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**FRACTAL THEORY**
**Discussion on the definition of fractals**

Technical Report
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# Fractal Theory
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## Abstract

This report presents a discussion on the definition of fractals. The main types, traditional definitions and theoretical elements necessary to them. Then the concept of fractal and its main points are discussed in the conclusion a new definition of fractal is presented to circumvent the problems of the previous ones.

# Contents

# Introduction

Fractals are widely used and fundamental to many applications, but they're not as well understood in their principles. The definitions used today are much more intuitive than objective, this have been enough so far, but the lack of definition consequences are beginning to be felt.

Here we wish to bring this point to sight, and reevaluate the position taken by Mandelbrot [MAN77] that the concept should remain open. Also a discussion on the definitions and the main theoretical elements are presented. Then we discuss the concept of fractal, what is out 'ideal' fractal and what should it satisfy. Finally a new definition is presented and discussed as an alternative to focus the development and research efforts in the field.

To allow a better comprehension of seasoned researchers and newcomers alike, we present a typology and the 'schools' of thought of the field to see how they understand the concept of fractal.

Moreover, most of the researchers are related to specific areas of knowledge, like physics, biology etc., so it is interesting to keep a view detached of bias towards any of these. Allowing us to discuss and present something understandable, usable and useful to all areas that deal with fractals.

After a mental portrait of the concepts and feelings involved we can see and criticize the definitions, their strong and weak points and focus ourselves in an attempt to build a new definition, that is better suited for the field.

# Chapter 1

# Typology

What is a fractal if we do not have an ultimate definition? Basically: we do not know. At least formally (we'll discuss this later). The term is intuitive, a suggestion of features that put together objects of totally different nature.

To capture the intuitive concept, and go beyond defining them formally it is interesting to classify the actually known fractals. Therefore follows a brief typology, it is not closed nor finished, like the fractal field itself. Moreover we do not intend to make disjoint categories to separate the objects. The goal is to present a classification that presents fractals as they're found in different areas.

We can make five main types of fractals:

1. Algebraic or Symbolic
2. Affines
3. Non-linear
4. IFS
5. Probabilistic

Many members can be found on different categories at the same time, because they have equivalents in other types. These are very common (usually made by bijections) and allow better understanding of many 'hard' problems transforming them into well-known ones. Sometimes, though, the conversion between types is very difficult, so that it is hard to call both instances the same object. And there are cases where no transformation is known.

It is not unreasonable to think that future work will reveal that any given fractal has equivalents in any type, then we should look to this classifications as the possible interpretations of the objects.

## 1.1 Algebraic or Symbolic

Encompasses all objects created by transformation rules over strings of simbols[1] (like formal languages), automata, etc. Therefore the 'symbolic' name.

Many times the symbol strings are related to real (or complex) numbers in some adequate base to facilitate the use of $\mathcal{R}$ properties and from this use comes the 'algebraic' name.

A very popular example is the Cantor set defined by the iteration of the transformations:

$$\begin{aligned} 0 &\mapsto 000 \\ 1 &\mapsto 101 \end{aligned}$$

Starting with the seed '1' in the infinite symbol strings space of '0' and '1' we have:

| Iteration | String |
|---|---|
| 0 | 1 |
| 1 | 101 |
| 2 | 101000101 |
| 3 | 101000101000000000101000101 |
| ⋮ | ⋮ |

Taking the iteration of the rules to the limit, the infinite string produced is a symbolic version of the cantor set. If we start the iterations with '0' the result is a infinite string of zeros, but with other seed strings things are more interesting:

| Iteration | String |
|---|---|
| 0 | 1011 |
| 1 | 101000101101 |
| 2 | 101000101000000000101000101101000101 |
| 3 | 1010001010000000001010001010000000000000··· |
| ⋮ | ⋮ |

All strings generated have a similar aspect: they contain copies of themselves. This feature is named: self-similarity, although it is difficult to make a formal definition applied to other sets that contain approximate copies of themselves.

[1]Given a finite symbol alphabet $\Sigma$, the rules operate over $\Sigma^* = \cup_n \Sigma^n$ where $\Sigma^n$ is the set of all strings of length $n$ formed by concatenation of symbols in $\Sigma$.

When a affine transformations[2] maps the set into its subsets exactly, there is strict self-similarity. When the subsets are deformed through any non linear transformation (known or not), but are still similar to the whole only self-similarity is used.

Bijections between other spaces and the string space are very useful by their capacity of addressing and classification. It is, though, a little far from the scope of this work. For a very clear introduction refer to [PJS92].

## 1.2 Affine Fractals

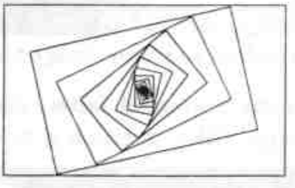
Working with iterated affine functions, that is $f(x) = Ax + b$ in some vector space (where $A$ is a linear transformation and $b$ a vector of the space) we have the affine fractals. These are normally studied in $\mathcal{R}^n$ and may have one or many component functions.

In Euclidean spaces (like $\mathcal{R}^n$) a matrix of transformations $A$ is used together with the translation vector $b$. Then $f$ can be interpreted as a composition of rotation, scaling, tilting and translation applied repetitively.

