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ON FAMILIES OF LINDELÖF AND RELATED SUBSPACES OF 2^{ω_1}

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ABSTRACT. We consider the families of all subspaces of size ω_1 of 2^{ω_1} (or of a compact zero dimensional space X of weight ω_1 in general) which are normal, have the Lindelöf property or are closed under limits of convergent ω_1-sequences. Various relations among these families modulo the club filter in $[X]^{\omega_1}$ are shown to be consistently possible. One of the main tools is dealing with a subspace of the form $X \cap M$ for an elementary submodel M of size ω_1. Various results with this flavor are obtained. The other tool used is forcing and in this case various preservation or non-preservation results of topological and combinatorial properties are proved. In particular we prove that there may be no c.c.c. forcing which destroys the Lindelöf property of compact spaces, answering a question of Juhász. Many related questions are formulated.

1. INTRODUCTION

The results of this paper are related to several closely located topics. First, one can draw a connection with the general topic of reflection. A classical theorem in logic, called the lower Löwenheim-Skolem theorem, states that for any infinite mathematical structure and any infinite cardinal κ, smaller than the cardinality of the structure, there is a substructure of cardinality κ which satisfies the same first order formulas as the entire structure. For instance, every uncountable field has a subfield of cardinality ω_1 with the same first order properties as the entire field. Most topological and many other properties can not be expressed by first order formulas, which makes the problem of reflection of these properties to smaller substructures more subtle. What we consider a small or a big substructure is a fundamental factor determining the flavor of the methods of research and the results. For example, here we consider small or big in the sense of infinite cardinality, and thus

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set-theoretic methods are central. Many deep results with this flavor have been obtained. Probably the most famous one is *Shelah's compactness theorem* for certain algebraic structures of singular sizes (see [23]). But there are also well known topological results of this sort, for example: if a compact first countable space is non-metrizable, then it has a nonmetrizable subspace of size ω_1 (Dow [6]); or if a regular space does not have a countable base, then it has a subspace of size ω_1 without a countable base (Tkačenko [28], see also [16]).

In this paper we focus on the following three properties: normality, the Lindelöf property, and the property of *being closed under limits of* ω_1-*sequences which are convergent.* We are focusing on the *basic* compact space 2^{ω_1} but we make some detours into compact spaces in general or with some extra properties.

For a cardinal κ and a topological space X of cardinality bigger than κ we define the following subfamilies of $[X]^\kappa$:

$$[X]^\kappa_L = \{Y \in [X]^\kappa : Y \text{ has the Lindelöf property}\};$$

$$[X]^\kappa_N = \{Y \in [X]^\kappa : Y \text{ is normal}\};$$

$$[X]^\kappa_\circ = \{Y \in [X]^\kappa : Y \text{ is closed under limits of } \omega_1\text{-sequences}\}.$$

The main focus of this paper is to investigate how big the above families can be and how they are interrelated. To measure the size of the above families one may consider the relations between $[X]^{\omega_1}_L$, $[X]^{\omega_1}_N$ and $[X]^{\omega_1}_\circ$ in the usual Boolean algebra of subsets of a compact space X. For example we always (i.e., for regular spaces) have $[X]^{\omega_1}_L \subseteq [X]^{\omega_1}_N, [X]^{\omega_1}_\circ$ and for metrizable compact spaces we have $[X]^{\omega_1}_L = [X]^{\omega_1}_N = [X]^{\omega_1}_\circ = [X]^{\omega_1}$. For $X = 2^{\omega_1}$ the finite Boolean algebra generated in $\wp(\wp(2^{\omega_1}))$ by the families $[2^{\omega_1}]^{\omega_1}_L$, $[2^{\omega_1}]^{\omega_1}_N$ and $[2^{\omega_1}]^{\omega_1}_\circ$ is completely determined in ZFC as shown in the following fact.

Fact 1.1. *The following statements hold:*

(a) $[2^{\omega_1}]^{\omega_1}_N \setminus [2^{\omega_1}]^{\omega_1}_L \neq \emptyset$;
(b) $[2^{\omega_1}]^{\omega_1}_N \setminus [2^{\omega_1}]^{\omega_1}_\circ \neq \emptyset$;
(c) $[2^{\omega_1}]^{\omega_1}_\circ \setminus [2^{\omega_1}]^{\omega_1}_L \neq \emptyset$;
(d) $[2^{\omega_1}]^{\omega_1}_\circ \setminus [2^{\omega_1}]^{\omega_1}_N \neq \emptyset$;
(e) $\emptyset \neq [2^{\omega_1}]^{\omega_1}_L$;
(f) $[2^{\omega_1}]^{\omega_1}_\circ \cup [2^{\omega_1}]^{\omega_1}_N \neq [2^{\omega_1}]^{\omega_1}$;
(g) $([2^{\omega_1}]^{\omega_1}_N \cap [2^{\omega_1}]^{\omega_1}_\circ) \setminus [2^{\omega_1}]^{\omega_1}_L \neq \emptyset$.

Proof. It is clear that b) implies a) and d) implies c). To see b), we define $f_\alpha \in 2^{\omega_1}$, for $\alpha < \omega_1$ by $f_\alpha(\beta) = 0$ if $\beta \neq \alpha$ and $f_\alpha(\alpha) = 1$. Then $Y = \{f_\alpha : \alpha < \omega_1\}$ witnesses that $[2^{\omega_1}]^{\omega_1}_N \setminus [2^{\omega_1}]^{\omega_1}_\circ \neq \emptyset$.
We now prove (d). The proof is a version of the proof of Stone's theorem that ω^{ω_1} is not normal. In this version we use an Aronszajn tree. It is well known that if $\{e_\alpha : \alpha < \omega_1\}$ are such that $e_\alpha : \alpha \longrightarrow \omega$ is a one-to-one mapping, $e_\alpha \restriction \beta =^* e_\beta$ for $\beta < \alpha$, then $T' = \{e_\alpha \restriction \beta : \beta \leq \alpha\}$ is an Aronszajn tree as a subtree of $\omega^{<\omega_1}$ ([20], theorem 5.9). Construct $(f_\alpha)_{\alpha<\omega_1}$ similarly, by induction, such that $f_\alpha : \omega \cdot \alpha \longrightarrow 2$, and for every $\beta < \alpha$ there is $e_\alpha(\beta)$ such that $f_\alpha(\omega \cdot \beta + n) = 1$ if $n < e_\alpha(\beta)$ and $f_\alpha(\omega \cdot \beta + n) = 0$ if $n \geq e_\alpha(\beta)$, for $n \in \omega$ (here, we are really embedding ω^{ω_1} into 2^{ω_1}). By the properties of the family $\{e_\alpha : \alpha < \omega_1\}$ mentioned above and by the construction of the family $\{f_\alpha : \alpha < \omega_1\}$, we have that $T = \{f_\alpha \restriction \beta : \beta \leq \omega \cdot \alpha\}$ is an Aronszajn tree as a subtree of $2^{<\omega_1}$. It is also clear that T^* is an Aronszajn tree, where T^* denotes the set of all finite modifications of elements of T. Now we define

$$X_T = \{(t^\frown i) \in 2^{\omega_1} : t \in T \text{ and } i \in \{0,1\}\},$$

where $t^\frown i$ for $i = 0,1$ are given by $t^\frown 0(\alpha) = t(\alpha)$ if $\alpha \in dom(t)$ and $t^\frown 0(\alpha) = 0$ otherwise; $t^\frown 1(\alpha) = t(\alpha)$ if $\alpha \in dom(t)$ and $t^\frown 1(\alpha) = 1$ if α is a limit ordinal and $\alpha \notin dom(t)$ and $t^\frown 1(\alpha) = 0$ if α is a successor ordinal and $\alpha \notin dom(t)$. We consider $X_T^* \subseteq 2^{\omega_1}$ the set of all finite modifications of the functions from $X_T \subseteq 2^{\omega_1}$. We will first show that $X_T^* \in [2^{\omega_1}]^{\omega_1}_\circ$ by showing that there are no (non-trivial) converging ω_1-sequences with terms in X_T^*. Let $\{x_\xi : \xi < \omega_1\}$ be such that $x_\xi \in X_T^*$ and $x_\xi \neq x_{\xi'}$ if $\xi \neq \xi'$. Suppose $x_\xi = (t_\xi^\frown i)^*$, where $i \in \{0,1\}$ and $*$ denote some finite modification. Since the levels of the tree are countable, we can assume that $t_\xi \in Lev_{\alpha_\xi}(T)$ and $\alpha_\xi < \alpha_{\xi'}$ for $\xi < \xi'$. If x is an ω_1-limit of the ω_1-sequence $\{x_\xi : \xi < \omega_1\}$, we have that for every $\alpha < \omega_1$, there is $\xi < \omega_1$ such that $\alpha_\xi > \alpha$ and $x \restriction \alpha = x_\xi \restriction \alpha$. But $x_\xi \restriction \alpha$ is a finite modification of $t_\xi \restriction \alpha$, and then x is a branch of T^*, a contradiction.
We now have to show that $X_T^* \notin [2^{\omega_1}]^{\omega_1}_N$. We consider the sets

$$F_i = \{(t^\frown i) \in 2^{\omega_1} : t \in T\},$$

for $i = 0,1$ and apply lemma 2.3 from section 2. We considered X_T^* instead of X_T because X_T^* is a dense subspace of 2^{ω_1}. Therefore, by lemma 2.3, it is enough to show that F_0 and F_1 have disjoint closures in X_T^*. For this we note that for every $x \in X_T^*$, there is an open set V such that $x \in V$ and there is $i \in \{0,1\}$ satisfying $V \cap F_i = \emptyset$; indeed, let $x \in X_T^*$, then $x = (t^\frown i)^*$ for some $t \in T$ and $i \in \{0,1\}$. We can pick $\beta < \omega_1$ such that $V_0 = [(\omega \cdot \beta, 0), (\omega \cdot (\beta + 1), 0)]$ or

$V_1 = [(\omega \cdot \beta, 1), (\omega \cdot \beta + 1, 0), (\omega \cdot (\beta + 1), 1)(\omega \cdot (\beta + 1) + 1, 0)]$ is an open neighborhood of x. Since each e_ξ is a one-to-one mapping, we have that $V_i \cap F_{1-i} = \emptyset$, and we are done.
The clause e) is witnessed by an ω_1-convergent sequence with a limit, f) can be witnessed by a disjoint union of a subspace witnessing d) with a subspace witnessing b). Finally for g), we find a discrete subspace which has no converging ω_1 subsequence. Consider e_α's and f_α's as before. Take $m > 0$ such that the set $A = \{\beta < \omega_1 : \ a_{\beta+1}(\beta) = m\}$ is uncountable, and then consider $x_\beta = f_{\beta+1} \frown 0$. Then $\{x_\beta : \beta \in A\}$ is a discrete space and the proof of the fact that it has no convergent ω_1-sequence is similar to the proof of the same property for X_T^* in d). □

The best known open problem related here is whether $[X]_L^{\omega_1}$ is nonempty for all compact uncountable X (see for example [12]).

