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Automorphic measures and invariant distributions for circle dynamics

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Abstract
Let f be a $C^{1+\mathrm{bv}}$ circle diffeomorphism with irrational rotation number. As established by Douady and Yoccoz in the eighties, for any given $s > 0$ there exists a unique *automorphic measure* of exponent s for f. In the present paper we prove that the same holds for *multicritical circle maps*, and we provide two applications of this result. The first one, is to prove that the space of *invariant distributions of order* 1 of any given multicritical circle map is one-dimensional, spanned by the unique invariant measure. The second one, is an improvement over the *Denjoy–Koksma inequality* for multicritical circle maps and absolutely continuous observables.

Keywords Critical circle maps · Automorphic measures · Invariant distributions · Denjoy–Koksma inequality

Mathematics Subject Classification Primary 37E10; Secondary 37C40

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1 Introduction

Smooth one-dimensional dynamical systems can be studied from various viewpoints, such as their topological classification, their smooth rigidity properties, the behaviour of their individual orbits or their measure-theoretic and ergodic properties. A specific class of such systems that has received a great deal of attention in recent years is the class of *multicritical circle maps*.

A *multicritical circle map* is a C^3 circle homeomorphism $f : S^1 \to S^1$ having $N \geq 1$ critical points (all of which are *non-flat*, see Sect. 2). We are only interested in maps of this type having no periodic points, in other words, only in those maps that have irrational rotation number. The classification of such maps up to topological conjugacy goes back to Yoccoz [38], who proved that they are always minimal, hence topologically equivalent to a rotation of the circle (see Theorem 2.4 below). The smooth rigidity of such maps—including the preliminary step known as *quasi-symmetric rigidity*—has been the object of intense research in recent decades; it is by now fairly well-understood, at least in the unicritical case, thanks to the combined efforts of several mathematicians, see [3, 6, 11, 12, 16, 19, 20, 25, 26, 33–36], or the book [8] and references therein (we note *en passant* that, quite recently, some rigidity results for maps with *more than one* critical point have been established, see [4, 5, 37] and the recent preprint [17]). The geometric behaviour of individual orbits of such maps was examined in the recent paper [10].

From the measure-theoretic viewpoint, multicritical circle maps have also been studied in detail. Having irrational rotation number, they are uniquely ergodic. Their unique invariant probability measure was shown to be purely singular with respect to Lebesgue (Haar) measure by Khanin [24] (see also [18]), and later it was shown to have zero Lyapunov exponent in [7]. In [9], the authors went a bit further and showed that such maps do not admit even σ-finite absolutely continuous invariant measures.

In the present paper, we are interested in further ergodic-theoretic properties of multicritical circle maps. In particular, we are interested in the question: "Does a (minimal) multicritical circle map admit other invariant *distributions* besides its unique invariant probability measure?". The analogous question in a more general setting seems to have been first asked by Katok (for a general reference, see [23]). However, our main source of inspiration is the remarkable paper by Avila and Kocsard [1], in which they give a fairly complete answer to the corresponding question for smooth circle *diffeomorphisms*.

Here, we give a (partial) answer to the above question (see Theorem B below) by relating it (following the paper [28], by Navas and Triestino) to the question of existence and uniqueness of so-called *automorphic measures* for multicritical circle maps—a question to which we give here a full answer (Theorem A).

Given $s \in \mathbb{R}$, an *automorphic measure* of exponent s for f is a Borel probability measure ν on S^1 whose pullback $f^*\nu$ is equivalent to ν, with Radon–Nikodym derivative given by $(Df)^s$. This concept is the analogue, for real one-dimensional maps, of the concept of *conformal measure* introduced by Sullivan in [31] in the context of rational maps—which in turn is inspired by a similar notion introduced by Patterson [29] and Sullivan himself [30] in the context of Fuchsian and Kleinian groups.

The precise definition of automorphic measure is given in Sect. 3 (Definition 3.1). In the eighties, it was proved by Douady and Yoccoz (but only published some years later, in [2]—see also [13]) that, for every minimal $C^{1+\mathrm{bv}}$ circle diffeomorphism and every real number s, there exists a unique automorphic measure of exponent s. In the present paper we prove the following result.

Theorem A (Existence and uniqueness of automorphic measures) *Let f be a multicritical circle map. For any given $s \geq 0$ there exists a unique automorphic measure of exponent s for f. This measure has no atoms, is supported on the whole circle and it is ergodic under f.*

1.1 Applications

In Sect. 7 of the present paper we provide a couple of applications of Theorem A, that we now describe.

As usual, a 1-*distribution* is a continuous linear functional defined on the space of C^1 real-valued functions of the circle (see Sect. 7.1 for precise definitions). As a consequence of Theorem A, we have the following result.

Theorem B (No invariant distributions) *Let f be a multicritical circle map with irrational rotation number and unique invariant measure μ. Then the space $\mathcal{D}'_1(f)$ of f-invariant distributions of order at most 1 is spanned by μ, that is,*

$$\mathcal{D}'_1(f) = \mathbb{R}\mu.$$

In other words, f admits no invariant distributions of order at most 1 different from (a scalar multiple of) its unique invariant measure. The proof of Theorem B will be given in Sect. 7.1, and will follow the approach of Navas and Triestino developed in [28] for $C^{1+\mathrm{bv}}$-diffeomorphisms. We would like to remark that this approach deals with distributions of order at most 1. Since multicritical circle maps are assumed to be C^3 smooth, it would be desirable to also rule out invariant distributions up to order 3. Unfortunately, we do not know how to do this. Moreover, if we consider C^∞, or C^ω, multicritical circle maps, we do not know how to deal with higher order distributions. Let us be more precise: a C^∞ dynamical system is *distributionally uniquely ergodic* if it admits a single invariant distribution (up to multiplication by a constant). In [1], Avila and Kocsard proved that every C^∞ circle diffeomorphism with irrational rotation number is distributionally uniquely ergodic. We believe that C^∞ multicritical circle maps are distributionally uniquely ergodic too but, as already mentioned, we do not know how to prove that. Nevertheless, to the best of our knowledge, Theorem B provides the first examples of dynamics with no invariant distributions of order at most 1 outside the realm of flows and diffeomorphisms. Finally, we would like to remark that the non-flatness condition on each critical point of f (see Sect. 2 below) is crucial in order to prove Theorem B. Indeed, the following holds.