For instance:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

As a affine transformation in $\mathcal{R}^2$, with appropriate values of $\alpha, \beta, \gamma, \delta, a$ and $b$, we could obtain:



Exemple of afine transform

## 1.3 Non–Linear Fractals

In this class we have all kinds of non–linear functions of one space into itself. The division of affine and non–linear is important because of the uniform behavior and the stability of systems based on affine functions.

[2] Here viewed as the composition of a linear transformation and a translation.

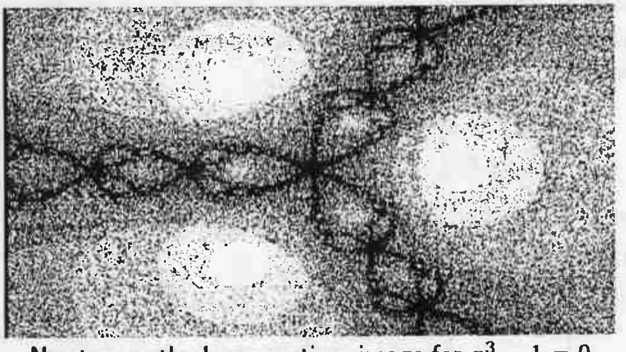
In some cases we have non–linear functions considered as IFS, but this is due to additional restrictions. In this category there are no such restrictions (see the IFS discussion bellow).

**Exemplo 1.3.1** Newton's method

Let's look to the equation: $x^3 - 1 = 0$ and apply the Newton method to evaluate it's roots. It is basically the iteration of the formula:

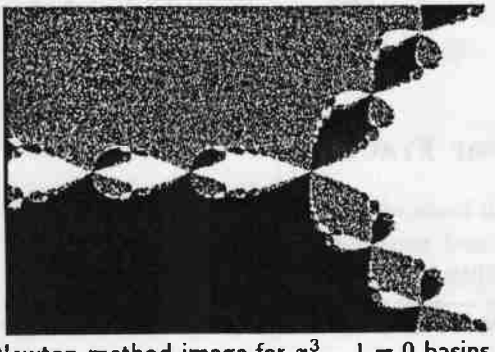
$$x = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^3 - 1}{3x_0^2}$$

Now we ask: if we pick each point in the complex plane as a starting point will the method converge, in a given number of iterations, to some root of $x^3 - 1 = 0$? (and we could go further asking to which one it converged). How long then each point will take to converge? Let the image below show:



Newton method escape time image for $x^3 - 1 = 0$

Points in the lighter areas converged faster, while the points in the dark areas took longer to converge. Other question is to which root the point goes:



Newton method image for $x^3 - 1 = 0$ basins of the roots

Each root has one shade showing the delicate patterns of this fractal.

## 1.4 Iterated Function Systems (IFS)

Contractions from a complete space into itself are named IFS when seen from the iterative point of view. But the composition of such functions is also a contraction. Given $f : E \to E$, $f$ is a contraction iif for any $x, y \in E$ we have $s.d(x,y) \geq d(f(x), f(y))$ where $d$ is the metric of the space and $s$ is a real number larger than 0 and less than 1.

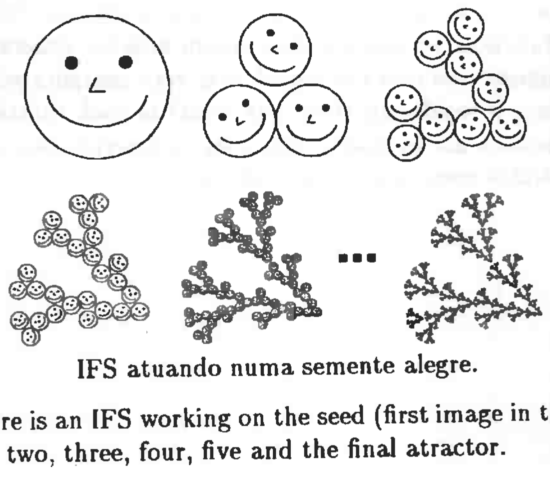
If we have several contractions $f_i : E \to E$ then

$$F(X) = \bigcup_{\forall i} \{f_i(x) / x \in X\}$$

is a contraction in $P(E)$ (the set of parts of $E$). Considering the Hausdorff metric based on the original space metric $d$. (as shown in [BAR88]). The contraction mapping principle is applicable then, meaning that the iteration of $F$ will converge to exactly one element $A \in P(E)$ and that this element will be its fixed point: $A = F(A)$.

Normally the IFS are affine functions, but all results do not need this, any contractive function will do. The reason for this is that affine functions are much more amenable to computational and mathematical treatment. The results obtained with this type of IFS are producing a wide field of applications like fractal image compression [SH94].

One example:



IFS atuando numa semente alegre.

Above there is an IFS working on the seed (first image in the upper left), one iteration, two, three, four, five and the final atractor.

Although Barnsleys work is very interesting and powerful he does not contribute much for the fractal definition, proposing a new definition that is too broad (see next section).