In this paper we will consider the above families $[X]_L^{\omega_1}$, $[X]_N^{\omega_1}$, $[X]_\circ^{\omega_1}$ in the Boolean algebra of subsets of X modulo the nonstationary ideal (or the club filter) in $[X]^{\omega_1}$ (this was first investigated by Jech [13] and Kueker [19]; see [3] for some introduction). Thus $\equiv$ will mean *club equivalence*, i.e., A and B are club equivalent if and only if there is a club $C \subseteq [X]^{\omega_1}$ such that $C \cap A = C \cap B$. Also, 0, 1 and $\leq$ will denote respectively the zero, the unit and the Boolean order of the algebra. This approach turned out to be natural and fruitful in several contexts. For instance, Shelah bases the main results of [26] on a lemma which states that $[X]_L^{\omega_1}$ is co-stationary if X is Lindelöf with points G_δ (also [27] uses it in a fundamental way); or Dow uses the property of being or not being a stationary subset of the set $\{Y \in [X]^\omega : cl(Y) = Y\}$ to obtain reflection or non-reflection of metrizability ([7]). Our results in this paper show that

a) $0 \leq [2^{\omega_1}]_L^{\omega_1} \equiv [2^{\omega_1}]_N^{\omega_1} \leq [2^{\omega_1}]_\circ^{\omega_1} < 1$ (corollary 2.9).

b) It is consistent that $0 \equiv [2^{\omega_1}]_L^{\omega_1} \equiv [2^{\omega_1}]_N^{\omega_1} \equiv [2^{\omega_1}]_\circ^{\omega_1} < 1$ (corollary 3.4).

c) Assuming the existence of an inaccessible cardinal it is consistent that $0 < [2^{\omega_1}]_L^{\omega_1} \equiv [2^{\omega_1}]_N^{\omega_1} \equiv [2^{\omega_1}]_\circ^{\omega_1} < 1$ (theorem 4.2).

d) Assuming the existence of a Mahlo cardinal it is consistent that $0 \leq [2^{\omega_1}]_L^{\omega_1} \equiv [2^{\omega_1}]_N^{\omega_1} < [2^{\omega_1}]_\circ^{\omega_1} < 1$ (theorem 5.14).

We do not know the answer to the following:

Question 1. *Does the statement* $[2^{\omega_1}]^{\omega_1}_L < [2^{\omega_1}]^{\omega_1}_\circ$ *decide whether* $0 < [2^{\omega_1}]^{\omega_1}_L$ *or* $0 \equiv [2^{\omega_1}]^{\omega_1}_L$ *?*

Another aspect of our results is more logical. It was Shelah ([24]) who suggested considering a generalized second order quantifier *"For club many countable subsets X..."*; this logic was later investigated in [2]. However when talking about a topological spaces we do not have at hand any structure in the sense of universal algebra. Nevertheless one can note (Lemma 2.1) a link between a general quantifier in the sense *" for club many subsets"* and the general quantifier over models meaning *"For any model* $M \prec H(\theta)$ *for* θ *big enough such that all relevant objects are in M..."*. In this sense the existential quantifier (i.e., the existence of a model) corresponds to the existence of stationary many subsets. Thus to prove our results about relations between the families we are considering, one needs to prove topological properties of $2^{\omega_1} \cap M$ for an elementary submodel M.
Actually the elementary submodels were the source of the research leading to this paper. In [18] the topological operation *taking a submodel* of a topological space was introduced. Roughly, given a topological space (X, τ) and an elementary submodel $M \prec H(\theta)$ for a sufficiently big θ we can consider the space $(X \cap M, \tau \cap M)$ as a *submodel of* X. In our case, when $\omega_1 \subseteq M$, this space is simply the set $M \cap 2^{\omega_1}$ with the subspace topology (since 2^{ω_1} has a base of size ω_1). So using lemma 2.1. the results a) - d) above are really about the existence or non-existence of topological submodels of the space 2^{ω_1} with given topological properties. For instance, we will prove: *it is independent of ZFC whether* $2^{\omega_1} \cap M$ *is not normal for every elementary submodel* M *such that* $\omega_1 \subseteq M$ *and* $|M| = \omega_1$ (corollary 3.3. and theorem 4.2).

The third aspect of our work is related to forcing, namely to the question of preservation of various properties by forcing. We consider three properties:
a) X is the set of all ω_1-branches of a tree T of height ω_1. (Note that this is a property of a pair (X, T).)
b) A topological space X has the Lindelöf property.
c) Forcing Q is c.c.c.

We say that P *does not add* ω_1*-branches* if it preserves a), i.e., if a) holds in the extension V^P for every pair (X, T) (in the ground model) satisfying a) in the ground model. We say that P *preserves the Lindelöf property* (for compact spaces) if b) holds in V^P for every Lindelöf (compact) space in the ground model. Finally, we say that P *preserves*

c.c.c. if any c.c.c. forcing in the ground model stays c.c.c. in the extension V^P.

Preservation of properties a) and c) by c.c.c. forcing have been extensively investigated. For example, we have the following combinatorial characterizations: a c.c.c. forcing P does not add ω_1-branches if and only if the completion of P does not contain a Souslin tree with dense levels (see [5]); and P preserves c.c.c if and only if $P \times Q$ is c.c.c. for each c.c.c. forcing Q. Also iterations with finite supports of forcings which preserve property a) or property d), preserve property a) or d) respectively (see [20], [5]). The situation with the preservation of the property b) is different, e.g. at the moment we do not have combinatorial characterizations. For the sake of our consistency proofs, we prove results from which we can conclude the following:

1) If a c.c.c. forcing P does not preserve the Lindelöf property of compact spaces, then there is an integer n such that P^n is not c.c.c. Thus it is consistent that all c.c.c. forcings preserve the Lindelöf property of compact spaces (theorem 4.4).

2) Under CH there is a c.c.c. forcing P which does not preserve the Lindelöf property of compact spaces and which does not add ω_1-branches, and Cohen forcing preserves that it does not add ω_1-branches (corollary 5.10).

3) There is a proper forcing which does not preserve the Lindelöf property of compact spaces and which does not add ω_1-branches (corollary 5.11).

On the other hand it is easy to see that

4) Every forcing which preserves ω_1 and adds ω_1-branches destroys the Lindelöf property of a compact space (fact 5.12).

In the light of these results it is natural to ask the following questions which we were unable to answer:

Question 2. *Is there $n \in \omega$ such that if P does not preserve the Lindelöf property of compact spaces, then P^n is not c.c.c.? Is this n=2?*

Question 3. *Characterize, in a combinatorial language, forcings which preserve the Lindelöf property of compact spaces.*

The notation used in this paper is standard and often follows e.g., [20] or [14].

Let us fix some more specific notation related to the partial order of the Cohen forcing $C(\alpha)$ or to the product topology on 2^α. For an ordinal

γ, let:

$$C(\gamma) = \{p : p \subset f\ , f \in 2^\gamma \text{ and } |p| < \omega\}.$$

The elements of the forcing $C(\gamma)$ can be considered as the codes for basic clopen subsets in the product topology on 2^γ. The coding is defined in the natural way: for $d \in C(\beta)$ and $D \subset C(\beta)$, let

$$[d]_\gamma = \{f \in 2^\gamma : d \subset f\}$$

and

$$[D]_\gamma = \bigcup\{[d]_\gamma : d \in D\}.$$

When $\gamma = \omega_1$ we will simply write $[d]$ or $[D]$. Note that these sets may be empty, for instance $[d]_\gamma = \emptyset$ when $d \in C(\beta) - C(\gamma)$. Considering this unnatural case of $\gamma \leq \beta$ will simplify some statements. It is clear that predense sets in $C(\gamma)$ code dense open sets in 2^γ.

The other group of symbols is related to the internal structure of the set-theoretic universes. By the axiom of regularity, the relation $\in$ is well-founded and so for each set x we can define its *transitive closure* $tc(x)$, i.e. the set consisting of all elements of x and their elements and their elements, etc. The cardinality of $tc(x)$ is called the *hereditary cardinality* of x. It turns out that the class $H(\theta)$ of all sets of hereditary cardinality less than θ is a set and a set which can be used to represent any given phenomenon happening in the set-theoretic universe provided θ is large enough (see for example [20] theorem IV 7.5.).

Finally, let us mention that intervals like $[\alpha, \beta)$ or (α, β) are considered with respect to the usual order of the ordinals.

2. Models and subfamilies of $[2^{\omega_1}]^{\omega_1}$.

The link between models and families of $[X]^{\omega_1}$ modulo the club filter is explained by the following lemma:

Lemma 2.1. *Suppose ϕ is a formula of ZFC. The set*

$$\{Y \in [2^{\omega_1}]^{\omega_1} : \phi(Y)\}$$

includes a club set in $[2^{\omega_1}]^{\omega_1}$ if and only if there is θ large enough and $B \in [H(\theta)]^{\omega_1}$ such that whenever $M \prec H(\theta)$, $B \subseteq M$, $|M| = \omega_1$, then $\phi(M \cap 2^{\omega_1})$. Equivalently, the set

$$\{Y \in [2^{\omega_1}]^{\omega_1} : \phi(Y)\}$$

is stationary in $[2^{\omega_1}]^{\omega_1}$ if and only if for any θ and $B \in [H(\theta)]^{\omega_1}$ there exists $M \prec H(\theta)$, $B \subseteq M$, $|M| = \omega_1$, such that $\phi(M \cap 2^{\omega_1})$.

Proof. It is clear that the two statements are equivalent. Also the reverse implication of the first equivalence follows from the fact that the collection of elementary submodels of $H(\theta)$ (including B) forms a club in $[H(\theta)]^{\omega_1}$ and the intersections of this club with 2^{ω_1} forms a club in $[2^{\omega_1}]^{\omega_1}$ (see [3]).
For the forward implication of the first equivalence, let $C \subseteq [2^{\omega_1}]^{\omega_1}$ be a club as in the assumption and let θ be such that $2^{\omega_1}, C \in H(\theta)$. We use the idea of the proof of Kueker result (see [3]), so let $g : [2^{\omega_1}]^{<\omega} \to C$ be a function such that $g(x) \supseteq x \cup \bigcup\{g(y) : y \subset x\}$ and $g(\emptyset)$ is anything. Let $M \prec H(\theta)$ be such that $g, 2^{\omega_1}, C \in M$, $\omega_1 \subseteq M$ and $|M| = \omega_1$. Note that for any countable $A \subseteq 2^{\omega_1}$ the family $C_A = \{g(x) : x \in [A]^{<\omega}\}$ is countable, directed and included in C, so there is a cofinal chain in this family and consequently $\bigcup C_A \in C$. Let $(A_\alpha)_{\alpha<\omega_1}$ be an increasing chain of countable subsets of $M \cap 2^{\omega_1}$ such that its union is equal to $M \cap 2^{\omega_1}$. Then $(\bigcup C_{A_\alpha})_{\alpha<\omega_1}$ forms an increasing chain in C and so its union C_M is in C. To note that $C_M = M \cap 2^{\omega_1}$ it is enough to see that $x \subseteq g(x) \subseteq M \cap 2^{\omega_1}$ for each $x \in [M \cap 2^{\omega_1}]^{<\omega}$ which follows from the fact that $g \in M$ and $\omega_1 \subseteq M$. Thus $\phi(M \cap 2^{\omega_1})$ holds which completes the proof of the lemma. □

We will now investigate normality and the Lindelöf property of certain *special* subspaces of 2^{ω_1}. We will actually show that these two properties are equivalent for such subspaces and we will give a characterization for it. ¿¿¿From now on M will always be a model of sufficiently large fragments of ZFC (or ZFC without the power set axiom), such that $\omega_1 \subseteq M$ and with all relevant objects in M. This in particular implies that $M \cap 2^{\omega_1}$ is dense in 2^{ω_1}. The typical cases we are interested on are when M is an elementary submodel of $H(\theta)$ for some θ and when M is the ground model in a generic extension. For a set $K \subseteq 2^{\omega_1}$ and an ordinal $\alpha < \omega_1$ we let $K_\alpha = \{f \restriction \alpha : f \in K\}$. We can obtain the following characterization (discussions of the first author with S. Watson helped getting it):

Theorem 2.2. *$2^{\omega_1} \cap M$ is normal if and only if for every compact space $K \subseteq 2^{\omega_1}$ satisfying $K \cap M = \emptyset$ there is $\alpha < \omega_1$ such that $K_\alpha \cap M = \emptyset$.*

We first show

Lemma 2.3. *Suppose that $Y \subseteq 2^{\omega_1}$ is a dense subspace. Also suppose that there are F_0 and F_1 closed disjoint subsets of Y such that for every $\alpha < \omega_1$, $(F_0)_\alpha \cap (F_1)_\alpha \neq \emptyset$. Then F_0 and F_1 can not be separated by open sets in Y and therefore Y is not normal.*

Proof. We have to show that F_0 and F_1 can not be separated in Y. Note that, since Y is dense in 2^{ω_1}, it is enough to show that we can not separate F_0 and F_1 by open sets in 2^{ω_1}. Suppose that there are open disjoint subsets U and V of 2^{ω_1} such that $F_0 \subseteq U$ and $F_1 \subseteq V$. By our assumptions on F_0 and F_1, for each $\alpha < \omega_1$, we can fix $g_\alpha \in (F_0)_\alpha \cap (F_1)_\alpha$. Let $h_\alpha \in F_0$ be such that $h_\alpha \restriction \alpha = g_\alpha$. For each $\alpha < \omega_1$, fix a basic open set $[p_\alpha] = \{f \in 2^{\omega_1} : p_\alpha \subseteq f\}$ such that $h_\alpha \in [p_\alpha] \subseteq U$. Note that $[p_\alpha] \subseteq U$ implies $[p_\alpha] \cap F_1 = \emptyset$.

