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Submersions Maps of Constant Rank Submersions with Folds and Immersions

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Submersions, Maps of Constant Rank Submersions with Folds and Immersions

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1 Introduction

The aim of these notes is to give global characterizations of maps of constant rank and submersions with definite folds by means of their Stein factorizations. We also stablish a relation between the problem of existence of these maps and the problem of extensions of immersions. A theorem on the existence of submersions with definite folds is proved.

Let M and N be connected differentiable manifolds of dimensions m and n. Assume that M is closed and $m \neq n$. Let $f : M \longrightarrow N$ be differentiable. The Stein factorization of f is given by the commutative diagram

$$\begin{array}{ccc} M & & \\ q \downarrow & \overset{f}{\searrow} & \\ W & \xrightarrow{g} & N \end{array}$$

where W is the quotient space of M by the equivalence relation that identifies to a point each connected component of each fiber of f.

As an example, let N be $\mathbb{R}$ and let $f : M \longrightarrow \mathbb{R}$ be a Morse function. Then W is the Reeb's graph of f.

Let f be a submersion. Then the local form of f as a projection is known. Its Stein factorization provides a global characterization of f as the composite of the projection of a bundle with connected fiber with a covering map.

When f is a map of constant rank k, $0 < k < m, n$ its local form is also known.

f is locally the composite of a projection with an inclusion map. In this case, the Stein factorization of f also gives a global version of this local characterization of f.

Let now f be a submersion with definite folds. Then f is a quotient of a submersion $h : E \longrightarrow N$, where E is an m-manifold with boundary ∂E and the restriction $h|_{\partial E}$ has rank $n-1$. There are several recent results in the study of submersions with folds [1], [3], [4], [6]. We also prove a result in this direction.

2 Maps of constant rank

Let $f : M \longrightarrow N$ be a submersion. Then $m > n$, f is onto N and M must also be closed. In this case g is a local homeomorphism thus W is a closed n-manifold with the differentiable structure induced by g. It follows that g is an immersion. As $\dim N = \dim W$ and g is onto then g is a covering map. Also q is a submersion. It follows that $M \xrightarrow{q} W$ is a differentiable bundle with connected fiber. Conversely, if $M \xrightarrow{q} W$ is a bundle with connected fiber and $W \xrightarrow{g} N$ is a differentiable covering map then $f = g \circ q$ is a submersion.

We have many non-trivial examples. When N is simply-connected g is a diffeomorphism. Take $N = S^4$. Then there is an S^7-bundle over S^4 that is not orthogonal [8]. If $N = S^1$ then W is diffeomorphic to S^1 and $g : W \longrightarrow S^1$ is an r-fold covering, where r = number of connected components of a fiber of f. In this case the bundles over S^1 are determined by $\pi_0(\mathrm{Diff}\, F)$. We have for example the non-orthogonal S^6-bundle over S^1 obtained from $S^6 \times [0,1]$ under identification of $S^6 \times \{0\}$ with $S^6 \times \{1\}$ by a diffeomorphism not isotopic to an orthogonal one [2].

Now let $f : M \longrightarrow N$ be of constant rank k, $0 < k < m, n$. In this case, $f(M)$ is locally given by submanifolds of dimension k of N. Thus $g : W \longrightarrow N$ is locally given by topological embeddings with submanifolds as images. As W is compact, we may give to W a differentiable structure of dimension k so that W becomes a closed k-manifold and g becomes an immersion. Here also $M \xrightarrow{q} W$ is a differentiable bundle with connected fiber. Thus we have proved the following proposition.

Proposition *Let $f : M \longrightarrow N$ have constant rank k, $0 < k < m,n$. Then W is a closed k-manifold, $M \xrightarrow{q} W$ is a bundle with connected fiber and $g : W \longrightarrow N$ is an immersion, where $f = g \circ q$ is the Stein factorization of f.*

The converse is also true.