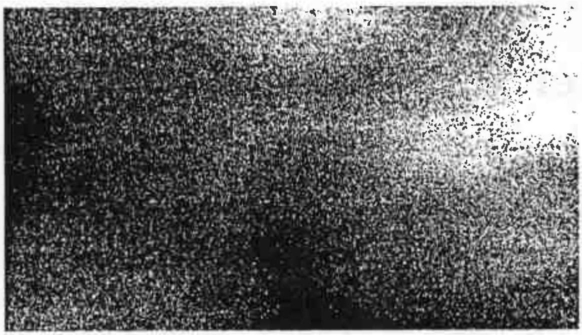
One of the most important results of Barnsley's work is worth mentioning here: the Collage Theorem. A 'collage' is understood as the first application of an IFS $F$. The theorem gives a bound between any set and the fixed point of the IFS, based on the distance of the set to it's collage and the contraction factor of the IFS.

The problem with IFS is (normally) the explosion of points to be evaluated. For instance if $F$ is composed of $k$ functions and we start with a single point then the number of points will grow with $k^n$ ($n$ is the number of iterations). This is dealt with in two ways: minimizing the number of iterations or using the 'chaos game' algorithm. This is based on the observation that if we randomly chose an index $i$ and apply only the $f_i$ to each iteration, the orbit formed will eventually cover the fixed point of $F$.

## 1.5 Random Fractals

When a given iterated function has a parameter that is in fact a random variable we have a random fractal. This variable is the source of non-determinism in each iteration step. These fractals are normally used to model natural phenomena, but comparing them with non-random fractals remains the doubt: are they really random? Considering the complexity and unpredictability of other deterministic systems we are inclined to say no to the last question.

Let's consider $f(x) = Ax + c$ where $c$ is fixed and $A$ is a random variable that changes in each iteration following some distribution. To simulate and evaluate this function we use pseudo-random number generators, that are in fact deterministic iterated functions! It is very complicated to use really random numbers (considering that they exist) in such functions. If seems that these functions are in reality disguises to deterministic ones, but this approach facilitates some statistical analysis.



Plasma clouds created by the random midpoint displacement technique.

One striking example of this is the use of the 'chaos game' to evaluate the attractor of a deterministic IFS.

# Chapter 2

# Dimensions

Due to the relation between self-similarity and the concept of dimension Mandelbrot [MAN91] (and then in [MAN83]) uses the Hausdorff-Besicovich dimension as basis for the first definition of the term. Despite of its problems (and misinterpretations) it is still one of the main source of tools to the field. In this chapter we wish to give a general outline and present the main definitions. This will prepare for the discussion in the next chapter.

Unfortunately dimension evaluation is not easily done (and to complicate matters further there are many different definitions) and the implementation is unfeasible. This normally forces the uses of weakened definitions that are not as accurate or informing as the Hausdorff-Besicovich one. Even so the evaluation can reveal important features of objects geometry.

Dimension is a notion that seems to have appeared first among the Greek (as number of measures needed to limit an object, as in Euclid, book I). Though normally stated the concept was intuitive. Poincaré brings the matter back to light and offers a more structured definition[1] (as in [MAN83, pg. 410], my remarks):

> "When we say that space has dimension three, what do we mean? If to divide a continuum $C$ it suffices to consider as cuts a certain [finite] number of distinguishable elements, we say that this continuum is of *dimension one*... If on the contrary,... to divide a continuum it suffices to use cuts which form one or [finitely] several continua of dimension one, we say that $C$ is of *dimension two*. If cuts which form one or [finitely] several continua of at most dimension two suffice, we say that $C$ is a continuum of *dimension three* and so on."

[1]There is a brief history in [MAN83, pg. 409]

This is the base of the topological dimension (though its first definition was a bit different there is another one, equivalent, that is the same as the one by Poincaré). One difference between this definition and the Greek one was to think of the dimension of a space instead of objects.

## 2.1 Topological dimension

The Greek notion of the number of measures needed to limit an object was used (as the least number of numbers needed to locate a point in some space) without much worry (and rigour) until Cantor and Peano presented two annoying examples. Cantor built a one-to-one correspondence between the points in a line and a plane (a bijection between plane and line). Peano showed a curve that fills the plane (continuous function between line and plane). If the two examples were united in one the dimension would not be a topological invariant!

Brower showed that this could not happen, but the concepts needed a formal presentation and definition! Lebesgue offers the first demonstration that dimension is a topological invariant, as the first definition of topological dimension (normally used as the space 'dimension'). According to [NAG65, pg. 9]:

**Definição 2.1.1** *Topological dimension (Covering or Lebesgue's): Let $U$ be a countable open cover of $A$. $Ord(A, U) = \sup\{ord_p U / p \in A\}$ and $ord_p U$ is the number of open sets of $U$ that contain $p$.*

*If given any finite cover $U$ of $A$ there is a finite cover $V$ of $A$ which refines[2] $U$, and $Ord(A, V) \leq n + 1$ for some $n \in \mathcal{N}$ then*

$$D_T(A) \leq n$$

*If $D_T(A) \leq n$ and $D_T(A) \nleq n - 1$ then $D_T(A) = n$ and if $D_T(A) \nleq n$ we say that $D_T(A) = \infty$.*