Theorem C *For any given irrational number $\rho \in (0, 1)$ there exists a C^∞ homeomorphism $f: S^1 \to S^1$, with rotation number $\rho(f) = \rho$, having invariant distributions of order 1 (different from a scalar multiple of its unique invariant measure).*

The examples of Theorem C are those constructed by Hall in [21], see Sect. 7.2 for the details. They are uniquely ergodic, but they are not distributionally uniquely ergodic.

As it turns out, Theorem A implies the following improvement over the Denjoy–Koksma inequality for absolutely continuous observables.

Theorem D (Improved Denjoy–Koksma) *Let f be a multicritical circle map with irrational rotation number ρ and unique invariant measure μ, and let $\phi: S^1 \to \mathbb{R}$ be absolutely continuous. If $\{q_n\}$ is the sequence of denominators for the rational approximations of ρ, we have that*

$$q_n \left\| \frac{1}{q_n} \sum_{i=0}^{q_n-1} \phi \circ f^i - \int_{S^1} \phi \, d\mu \right\|_{C^0(S^1)} \longrightarrow 0 \quad \text{as } n \text{ goes to } \infty.$$

The proof of Theorem D will be given in Sect. 7.3, following the very same lines as that of Avila and Kocsard [1, Section 3] and Navas [27, Section 2] for circle diffeomorphisms.

The paper is organized as follows: in Sect. 2, we present a recap of some topics from the theory of critical circle maps. In Sect. 3, we define automorphic measures of positive exponent, and explore some of their elementary properties; we remark that, at that point in the paper, it is still not clear whether automorphic measures actually exist. This is done in Sect. 4, where we show that automorphic measures indeed exist for all positive exponents. In Sect. 5, we obtain fundamental bounds on automorphic measures, which may themselves be of interest in future works. In Sect. 6, we use said bounds to show that any automorphic measure is ergodic, and as a consequence, easily derive the uniqueness part of Theorem A. Finally, in Sect. 7, we prove that Theorem A implies both Theorems B and D, and we briefly explain the proof of Theorem C.

2 Multicritical circle maps

Let us now define the maps which are the main object of study in the present paper. We start with the notion of *non-flat critical point*.

Definition 2.1 We say that a critical point c of a one-dimensional C^3 map f is *non-flat* of degree $d > 1$ if there exists a neighborhood W_c of the critical point and a C^3 diffeomorphism $\phi_c \colon W_c \to \phi_c(W_c) \subset \mathbb{R}$ such that $\phi_c(c) = 0$ and, for all $x \in W_c$,

$$f(x) = f(c) + \phi_c(x)\ |\phi_c(x)|^{d-1}.$$

This local form easily implies the following estimate (see [8, ch. 5]).

Proposition 2.2 *Let c be a non-flat critical point of degree d of a one-dimensional C^3 map f. There exists an interval $U = U_c \subset W_c$ that contains c such that, for any non-empty interval $J \subset U$ and $x \in J$,*

$$Df(x) \leq 3d\,\frac{|f(J)|}{|J|}, \tag{2.1}$$

where $|J|$ denotes the Euclidean length of an interval J.

Definition 2.3 A *multicritical circle map* is an orientation-preserving C^3 circle homeomorphism having finitely many critical points, all of which are non-flat.

We refer the reader to [8, ch. 6], where examples of multicritical circle maps are discussed. Being an orientation-preserving circle homeomorphism, a multicritical circle map f has a well defined rotation number. We will focus on the case that f has no periodic orbits (i.e., $\rho(f) \notin \mathbb{Q}$). As it turns out, these maps have no wandering intervals. More precisely, we have the following fundamental result.

Theorem 2.4 *Let f be a multicritical circle map with irrational rotation number ρ. Then f is topologically conjugate to the rigid rotation R_ρ, i.e., there exists a homeomorphism $h \colon S^1 \to S^1$ such that $h \circ f = R_\rho \circ h$.*

Theorem 2.4 was proved by Yoccoz in [38], see also [8, ch. 6].

2.1 The Koebe distortion principle

Given two circle intervals $M \subset T \subset S^1$ with M compactly contained in the interior of T (written $M \Subset T$), we denote by L and R the two connected components of $T \setminus M$. The *space of M inside T* is defined to be the number

$$\tau = \min\left\{ \frac{|L|}{|M|}, \frac{|R|}{|M|} \right\}. \tag{2.2}$$

Given circle intervals M, T with $M \Subset T$ and $k \geq 1$ such that $f^k \colon T \to f^k(T)$ is a C^1 diffeomorphism onto its image, one can bound the distortion of f^k inside M independently of k as long as the intermediate images $T, f(T), \ldots, f^{k-1}(T)$ satisfy a mild summability condition and the space of $f^k(M)$ inside $f^k(T)$ is bounded from below independently of k. This is the content of the Koebe distortion principle, and, as one can expect, it is of fundamental importance in controlling the geometric behavior of large iterates of the map f.

