K obtained from a collection of six torus joint by tubes. Observe that $K = K_1 \cup K_2 \cup \dots \cup K_{21}$, the torus are $K_1 \cup K_2, K_3 \cup K_4, \dots, K_{11} \cup K_{12}$, and the tubes $K_{13}, K_{14}, \dots, K_{21}$ either have empty intersection or intercept in a boundary circle, see Figure III for an immersed model of K . Notice also that the decomposition of K as the union of the K'_i 's, $i = 1, 2, \dots, 21$, is the one defined in the preceding section.

It is not hard to see that for $C = \{K_1, K_2, \dots, K_{21}\}$, the minimal number of admissible subsets that cover C , $\ell(C)$, is 7 and this covering is realized by the subsets:

$$\{K_1, K_2, K_{14}\}, \{K_3, K_4, K_{20}\}, \{K_5, K_6, K_{21}\}, \{K_7, K_8, K_{19}\}, \\ \{K_9, K_{10}, K_{18}\}, \{K_{11}, K_{12}, K_{13}\}, \{K_{15}, K_{16}, K_{17}\}$$

Consider now a pair of maps $(f, g): K \rightarrow S^2$. Since S^2 is simply connected, we know that the obstruction to deform (f, g) to coincidence free can be represented by a sum of elementary cocycles so that no two of them have supports in the same K_i . It is not hard to see that with techniques developed by H. Schirmer in [Sc2] and in [BG] we may, by adding suitable coboundaries to this cocycle, assume that the cocycle is composed by elementary ones with supports lying in some, if not all, of the complexes K_1, K_2, \dots, K_{12} . Therefore (f, g) can be made homotopic to a pair with at most 12 coincidences lying in different K_i 's, $i = 1, 2, \dots, 12$. Since any two coincidences lying in the torus $K_i \cup K_{i+1}$ can be joint to one, we end up with $MC[f, g] \leq 6$. It is also clear that we can construct a pair (f, g) such that $MC[f, g]$ is, in fact, 6.

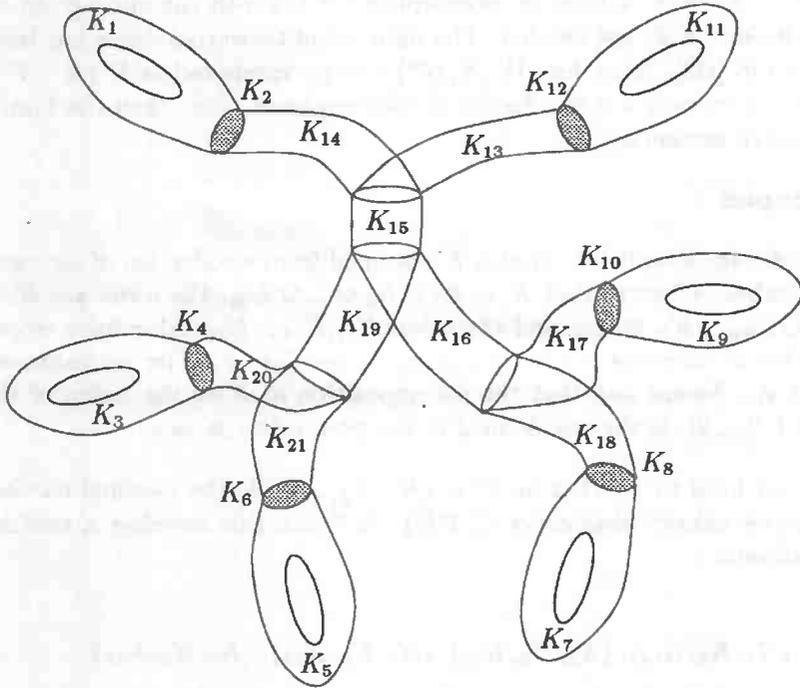


Figure III

We have therefore an example where

$$MC[f, g] \leq 6 < 7 = \ell(C) \text{ and } W\{K, N^n\} = 6 .$$

Let us represent the complex K by the graph in Figure IV. So each tube is represented by a segment and each torus by a circle.

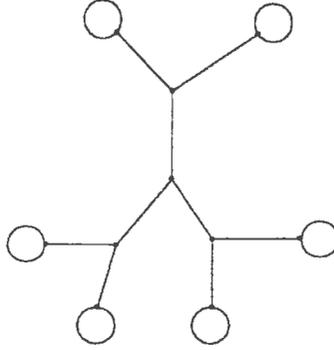


Figure IV

With this in mind, we can produce more examples as represented in Figure V.

In general we have, for the n^{th} step, the following:

Cardinality of C is $3 \cdot 2^0 + 3 \cdot 2^1 + \dots + 3 \cdot 2^n + 3 \cdot 2^n + 3 \cdot 2^n$,

$$\ell(C) = \begin{cases} 3 \cdot 2^n + 3 \cdot 2^{n-2} + \dots + 3 \cdot 2^1 + 3 \cdot 2^0, & \text{for } n \text{ even} \\ 3 \cdot 2^n + 3 \cdot 2^{n-2} + \dots + 3 \cdot 2^1 + 1, & \text{for } n \text{ odd} \end{cases}$$

$MC[f, g] \leq 3 \cdot 2^n$, for all $(f, g): K \rightarrow S^2$, and we have examples of pairs of maps for which $MC[f, g] = 3 \cdot 2^n$.