This definition is not concerned with the geometric structure of any object (or space), but is a formal definition of the topological invariant concept. Two other definitions appeared to share the title of "topological dimension" the weak and the strong inductive dimensions. The last one is equivalent to Lebesgue's in normal topological spaces. Here we present it because it is a beautiful analogy to Poincaré's idea (more details in [HW41] and [NAG65]):

[2]That is: $\forall u \in U \implies \exists v \in V / v \subset u$

**Definição 2.1.2** *Strong Inductive dimension (Large or Čech-Brower's): If $A = \emptyset$ then $Ind(A) = -1$, otherwise if for any disjoint open sets $F, G \subset A$ there is an open set $U$ such as: $F \subset U \subset A - G$ and $Ind(Border(U)) \leq n-1$ then:*

$$Ind(A) \leq n$$

*The same observations made in the last definition apply here.*

## 2.2 Self-Similarity dimension

The key to the use of dimensions to describe fractals is the notion of self-similarity: possessing parts that are similar to the whole. If an object is made up of small copies of itself (exact copies except for scaling, rotating and translation) then there is a simpler way of defining dimension.

**Definição 2.2.1** *Self-Similarity dimension: If an object $A$ is divided in $N$ parts, each equal to the whole, but scaled by $L$ then the Self-Similarity dimension $D_{AS}$ is:*

$$D_{AS} = -\frac{\log N}{\log L}$$

For instance, the ternary Cantor set self-similarity dimension is calculated noting that each half of the set is exactly equal to the whole, but reduced to 1/3 of the original size:

Cantor middle-thirds set (enhanced to be seen).

Then the dimension is $D_{AS} = -\frac{\log 2}{\log(1/3)} = \log_3 2$, between 0 and 1 as we could expect, for the set is totally disconnected (length 0) – therefore has topological dimension 0 – but has as many numbers as the interval $[0;1]$ and therefore as the real line $\mathcal{R}$.

## 2.3 Hausdorff-Besicovich dimension

To generalize the notion of dimension to arbitrary sets (not necessarily self-similar) we need a measure indexed by the scale (size range) to obtain a measure of the object's geometry in all scales.

This is similar to the content measure (volume, area etc.) $C_r$ of a $n$ dimensional ball of radius $r$:

$$C_r = \gamma(n) r^n \quad \text{where} \quad \gamma(n) = \frac{(\Gamma(1/2))^n}{\Gamma(1+n/2)}$$

where $\Gamma$ is Eulers function[3]. For a bidimensional disc we would have, for instance: $C_r = \pi r^2$.

Hausdorff noted that this measure can be enhanced changing the $n$ by an arbitrary number $s$. This parameter is varied from 0 to $\infty$ using $C_r^s$ to evaluate a cover content. Formally with Lebesgue's definition we had the measure of a given set $A$:

$$\mu(A) = \lim_{r \to 0} \mu_r(A)$$

with

$$\mu_r(A) = \inf \left\{ \sum_{i=0}^{\infty} C_{r_i} \ / \ \begin{array}{l} \text{for all covers } (B_i)_{i=0}^{\infty} \text{ of } A, \\ \text{where } r_i \text{ is the diameter of } B_i \text{ and all } r_i \leq r \end{array} \right\}$$

That is, $(B_i)_{i=0}^{\infty}$ is an open countable cover of $A$, each open set has diameter[4] less or equal to $r$. Taking $r$ to zero and adding the contents of the 'smallest' covers we measure the set's content regarding the space dimension (topological). But Hausdorff's modifications leaded to:

$$h^s(A) = \lim_{r \to 0} h_r^s(A)$$

with

$$h_r^s(A) = \inf \left\{ \sum_{i=0}^{\infty} (r_i)^s \ / \ \begin{array}{l} \text{for all covers } (B_i)_{i=0}^{\infty} \text{ of } A, \\ \text{where } r_i \text{ is the diameter of } B_i \text{ and all } r_i \leq r \end{array} \right\}$$

And $h^s(A)$ is the $s$-Hausdorff measure[5] of the set $A$. Varying $s$ we get a decreasing function with $s$ that presents a 'jump' from 0 to $\infty$ in a particular value of $s$, this is the Hausdorff-Besicovich dimension.

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[3]Generalization of factorial made by Euler:

$$\Gamma(x) = \lim_{m \to \infty} \frac{m^x m!}{x(x+1)(x+2)\ldots(x+m)}$$

is defined for all real numbers except the negative integers where the limit does not exist.

[4]Defined by the largest distance between two points in the open set (according to some metric).

[5]Not all the set are measurable in this way, only those that fulfill: $h^s(A) = h^s(A \cap T) + h^s(A \setminus T)$ for any $T \subset E$. In normal applications we deal with closed sets, which are all $h^s$-measurable

**Definição 2.3.1** *Hausdorff-Besicovich dimension: given by:*

$$D_{HB}(A) = \inf\{s/h^s(A) = 0\} = \sup\{s/h^s(A) = \infty\}$$

With this in mind Mandelbrot's definition [MAN83, pg. 15] makes perfect sense: fractals are those sets whose Hausdorff-Besicovich dimensiondiffers from its topological dimension, that is are not measured accurately by the topological dimension.

As with the topological dimension, the $D_{HB}$ groups sets in equivalence classes, but we were able to find only a result valid for strict self-similar sets ([DK89, in *An Introduction to Fractals*]). Therefore there is a strong theoretical indication to use $D_{HB}$ as a classification method.