Lemma 2.5 (Koebe distortion principle) *For each $\ell, \tau > 0$ and each multicritical circle map $f \colon S^1 \to S^1$ there exists a constant $K = K(\ell, \tau, f) > 1$ with the following property. If $k \geq 1$, $M \subset T \subset S^1$ are intervals, with M compactly contained in the interior of T, are such that the intervals $T, f(T), \ldots, f^{k-1}(T)$ contain no critical point of f,*

$$\sum_{j=0}^{k-1} \left| f^j(T) \right| \leq \ell \tag{2.3}$$

and the space of $f^k(M)$ inside $f^k(T)$ is at least τ, then

$$K^{-1} \leq \frac{Df^k(x)}{Df^k(y)} \leq K \quad \textit{for all } x, y \in M. \tag{2.4}$$

A proof of the Koebe distortion principle can be found in [14, p. 295].

Remark 2.6 Given a family of intervals $\mathcal{F}$ on S^1 and a positive integer m, we say that $\mathcal{F}$ has *multiplicity of intersection at most m* if each $x \in S^1$ belongs to at most m elements of $\mathcal{F}$. For our purposes, the following (elementary) observation relating the hypotheses of the Koebe distortion principle to multiplicity of intersection will be crucial: if the family $T, f(T), \ldots, f^{k-1}(T)$ has multiplicity of intersection at most m, then (2.3) holds with $\ell = m$. This observation also holds in the context of arbitrary finite measures on S^1: if ν is a finite measure on the circle, $m \geq 1$ and $\mathcal{F}$ is a family of circle intervals with intersection multiplicity at most m, then

$$\sum_{I \in \mathcal{F}} \nu(I) \leq m\, \nu(S^1).$$

2.2 Combinatorics and real bounds

Throughout this paper, $f \colon S^1 \to S^1$ will be a C^3 multicritical circle map with irrational rotation number. Furthermore, $N \geq 1$ will be the number of critical points of f, $\mathrm{Crit}\,(f) = \{c_1, \ldots, c_N\}$ will be the set of critical points of f, and $d_1, \ldots, d_N$ their corresponding criticalities.

Let ρ be the rotation number of f. Being irrational, it has an infinite continued fraction expansion, say

$$\rho(f) = [a_0, a_1, \ldots] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}}}.$$

Truncating the expansion at level $n-1$, we obtain a sequence of fractions p_n/q_n which are called the *convergents* of the irrational ρ.

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_{n-1}] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1}}}}}.$$

Since each p_n/q_n is the best possible approximation to ρ by fractions with denominator at most q_n, we have

$$\text{If } 0 < q < q_n \text{ then } \left|\rho - \frac{p_n}{q_n}\right| < \left|\rho - \frac{p}{q}\right|, \quad \text{for any } p \in \mathbb{Z}.$$

The sequence of numerators satisfies

$$p_0 = 0, \quad p_1 = 1, \quad p_{n+1} = a_n p_n + p_{n-1} \text{ for } n \geq 1.$$

Analogously, the sequence of the denominators, which we call the *return times*, satisfies

$$q_0 = 1, \quad q_1 = a_0, \quad q_{n+1} = a_n q_n + q_{n-1} \text{ for } n \geq 1.$$

For each point $x \in S^1$ and each non-negative integer n, let $I_n(x)$ be the closed interval with endpoints x and $f^{q_n}(x)$ containing $f^{q_{n+2}}(x)$ (note that $I_n(x)$ contains no other iterate $f^j(x)$ with $1 \leq j \leq q_n - 1$). We write $I_n^j(x) = f^j(I_n(x))$ for all j and n.

Lemma 2.7 *For each $n \geq 0$ and each $x \in S^1$, the collection of intervals*

$$\mathcal{P}_n(x) = \left\{ f^i(I_n(x)) : \ 0 \leq i \leq q_{n+1} - 1 \right\} \bigcup \left\{ f^j(I_{n+1}(x)) : \ 0 \leq j \leq q_n - 1 \right\}$$

is a partition of the unit circle (modulo *endpoints*), *called the n*-th dynamical partition *associated to the point x.*

For a proof of this lemma, see [8, ch. 6]. The intervals of the form $f^i(I_n(x))$ in $\mathcal{P}_n(x)$ are called *long*, while the intervals of the form $f^j(I_{n+1}(x))$ are called *short*. This nomenclature is inspired by the rigid rotation, for which the long intervals indeed have longer (Lebesgue) length than the short ones.

Note that, for each n, the partition $\mathcal{P}_{n+1}(x)$ is a (non-strict) refinement of $\mathcal{P}_n(x)$ (see Fig. 1 below): the short intervals of $\mathcal{P}_n(x)$ become long intervals of $\mathcal{P}_{n+1}(x)$, while each of the long intervals of $\mathcal{P}_n(x)$ are partitioned into one short interval at level $n+1$ (an iterate of $I_{n+2}(x)$) and a_{n+1} long intervals at level $n+1$ (iterates $f^j(I_{n+1}(x))$ for $q_n \leq j < q_{n+2}$). Meanwhile, the partition $\mathcal{P}_{n+2}(x)$ is a strict refinement of $\mathcal{P}_n(x)$.

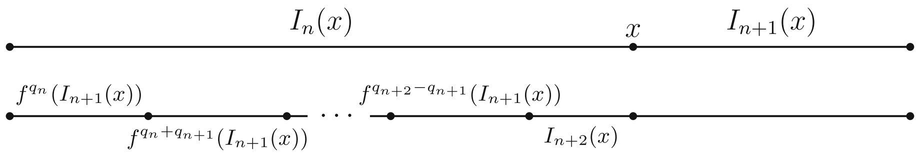

Fig. 1 Moving down levels in dynamical partitions, short intervals become long, while long intervals are subdivided

Theorem 2.8 (Real bounds) *There exists a constant $C = C(f) > 1$, depending only on f, such that the following holds for every critical point c of f. For all $n \geq 0$ and for each pair of adjacent atoms $I, J \in \mathcal{P}_n(c)$ we have*

$$C^{-1}|J| \leq |I| \leq C|J|. \tag{2.5}$$

Note that for a rigid rotation we have $|I_n| = a_{n+1}|I_{n+1}| + |I_{n+2}|$. If a_{n+1} is large, then I_n is much larger than I_{n+1}. Thus, even for rigid rotations, real bounds do not hold in general.