Claim . *We need uncountably many $[p_\alpha]$'s to cover $\{h_\alpha : \alpha < \omega_1\}$.*

Proof. Fix $\{[p_{\alpha_n}] : n \in \omega\}$ and take $\delta > sup \bigcup \{dom(p_{\alpha_n}) : n \in \omega\}$. We will show that $h_\delta \notin [p_{\alpha_n}]$, for every $n \in \omega$. Suppose $h_\delta \in [p_{\alpha_n}]$. Then $p_{\alpha_n} = h_\delta \restriction dom(p_{\alpha_n})$. Since $\delta > max\, dom(p_{\alpha_n})$, we have that $p_{\alpha_n} = g_\delta \restriction dom(p_{\alpha_n})$. But now take $f_\delta \in F_1$ such that $f_\delta \restriction \delta = g_\delta$. Then $f_\delta \in [p_{\alpha_n}] \cap F_1$, a contradiction. □

We can then fix $\{p_\alpha : \alpha < \omega_1\}$ such that $\{[p_\alpha] : \alpha < \omega_1\}$ covers $\{h_\alpha : \alpha < \omega_1\}$. Without loss of generality, we can suppose that $\{d_\alpha : \alpha < \omega_1\}$, where $d_\alpha = dom(p_\alpha)$ for each $\alpha < \omega_1$, forms a Δ-system with root d. Since d is finite, $2^{|d|}$ is finite. So we can also suppose that $p_\alpha \restriction d = p$, for every $\alpha < \omega_1$, for some fixed p.
Pick $\delta > max\, d$. Note that $g_\delta \restriction d = p_\delta \restriction d = p$. If $f_\delta \in F_1$ is such that $f_\delta \restriction \delta = g_\delta$, we have that there is a basic open set $[q] \subseteq V$ such that $f_\delta \in [q]$. But then $[q] \cap [p_\alpha] = \emptyset$, for every $\alpha < \omega_1$, which implies that $dom(q) \cap dom(p_\alpha) \neq \emptyset$, for every $\alpha < \omega_1$.
Now, if $\beta > \delta$, then $p_\beta \restriction d = p_\delta \restriction d = g_\delta \restriction d = q \restriction d$. We then must have that $(dom(q) \cap dom(p_\beta)) \setminus d \neq \emptyset$ for every $\beta > \delta$, a contradiction. □

Proof of the Theorem. Here we will only prove one direction. For the other direction, we can actually get a stronger result, which will be given in the next theorem. Suppose there is a compact K such that $K \cap M = \emptyset$ but $K_\alpha \cap M \neq \emptyset$, for every $\alpha < \omega_1$. By the previous lemma we just have to define the two closed sets satisfying its hypothesis. For $i = 0, 1$ define F_i to be the set

$$\{f \in 2^{\omega_1} \cap M : \exists \alpha < \omega_1 \exists\, g \in K_\alpha \cap M\ f \restriction \alpha = g \restriction \alpha \,\&\, \forall \beta > \alpha\ f(\beta) = i\}.$$

We first show that $\overline{F_0} \cap \overline{F_1} = \emptyset$, with the closures taken in $2^{\omega_1} \cap M$. For that we fix $f \in M$. Then, by our assumption on K, $f \notin K$. Therefore there is a basic open set $[p]$ such that $f \in [p]$ and $[p] \cap K = \emptyset$, where p is a finite function from ω_1 to 2. Pick $\alpha > max\, dom(p)$ and define $q = p \cup \{\alpha, f(\alpha)\}$.

Claim . *$f(\alpha) = 1 - i$ implies that $[q] \cap F_i = \emptyset$.*

Proof. Fix $h \in F_i$. By the definition of F_i, there is $\beta < \omega_1$ and $g \in K_\beta \cap M$ such that $h \restriction \beta = g \restriction \beta$ and $h(\delta) = i$, for every $\delta > \beta$. We have two possibilities:
Case 1: $\beta \geq \alpha$. Then, since $g \in K_\beta \cap M$, $[p] \cap K = \emptyset$ and $\beta > max\, dom(p)$, we have that $h \notin [p]$. Therefore $h \notin [q]$.
Case 2: $\beta < \alpha$. Then, since $h(\delta) = i$ for every $\delta > \beta$ and $\beta < \alpha$, we have that $h(\alpha) = i$ and therefore $h \notin [q]$. □

We thus have that that for every $f \in M$ there is $\alpha < \omega_1$ such that $f(\alpha) = 1 - i$ implies $f \notin \overline{F_i}$, which implies that $\overline{F_0} \cap \overline{F_1} = \emptyset$.

□

We now look at the Lindelöf property. In particular, the next theorem finishes the proof of the previous theorem.

Theorem 2.4. *Let Y be any subspace of 2^{ω_1}. Then Y is Lindelöf if and only if for every compact $K \subset 2^{\omega_1}$ with $K \cap Y = \emptyset$ there is $\alpha < \omega_1$ such that $K_\alpha \cap Y_\alpha = \emptyset$.*

Proof. For one direction, we want to show that Y is Lindelöf. Let $\mathcal{C}$ be an open cover of Y. If $\bigcup \mathcal{C} = 2^{\omega_1}$, it has a finite subcover (since 2^{ω_1} is compact) and we are done. If not, let $K = 2^{\omega_1} \setminus \bigcup \mathcal{C}$. Then K is a compact set disjoint from Y, so fix $\alpha < \omega_1$ such that $K_\alpha \cap Y_\alpha = \emptyset$. Since K_α is a closed subset of 2^α and α is countable, we have that K_α is a G_δ set in 2^α. So fix open sets (in 2^α) W_n, for $n \in \omega$, such that $K_\alpha = \bigcap_{n \in \omega} W_n$. Let V_n be the open set of 2^{ω_1} which is the inverse image of W_n by the projection map onto 2^α, i.e., $V_n = proj_\alpha^{-1}(W_n)$. Let $F = \bigcup_{n \in \omega}(2^{\omega_1} \setminus V_n)$. Note that F is an F_σ in 2^{ω_1}, and therefore it has the Lindelöf property. Also, since $K_\alpha \cap Y_\alpha = \emptyset$, we have that $Y \subseteq F$. Finally, note that by the definition of F and the V_n's, we have that $F \subseteq \bigcup \mathcal{C}$. These three facts imply that there is $\mathcal{C}' \subseteq \mathcal{C}$ countable such that $Y \subseteq \bigcup \mathcal{C}'$, and therefore Y is Lindelöf.

For the other implication, suppose that a compact set $K \subset 2^{\omega_1}$ is such that $K \cap Y = \emptyset$ but $K_\alpha \cap Y_\alpha \neq \emptyset$, for every $\alpha < \omega_1$. Then for each $\alpha < \omega_1$, we can fix $f_\alpha \in 2^{\omega_1}$ such that $f_\alpha \restriction \alpha \in K_\alpha \cap Y_\alpha$. Clearly, these functions can be assumed to be distinct. Let g be any complete accumulation point of $\{f_\alpha : \alpha < \omega_1\}$.
We will show that g is a complete accumulation point of K. Then since K is closed we will have $g \in K$. But this implies that $g \notin Y$ and therefore $\{f_\alpha : \alpha < \omega_1\}$ do not have any complete accumulation point in Y, which completes the proof.
To show that g is a complete accumulation point of K, fix $p \subset g$ finite and let $[p]$ be a basic open set. Since g is a complete accumulation point of $\{f_\alpha : \alpha < \omega_1\}$, we have that $|[p] \cap \{f_\alpha : \alpha < \omega_1\}| = \omega_1$. Pick

$\beta > max\, dom(p)$. Then $f_\beta \in [p]$ implies that $f_\beta \restriction \beta \supseteq g \restriction dom(p)$. But there is $g_\beta \in K$ such that $g_\beta \restriction \beta = f_\beta \restriction \beta$. Thus, since $\beta > max\, dom(p)$, we have $g_\beta \in [p]$. Since this is true for every β such that $f_\beta \in [p]$, we can conclude that $|[p] \cap K| = \omega_1$. □

Corollary 2.5. *For M as before, we have that $2^{\omega_1} \cap M$ is Lindelöf if and only if it is normal. In particular:*
(a) if M is an elementary submodel of $H(\theta)$, $\theta \geq (2^{\omega_1})^+$ such that $\omega_1 \subset M$, then the space $2^{\omega_1} \cap M$ is Lindelöf if and only if it is normal;
(b) if G is a P-generic, for a partial order P which preserves ω_1, then in V^P the space $2^{\omega_1} \cap V$ is Lindelöf if and only if it is normal.

Corollary 2.6. *$[2^{\omega_1}]^{\omega_1}_L$ and $[2^{\omega_1}]^{\omega_1}_N$ are club equivalent, i.e., $[2^{\omega_1}]^{\omega_1}_L \equiv [2^{\omega_1}]^{\omega_1}_N$.*

The proof of next theorem is in [17], or in a more general form, in [18]:

Proposition 2.7. *For a cardinal $\theta \geq (2^{\omega_1})^+$, there is a model $M \prec H(\theta)$ such that $|M| = \omega_1$ and $2^{\omega_1} \cap M$ is not closed under ω_1-limits.*

Corollary 2.8. *$[2^{\omega_1}]^{\omega_1}_\circ$ and so $[2^{\omega_1}]^{\omega_1}_L, [2^{\omega_1}]^{\omega_1}_N$ are co-stationary .*

Corollary 2.9. $0 \leq [2^{\omega_1}]^{\omega_1}_L \equiv [2^{\omega_1}]^{\omega_1}_N \leq [2^{\omega_1}]^{\omega_1}_\circ < 1$.

3. Subspaces of 2^{ω_1} and Kurepa trees.

Note that in Theorem 2.2, to have that $2^{\omega_1} \cap M$ is not normal, we could have, in particular, that the compact set K is a single point. Actually, a similar proof (but simpler) yields the following result:

Theorem 3.1. *Let M be as in the previous section. If $\kappa \subseteq M$ and there is $f \in 2^\kappa$ such that for some ordinal α of uncountable cofinality we have $f \restriction \beta \in M$ for every $\beta < \alpha$ and $f \restriction \alpha \notin M$, then $2^\kappa \cap M$ is not normal.*

It is natural to ask when there is such a function. For the case $\kappa = \omega_1$ we have the following characterization:

Theorem 3.2. *The statement "for every elementary submodel M of $H(\theta)$, $\theta \geq (2^{\omega_1})^+$ with $\omega_1 \subseteq M$, $|M| = \omega_1$, there is $f \notin M$ such that $f \restriction \alpha \in M$ for every $\alpha < \omega_1$" is true if and only if there is a weak Kurepa tree, i.e., a tree of height ω_1 with levels of cardinality $\leq \omega_1$ and with more then ω_1 uncountable branches.*

Proof. First, suppose there are no weak Kurepa trees. Then, in particular, there are no weak Kurepa trees $T \subseteq 2^{\omega_1}$. We will show that there is an elementary submodel M of $H(\theta)$, with $\omega_1 \subseteq M$, $|M| = \omega_1$ such that for every $f \notin M$ there is $\alpha < \omega_1$ such that $f \restriction \alpha \notin M$. We will

identify a cofinal branch b through the tree $2^{<\omega_1}$ with the corresponding function $f_b \in 2^{\omega_1}$. We can construct a chain of elementary submodels $\{M_n : n \in \omega\}$ such that for every $n \in \omega$, M_n is an elementary submodel of M_{n+1}, $|M_n| = \omega_1$, $\omega_1 \subseteq M_n$ and

$$\{f \in 2^{\omega_1} : f \restriction \alpha \in M \text{ for every } \alpha \in \omega_1\} \subseteq M_{n+1}.$$

Note that the construction is possible since there are no weak Kurepa trees, and therefore $2^{<\omega_1} \cap M$ has at most ω_1 cofinal branches. We claim that $M = \bigcup_{n\in\omega} M_n$ is as we want. Suppose $f \in 2^{\omega_1} \setminus M$. Then for each $n \in \omega$ we can take $\alpha_n \in \omega_1$ minimum such that $f \restriction \alpha_n \notin M_n$ (such α_n exists because otherwise we would have $f \in M_{n+1} \subseteq M$). Take $\alpha = sup\{\alpha_n : n \in \omega\}$. Since $f \restriction \alpha \in M_n$ implies $f \restriction \alpha_n \in M_n$, we have that $f \restriction \alpha \notin M_n$ for every $n \in \omega$. Thus $f \restriction \alpha \notin M$.