3 Submersions with folds

Let $f : M \longrightarrow N$ be a submersion with definite folds. This means that all the singularities of f are definite fold points. At those points f is locally given by $y_i = x_i$, $i = 1, \ldots, n-1$, $y_n = x_n^2 + \cdots + x_m^2$. Let S_0 denote the singular set of f. This is an $(n-1)$-submanifold of M. In this case, $g\,|_{q(M-S_0)}$: $q(M - S_0) \longrightarrow N$ is a local homeomorphism. It follows from this and from the normal forms of f at fold points that W is an n-manifold with boundary ∂W diffeomorphic to S_0 and that $g : W \longrightarrow N$ is an immersion. We also have that $q\,|_{M-S_0}$ is a proper submersion. Thus $q\,|_{M-S_0}$: $M - S_0 \longrightarrow W - \partial W$ is a bundle map. This bundle extends to a bundle $E \xrightarrow{p} W$ where E is a manifold with boudary ∂E. That follows from the behavior of the bundle on a collar neighbourhood of ∂W. We also have a projection map $j : E \longrightarrow M$ such thap $p = q \circ j$. Now it follows from the normal forms of f at points of S_0 that the fiber of those bundles is diffeomorphic to S^{m-n} and that $\partial E \xrightarrow{p|\partial E} \partial W$ extends to a bundle of open discs over ∂W. This implies that $\partial E \xrightarrow{p|\partial E} \partial W$ is an orthogonal bundle.

There are many recent results on the study of manifolds which admit submersions with folds into $\mathbb{R}^n$. We present here a contribution.

Let $f : M^{n+1} \longrightarrow \mathbb{R}^n$ be a submersion with folds such that M is closed and $n \geq 6$. Assume that M is simply connected and that S_0 is the disjoint union of two simply connected components S_1 and S_2. Also assume that a closed tubular neighbourhood N of

S_1 with boundary ∂N is such that $\pi_*(M - S_0,\ \partial N) = 0$.

Theorem *On the above conditions M is a differentiable S^2-bundle over S_1.*

Proof

The boundary ∂W of W is the union of two components $\partial_1 W = q(S_1)$ and $\partial_2 W = q(S_2)$. For $i = 1, 2$ let $c_i : \partial_i W \times [0, \varepsilon) \longrightarrow W$ be a diffeomorphism into W such that $C_i = c_i(\partial_i W \times [0, \varepsilon/2])$ is a closed collar neighbourhood of $\partial_i W$ in W. Then $N_i = q^{-1}(C_i)$ is a tubular neighbourhood of S_i in M. It follows that $\pi_*(M - S_0,\ \partial N_1) = 0$. Assume that $C_1 \cap C_2 = \emptyset$. There is a strong deformation retraction from $M - S_0$ onto the closure E_0 of $M - (N_1 \cup N_2)$. Thus we have $\pi_*(E_0,\ \partial N_1) = 0$. Set W_0 = closure of $W - (C_1 \cup C_2)$. Then $E_0 \xrightarrow{q|E_0} W_0$ is a fibration that maps ∂N_1 onto ∂C_1. It follows that $\pi_*(W_0,\ \partial C_1) = 0$. From the collar neighbourhoods we get a diffeomorphism between W and W_0. This implies $\pi_*(W,\ \partial W_1) = 0$. Now for $i = 1, 2$ $\partial_i W$ is diffeomorphic to S_i. Thus $\partial_1 W$ and $\partial_2 W$ are simply connected. From the fibration $M - S_0 \longrightarrow W - \partial W$ and the inclusion map $M - S_0 \longrightarrow M$ we get $\pi_1(W) = \pi_1(M)$ so that W is also simply connected. As $n \geq 6$ we have from h-cobordism that W is diffeomorphic to $\partial_1 W \times [0, 1]$. We may now assume that $W = \partial_1 W \times [0, 1]$. The composite map $p_1 \circ q : M \longrightarrow \partial_1 W$ is a submersion, where $p_1 : W \longrightarrow \partial_1 W$ is the usual projection on the first factor. This follows from the normal forms of f on S_0. This means that $M \xrightarrow{p_1 \circ q} \partial_1 W$ is a differentiable bundle. Now for any $x \in \partial_1 W$ $Q_x = (p_1 \circ q)^{-1}(x)$ is a differentiable manifold. If $p_2 : \partial_1 W \times [0, 1] \longrightarrow [0, 1]$ denotes the projection on the second factor then $p_2 \circ q\,|_{Q_x}\colon Q_x \longrightarrow [0, 1]$ is a Morse function with two critical points. This implies that Q_x is diffeomorphic to S^2.