## 2.4 Generalized dimensions

It is possible to create several dimensions based on Hausdorff's mechanism. We choose a function $m_s(r)$ that goes to zero with $r$ and we search for the same cutting point in the real line:

$$M_s(A) = \lim_{r \to 0} M_{r,s}(A)$$

with

$$M_{r,s}(A) = \inf \left\{ \sum_{i=0}^{\infty} m_s(r_i) \;/\; \begin{array}{l} \text{for all covers } (B_i)_{i=0}^{\infty} \text{ of } A, \\ \text{where } r_i \text{ is the diameter of } B_i \text{ and all } r_i \leq r \end{array} \right\}$$

The function $m_s(r)$ can be adjusted to each set choosing the critical value of $s$. For instance we have sets with good fit with: $m(r) = 1/\log|r|$. Mandelbrot calls this function the intrinsic test function of a given set [MAN83, pg. 364]. One final example: the brownian movement in the plane has as test function: $m(r) = r^2 \log\log(1/r)$

## 2.5 Bouligand-Minkowsky dimension

Similar to the Hausdorff-Besicovich definition, we have the idea of Minkowsky refined by Bouligand. Take balls of radius exactly $r$ and use the Hausdorff measure for this type of covers:

$$BM_s(A) = \lim_{r \to 0} BM_{r,s}(A)$$

with

$$BM_{r,s}(A) = \inf \left\{ \sum_{i=0}^{\infty} r^s \;/\; \begin{array}{l} \text{for all covers } (B_i)_{i=0}^{\infty} \text{ of } A, \\ \text{where the diameter of } B_i \text{ is } r \end{array} \right\}$$

**Definição 2.5.1** *Bouligand-Minkowsky dimension: defined by:*

$$D_{BM}(A) = \inf\{s/BM_s(A) = 0\} = \sup\{s/BM_s(A) = \infty\}$$

As we are restricted to particular covers, we can have larger values than those obtained with $D_{HB}$. And the topological dimension is still less (being the largest integer less or equal to $D_{HB}$) we have the relations ($D_E$ is the embedding space topological dimension):

$$D_T \leq D_{HB} \leq D_{BM} \leq D_E$$

A set $A$ such as $D_{HB} = D_{MB}$ is a *regular set* and can be used to decompose more complex sets, for $D_{HB}(A \times B) = D_{HB}(A) + D_{HB}(B)$ for any set $B$

## 2.6 Box Counting dimension

The previous definitions are very good in the theoretical point of view, but are very difficult to evaluate numerically. Phrases like "all covers of..." are simply untractable (unless another result allows us to bypass them) in computers, but restricting the cover class and allowing its elements to shrink to zero we can evaluate a new dimension $D^*$ as a upper bound to $D_{HB}$.

The most popular technique (although sometimes it may give very bad results) is known as 'box counting dimension', some authors call it 'grid dimension'. Its origin is the entropy dimension or Schnidelman-Komolgorov dimension.

It is a simple counting problem, we divide the space in a grid of hyper-cubes of length $r$ and count the number of cubes that contain at least one point of the set being evaluated. Then the size of the cubes is reduced and the process is repeated. The dimension is given by:

$$D_{BC}(A) = \lim_{r \to 0} \frac{\log N(r)}{\log(1/r)}$$

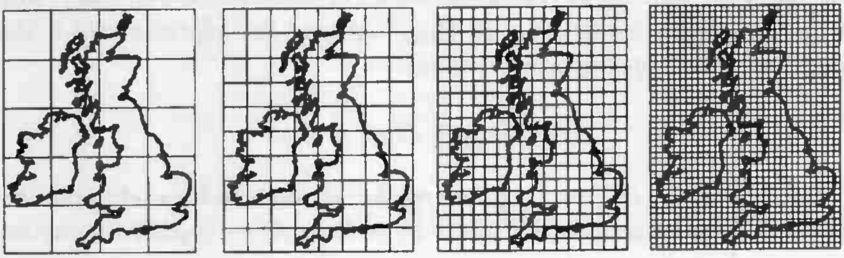
where $N(r)$ is the number of hyper-cubes containing points of $A$.

How this definition relates to the previous ones? It is based on the measure $r^d N(r)$, and noting that: $d = inf\{d/lim_{r \to 0} r^d N(r) = 0\}$ we see how they relate.

Normally the method is used with successive divisions of $r$ (by powers of 2, for instance) and a linear regression on a log-log plot of $N(r)$ against $r$ is used to estimate the box counting dimension.

**Exemplo 2.6.1** Box counting England's perimeter

If we take the England's coast we can evaluate it's box-counting dimension using finer and finer grids, and for each counting the number of boxes that cover the outline:

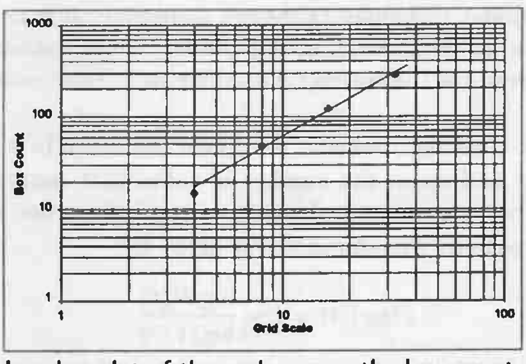


England coastline under grids of size 1/4, 1/8, 1/16 and 1/32.