Theorem 2.8 was obtained by Herman [22], based on estimates by Świątek [32]. A detailed proof can be found in [8, ch. 6].

Theorem 2.8 has the following consequence (see [8, ch. 8]).

Lemma 2.9 *There exists $C_1 = C_1(f) > 0$ such that, for each $x \in S^1$ and all $n \geq 0$, we have $Df^{q_n}(x) \leq C_1$.*

Yet another consequence of the real bounds that will be useful in the present paper (see Sect. 4 below) is the following.

Lemma 2.10 (Zero Lyapunov exponent) *Let f be a multicritical circle map with irrational rotation number and unique invariant measure μ. Then $\log Df \in L^1(\mu)$ and*

$$\int_{S^1} \log Df \, d\mu = 0.$$

A proof of Lemma 2.10 can be found in [7] (see also [8, section 8.3]).

2.2.1 On the notions of domination and comparability

To simplify both the understanding of and future calculations involving the real bounds, we introduce the notions of domination and comparability *modulo f*.

Given two circle intervals I, J, we will say that I *dominates* J *modulo* f, and write $I \succeq J$, if there exists a constant $K > 1$ depending only on f such that $|J| \leq K|I|$. If both $I \succeq J$ and $J \succeq I$ (i.e. if there is $K = K(f) > 1$ such that $K^{-1}|I| \leq |J| \leq K|I|$), we will say that *$I$ and J are comparable modulo f* (and write $I \asymp J$).

Thus, Theorem 2.8 states precisely that adjacent atoms of a dynamical partition are always comparable.

Observe that neither domination or comparability are transitive relations: if we are given a domination chain $I_1 \succeq I_2 \succeq \cdots \succeq I_k$, we can only say that $I_1 \succeq I_k$ if the length k of the chain is bounded by a constant that depends only on f (and similarly for comparability).

3 Automorphic measures

In this section we define automorphic measures of non-negative exponent for multicritical circle maps. We further prove that they have full support on the circle (Proposition 3.3) and are non-atomic (Lemma 3.4).

Definition 3.1 (*Automorphic measures*) Let $s \geq 0$. An *automorphic measure of exponent s for f* (or f-automorphic measure of exponent s) is a Radon probability measure ν on S^1 such that, for all continuous functions $\phi \in C^0(S^1)$,

$$\int_{S^1} \phi \, d\nu = \int_{S^1} (\phi \circ f) \, (Df)^s \, d\nu. \tag{3.1}$$

We denote the set of f-automorphic measures of exponent s by $\mathcal{A}_s$.

Equivalently (see Proposition 3.2 below), a Radon probability measure ν on S^1 is f-automorphic of exponent s if, and only if, the pullback measure $f^*\nu$ is equivalent to ν, with Radon–Nikodym derivative

$$\frac{df^*\nu}{d\nu} = (Df)^s.$$

Observe that we leave out the possibility of negative exponents, i.e., $s < 0$. Though automorphic measures of negative exponent make perfect sense and indeed always exist in the case of diffeomorphisms,[1] they are significantly more difficult to work with in the critical case. Indeed, if $s < 0$, then $(Df)^s$ blows up at the critical points of f, so we cannot take for granted that $(\phi \circ f) \, (Df)^s$ will be ν-integrable for any $\phi \in C^0(S^1)$ and Radon probability measure ν on S^1.

We further remark that the notion of automorphic measures makes perfect sense on any dimension, provided the one-dimensional derivative $Df(x)$ is replaced by the Jacobian of f at x, i.e., the absolute value of the determinant of the matrix $Df(x)$. As we mentioned in the introduction, for complex one-dimensional systems this is exactly the same as the notion of *conformal measure* introduced by Sullivan in [31]. In the present paper, however, we will of course only treat the real one-dimensional case.

Finally, observe that, in the case $s = 0$, an f-automorphic measure of exponent 0 is simply an f-invariant probability measure. Therefore, the case $s = 0$ is well understood, and Theorem A in this case is precisely the statement that f (just like any circle homeomorphism with irrational rotation number) is uniquely ergodic. Therefore, for the rest of this paper, we will focus on positive exponents. Thus, let $s > 0$ and $\nu \in \mathcal{A}_s$ be fixed.

Proposition 3.2 *For all $\phi \in L^1(\nu)$ and $n \geq 1$, $(\phi \circ f^n)(Df^n)^s \in L^1(\nu)$ and*

$$\int_{S^1} \phi \, d\nu = \int_{S^1} (\phi \circ f^n)(Df^n)^s \, d\nu. \tag{3.2}$$

Proof Observe that (3.2) holds trivially if ϕ is continuous, by applying (3.1) inductively. The extension to L^1 functions ϕ now follows from a standard argument, with the main difficulty being to show that $(\phi \circ f^n)(Df^n)^s \in L^1(\nu)$ for all $\phi \in L^1(\nu)$. □

[1] As it happens, the case $s = -1$ is suitable to understand both the variation of the rotation number along generic 1-parameter families of circle diffeomorphisms [13], as well as to build the tangent space of the set of C^2 diffeomorphisms with a given irrational rotation number [2, Théorème 2]. See also the recent preprint [15].

Note in particular that, under forward iteration, the ν-measure of a Borel set $A \subset S^1$ behaves according to the following rule:

$$\nu\big(f^n(A)\big) = \int_A (Df^n)^s \, d\nu, \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

Proposition 3.3 *The measure ν is supported on the entire circle.*

Proof Since the pullback measure $f^*\nu$ is equivalent to ν, ν-null sets are mapped under f into ν-null sets (see (3.3) above). But, since f is topologically conjugate to an irrational rotation, the positive orbit of an open interval eventually covers the whole circle, and then this interval must have positive ν-measure. □

We denote by

$$\mathrm{Crit}^{\pm}(f) = \bigcup_{j=1}^{N} \mathcal{O}_f(c_j)$$

the union of the critical orbits of f, and its complement $S^1 \setminus \mathrm{Crit}^{\pm}(f)$ by Λ. Observe that Λ is f-invariant and its complement is countable (but dense!).