Suppose now that there is a weak Kurepa tree. Then there is one which is a subtree of $2^{<\omega_1}$. Let $M \prec H(\theta)$ be such that $\omega_1 \subseteq M$ and $|M| = \omega_1$. By elementarity, there is $T \in M$ such that $T \subseteq 2^{<\omega_1}$ and T is a weak Kurepa tree. Note that $|T| = \omega_1$ and therefore $T \subseteq M$ (since $\omega_1 \subseteq M$). We then have that $2^{<\omega_1} \cap M$ contains a weak Kurepa tree, so it has $> \omega_1$ branches. Since $|M| = \omega_1$, this implies that there must be a branch b of $2^{<\omega_1} \cap M$, which is not in M. Then the corresponding function $f_b \notin M$, but $f_b \restriction \alpha \in M$, for every $\alpha \in \omega_1$. □

Corollary 3.3. *Suppose that wKH holds. If M is an elementary submodel of $H(\theta)$, $\theta \geq (2^{\omega_1})^+$ of size ω_1 and $\omega_1 \subseteq M$, then $2^{\omega_1} \cap M$ is not closed under limits of ω_1-sequences and therefore it is non-normal and non-Lindelöf.*

Corollary 3.4. *wKH implies that $0 \equiv [2^{\omega_1}]^{\omega_1}_L \equiv [2^{\omega_1}]^{\omega_1}_N \equiv [2^{\omega_1}]^{\omega_1}_\circ < 1$.*

Corollary 3.5. *$\neg wKH$ implies that $[2^{\omega_1}]^{\omega_1}_\circ$ is stationary, i.e., that $0 < [2^{\omega_1}]^{\omega_1}_\circ < 1$.*

4. Forcing results by preservation of the Lindelöf property.

For the first result of this section we will use Mitchell model and a lemma due to A. Dow. The Mitchell forcing Mi_κ (see [22] for details) is an iteration with mixed supports of forcings that add Cohen reals and forcings which collapse 2^ω to ω_1. The iteration $(Mi_\alpha, \dot{Mi}^\alpha)_{\alpha<\kappa}$ is of length κ, where κ is an inaccessible cardinal. The Cohen part is taken with finite supports and the collapse part is taken with countable supports. It is true (see [22] or [8]) that in V^{Mi_α} for $\alpha < \kappa$ the remaining part of the forcing looks like the entire Mitchell forcing. For

the convenience of the reader we list after [8] the properties of Mitchell forcing:

(i) if G is Mi_κ-generic, for κ strongly inaccessible, then $V[G]$ satisfies $\omega_1 = \omega_1^V$, $2^\omega = \omega_2 = \kappa$.
(ii) if $\lambda < \kappa$ is even, then Mi_κ is a forcing isomorphic to $Mi_\lambda * Mi_{\kappa'}$ where $\lambda + \kappa' = \kappa$;
(iii) there is a Mi_κ-name, $\dot{R}$, of a poset such that $Mi_\kappa * \dot{R}$ is a forcing isomorphic to $C(\mu) \times Q$ for some ω_1-closed poset Q and some uncountable μ;
(iv) there is a Mi_κ-name of a poset $\dot{R}$ and a $C(\omega_1)$-name of a poset $\dot{Q}$ such that $Mi_\kappa * \dot{R}$ is a forcing isomorphic to $C(\omega_1) * (\dot{Q} \times C(\kappa - \omega_1))$ such that $C(\omega_1) \Vdash \dot{Q}$ *is* ω_1*-closed.*

We will also need (see [22], lemma 3.3.)
(v) if κ is an inaccessible cardinal, then Mi_κ satisfies the κ-c.c.

Alan Dow proved that forcing with Mitchell forcing does not destroy the Lindelöf property ([8]), so $2^{\omega_1} \cap V^{Mi_\alpha}$, which is a compact space in V^{Mi_α}, remains Lindelöf in V^{Mi_κ}. On the other hand for $\alpha < \kappa$ $|V^{Mi_\alpha} \cap 2^{\omega_1}| < \kappa$ by the inaccessibility of κ, so by (i) this set is in $[2^{\omega_1}]^{\omega_1}$ (in V^{Mi_κ}). This is sufficient for describing the relationships that hold in the Mitchell model among the families we are considering. Let κ be an inaccessible cardinal. We define by induction, in V, a sequence $(\dot{X}_\alpha : \alpha < \kappa)$ of Mi_κ-names for subsets of 2^{ω_1} of cardinality ω_1 and a sequence $(\gamma_\alpha : \alpha < \kappa)$ of ordinals in the following way: $Mi_\kappa \Vdash \dot{X}_\alpha = \bigcup_{\beta<\alpha} \dot{X}_\beta$ and $\gamma_\alpha = \sup\{\gamma_\beta : \beta < \alpha\}$ for α limit and if $\alpha = \beta + 1$ for some β we require $Mi_\kappa \Vdash \dot{X}_\alpha = 2^{\omega_1} \cap V^{Mi_{\gamma_\alpha}}$ for γ_α minimal bigger than γ_β such that $Mi_\kappa \Vdash \dot{X}_\beta \subset V^{Mi_{\gamma_\alpha}}$. Note that this γ_α is well defined by (v) as X_β is of cardinality ω_1 in V^{Mi_κ} (by (i)).

Lemma 4.1. *In* V^{Mi_κ}, *if* $cf(\alpha) \leq \omega$ *or* $X_\alpha = 2^{\omega_1} \cap V^{Mi_{\gamma_\alpha}}$, *then* $X_\alpha \in [2^{\omega_1}]^{\omega_1}_L$. *If* $cf(\alpha) > \omega$ *and* $X_\alpha \neq 2^{\omega_1} \cap V^{Mi_{\gamma_\alpha}}$, *then* $X_\alpha \notin [2^{\omega_1}]^{\omega_1}_o$.

Proof. If α is a successor or $X_\alpha = 2^{\omega_1} \cap V^{Mi_{\gamma_\alpha}}$, then $X_\alpha = 2^{\omega_1} \cap V^{Mi_{\gamma_\alpha}}$ and as $2^{\omega_1} \cap V^{Mi_{\gamma_\alpha}}$ is a compact space in $V^{Mi_{\gamma_\alpha}}$, by Dow's result we have that X_α is Lindelöf in V^{Mi_κ}.
If α is of countable cofinality, then X_α is the union of a countable family of Lindelöf subspaces of the form $X_{\beta+1} = 2^{\omega_1} \cap V^{Mi_{\gamma_{\beta+1}}}$ for $\beta < \alpha$, so it is Lindelöf as well.
If $cf(\alpha) > \omega_1$ and $X_\alpha \neq 2^{\omega_1} \cap V^{Mi_{\gamma_\alpha}}$, then pick $x \in 2^{\omega_1} \cap (V^{Mi_{\gamma_\alpha}} \setminus \bigcup_{\beta<\alpha} V^{Mi_{\gamma_\beta}})$. As conditions of Mi_κ have at most countable supports, each condition of the generic set deciding the $x(\xi)$ for $x \in 2^{\omega_1} \cap V^{Mi_{\gamma_\alpha}}$

and $\xi \in \omega_1$ is in Mi_δ for some $\delta < \gamma_\alpha$. As $cf(\alpha) > \omega$, it stays such in the generic extension (as ω_1 is preserved), so any countable many of these conditions are in some Mi_δ for $\delta < \gamma_\alpha$ and so we have that $x \restriction \xi^\frown 0 \in X_\alpha$ for every countable ξ. This implies that $X_\alpha \notin [2^{\omega_1}]^{\omega_1}_\circ$ as required.

□

Theorem 4.2. *Assuming the existence of an inaccessible cardinal it is consistent that*

$$0 < [2^{\omega_1}]^{\omega_1}_L \equiv [2^{\omega_1}]^{\omega_1}_N \equiv [2^{\omega_1}]^{\omega_1}_\circ < 1.$$

Proof. Let us work in the Mitchell model and let us use the previous lemma. Clearly the X_α's form a club subset of $[2^{\omega_1}]^{\omega_1}$. The previous lemma and corollary 2.9, imply all the required relations.

□

Corollary 4.3. *Assuming the existence of an inaccessible cardinal it is consistent that there is an elementary submodel M of size ω_1 such that $\omega_1 \subseteq M$ and $2^{\omega_1} \cap M$ is Lindelöf and therefore normal.*

Using the same preservation argument of A. Dow ([8]), it was noted by Baumgartner and Tall ([4]) (using weakly compact cardinals) that it is consistent that Lindelöf spaces of character $\leq \omega_1$ and weight $\leq \omega_2$ have Lindelöf subspaces of size ω_1. Restricting the attention to compact spaces of weight ω_1, one can get a consistency reflection result just from the existence of an inaccessible. Indeed, the proof of the previous lemma works in the case of any compact space of weight ω_1, as we can code it with a family of clopen subsets of $[0,1]^{\omega_1}$ of cardinality ω_1, i.e., in the Mitchell model, we can obtain $0 < [X]^{\omega_1}_L$ for any compact space of weight ω_1 and cardinality bigger than ω_1.

One can eliminate the large cardinals completely for the price of the assumption of first countability of the space. By the Arhang'elskii theorem, it makes sense to talk about $[X]^{\omega_1}_L$, for X first countable, only in the presence of the negation of CH. Various models of $\neg$CH where all first-countable compact spaces X of weight ω_1 have $[X]^{\omega_1}_L$ stationary can be provided, the easiest being the Cohen model (using the fact that the Cohen forcing preserves the Lindelöf property; see [8], Lemma 3.4.). To build other models one needs stronger preservation lemmas, e.g. random reals also preserve the Lindelöf property (see [10]). But stronger (than Dow's lemma) preservation results can be proved for the Lindelöf property and compact spaces in general. Here is our strongest preservation result of this form which answers a question of Juhász

(whether there is in ZFC a compact space and a c.c.c. forcing which destroys the Lindelöf property of the space [15]).