The bundle $M \longrightarrow S_1$ in the theorem must be orthogonal, as the fiber is S^2. The bundle $E \xrightarrow{p} \partial_1 W \times [0,1]$ must be equivalent to $\partial_1 E \times [0,1] \xrightarrow{p|_{\partial_1 E} \times identity} \partial_1 W \times [0,1]$, where $\partial_1 E = p^{-1}(\partial_1 W)$. Now $j: E \longrightarrow M$ identifies to a point in S_0 each fiber over a point of ∂W. This fiber is diffeomorphic to S^1. Thus the diffeomorphism of E that corresponds to $(y,t) \longmapsto (y, 1-t)$ on $\partial_1 E \times [0,1]$ extends to a homeomorphism of M that interchanges S_1 and S_2. This implies that S_1 and S_2 are homeomorphic.

Let $n = 6$ and $S_1 = S^5$. Then M is an S^2-bundle over S^5. As $\pi_4(SO(3))$ is cyclic of order 2 this implies that M is either diffeomorphic to $S^5 \times S^2$ or to the non-trivial S^2-bundle over S^5. On the other hand the bundle $E \xrightarrow{p} S^5 \times [0,1]$ is equivalent to the trivial bundle $S^5 \times S^1 \times [0,1] \longrightarrow S^5 \times [0,1]$. In fact its restriction to $S^5 \times \{0\}$, $\partial_1 E \longrightarrow S^5$ say, is trivial for it is an S^1-bundle over S^5. Its implies that M is homeomorphic to $S^5 \times S^2$. Thus M must be diffeomorphic to $S^5 \times S^2$. The knots $S^5 \longrightarrow S_1 \subset M$ and $S^5 \longrightarrow S_2 \subset M$ are then equivalent in the sense that there is a homeomorphism of M that maps S_1 onto S_2. Indeed they are the sections of $S^5 \times S^2 \longrightarrow S^5$ given by south pole and north pole of S^2.

4 Immersions

In all previous cases of maps $f: M \longrightarrow N$ an immersion $g: W \longrightarrow N$ is involved. We may ask several questions about extensions. For example, let S_1 and S_2 be $(n-1)$-manifolds and let immersions $g_1: S_1 \longrightarrow \mathbb{R}^n$ and $g_2: S_2 \longrightarrow \mathbb{R}^n$ as well as disjoint embeddings $i_1: S_1 \longrightarrow M^{n+1}$ an $i_2: S_2 \longrightarrow M^{n+1}$ be given. We may ask if there is a submersion with definite folds $f: M^{n+1} \longrightarrow \mathbb{R}^n$ such that $S_0 = i_1(S_1) \cup i_2(S_2)$ and $g_1 = f \circ i_1$, $g_2 = f \circ i_2$. The answer is negative in general. If $n = 6$, $S_1 = S_2 = S^5$ and $i_1(S_1)$ satisfies the conditions of the theorem in section 3 then M must be $S^5 \times S^2$. On the other hand, there must be an immersion $g: S^5 \times [0,1] \longrightarrow \mathbb{R}^6$ such that $g|_{S^5 \times \{0\}} = g_1$ and $g|_{S^5 \times \{1\}} = g_2$. This is not always true.

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