Lemma 3.4 *The measure ν has no atoms. In particular, Λ has full ν-measure on the circle.*

Proof Arguing by contradiction, suppose there is some $x_0 \in S^1$ such that $\nu(\{x_0\}) = \delta > 0$, and note that (3.3) implies

$$\nu(\{x_0\}) = \nu\Big(\big\{f^{-n}(x_0)\big\}\Big)\,\big(Df^n(f^{-n}(x_0))\big)^s \quad \text{for all } n \in \mathbb{N}.$$

In particular, x_0 cannot be in the forward orbit of any critical point of f. Moreover, since ν is a probability measure and f has no periodic orbits,

$$1 \geq \sum_{n=0}^{\infty} \nu\left(\big\{f^{-n}(x_0)\big\}\right) = \delta \sum_{n=0}^{\infty} \frac{1}{\big(Df^n(f^{-n}(x_0))\big)^s} \geq \delta \sum_{n=0}^{\infty} \frac{1}{\big(Df^{q_n}(f^{-q_n}(x_0))\big)^s}.$$

However, by Lemma 2.9, we have $Df^{q_n}(f^{-q_n}(x_0)) \leq C_1$ for all $n \geq 0$. Thus we obtain

$$1 \geq \delta \sum_{n=0}^{\infty} C_1^{-s} = \infty,$$

which is the desired contradiction. □

4 Existence

In this section we show that, for all $s > 0$, $\mathcal{A}_s$ is non-empty (the existence part of Theorem A). For the entire section, s will be a fixed positive number.

Let $P_{s,f} \colon S^1 \to [0, \infty]$ be the *Poincaré series* defined by[2]

$$P_{s,f}(x) = \sum_{n=0}^{\infty} \big(Df^n(x)\big)^s. \tag{4.1}$$

[2] We use the expression *Poincaré series* by analogy with a similar series appearing in the study of Fuchsian or Klenian groups (see for instance [30]).

Observe that if $f^k(x) \in \operatorname{Crit}(f)$ for some $k \geq 0$, then $P_{s,f}(x) < \infty$, since it is just a finite sum. Therefore, there is a dense subset of S^1 (the union of the backward orbits of the critical points) on which $P_{s,f}$ is finite. However, as the following lemma shows, there are plenty of points on the circle where $P_{s,f}$ diverges.

Lemma 4.1 *The Poincaré series $P_{s,f}$ diverges μ-almost everywhere.*

Proof We show that the set

$$A := \left\{ x \in \Lambda \;\middle|\; P_{s,f}(x) = \infty \right\} \tag{4.2}$$

has full μ-measure. To do this, first observe that, for all $x \in S^1$,

$$P_{s,f}(x) = 1 + (Df(x))^s \, P_{s,f}(f(x)).$$

It follows that A is f-invariant, so, by the ergodicity of μ, it suffices to show that $\mu(A) > 0$.

For each $n \geq 0$, $(Df^n)^s \in C^2(S^1)$, so we may apply Jensen's inequality to obtain

$$\log\left(\int_{S^1} \left(Df^n\right)^s \, d\mu\right) \geq \int_{S^1} \log\left(Df^n\right)^s \, d\mu = s \sum_{i=0}^{n-1} \int_{S^1} \log Df \circ f^i \, d\mu.$$

Since μ is f-invariant,

$$\log\left(\int_{S^1} \left(Df^n\right)^s \, d\mu\right) \geq s \sum_{i=0}^{n-1} \int_{S^1} \log Df \, d\mu = 0, \tag{4.3}$$

where we have used Lemma 2.10. Thus,

$$\int_{S^1} \left(Df^n\right)^s \, d\mu \geq 1$$

for all $n \geq 0$, which implies in particular that

$$\int_{S^1} \sum_{k=0}^{n-1} (Df^{q_k})^s \, d\mu \geq n \tag{4.4}$$

for all $n \geq 1$.

We argue by contradiction. Suppose $\mu(A) = 0$; then $P_{s,f}$ must be finite μ-almost everywhere. Now, for each $m \geq 1$, let

$$X_m := \left\{ x \in S^1 \;\middle|\; P_{s,f}(x) \leq m \right\}. \tag{4.5}$$

Since we are assuming that $P_{s,f}$ is finite μ-almost everywhere,

$$\lim_{m \to \infty} \mu(X_m) = 1. \tag{4.6}$$

Let $0 < \epsilon < C_1^{-s}$, where $C_1 = C_1(f)$ is the constant of Lemma 2.9. From (4.6), there exists $m_0 \in \mathbb{N}$ such that, for all $m \geq m_0$, $\mu(S^1 \setminus X_m) \leq \epsilon$.

But then, from (4.4), (4.5) and Lemma 2.9, we have that, for all $n \geq 1$ and $m \geq m_0$,

$$\begin{aligned} n &\leq \int_{S^1} \sum_{k=0}^{n-1} (Df^{q_k})^s \, d\mu = \int_{X_m} \sum_{k=0}^{n-1} (Df^{q_k})^s \, d\mu + \int_{S^1 \setminus X_m} \sum_{k=0}^{n-1} (Df^{q_k})^s \, d\mu \\ &\leq \int_{X_m} P_{s,f} \, d\mu + \sum_{k=0}^{n-1} \int_{S^1 \setminus X_m} (Df^{q_k})^s \, d\mu \leq m \, \mu(X_m) + n \, C_1^s \, \mu(S^1 \setminus X_m) \\ &\leq m + n \, C_1^s \, \epsilon, \end{aligned} \tag{4.7}$$

which implies that $m \geq n\,(1 - C_1^s\,\epsilon)$. Since $1 - C_1^s\,\epsilon > 0$, the contradiction arises by letting $n \to \infty$ while keeping m fixed. □

As it turns out, the only thing we will need in the proof below from the set A is the fact that it is non-empty, which certainly follows from Lemma 4.1.