Hence, we have that $W\{K, N^n\} = 3 \cdot 2^n - 1$ which is strictly less than $\ell(C)$. Also, as K varies with n , both $\ell(C)$ and $\ell(C) - W\{K, N^n\}$ go to infinity.

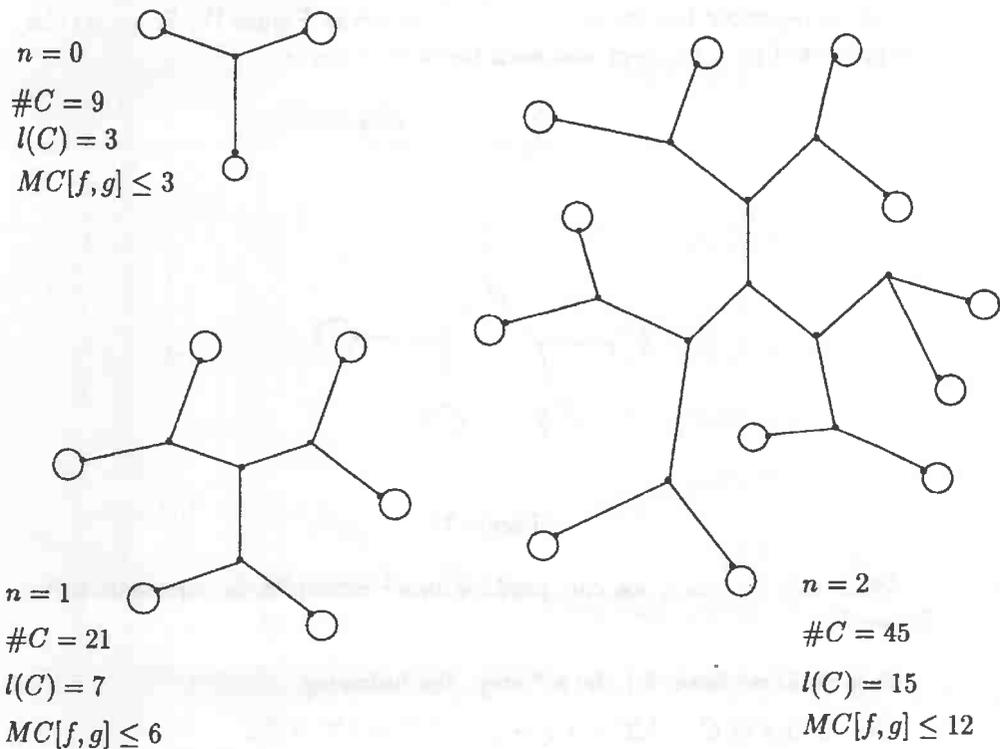


Figure V

The case where K is a union of closed manifolds

Let us consider the minimizing problem for $(f, g): K \rightarrow N^n$, where K is a union of n -dimensional manifolds M_1, \dots, M_r with no boundary. We can always assume that this covering has no proper subcovering. We make the assumption that $M_0 = \bigcup_{i \neq j} M_i \cap M_j$ is by-passing in each M_i . This notion of by-passing does not depend on the decomposition of K as a union of closed

manifolds. The results obtained here are a generalization of what we have done in [BG].

For a pair of maps $(f, g): K \rightarrow N^n$, consider $(f_i, g_i): M_i \rightarrow N^n$ be the restriction of (f, g) to M_i . For each (f_i, g_i) we consider the usual essential Nielsen classes.

Let F_1, F_2, \dots, F_r be a set of essential Nielsen classes so that each F_i belongs to a different manifold M_j . We say that this set is combinable if the following hold:

- a) The intersection of the manifolds M_j , to where the classes F_i belong to, is non-empty.
- b) There is a point a in the intersection mentioned above, and paths α_i in M_i from any point in F_i to a so that $f(\alpha_1 * \alpha_i^{-1}) \cong g(\alpha_1 * \alpha_i^{-1})$, see Figure VI.

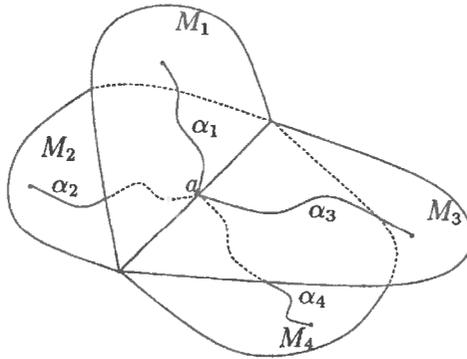


Figure VI

This relation enable us to partition the set of Nielsen classes of $(f_i, g_i): M_i \rightarrow N^n$, for $i = 1, 2, \dots, r$ into combinable subsets. These can be done in various manners. We are interested in doing it in a way that we end up with the least number of combinable subsets. Therefore we consider all possible coverings of the set of Nielsen classes by combinable subsets and we set

Definition 5.4 *The number $N(f, g; K)$ corresponds to the smallest cardinality of combinable subsets among all possible coverings.*

Just as we have done in [BG] we obtain

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