Theorem 4.4. *Suppose that X is a compact space and P is a forcing notion such that P forces that X is not Lindelöf. Then there is $n < \omega$ such that P^n does not satisfy the c.c.c. In particular MA+$\neg$CH implies that there are no such X and P.*

Proof. Let $\mathcal{B}$ be a basis for the topology of X. Without loss of generality, we can assume that there is an uncountable cardinal κ and a sequence of P-names $\{\dot{U}_\xi : \xi < \kappa\}$ for elements of $\mathcal{B}$ such that P forces that $\check{X} \subseteq \bigcup\{\dot{U}_\xi : \xi < \kappa\}$ and that there is no countable subcover of this cover.
For $p \in P$ define

$$V_p = \bigcup\{U : \exists \xi < \kappa \; p \Vdash \check{U} = \dot{U}_\xi\}.$$

Note that, since for every $p \in P$ and $x \in X$ there is $\xi < \kappa$, $U \in \mathcal{B}$ and $q \leq p$ such that

$$q \Vdash x \in \check{U} = \dot{U}_\xi,$$

we have that

$$X \subseteq \bigcup\{V_q : q \leq p\}$$

for any $p \in P$. Using the fact that X is compact we can find a finite subcover of the cover $\{V_q : q \leq p\}$ and using the compactness of X once again for each $p \in P$ we can obtain $n_p \in \omega$, $q_1(p), ..., q_{n_p}(p) \leq p$, $U_1^p, ..., U_{n(p)}^p \subseteq \mathcal{B}$ and $\xi_1(p), ..., \xi_{n(p)}(p) < \kappa$ such that

1) $$X \subseteq U_1^p \cup ... \cup U_{n(p)}^p$$

and

2), $$q_i(p) \Vdash \check{U}_i^p = \dot{U}_{\xi_i(p)}$$

for all $i \leq n(p)$.
Now by induction on $\alpha < \omega_1$ construct a sequence $(p_\alpha : \alpha < \omega_1)$ of conditions of P and a sequence $(x_\alpha : \alpha < \omega_1)$ of points of X such that

3) $$p_\alpha \Vdash \check{x}_\alpha \notin \bigcup\{\dot{U}_{\xi_i(p_\beta)} : \; i \leq n(p_\beta), \; \beta < \alpha\}.$$

Given $(p_\beta : \beta < \alpha)$ and $(x_\beta : \beta < \alpha)$, we can easily find the next p_α and x_α as P forces that $\{\dot{U}_\xi : \xi < \kappa\}$ has no countable subcover, so in particular $\{\dot{U}_{\xi_i(p_\beta)} : \; i \leq n(p_\beta), \; \beta < \alpha\}$ is not a cover of X.
Clearly without loss of generality we can assume that $n(p_\alpha) = n$ for all $\alpha < \omega_1$ and some $n \in \omega$. We now consider the sequence $\{(q_1(p_\alpha), ..., q_n(p_\alpha)) : \alpha < \omega_1\} \subseteq P^n$ and let us observe that it is an uncountable antichain of P^n, which will complete the proof. Let

$\beta < \alpha < \omega_1$. Let $i \leq n$ be such that $x_\alpha \in U_i^{p_\beta}$ (by 1). By 2) we have that

$$q_i(p_\beta) \Vdash \check{U}_i^{p_\beta} = \dot{U}_{\xi_i(p_\beta)}.$$

On the other hand by 3)

$$q_i(p_\alpha) \leq p_\alpha \Vdash \ x_\alpha \notin \dot{U}_{\xi_i(p_\beta)}$$

and so this implies that $q_i(p_\beta)$ and $q_i(p_\alpha)$ are incompatible. □

We can use this theorem for the following simple reflection result (without large cardinals). Recall that MAK is Martin's axiom for partial orders which satisfy property K.

Theorem 4.5. *MAK + $\neg CH$ is consistent with "$[X]_L^{\omega_1}$ is stationary for any compact first countable space of weight $< 2^{\omega}$".*

Proof. Iterate with finite supports all c.c.c. forcings which are powerfully c.c.c. i.e., whose finite powers satisfy the c.c.c. (in particular they have property K). This preserves the Lindelöf property of compact spaces (by the previous theorem). Make the iteration of length ω_2 consisting of forcings of size ω_1 and start with a model of CH. Let X be a compact first countable space of weight ω_1. Without loss of generality we can assume that $X = [0,1]^{\omega_1} - \bigcup \mathcal{U}$ where $\mathcal{U}$ is a family of basic open subsets of $[0,1]^{\omega_1}$ determined by finitely many coordinates and open intervals with rational end-points. A code for $\mathcal{U}$ is in V^{P_α} for every $\alpha < \omega_2$ bigger than some $\beta < \omega_2$ and $X_\alpha = ([0,1]^{\omega_1} - \bigcup \mathcal{U}) \cap V^{P_\alpha}$ is a compact space in V^{P_α}.

Claim . *In V^{P_α} the space X_α is first countable.*

Proof. Let $x \in X_\alpha$. Working in V^P and using the first countability of X, we can find $\beta < \omega_1$ such that the family $\mathcal{B}_x^\beta$ of all basic open sets of $[0,1]^{\omega_1}$ containing x determined by finitelly many intervals with rational end-points rises (by intersecting its elements with X_α) to a local basis of x in X_α. But this family is definable from x, thus belongs to V^{P_α}. □

Now by Arhangelskii's theorem, the cardinality of X_α is at most continuum in V^{P_α}, which is ω_1. By the previous theorem, X_α has the Lindelöf property as iterations of c.c.c. forcings whose powers are c.c.c. are powerfully c.c.c. as well. Now the X_α's and their countable unions are in $[X]_L^{\omega_1}$ and so $[X]_L^{\omega_1}$ is stationary, as required. □

Here we do not know how to add full MA without destroying the Lindelöf property of compact spaces. For example, one could do it if the answer to the following question were positive.

Question 4. *Suppose P is c.c.c. and destroys the Lindelöf property of a compact space (in particular 2^{ω_1}). Is there a forcing Q which does not destroy the Lindelöf property of compact spaces (in particular 2^{ω_1}) such that Q destroys the c.c.c. of P?*

The possibility of adding MA by a c.c.c. forcing without destroying the Lindelöf property of compact spaces would mean by theorem 4.4 that one can make (using a c.c.c. forcing) the Lindelöf property of 2^{ω_1} indestructible by c.c.c. forcings. This could facilitate the answer to question 1. Another natural question related to theorem 4.5. is:

Question 5. *Is it consistent that $[X]^{\omega_1}_L$ is nonstationary for some first countable compact space X of cardinality bigger than ω_1?*

5. Forcing results by destroying the Lindelöf property.

In this section instead of preservation results we will be more interested in destroying the Lindelöf property. This will be combined with a technology essentially due to Devlin (see [5]) and designed in our particular case by Todorčević (see [29]) to obtain the consistency of MA + $\neg$wKH. First, using CH we will construct a c.c.c. forcing which destroys the Lindelöf property of 2^{ω_1}. Moreover, we need this forcing to satisfy extra properties like, for example, do not add uncountable branches. The idea relies on a phenomenon discovered by Luzin [21], fully combinatorially exploited by the Hungarian school e.g., [11] or [9], and later in the forcing context by Shelah in e.g., [25] and Todorčević in e.g., [30]. First we will need some preparatory definition and facts (about the notation, see the end of the introduction).

Definition 5.1. $D \subseteq C(\alpha)$ is called *powerfully predense* in $C(\alpha)$ if and only if for all $n < \omega$ the collection $\Delta_n(D) = \{\langle d, ..., d\rangle \in C(\alpha)^n : d \in D\}$ is predense in $C(\alpha)^n$.

Fact 5.2. *Suppose that $\alpha < \beta$ are ordinals. If D is powerfully predense in $C(\alpha)$, then D is powerfully predense in $C(\beta)$.*

Fact 5.3. *Suppose that D is powerfully predense in $C(\omega_1)$. Suppose that $N \prec H(\theta)$ for $\theta \geq (2^{\omega_1})^+$ is such that $|N| = \omega_1$ and $D \in N$. Then there is $D' \in [D]^\omega$ such that D' is powerfully predense in $C(\omega_1)$ and $D' \in N \cap H(\omega_1)$.*

Proof. By the c.c.c. of $C(\omega_1)^n$, there are countable $D_n \subseteq D$ such that $\Delta_n(D_n)$ are predense in $C(\omega_1)^n$. Take a countable $D' \subseteq C(\omega_1)$ which includes all of them. D' is as required. □

Our forcing will depend on two sequences, $\mathcal{M} = (M_\alpha)_{\alpha<\omega_1}$ and $\mathcal{L} = (l_\alpha)_{\alpha<\omega_1}$ where M_α is a countable set such that $M_\alpha \cap \omega_1 = \gamma_\alpha \in \omega_1$, $\gamma_\alpha < \gamma_{\alpha+1}$ and $l_\alpha \in 2^{\gamma_\alpha}$ for all $\alpha < \omega_1$. Usually we will require more properties but for the definition of the forcing and the following lemma we just need these weak assumptions.
We define the forcing $P = P_{\mathcal{M},\mathcal{L}}$, where $\mathcal{M}$ and $\mathcal{L}$ are as above, as follows: the conditions $p \in P$ are of the form (a_p, U_p) where
a) $a_p \in [\omega_1]^{<\omega}$;
b) $U_p : a_p \to C(\omega_1)$;
c) for all $\alpha \in a_p$ we have $dom(U_p(\alpha)) \subseteq [\gamma_\alpha, \gamma_{\alpha+1})$;
d) for all $\alpha < \beta$ such that $\alpha, \beta \in a_p$ we have $l_\beta \notin [U_p(\alpha)]_{\gamma_\beta}$;
e) $2^{\omega_1} - \bigcup\{[U_p(\alpha)] : \alpha \in a_p\} \neq \emptyset$ (or equivalently, by c), $U_p(\alpha) \neq \emptyset$, for all $\alpha \in a_p$).
We define the order as $p \leq q$ if and only if $a_p \supseteq a_q$ and $U_p \restriction a_q = U_q$.

Lemma 5.4. *Let M be a model as in section 2 such that $\omega_1 \subseteq M$ and $\omega_1 \in M$. Let $(\mathcal{M}, \mathcal{L})$ be as above and let $\mathcal{L} \subseteq M$. Suppose that for all $d \in C(\omega_1)$ the set $\{\alpha : l_\alpha \in [d]_{\gamma_\alpha}\}$ is uncountable and that $P_{\mathcal{M},\mathcal{L}}$ satisfies the c.c.c. Then $P_{\mathcal{M},\mathcal{L}}$ forces that $2^{\omega_1} \cap M$ does not have the Lindelöf property. In particular, in the generic extension $V^{P_{\mathcal{M},\mathcal{L}}}$, we have that $2^{\omega_1} \cap V$ does not have the Lindelöf property.*

Proof. Let $G \subseteq P$ be a P-generic over V. Let

$$\mathcal{U} = \{[U_p(\alpha)] : p \in G \text{ and } \alpha \in a_p\}.$$

We will show that $\mathcal{U}$ is an open cover of $2^{\omega_1} \cap M$ without a countable subcover.
Let $x \in 2^{\omega_1} \cap M$ and $q \in P$. We have to show that there is $p \leq q$ and there is $\alpha \in a_p$ such that $x \in [U_p(\alpha)]$. For this let $\gamma \in \omega_1$ be such that $\gamma \geq \max(dom(U_q(\alpha'))$ for all $\alpha' \in a_q$. Let $\alpha \in \omega_1$ be such that $\alpha > \gamma$ and

$$l_\alpha \in 2^{\gamma_\alpha} - \bigcup\{[U_q(\alpha')]_{\gamma_\alpha} : \alpha' \in a_q\}.$$

We can find such an α using the assumption of the lemma and e).