Proof of Theorem A, existence part Consider the Poincaré series $P_{s,f}$ and the set A from Lemma 4.1. Fix $x \in A$ and, for each $n \geq 1$, let

$$S_n(x) = \sum_{i=0}^{q_n - 1} (Df^i(x))^s.$$

Consider

$$\mu_{s,x,n} := \frac{1}{S_n(x)} \sum_{i=0}^{q_n - 1} (Df^i(x))^s\,\delta_{f^i(x)},$$

which is an atomic probability measure. By compactness, there is a monotone sequence $(n_k) \subset \mathbb{N}$ and $\mu_{s,x} \in \mathcal{P}(S^1)$ such that, for all $\phi \in C^0(S^1)$,

$$\int_{S^1} \phi\, d\mu_{s,x,n_k} \longrightarrow \int_{S^1} \phi\, d\mu_{s,x}.$$

In particular, since $(\phi \circ f)(Df)^s \in C^0(S^1)$,

$$\int_{S^1} (\phi \circ f)(Df)^s\, d\mu_{s,x,n_k} \longrightarrow \int_{S^1} (\phi \circ f)(Df)^s\, d\mu_{s,x}$$

for all $\phi \in C^0(S^1)$. We claim that $\mu_{s,x}$ is automorphic of exponent s under f. Indeed, for all $k \geq 1$ and $\phi \in C^0(S^1)$, we have

$$\begin{aligned}
&\left| \int_{S^1} [\phi - (\phi \circ f)(Df)^s]\, d\mu_{s,x,n_k} \right| \\
&= \frac{1}{S_{n_k}(x)} \left| \sum_{i=0}^{q_{n_k} - 1} (Df^i(x))^s\, [\phi(f^i(x)) - \phi(f^{i+1}(x))(Df(f^i(x)))^s] \right| \\
&= \frac{1}{S_{n_k}(x)}\, \left| \phi(x) - \phi(f^{q_{n_k}}(x))\,(Df^{q_{n_k}}(x))^s \right| \\
&\leq \|\phi\|_{C^0}\, [1 + (Df^{q_{n_k}}(x))^s]\, \frac{1}{S_{n_k}(x)} \leq \|\phi\|_{C^0}\, (1 + C_1^s)\, \frac{1}{S_{n_k}(x)},
\end{aligned}$$

where we have used Lemma 2.9. Consequently,

$$\begin{aligned}
&\left| \int_{S^1} \phi\, d\mu_{s,x} - \int_{S^1} (\phi \circ f)(Df)^s\, d\mu_{s,x} \right| = \lim_{k \to \infty} \left| \int_{S^1} [\phi - (\phi \circ f)(Df)^s]\, d\mu_{s,x,n_k} \right| \\
&\leq \|\phi\|_{C^0}\, (1 + C_1^s)\, \lim_{k \to \infty} \frac{1}{S_{n_k}(x)} = 0,
\end{aligned}$$

since $x \in A$. Thus, $\mu_{s,x}$ is f-automorphic of exponent s, which concludes the proof. □

Remark 4.2 In [2, Section 3.2], Douady and Yoccoz prove the existence part of Theorem A in the context of diffeomorphisms through a different approach. First, they define a continuous operator $U_{s,f}\colon \mathcal{M}(S^1) \to \mathcal{M}(S^1)$ on the space $\mathcal{M}(S^1)$ of signed finite Radon measures on the circle (equipped with the weak* topology) by

$$\int_{S^1} \phi \, d(U_{s,f}\nu) := \frac{1}{\int_{S^1} (Df)^s \, d\nu} \int_{S^1} (\phi \circ f)(Df)^s \, d\nu.$$

Clearly, the operator $U_{s,f}$ leaves invariant the convex compact set $\mathcal{P}(S^1)$ of Radon probability measures on the circle. The authors then use the Schauder–Tychonoff fixed point theorem to conclude that $U_{s,f}$ has a fixed point $\mu_s \in \mathcal{P}(S^1)$, and through some estimates, they conclude that this fixed point μ_s must be f-automorphic of exponent s, i.e., $\int (Df)^s \, d\mu_s = 1$.

In the critical case, this approach fails. Indeed, if $\nu = \delta_c$ is a point mass on a critical point c of f, then $U_{s,f}\nu$ is ill-defined, since

$$\int_{S^1} (Df)^s \, d\nu = (Df(c))^s = 0.$$

Furthermore, if we remove from $\mathcal{P}(S^1)$ the point masses at the critical points of f, then we lose compactness, which is essential to apply the Schauder–Tychonoff fixed point theorem.

5 Bounds for automorphic measures

In the previous section we have proved the *existence* part of Theorem A. Sections 5 and 6 are devoted to *uniqueness*. In this section, we dive further into the fine-scale structure of f-automorphic measures of exponent $s > 0$. For this entire section, fix some $s > 0$ and $\nu \in \mathcal{A}_s$, and let $E \subset S^1$ be an arbitrary Borel f-invariant set.

In what follows, the ratio

$$\frac{\nu(I \cap E)}{|I|^s} \tag{5.1}$$

where I is an interval, will play a fundamental role. Hence we introduce the special notation

$$\omega(I) := \frac{\nu(I \cap E)}{|I|^s}. \tag{5.2}$$

The following theorem is the main result of this section.