And the number of boxes that cover the coastline in each grid are:

| Scale | 1/4 | 1/8 | 1/16 | 1/32 |
|---|---|---|---|---|
| Boxes covering | 15 | 48 | 122 | 283 |

Plotting this in a log-log graph:




Log-log plot of the scale versus the box count.

The slope of the interpolated line is 1.37, that is the box-counting dimension of England's coastlines (on the scale range considered here) is $D_{BC} \approx 1.37$.

This method has some intrinsic limitations, for instance it will evaluate dense subsets of $\mathcal{R}$ to have a non null dimension, while their Hausdorff-Besicovich dimension is zero (for instance the set of rational numbers).

## 2.7 Information dimensions

Also known as generalized Renji dimensions, are very popular in physics ([FFdP94]), and actually in mathematics in the study of Multifractals (for an introduction: [PJS92, in Mandelbrot's article])

First we take an invariant measure on a grid of size $\epsilon$ in which each box has a probability $p_i$ attached (related to the 'amount' of points in the box). Then we define:

**Definição 2.7.1** *Generalized Renji dimension of $q$ order, where $q \in \mathcal{R}$ and $q \neq 1$ is defined:*

$$D_q = \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\log_2 \sum_i p_i^q}{\log_2 \epsilon}$$

This gives a spectrum of values when $q$ varies. Normally used with dynamical systems that have attractors or are bounded, taking orbits we define:

$$p_i = \lim_{\epsilon \to 0} \frac{N_i}{N}$$

Where $N_i$ is the number of points of the orbit that fall in the $i$th. box. When $N$ is large (very) the orbit goes around the attractor several times, so the density of the attractor is well represented by $p_i$ then the previous definition is used to get the Renji dimensions of the system.

Some values of $q$ are particularly relevant: 0, 1 e 2.

For $q = 0$ we have $D_0 = \lim_{\epsilon \to 0} \frac{\log_2 \sum_i 1}{\log_2 \epsilon}$ but $\sum_i 1$ is exactly the number of points in each box $N_i(\epsilon)$, therefore:

$$D_0 = \lim_{\epsilon \to 0} \frac{\log_2 N_i(\epsilon)}{\log_2 \epsilon}$$

That is exactly the box counting dimension. $D_{BC} = D_0$!

With $q = 1$ we have a slight technical problem with the definition that can be solved. Let's rewrite $D_q$ as:

$$D_q = -\lim_{\epsilon \to 0} \frac{I_q(p)}{\log_2 \epsilon}$$

where $I_q(p)$ is the generalized Renji information of order $q$[6] defined by:

[6] Informarion theory is extremely beautiful and rich, but out of the scope of this work. For general results [HAM80] and for fractal related results: [FFdP94, in parts II.II and III.III.C].

$$I_q(p) = \frac{1}{1-q} \log_2 \sum_i p_i^q$$

That have limit when $q \to 1$ [FFdP94, pg. 151]: $-\sum p_i \log_2 p_i$, this is the information (as defined by Shannon) of the system. And using this we have:

$$D_1 = \lim_{\epsilon \to 0} \frac{\sum_i p_i \log_2 p_i}{\log_2(1/\epsilon)}$$

That is the information dimension.

Finally the case $q = 2$:

$$D_2 = \lim_{\epsilon \to 0} \frac{\log_2 \sum_i p_i^2}{\log_2 \epsilon}$$

is the correlation dimension easy to compute in complex systems by the Grassemberger-Procaccia algorithm.

A interesting property of the Renji dimensions is:

$$D_q \leq D_{q'} \text{ iif } q \geq q' \text{ where } q, q' \in \mathcal{R}$$

The equality holds in equiprobable self-similar sets.

# Chapter 3

# Definition

With the previous chapter and some basic facts in mathematics we can now proceed to the discussion of the old definitions of fractal and build up to a new one.

## 3.1 Old definitions

Why several dimensions definitions? One wouldn't suffice? Unfortunately no, Mandelbrot tries, at first, to define fractals using the most general dimension: Hausdorff-Besicovich but he was already aware (and said it) of the 'tentative' nature of this definition. After, he admits that he shouldnt have defined one dimension in particular relying on the intuitive idea of dimension and leaving its interpretations to a per-case basis.

This does not help much, some authors have tried to enhance this. For instance James Taylor presents an enhanced definition based on Mandelbrot's. Then Barnsley [BAR88] made up a new definition. Exactly what he needed for his work, but it encompasses too much to be useful as definition.

### 3.1.1 Mandelbrot

Let's review Mandelbrot's definition:

> "Whenever we work in the Euclidean space $R^E$, both $D_T$ [topological dimension and $D$ [Hausdorff-Besicovitch dimension] are at least 0 and to most $E$. But the resemblance ends here. The dimension $D_T$ is always an integer, but $D$ need not to be an integer. And the two dimensions need not coincide; they only satisfy the Szpilrajn inequality $D \geq D_T$. For All Euclid, $D = D_T$. But nearly all sets in this Essay satisfy $D > D_T$.