Now we define $p = (a_p, U_p)$ where $a_p = a_q \cup \{\alpha\}$ and

$$U_p = U_q \cup \{\langle \alpha, \langle \gamma_\alpha, x(\gamma_\alpha)\rangle\rangle\}$$

It is clear that p satisfies (a) and (b). We have that (c) is satisfied because $dom(U_p(\alpha)) = \{\gamma_\alpha\}$. The condition (d) holds because $l_\alpha \in 2^{\gamma_\alpha} - \bigcup\{[U_q(\beta)]_{\gamma_\alpha} : \beta \in a_q\}$ and finally (e) follows from the independence of $U_p(\alpha)$ from all $U_q(\beta)$ for $\beta \in a_q$.
Now we show that $\mathcal{U}$ does not have a countable subcover in the generic extension $V[G]$. Suppose that $\mathcal{V} \subseteq \mathcal{U} \subseteq C(\omega_1)$ is countable. Let α_0 be a countable ordinal such that $supp(U) \subseteq \alpha_0$ for all $U \in \mathcal{V}$. Since $P_{\mathcal{M},\mathcal{L}}$ is c.c.c., we can decide α_0 in the ground model. Now work in the ground model. As before it is easy to see that there is $p \in G$ such that $max(a_p) > \alpha_0$. Let $\alpha \in a_p - \alpha_0$. By (d), this implies that $l_\alpha \notin U$ for all $U \in \mathcal{V}$. Let $l \in 2^{\omega_1}$ be in M so that $l|\gamma_\alpha = l_\alpha$ (this follows from the fact that $l_\alpha \in M$). Now $l \notin \bigcup \mathcal{V}$ and so $\mathcal{V}$ is not a cover of $2^{\omega_1} \cap M$. □

We will often be interested in some special pairs $(\mathcal{M}, \mathcal{L})$ introduced by the following definition:

Definition 5.5. Suppose that $\mathcal{M} = (M_\alpha)_{\alpha<\omega_1}$ and $\mathcal{L} = (l_\alpha)_{\alpha<\omega_1}$ are as before and let $M_\alpha \cap \omega_1 = \gamma_\alpha$. We say that $(\mathcal{M}, \mathcal{L})$ is a *Luzin pair* if and only if the following conditions are satisfied:
f) for all $\alpha < \beta < \omega_1$ we have $|M_\alpha| = \omega$, $M_\alpha \in M_\beta \prec H(\omega_1)$ and $\bigcup_{\beta<\alpha} M_\beta = M_\alpha$ for α limit;
g) $\bigcup \mathcal{M} = H(\omega_1)$;
h) $l_\alpha \in M_{\alpha+1} \cap 2^{\gamma_\alpha}$;
i) $l_\alpha \in \bigcap\{[D]_{\gamma_\alpha} : D \in M_\alpha, D \in [C(\gamma_\alpha)]^\omega,\ D$ is predense in $C(\omega_1)\}$;
j) for all $d \in C(\omega_1)$, the set $\{\alpha : l_\alpha \in [d]_{\gamma_\alpha}\}$ is uncountable.

The point is that when $(\mathcal{M}, \mathcal{L})$ is a Luzin pair, $P_{\mathcal{M},\mathcal{L}}$ has nice properties, for example it satisfies the c.c.c.

Lemma 5.6. *Suppose M is a model as in section 2 such that $\omega_1 \subseteq M$ and $2^\omega \cap M$ is not of first Baire category in 2^ω. Suppose CH holds. Then there is a Luzin pair $(\mathcal{M}, \mathcal{L})$ such that $\mathcal{L} \subseteq M$.*

Proof. The construction of M_α and l_α is by induction on $\alpha < \omega_1$. Since $\omega_1 \subseteq M$ and $|H(\omega_1)| = 2^\omega$ (see [20] exercise III 4.), it is clear that CH provides the tools for the construction, in M, of $(\mathcal{M}, \mathcal{L})$ such that g), h) and j) are satisfied. So let us focus on the construction of $l_\alpha \in M$, given M_α, so that i) is satisfied. In other words, we need to find $l_\alpha \in M$ which is in the intersection of a given countable collection of predense subsets of $C(\gamma_\alpha)$. Using the equivalence (in M, because $\omega_1 \subseteq M$) of 2^ω

and 2^{γ_α} we can work in 2^ω. Here, by our assumption, the intersection of any countable family of dense open sets must contain a point of $2^\omega \cap M$, so using the equivalence of 2^ω and 2^{γ_α} back we obtain the required l_α. □

Lemma 5.7. *Suppose that $(\mathcal{M}, \mathcal{L})$ is a Luzin pair and let $D \subseteq C(\omega_1)$ be powerfully predense in $C(\omega_1)$. Let $(F_\xi)_{\xi<\omega_1}$ be a family of finite and pairwise disjoint subsets of ω_1 of the same cardinality $n \in \omega$. Then*

$$\{\xi < \omega_1 : \exists d \in D \text{ such that } \forall \alpha \in F_\xi \ l_\alpha \in [d]_{\gamma_\alpha}\}$$

is co-countable.

Proof. Let N be a countable elementary submodel of $H((2^{\omega_1})^+)$ such that $D \in N$, $(F_\xi)_{\xi<\omega_1} \in N$ and $N \cap H(\omega_1) = M_\alpha$ for some $\alpha < \omega_1$ (the last property can be easily achieved, for example, if $\mathcal{M} \in N$ and taking $\alpha = N \cap \omega_1$). Using fact 5.3, we can find $D_1 \in [D]^\omega$ such that D_1 is powerfully predense in $C(\omega_1)$ and such that $D_1 \in M_\alpha$. So without loss of generality we can assume that D is countable and that it is an element of M_α.
Let $\xi > \alpha$ and let $\alpha_1, ..., \alpha_n$ be elements of F_ξ listed in the increasing order. In particular $\alpha < \alpha_i$ for $i = 1, ..., n$. Let us show that there is $d \in D$ such that $l_{\alpha_i} \in [d]_{\gamma_{\alpha_i}}$ for all $i = 1, ..., n$. For this, we prove by induction on $0 \leq i < n$ that

$$E_i = \{\langle d, ..., d\rangle \in \Delta_{n-i}(D) : \ \forall j = 1, ..., i \ \ l_{\alpha_j} \in [d]_{\gamma_j}\}$$

is predense in $C(\gamma_\alpha)^{n-i}$. For $i = 0$ this is just the fact that $\Delta_n(D)$ is predense in $C(\gamma_\alpha)^n$, which follows from the assumption that D is powerfully predense in $C(\omega_1)$ and included in $C(\gamma_\alpha)$. So assume that we are done below $i < n$ and let us show that E_i is predense in $C(\gamma_\alpha)^{n-i}$. Fix $d_{i+1}, ..., d_n$ in $C(\gamma_\alpha)$. Using the inductive hypothesis we have that

$$D_i(d_{i+1}, ..., d_n) = \{d \in D : \forall j > i \ d \parallel d_j \ \text{ and } \forall j < i \ l_{\alpha_j} \in [d]_{\gamma_{\alpha_j}}\}$$

is predense in $C(\gamma_\alpha)$ (this is a straightforward consequence of the inductive assumption that E_{i-1} is predense in $C(\gamma_\alpha)^{n-(i-1)}$). Thus, by fact 5.2, it is predense in $C(\gamma_{\alpha_i})$. Also, the above set is an element of M_{α_i}, so by (i) the element l_{α_i} is in $[D_i(d_{i+1}, ..., d_n)]_{\gamma_{\alpha_i}}$. This implies that there is $d \in D_i(d_{i+1}, ..., d_n)$ such that $l_{\alpha_i} \in [d]_{\gamma_{\alpha_i}}$. But then this d witnesses that there is an element of E_i compatible with $\langle d_{i+1}, ... d_n\rangle$, which completes the inductive step. In particular we have E_{n-1} predense in $C(\gamma_\alpha)$ so, using (i) again, we conclude the lemma. □

In what follows we will often use a result of Devlin from [5]. For the convenience of the reader we quote it below:

Lemma 5.8. *([5], Corollary 3.15) Let B be a complete c.c.c. Boolean algebra. The following are equivalent:*
i) B adds an ω_1-branch;
ii) there is a subset $T \subseteq B$ such that
a) $\langle T, \geq_B \rangle$ is a Souslin tree and
b) for each $\alpha < \omega_1$, $Lev_\alpha T$ is a maximal antichain of B.

Proposition 5.9. *Let $(\mathcal{M}, \mathcal{L})$ be a Luzin pair. Then*
1) $P_{\mathcal{M},\mathcal{L}}$ is c.c.c.;
2) $P_{\mathcal{M},\mathcal{L}}$ does not add ω_1-branches;
3) $P_{\mathcal{M},\mathcal{L}}$ has property 2) in all Cohen extensions.

Proof. Put $P = P_{\mathcal{M},\mathcal{L}}$. First, let us see that 3) is satisfied if P has property 2) in the forcing extension obtained by $C(\omega_1)$. For this, suppose that P does not satisfy 2) in the extension obtained by $C(\kappa)$ for some cardinal $\kappa > \omega_1$. By lemma 5.8, this means that there is a $C(\kappa)$-name $\dot{T}$ for a Souslin tree so that, in $V^{C(\kappa)}$, forcing with P adds an ω_1-branch in T, where T is the value of the name $\dot{T}$. Because the cardinality of T is ω_1, we have that $T \in V^{C(A)}$ for some $A \subseteq \kappa$ of cardinality ω_1, with A is in the ground model. In $V^{C(A)\times C(\kappa-A)\times P}$ the tree T has an ω_1-branch. As $V^{C(A)\times C(\kappa-A)\times P} = V^{C(A)\times P\times C(\kappa-A)}$ and $C(\kappa - A)$ does not add ω_1-branches (because, for example, by 5.8. it would contradict the precaliber ω_1 of the Cohen forcing), T has an ω_1-branch in $V^{C(A)\times P}$. But T does not have an ω_1-branch in $V^{C(A)}$ because it is a Souslin tree in $V^{C(\kappa)}$. In other words, $C(A)$, which is equivalent to $C(\omega_1)$, destroys property 2) of P.
To show 1), 2), and 3) restricted to $C(\omega_1)$ it is enough to prove the following:

Claim . *Let $\dot{T}$ be a $C(\omega_1)$-name for a Souslin tree and $(\sigma_\alpha, \dot{t}_\alpha, p_\alpha)_{\alpha<\omega_1}$ be such that $\sigma_\alpha \in C(\omega_1)$, $\sigma_\alpha \Vdash \dot{t}_\alpha \in Lev_\alpha(\dot{T})$ and $p_\alpha \in P$ for all $\alpha < \omega_1$, Then there are $\alpha, \beta \in \omega_1$, $\alpha < \beta$, such that $\sigma_\alpha \cup \sigma_\beta \in C(\omega_1)$, $p_\alpha \parallel p_\beta$ and*

$$\sigma_\alpha \cup \sigma_\beta \nVdash \dot{t}_\alpha \leq_{\dot{T}} \dot{t}_\beta.$$

First, let us prove that the proposition follows from the claim. Clearly 1) would be proved because we can choose $\sigma_\alpha = \emptyset$. Now let us see that, assuming 3) and 1), the negation of 2) leads to a contradiction. Clause 1) and the negation of 2), by 5.8., imply that there is a Souslin tree with predense levels included in the completion of P. Note that

Cohen forcing preserves this Souslin tree as well as the c.c.c. of P and so 3) can not hold in the Cohen extension. Finally, assume the claim and that 3) is false and let us show we get a contradiction. Using again 5.8., we can assume that there is a $C(\omega_1)$-name $\dot{T}$ for a Souslin tree and a $C(\omega_1) \times P$-name $\dot{b}$ for an ω_1-branch of $\dot{T}$.
Let us choose σ_α, p_α and $\dot{t}_\alpha$ so that

$$(\sigma_\alpha, p_\alpha) \Vdash \dot{t}_\alpha = \dot{b}(\alpha).$$

Let α, β be as in the claim, $\sigma \leq \sigma_\alpha \cup \sigma_\beta$ be such that $\sigma \Vdash \dot{t}_\alpha \not\leq_{\dot{T}} \dot{t}_\beta$ and $p \in P$ be such that $p \leq p_\alpha, p_\beta$. Now

$$(\sigma, p) \Vdash \dot{b}(\alpha) \not\leq_{\dot{T}} \dot{b}(\beta),$$

a contradiction.