Theorem 5.1 *There exists a constant $B = B(f, s) > 1$ with the following property. For any critical point c of f, sufficiently large n and $\Delta_1, \Delta_2 \in \mathcal{P}_n(c)$, we have*

(a) *If Δ_1, Δ_2 are both long atoms or both short atoms of $\mathcal{P}_n(c)$, then*

$$B^{-1}\,\omega(\Delta_2) \le \omega(\Delta_1) \le B\,\omega(\Delta_2). \tag{5.3}$$

(b) *If Δ_1 is a short atom and Δ_2 is a long atom of $\mathcal{P}_n(c)$, then*

$$\omega(\Delta_1) \le B\,\omega(\Delta_2). \tag{5.4}$$

5.1 Fundamental estimates on distortion

We must now introduce a bit of notation. For the rest of this paper, we fix a critical point c of f, and we write simply $\mathcal{P}_n$ in place of $\mathcal{P}_n(c)$. Furthermore, if $I \subset S^1$ is an interval, we

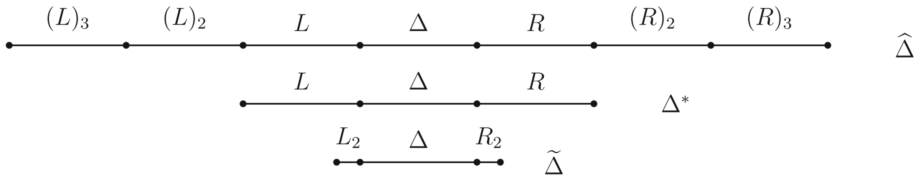

Fig. 2 The intervals Δ^*, $\widetilde{\Delta}$ and $\widehat{\Delta}$

write I^k for $f^k(I)$. For any $n \geq 0$ and any atom $\Delta \in \mathcal{P}_n$, we write Δ^* for the reunion of Δ with its two adjacent atoms, L and R, in $\mathcal{P}_n$. For example, if $\Delta = I_n$, then

$$\Delta^* = I_{n+1} \cup I_n \cup I_n^{q_n}.$$

We also write $\widetilde{\Delta}$ for the following interval. First write Δ as a reunion of atoms of $\mathcal{P}_{n+2}$ and let L_1, R_1 be the leftmost and rightmost atoms of $\mathcal{P}_{n+2}$ in this reunion, respectively; we then take $\widetilde{\Delta} = L_2 \cup \Delta \cup R_2$, where L_2, R_2 are the atoms of $\mathcal{P}_{n+2}$ left-adjacent to L_1 and right-adjacent to R_1, respectively. For example, if $\Delta = I_n$, then

$$\widetilde{\Delta} = I_{n+3} \cup I_n \cup I_{n+2}^{q_n}. \tag{5.5}$$

Lastly, we write $\widehat{\Delta}$ for the following interval (Fig. 2). If $\Delta^* = L \cup \Delta \cup R$, $L^* = (L)_2 \cup L \cup \Delta$ and $R^* = \Delta \cup R \cup (R)_2$, we will write

$$\widehat{\Delta} = (L)_2^* \cup \Delta \cup (R)_2^* = (L)_3 \cup (L)_2 \cup L \cup \Delta \cup R \cup (R)_2 \cup (R)_3 \supset \Delta^*. \tag{5.6}$$

For example, if $\Delta = I_n$ and $a_n \geq 5$, then

$$\widehat{\Delta} = I_n^{q_{n+1}-2q_n} \cup I_n^{q_{n+1}-q_n} \cup I_{n+1} \cup I_n \cup I_n^{q_n} \cup I_n^{2q_n} \cup I_n^{3q_n}.$$

Of course, if n is small, it may be that $\mathcal{P}_n$ has at most 7 atoms, so in this case we would have $\widehat{\Delta} = S^1$. Thus, when dealing with $\widehat{\Delta}$, we always assume that n is sufficiently large for $\widehat{\Delta}$ to be a proper interval.

To provide the bounds on distortion needed for the rest of this paper, we will need the following combinatorial facts. Although their proofs are somewhat involved, the techniques used are standard. Accordingly, we have decided to omit these proofs.

Recall from Sect. 2 that, given a family of intervals $\mathcal{F}$ on S^1 and a positive integer m, we say that $\mathcal{F}$ has multiplicity of intersection at most m if each $x \in S^1$ belongs to at most m elements of $\mathcal{F}$.

Lemma 5.2 *Let* $n \geq 0$, $\Delta \in \mathcal{P}_n$. *Then*:

(a) *the collection* $\{f^k(\Delta^*)\}_{k=0}^{q_{n+1}-1}$ *has intersection multiplicity at most* 3;
(b) *the collection* $\{f^k(\widetilde{\Delta})\}_{k=0}^{q_{n+1}-1}$ *has intersection multiplicity at most* 3;
(c) *the collection* $\{f^k(\widehat{\Delta})\}_{k=0}^{q_{n+1}-1}$ *has intersection multiplicity at most* 8.

We will need the following consequence of the Real Bounds.

Lemma 5.3 *There exists a constant* $C_2 = C_2(f) \geq C > 1$ *with the following property. For* $n \geq 0$, *let*

$$\mathcal{C}_n := \left\{ I_n^j \right\}_{j=0}^{2q_{n+1}} \cup \left\{ I_{n+1}^k \right\}_{k=0}^{q_n+q_{n+1}}$$

be the set of all atoms of $\mathcal{P}_n$, together with their forward images under f up to iterate $q_{n+1}+1$. Then, for any $J_1, J_2 \in \mathcal{C}_n$ that share a common endpoint,

$$C_2^{-1}\,|J_1| \le |J_2| \le C_2\,|J_1|\,. \tag{5.7}$$

The following two lemmas contain the bounds on distortion needed for the rest of this paper. Their proof is a standard application of Koebe's distortion principle, with Lemmas 5.2 and 5.3 guaranteeing that the corresponding hypotheses on summability and space are satisfied (recall Sect. 2.1).