There was no term to denote such sets, which led me to coin the term fractal, and to define it as it follows:

*A fractal is by definition a set for which the Hausdorff Besicovitch dimension strictly exceeds the topological dimension.*"

Now it is clear the intention behind $D_T(A) < D_{HB}(A)$, for this allows a geometric interpretation of the objects: one is Euclidean the other is not. Then the need for a new term: fractal – of fractionated, irregular appearance.

How easy it is to classify an object as fractal in these terms? That depends on the object. Euclidean objects are easy, but in general it is necessary an individual analysis for each case. This because the Hausdorff measure is highly theoretical and general, the 'all possible covers' makes the direct use of the definition impractical.

The usual way of overcoming this is to use another dimension instead of $H_{HB}$, which can be more easily evaluated. But not always the proper care is taken and the readers do not know of the substitution (or of the several alternatives) nor that the number presented is really an upper bound. Fact regretted by Mandelbrot in the revised edition [MAN83]. Then he restates that the dimension idea and not its formal definition is to be used. After twenty years of research we should readdress this problem and, hopefully, be able to solve it.

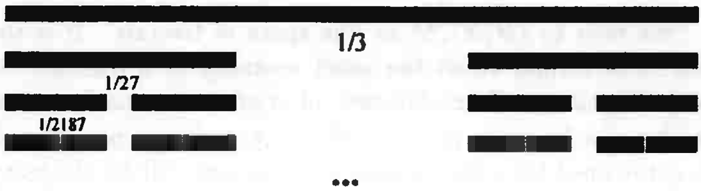
The dimension evaluation difficulties make it very hard to use it as a sieve to tell whether we have a fractal or not. Normally we hear (from respectable authors): 'the system presents fractal features', 'the object possess a fractal nature' etc. And some authors rush into classifying as fractal when any dimension is non-integer, though this do not guarantee that $D_{HB} > D_T$.

Worse even. We could argue that if the definition is general, what is needed is some good theoretical result to provide alternative (and hopefully easier) ways of evaluation. But even this is problematic, for there are objects built as fractals normally are, but have $D_{HB} = D_T$. How to say that they are Euclidean? Or why all tools and techniques we use for fractals work for them? Should we make up a hemi-fractal, and how it is different from a non-fractal?

One example is the devil's staircase and other are the fat Cantor sets (both well illustrated in [SCH91]):

**Exemplo 3.1.1** A fat Cantor set.

Here we have a fat Cantor set built removing the middle segments of relative (to each other) size decreasing with $1/3^{2^k}$, in the figure we have the first third removed then two segments of $1/9$ of the size of the initial one, then four segments $1/81$ the previous ones and so on:




First steps in constructing a fat Cantor set

In this case the set built has infinetely many holes but its final lenght is greater than zero:

$$L_n = 1 - \prod_{k=0}^{n} (1 - 3^{2^k})$$

And has a limit for $n \to \infty$: $L_\infty = 0.41482\ldots$

---

more annoying even: the Mandelbrot set (based on $z = z^2 + c$) border has dimension 2, this put one of the most popular examples in limbo!

This problems have not passed unnoticed, for instance Edgar [EDG90, pg. 180] quotes Taylor, that proposes one further restriction to the definition: "$F$ is a fractal iif $dim_T(F) < dim_{HB}(F) = dim_P$"[1]. This improvement fails to classify the objects above and others, so it inherits the same difficulties of the original definition, but it leaves out some objects that do not look or are built like fractals (Mandelbrot called them 'true geometric chaos').

Taking this in consideration it seems that dimensions and their values are 'side-effects' or consequences of fractals, but not necessarly fulfils the proposed inequalities in all of them. Dimensions are and will continue to be a fundamental tool and feature to be studied, but they are nor enough to define fractals

### 3.1.2 Barnsley

Barnsley in [BAR88] recognizes this situation and presents a new definition that abandons completely the dimension requirements. It is not a proposal to substitute the old one, though, but simply what the author needed in his work. Particularly what was best suited for the tools developed and used by him.

Let's see the definition:

[1] Here appears yet another dimension: $dim_P$ that is the packing dimension.

> "We refer to $(H(X), h)$ as 'the space of fractals'. It is too soon to be formal about the exact meaning of 'a fractal'. At the present stage of development of science and mathematics, the idea of a fractal is most useful as a broad concept. Fractals are not defined by a short legalistic statement, but by the many pictures and contexts which refer to them. For us [...] any subset of $(H(X), h)$ is a fractal. However as with the concept of 'space' more meaning is suggested than is formalized."

Formally what is said? $(H(X), h)$ is the complete metric space of compact subsets of a given complete metric space $X$ with the Hausdorff metric (based on the $X$ metric) and a fractal is any subset of $(H(X), h)$. But a single point in $X$ is a compact subset of $X$, then any union of points is a fractal! Conclusion: any subset of $X$ is a fractal.

Two things: this does not add anything to what is a fractal, for everything is a fractal. Furthermore we are restricted to complete metric spaces. It seems a not too useful definition.