Proof of the claim: Let $\dot{T}$ and $(\sigma_\alpha, \dot{t}_\alpha, p_\alpha)_{\alpha<\omega_1}$ be as in the assumptions of the claim. Let N be a countable elementary submodel of $H(\omega_2)$ such that all relevant objects are in N and such that there is an $\alpha_0 < \omega_1$ with $N \cap H(\omega_1) = M_{\alpha_0}$ and $\alpha_0 = \gamma_{\alpha_0}$ (again, $\mathcal{M} \in N$ would imply it for $\alpha_0 = N \cap \omega_1$).
Let $\xi < \omega_1$ be such that $\xi > \gamma_{\alpha_0}$, $a = a_{p_\xi} \cap \gamma_{\alpha_0}$ and $\mathcal{U} = \mathcal{U}_{p_\xi}|a$. Let $m_0 \in \omega$ be the cardinality of a_{p_ξ} Using the definition of P, it is easy to see that $(a, \mathcal{U})$ is a condition of P in M_{α_0} and that $p_\xi \leq (a, \mathcal{U})$.
Let $\sigma = \sigma_\xi|\alpha_0$. Note that for all $\eta < \gamma_{\alpha_0}$ such that $\eta > \max(a)$, $\max(dom(\sigma))$, the structure $H(\omega_2)$ satisfies the formula

$$\exists \xi > \eta \;\; p_\xi \leq (a, \mathcal{U}) \; a_{p_\xi} \cap \eta = a, \; \sigma = \sigma_\xi|\eta \; |a_{p_\xi}| = m_0.$$

So by the elementarity of the model N, by induction and by the Δ-system lemma (inside N) the model N satisfies the following: *there is a strictly increasing sequence $\{\delta_\xi : \xi < \omega_1\}$ and sequences $\{p_\xi : \xi < \omega_1\}$ in P and $\{\sigma_\xi : \xi < \omega_1\}$ in $C(\omega_1)$ such that $p_\xi \leq (a, \mathcal{U})$, $a_{p_\xi} \cap \delta_\xi = a$, $a_{p_\xi} - \delta_\xi \subseteq \delta_{\xi+1}$, $\sigma = \sigma_\xi|\delta_\xi$ $\sigma_\xi - \sigma \in C(\delta_{\xi+1} - \delta_\xi)$ and $|a_{p_\xi}| = m_0$ for all $\xi < \omega_1$.*
Let us see that this implies that $\mathcal{F}$ is powerfully predense in $C(\omega_1)$ where

$$\mathcal{F} = \{d \in C(\omega_1) : \exists \xi_1 < \xi_2 < \omega_1 \;\; p_{\xi_1}, p_{\xi_2} \leq (a, \mathcal{U}),$$

$$[d] \subseteq 2^{\omega_1} - [\{U_{p_{\xi_1}}(\alpha) : \alpha \in a_{p_{\xi_1}} - a\} \cup \{U_{p_{\xi_2}}(\alpha) : \alpha \in a_{p_{\xi_2}} - a\}],$$

$$\sigma \cup \sigma_{\xi_1} \cup \sigma_{\xi_2} \in C(\omega_1), \; \sigma \cup \sigma_{\xi_1} \cup \sigma_{\xi_2} \not\Vdash \dot{t}_{\xi_1} \leq_{\dot{T}} \dot{t}_{\xi_2}\}.$$

To see this, let us choose $d_1, ..., d_n \in C(\omega_1)$. We can find ξ_0 such that $dom(d_i) \subseteq \delta_{\xi_0}$ for all $i = 1, ..., n$. By the independence of the elements

of $C(\omega_1)$ with disjoint domains, it is clear that for any $\xi_1 > \xi_2 > \xi_0$ we have

$$[d_i] \cap (2^{\omega_1} - [\{U_{p_{\xi_1}}(\alpha) : \alpha \in a_{p_{\xi_1}} - a\} \cup \{U_{p_{\xi_2}}(\alpha) : \alpha \in a_{p_{\xi_2}} - a\}]) \neq \emptyset.$$

Also $\sigma_{\xi_1} \cup \sigma_{\xi_2} \in C(\omega_1)$. So the above complements are good candidates for d and really to see that $\mathcal{F}$ is powerfully predense we only need to find two distinct $\xi_0 < \xi_1 < \xi_2$ such that $\sigma_{\xi_1} \cup \sigma_{\xi_2} \not\Vdash \dot{t}_{\xi_1} \leq \dot{t}_{\xi_2}$. But if we can not find ξ_1, ξ_2 with these properties, it means that all pairs satisfy the negation of this property. As σ forces that for uncountably many ξ's the conditions σ_ξ are in the $C(\omega_1)$-generic set, this would mean that σ forces that

$$\{\dot{t}_{\delta_\xi} : \sigma_\xi \in \Gamma\}$$

is an uncountable branch in $\dot{T}$, where Γ is the canonical $C(\omega_1)$-name for the generic set. But this contradicts the fact that $C(\omega_1)$ forces that $\dot{T}$ is a Souslin tree, which implies that $\mathcal{F}$ is powerfully predense in $C(\omega_1)$.

Using fact 5.3, we can find $D \in [\mathcal{F}]^\omega$ such that D is powerfully predense in $C(\omega_1)$ and that $D \in M_{\alpha_0}$. Now using lemma 5.7 (in N), for D and $F_\xi = a_{p_\xi} - a$, we can find in N (and therefore in M_{α_0}) the upper limit of the complement of the co-countable set mentioned in the lemma. Let $\eta < \omega_1$ be outside M_{α_0} and so above this limit. The lemma implies that there is $d \in D$ such that $l_\alpha \in [d]_{\gamma_\alpha}$ for all $\alpha \in a_{p_\eta} - a$. Since $d \in D \in M_{\alpha_0}$ and $D \subseteq \mathcal{F}$, the ordinals ξ_1, ξ_2 mentioned in the definition of $\mathcal{F}$ are in M_{α_0}, so this also applies to derived objects like $\sigma_{\xi_1}, p_{\xi_2}$ etc. This in turn implies that $p_{\xi_i} \parallel p_\eta$ and $\sigma_\eta \cup \sigma_{\xi_i} \in C(\omega_1)$ for $i = 1, 2$. (For the compatibility, note that for every $\alpha \in a_{p_\eta} - a$, we have $l_\alpha \in [d]_{\gamma_\alpha}$ and $[d] \cap [U_{p_{\xi_i}}(\alpha')] = \emptyset$, for every $\alpha' \in a_{p_{\xi_i}} - a$.) So to finish the proof of the claim we only need to find $i \in \{1, 2\}$ so that $\sigma_\eta \cup \sigma_{\xi_i} \not\Vdash \dot{t}_{\xi_i} \leq_{\dot{T}} \dot{t}_\eta$. As $\sigma_{\xi_1} \cup \sigma_{\xi_2} \not\Vdash \dot{t}_{\xi_1} \leq_{\dot{T}} \dot{t}_{\xi_2}$ and $\sigma_{\xi_1}, \sigma_{\xi_2} \in M_{\alpha_0}$, there is $\sigma' \leq \sigma_{\xi_1}, \sigma_{\xi_2}$ in $C(\alpha_0)$ such that $\sigma' \Vdash \dot{t}_{\xi_1} \not\leq \dot{t}_{\xi_2}$.
As $\sigma' \in C(\alpha_0)$ and $\sigma_\eta \restriction \alpha_0 = \sigma$ we have that $\sigma' \cup \sigma_\eta \in C(\omega_1)$. Now let $\rho \leq \sigma' \cup \sigma_\eta$ and n_1, n_2, n_3 be such that

$$\rho \Vdash \dot{t}_{\xi_1} \leq_{\dot{T}} \dot{t}_{\alpha_0, n_1},\ \dot{t}_{\xi_2} \leq_{\dot{T}} \dot{t}_{\alpha_0, n_2},\ \dot{t}_{\alpha_0, n_3} \leq_{\dot{T}} \dot{t}_\eta,$$

where $\{\dot{t}_{\alpha_0, n} : n \in \omega\}$ are names for all and distinct elements of $Lev_{\alpha_0}(\dot{T})$. Now as $\rho \Vdash \dot{t}_{\xi_1} \not\leq \dot{t}_{\xi_2}$, we obtain that $n_1 \neq n_2$ and then we have $i \in \{1, 2\}$ such that $n_i \neq n_3$. Thus $\rho \Vdash \dot{t}_{\xi_i} \not\leq_{\dot{T}} \dot{t}_\eta$ and so $\sigma_{\xi_i} \cup \sigma_\eta \not\Vdash \dot{t}_{\xi_i} \leq \dot{t}_\eta$, which completes the proof of the claim. □

Corollary 5.10. *(CH) Suppose M is a model as in section 2 such that $2^\omega \cap M$ is of the second Baire category. Then there is a c.c.c. forcing*

P which does not add ω_1-branches and such that P forces that $2^{\omega_1} \cap M$ does not have the Lindelöf property.

Proof. Use lemmas 5.4 and 5.6 and proposition 5.9. □

We do not really understand the relation between the category of $M \cap 2^{\omega_1}$ and the possibility of destroying the Lindelöf property of $M \cap 2^{\omega_1}$ by a c.c.c. forcing. For example, after adding many random reals CH may fail and $V \cap 2^{\omega_1}$ is of the first category, but the forcing $P_{\mathcal{M},\mathcal{L}}$ which destroys the Lindelöf property of $V \cap 2^{\omega_1}$ (which exists before adding the random reals) stays c.c.c. (using the fact that the random forcing is σ-linked [1], lemma 3.1.1.). This is also related to question 1. We can also obtain the following corollary:

Corollary 5.11. *There is a proper forcing which does not add ω_1-branches and forces that $2^{\omega_1} \cap V$ does not have the Lindelöf property.*

Proof. The forcing is $P = C(\omega_2) * Coll * P_{\mathcal{M},\mathcal{L}}$ where $Coll$ is the $C(\omega_2)$-name for the σ-closed collapse of 2^ω (in $V^{C(\omega_2)}$) onto ω_1. Note that CH holds in $V^{C(\omega_2)*Coll}$ and $V \cap 2^\omega$ is not of the first category as the Cohen forcing preserves second category of the old continuum (see [1], lemma 3.3.17.) and $Coll$ does not add new codes for Borel subsets of 2^ω. So by lemma 5.6 we have a Luzin pair $(\mathcal{M}, \mathcal{L})$, $\mathcal{L} \subseteq V$, and therefore $P_{\mathcal{M},\mathcal{L}}$ has the properties stated in proposition 5.9 in $V^{C(\omega_2)*Coll}$. It is not difficult to see that $C(\omega_2) * Coll$ does not add branches (e.g. use the same argument as for the Mitchel forcing Mi, see e.g. [29] lemma 1.5). So proposition 5.9 b) implies that P does not add uncountable branches. On the other hand, lemma 5.4 implies that $2^{\omega_1} \cap V$ is not Lindelöf in V^P. □

On the other hand, recall our result from the previous section which says that it is independent that there is a c.c.c. forcing destroying the Lindelöf property of a compact space. The following observation shows some ZFC link between adding ω_1-branches and destroying the Lindelöf property.

Fact 5.12. *Suppose that P is a forcing which does not collapse ω_1 but adds an ω_1-branch. Then P destroys the Lindelöf property of a compact space.*

Proof. Without loss of generality we can assume that P is a complete Boolean algebra. Let X be the Stone space of P. Let $\dot{b}$ be a P-name

for the new uncountable branch of a tree T of height ω_1. Consider the following conditions of P or clopen subsets of X:

$$p(\alpha, t) = \sup\{p : p \Vdash \dot{b}(\alpha) = t\}.$$

Work in V^P and let b be the value of the name $\dot{b}$. Consider the open cover of X given by $\{X - p(\alpha, b(\alpha)) : \alpha < \omega_1\}$. This family covers X because any point $x \in X$ missed by the family would allow the definition

$$\{t \in T : \exists \alpha < \omega_1 \; x \in p(\alpha, t)\}$$

in V. But the above set is equal to b which is impossible as $b \notin V$. Also note that the cover can not have a countable subcover. This is because it is a non-decreasing sequence of clopen sets. So a countable subcover would imply that the sequence stabilizes from some point on (using the fact that ω_1 is not collapsed), e.g. $X - p(\alpha, b(\alpha))$. This would mean that $p(\alpha, b(\alpha))$ decides the entire branch b which again contradicts the fact that $b \notin V$. □

Finally let us make use of proposition 5.9 for a consistency result concerning the families $[2^{\omega_1}]^{\omega_1}_L$ and $[2^{\omega_1}]^{\omega_1}_\circ$. Before we need the following:

Lemma 5.13. *Suppose P is a forcing notion which is c.c.c and does not add ω_1-branches and for every α the forcing $C(\alpha)$ forces that P does not add ω_1-branches. Let S be a forcing notion for which there is an S-name $\dot{R}$ for a forcing notion such that $S * \dot{R}$ is equivalent to $C(\omega_1) * (\dot{Q} \times C(\kappa - \omega_1))$ for some cardinal κ, where $\dot{Q}$ is an $C(\omega_1)$-name for a σ-closed forcing. Then in V^S forcing P satisfies the c.c.c. and does not add ω_1-branches.*

Proof. First note that P satisfies the c.c.c. in V^S. Suppose not. Then an uncountable antichain of P in V^S would stay uncountable in $V^{S*\dot{R}}$ (R must preserve ω_1 as $C(\omega_1) * (\dot{Q} \times C(\kappa - \omega_1))$ which is equivalent to $S * \dot{R}$ is proper). This would mean that $C(\omega_1) * (\dot{Q} \times C(\kappa - \omega_1))$ adds an uncountable antichain in P. But Cohen forcings can not add it because they have precaliber ω_1 and σ-closed forcing can not add it, because we could decide some uncountable antichain in P in the ground model, so we obtained a contradiction.
Now let us see that P does not add ω_1-branches in V^S. Suppose it adds. By 5.8, it means that in V^S there is a Souslin tree in the completion of P with predense levels (we already know that P is c.c.c. in V^S, so we can use 5.8). Note that this Souslin tree stays Souslin in any extension that preserves ω_1 and that preserves the c.c.c. of P. The density of the levels also is not affected. Using the proof of the first part, we know

that R preserves ω_1 and preserves the c.c.c. of P, thus there is a Souslin tree with predense levels in the completion of P after forcing with $P * \dot{R}$ which is equivalent to $C(\omega_1) * (\dot{Q} \times C(\kappa - \omega_1))$. But by the assumption about P the forcing $C(\omega_1)$ does not add such a tree and Q preserves the property that the P does not add ω_1-branches in any Cohen extension, as any Q-name for a $C(\kappa - \omega_1)$-name for a Souslin subtree of P with dense levels would give rise to a $C(\kappa - \omega_1)$-name for a Souslin subtree of P with dense levels, so we obtained a contradiction. □

Theorem 5.14. *Assume the consistency of ZFC together with the existence of a Mahlo cardinal. Then it is consistent (with ZFC + ¬wKH +MA) that* $0 \leq [2^{\omega_1}]^{\omega_1}_L < [2^{\omega_1}]^{\omega_1}_\circ < 1$.

Proof. In [29] the construction of a model of MA+ ¬CH + ¬KH of K. Devlin ([5]) was enriched and MA + ¬wKH was shown to hold in any model of the following form:
We start with any model of GCH with an inaccessible (or larger cardinal if needed), we force with the Mitchell forcing Mi_κ *and then we force with an iteration* Q_κ. *In* V^{Mi_κ} *we have* $\kappa = \omega_2$ *and* $(Q_\alpha, \dot{Q}^\alpha)_{\alpha \leq \kappa}$ *is an iteration with finite supports of c.c.c. forcings such that* Q_α *forces that* $\dot{Q}^\alpha$ *is a c.c.c. forcing of cardinality* ω_1 *which does not add* ω_1*-branches. The forcings* $\dot{Q}^\alpha$ *can be arranged as we wish as long as the final model satisfies MA. For this purpose we have to consider all c.c.c. forcings of size* ω_1. *By the result of Devlin (lemma 5.8) we can restrict our attention to forcings of size* ω_1 *which do not add* ω_1*-branches (because we can kill the c.c.c.-ness of forcings without this property with forcings with this property (see* [5]*)).*
We will prove the theorem considering a model obtained by a forcing as above with some additional properties. One of them will be that κ is a Mahlo cardinal (see [14]), i.e., $S = \{\alpha < \kappa : \alpha \text{ is inaccessible }\}$ is a stationary set in κ. To express other properties, we need the following claims in which the iteration Q_κ is any iteration which satisfies the above description.

Claim . *There is a club set* $C \subseteq \kappa$ *such that for every* $\alpha \in C$ *we have* $Mi_\kappa \Vdash Q_\alpha \in V^{Mi_\alpha}$.

Proof. This follows from κ-c.c. of the forcing Mi_κ (see (v) of section 4.) and the fact that in V^{Mi_κ} the forcings Q_α for $\alpha < \kappa$ have cardinality less than κ. □

Claim . *Let* $C \subseteq \kappa$ *be as in the claim above. Then for* $\gamma \in C$ *the forcing* $Mi_\gamma * Q_\gamma$ *is completely embedded in* $Mi_\kappa * Q_\kappa$.

Proof. Use the product lemma (see [20]) in V^{Mi_γ} for $Mi_{[\gamma,\kappa)}$ and Q_γ. □

Claim . *Let $C \subseteq \kappa$ be as above. Suppose that $\gamma \in C \cup \{\kappa\}$ is an inaccessible cardinal which is a limit point of $C \cup \{\kappa\}$, then*

$$[2^{\omega_1}]^{\omega_1} \cap V^{Mi_\gamma * Q_\gamma} = \bigcup_{\beta \in C \cap \gamma} [2^{\omega_1}]^{\omega_1} \cap V^{Mi_\beta * Q_\beta}$$

Proof. Use the fact that Mi_γ satisfies the γ-c.c. (by v) section 4) and the fact that the supports of the conditions from Mi_γ have cardinality less than γ (they are countable). □

As in section 4, we define by induction, in V, a sequence $(\dot{X}_\alpha : \alpha \in \kappa)$ of Mi_κ-names for subsets of 2^{ω_1} of cardinality ω_1 and a sequence of ordinals $(\gamma_\alpha : \alpha < \kappa) \subseteq C$ in the following way: $Mi_\kappa \Vdash \dot{X}_\alpha = \bigcup_{\beta<\alpha} \dot{X}_\beta$ and $\gamma_\alpha = \sup(\gamma_\beta : \beta < \alpha)$ for α limit and if $\alpha = \beta + 1$ for some β we require that $Mi_\kappa \Vdash \dot{X}_\alpha = 2^{\omega_1} \cap V^{Mi_{\gamma_\alpha} * Q_{\gamma_\alpha}}$ for $\gamma_\alpha \in C$ minimal bigger than γ_β such that $Mi_\kappa \Vdash \dot{X}_\beta \subset V^{Mi_{\gamma_\alpha} * Q_{\gamma_\alpha}}$; this γ_α exists by the first claim above.

We are now ready to specify our iteration. We consider κ which is a Mahlo cardinal. The forcings Q^α's are arranged so that in $V^{Mi_\kappa * Q_\alpha}$ we have for $\alpha \in \kappa$:
1) $\dot{Q}^\alpha$ has the c.c.c.
2) $\dot{Q}^\alpha$ has cardinality ω_1.
3) $\dot{Q}^\alpha$ does not add ω_1-branches.
and additionally
4) MA holds in the final model.
5) For $\alpha \in C$ we define $\dot{Q}^\alpha$ in $V^{Mi_\kappa * Q_\alpha}$ to be the forcing of the form $P_{\mathcal{M},\mathcal{L}}$ for some pair $(\mathcal{M}, \mathcal{L})$ such that $\mathcal{L} \subseteq 2^{\omega_1} \cap V^{Mi_\alpha * Q_\alpha}$. Thus we need the following:

Claim . *Let $\alpha \in C$. There exists in $V^{Mi_\kappa * Q_\alpha}$ a pair $(\mathcal{M}, \mathcal{L})$ such that $\mathcal{L} \subseteq 2^{\omega_1} \cap V^{Mi_\alpha * Q_\alpha}$ and $Mi_\kappa * Q_\alpha \Vdash$ 1), 2) and 3) are satisfied for $\dot{Q}^\alpha = P_{\mathcal{M},\mathcal{L}}$.*

Proof. We have $2^\omega = \alpha$ in $V^{Mi_{\alpha+1}}$. Now, the forcing $Mi^{\alpha+2}$ is a collapse of α onto ω_1 which does not add new reals, i.e., $V^{Mi_{\alpha+2}}$ satisfies the CH and so $V^{Mi_{\alpha+2} * Q_\alpha}$ satisfies the CH by 2) above. Also note that $2^\omega \cap V^{Mi_\alpha * Q_\alpha}$ is not of the first Baire category in $V^{Mi_{\alpha+2} * Q_\alpha}$. This follows from the fact that $Mi_{\alpha+2} * Q_\alpha$ is equivalent to $Mi_\alpha * Q_\alpha \times C(\omega) * Coll(\alpha)$, where $Coll(\alpha)$ is a σ-closed collapse of α onto ω_1, and the fact that the Cohen forcing preserves the second category of the old continuum (see

[1], lemma 3.3.17.) and that the collapse does not add codes for new Borel sets. This means that we can apply lemma 5.6 to get the existence of a Luzin pair $(\mathcal{M}, \mathcal{L})$ in $V^{Mi_{\alpha+2}*Q_\alpha}$ such that $\mathcal{L} \subseteq 2^{\omega_1} \cap V^{Mi_\alpha * Q_\alpha}$. Now by 5.9, the forcing $P_{\mathcal{M},\mathcal{L}}$ is c.c.c. and does not add ω_1-branches and clearly is of cardinality ω_1 (as CH holds) in $V^{Mi_{\alpha+2}*Q_\alpha}$. We need to see that these properties are also satisfied in $V^{Mi_\kappa * Q_\alpha}$. For this we use properties ii) and iv), from section 4, of the forcing Mi_κ combined with 5.9 part 3) and the conclusion follows from 5.13, so the claim is proved. □

Now let us prove that the statement of the theorem holds in $V^{Mi_\kappa * Q_\kappa}$. We will denote by C° the set of limit points of C. Clearly $(X_\alpha : \alpha \in \kappa)$ is an increasing club sequence in $[2^{\omega_1}]^{\omega_1}$ and so $(X_\alpha : \gamma_\alpha \in C^\circ \cap S)$ forms a stationary set in $[2^{\omega_1}]^{\omega_1}$ (S is stationary in V and by the properness of the forcing $Mi_\kappa * Q_\kappa$ it stays stationary in $V^{Mi_\kappa * Q_\kappa}$). As the iteration does not add ω_1-branches (see [5]) by the second claim we have that $X_\alpha \in [2^{\omega_1}]^{\omega_1}_\circ$ for all $\gamma_\alpha \in C^\circ \cap S$. On the other hand $\{X_\alpha : \gamma_\alpha \in C^\circ \cap S\}$ is disjoint from $[2^{\omega_1}]^{\omega_1}_L$. This follows from lemma 5.4 in $V^{Mi_{\alpha+2}*Q_{\alpha+1}}$ and the fact that being not Lindelöf is preserved by proper forcings (this follows from the fact that if P is a proper forcing and A is a countable set of ordinals in V^P, then there is a countable set of ordinals B in V such that $A \subseteq B$). This completes the proof of the theorem. □

Corollary 5.15. *Assume the consistency of the existence of a Mahlo cardinal. It is consistent there is a model M as in section 2 such that $M \cap 2^{\omega_1}$ is not Lindelöf but there is no one-point K which witnesses the failure of 2.4.*

Proof. Let $M \cap 2^{\omega_1} \in [2^{\omega_1}]^{\omega_1}_\circ \setminus [2^{\omega_1}]^{\omega_1}_L$. The one-point K as in 2.4 would give rise to a convergent sequence in M with the limit outside of M which would contradict the choice of M □

Corollary 5.16. *Assume the consistency of the existence of a Mahlo cardinal. It is consistent that there is a dense subgroup $X \subseteq 2^{\omega_1}$ and a compact set $K \subseteq 2^{\omega_1}$ such that $K \cap X = \emptyset$, there is no G_δ-set G such that $K \subseteq G$ and $X \cap G = \emptyset$ but for each $y \in K$ there is a G_δ-set G such that $y \in G$ and $G \cap X = \emptyset$.*

Proof. Let $X = 2^{\omega_1} \cap M \in [2^{\omega_1}]^{\omega_1}_\circ \setminus [2^{\omega_1}]^{\omega_1}_L$. □

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