## 3.2 Definition

If the current definitions aren't adequate, the best path it to backtrack and revisit our notions and expectations of what is fractal to search for a new definition. What identifies a fractal before one thinks of dimension? Its self-similarity, having parts similar to the whole is essential.

### 3.2.1 Visualization

Here we make a little pause and note what Mandelbrot meant when he defends the 'return of geometry' as a important notion to the mathematical development.

How to know if a set is or not self-similar? By its definition? Normally no. Our vision is the most important tool in this process. The current way introduced by Mandelbrot is fundamentally important: computerized visualization[2]. Without 'seeing' a set it is very difficult to see it's geometrical properties, in particular sets originated in the study of iterated functions. This adds up to the importance of computerized visualization as a fundamental tool to mathematics.

[2]Not even Mandelbrot was prepare for the complexity and detail present in the first images, which he thought were attributed to some error in calculation. Soon he discovered that everything was correct, and that simple rules generated sophisticated behavior

This is a return to the 'visual' in mathematics. Where, clearly, our geometrical vision can have a important role. Other thing that comes from this is the suggestion of facts and properties to be proven (some examples: connectivity of $M$, universality of $\delta$ etc.) and the inspiration of researchers investigating images of fractals.

The work of mathematicians in research assisted by computers provides several benefits in relation to the traditional isolation work. One of this points is the feedback capacity of this work. Interesting systems can generate interesting images that suggest interesting properties to be studied. Point that will probably be a great attractive to students to become professional mathematicians.

### 3.2.2 Main Features

But there is another point common to all fractals – as much as self similarity –, and fundamental to their genesis: iteration. From this the computer becomes the ideal tool to exploration and study of iterated systems.

Practically all fractal sets are generated by iterated functions[3], and all were created from iterated functions. With this the importance of iterated functions becomes clearer, and it is why they are present with great care and rigour in the author's doctorate thesis (still in course).

Summing up the main features of fractals in order of appearance in normal use:

1. Iteration
2. Self-similarity
3. Dimensionality

Mandelbrot's idea is the last in the sequence: it is probably consequence from the previous ones. It has the problems we have already discussed and the advantage of being extremely compact and allows an easy geometric interpretation of the objects structure. The second is the fractal's essence but is of very difficult definition. Some possibilities are restricted to affine systems, to the general case we do not know what means non-strict self-similarity. The concept seems to be a geometric feature arising from the object with determined construction.

[3]One exception could be the triadic Cantor set, that can be defined in terms of the triadic expansion of numbers in $[0;1]$. All points in the set do not have the 1 digit when written in base 3 (except as the last non-null digit). This can also be considered a iterated system, but wheather this uses iteration is an argument that does not belong here.

### 3.2.3 A new proposal

Then there is iteration, as a seed to definition. We see two possibilities here.

First: call the iterated function a fractal. This would be good for the iterated systems (for they are defined in terms of iterated functions) study and would enhance the importance of the process in the objects creation. But the process, the transformation, the iteration does not capture nor allow a geometric approach as we defended a while ago.

As second possibility we have the concept of em locus as inspiration. Why? Because all fractals are sets of points that when used as seed to some iterated function generate orbit with some desired property: convergence, periodicity etc. The resulting set is the locus of all those seed that share this quality.

This is what is behind all fractals, as geometrical objects: sets of points that generate orbits with the same properties. Using this we are able to present a new definition and framework:

**Definição 3.2.1** *Fractal: Let $f : E \to E$ be a iterated function of an arbitrary space into itself; $M : \text{Orb}(f) \to R$ be an evaluation function from the orbits space $\text{Orb}(f)$ to some result space $R$. This is a function (normally an algorithm) that tells us which property the orbit has. And let $r$ be a value of $R$ (one property that we wish to study)*

*Then the fractal is the set $A$ such as: $A = \{x \in E / M(o(x)) = r\}$.*

*that is all the points that when used as seeds generate orbits with the same property.*

Therefore any set formed by points that have the same property (when iterated by $f^n$) is a fractal. This agrees with the definition of classical fractals and includes the 'strange' cases as the $M$ set border. And we have even another benefit: the fractal framework is organized and exposed.

This is a more broad definition than Mandelbrot's and more specific and adjusted then Barnsleys. The experience accumulated by researchers like these is what takes us to a definition that incorporates the genesis of fractals – the iteration – and the geometric notion that always motivated these studies – the locus. The self-similarity and dimentionality are features that are studied as consequences not causes.

# Chapter 4

# Conclusion

We hope to provide a new definition that is both useful, applicable and that provides for the geometric mathematics we are beginning to use more and more in the fractal field.

The iteration concept is in itself exceptional, because it can transcend its place and appear as a geometrical feature when used to create an object. This is one of several mysterious points regarding fractals, but that excites our imagination and abilities to discover the reason for this.

Regarding fractals we see theory and application, the last is well developed and makes up it's own tools borrowing from fractal theory and other fields expertise. But with fractal theory we do not have the same richness, partly because of the lack of definitions and structured frameworks to refer. Each researcher then develops and presents his own view, this has given us a great number of opinions and insights to the theory, but has also wasted time and built up a fractal babel. This is why we believe in a new definition and way of work to gather all this efforts in a same stream. Never limited, and always flowing strong but as convolutes as the object we study.

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