Lemma 5.4 *There exists $B_0 = B_0(f) > 1$ with the following property. If $\Delta \in \mathcal{P}_n$ and $0 \le j < k \le q_{n+1}+1$ are such that the intervals $f^j(\widetilde{\Delta}), f^{j+1}(\widetilde{\Delta}), \ldots, f^{k-1}(\widetilde{\Delta})$ do not contain any critical point of f, then the map $f^{k-j}\colon f^j(\Delta) \to f^k(\Delta)$ has distortion bounded by B_0, that is*

$$B_0^{-1} \le \frac{Df^{k-j}(x)}{Df^{k-j}(y)} \le B_0 \quad \textit{for all } x, y \in f^j(\Delta).$$

Lemma 5.5 *There exists $B_1 = B_1(f) > 1$ with the following property. If $\Delta \in \mathcal{P}_n$ and $0 \le j < k \le q_{n+1}$ are such that the intervals $f^j(\widehat{\Delta}), f^{j+1}(\widehat{\Delta}), \ldots, f^{k-1}(\widehat{\Delta})$ do not contain any critical point of f, then the map $f^{k-j}\colon f^j(\Delta^*) \to f^k(\Delta^*)$ has distortion bounded by B_1, that is*

$$B_1^{-1} \le \frac{Df^{k-j}(x)}{Df^{k-j}(y)} \le B_1 \quad \textit{for all } x, y \in f^j(\Delta^*).$$

We remark that $\widehat{\Delta}$ and Lemma 5.5 will not be mentioned further in this section, but will play a fundamental role in Sect. 6.

5.2 ω-domination and comparability

To simplify both the statement and the proof of the remaining results in this section, we introduce the notions of *ω-domination* and *ω-comparability* between intervals. If $I, J \subset S^1$ are intervals, we will say that I *ω-dominates* J (and write $I \succcurlyeq J$) if there is some constant $K = K(f, s) > 1$ (depending *only* on f and s, but not on ν or B) such that

$$\omega(J) \le K\omega(I). \tag{5.8}$$

Similarly, we say that I, J are *ω-comparable* (and write $I \sim J$) if $I \succcurlyeq J$ and $J \succcurlyeq I$; that is, if there is some constant $K = K(f, s) > 1$ such that

$$K^{-1}\omega(I) \le \omega(J) \le K\omega(I). \tag{5.9}$$

In what follows, when a constant $K(f, s)$ is written after an expression of ω-domination or comparability between intervals, it is to be inferred that (5.8) or (5.9) hold for said constant and intervals.

Definition 5.6 Let $\Delta \in \mathcal{P}_n$, $0 \le k < q_{n+1}$. We will say that k is a *critical time of type 1* for Δ if $f^k(\widetilde{\Delta}) \cap \operatorname{Crit}(f) \ne \varnothing$.

Since f has N critical points $c_1, \ldots, c_N$ and the collection $\left\{f^k(\widetilde{\Delta})\right\}_{k=0}^{q_{n+1}-1}$ has intersection multiplicity at most 3 (see Lemma 5.2), it follows that, for any $n \ge 0$ and $\Delta \in \mathcal{P}_n$, there are at most $3N$ critical times of type 1 for Δ.

Remark 5.7 It follows easily from the minimality of f that there exists some level $n_0 = n_0(f) \in \mathbb{N}$, depending only on f, such that, for all $n \geq n_0$, $\Delta \in \mathcal{P}_n$ and $0 \leq k < q_{n+1}$ a critical time of type 1 for Δ, we have that: (i) $f^k(\widetilde{\Delta})$ contains a *single* critical point of f; and (ii) $f^k(\widetilde{\Delta}) \subset U$, where U is the interval about the critical point of f in $f^k(\widetilde{\Delta})$ from Proposition 2.2.

The following lemma tells us what happens to the ratios $\omega(I)$ as we iterate f while staying (combinatorially) far away from the critical points of f. Recall our use of the simplifying notation $I^j := f^j(I)$ for intervals $I \subset S^1$.

Lemma 5.8 *Let $n \geq 0$ and $\Delta \in \mathcal{P}_n$. Then, for any interval $I \subset \Delta$ and $0 \leq j < k \leq q_{n+1}+1$ such that the intervals $f^j(\widetilde{\Delta}), \dots, f^{k-1}(\widetilde{\Delta})$ do not contain any critical point of f,*

$$B_0^{-s}\,\omega(I^j) \leq \omega(I^k) \leq B_0^s\,\omega(I^j). \tag{5.10}$$

Proof Indeed, by Lemma 5.4, the distortion of f^{k-j} in Δ^j is bounded by B_0. By the Mean Value Theorem, there exists $z \in I^j$ such that

$$Df^{k-j}(z) = \frac{\left|I^k\right|}{\left|I^j\right|}.$$

Thus, for any $x \in \Delta^j \supset I^j$,

$$B_0^{-1}\frac{\left|I^k\right|}{\left|I^j\right|} \leq Df^{k-j}(x) \leq B_0\frac{\left|I^k\right|}{\left|I^j\right|}. \tag{5.11}$$

From Eq. (3.3), we have

$$\omega(I^k) = \left|I^k\right|^{-s}\int_{I^j\cap E}(Df^{k-j}(x))^s\,d\nu(x)$$

so, from (5.11), we get

$$B_0^{-s}\frac{\left|I^k\right|^s}{\left|I^j\right|^s}\left|I^k\right|^{-s}\nu(I^j\cap E) \leq \omega(I^k) \leq B_0^s\frac{\left|I^k\right|^s}{\left|I^j\right|^s}\left|I^k\right|^{-s}\nu(I^j\cap E)$$

which is (5.10). □

The next result is now an easy corollary of Lemma 5.8:

Corollary 5.9 *Let $n \geq 0$, $\Delta \in \mathcal{P}_n$, and let $0 \leq k_1 < k_2 < \cdots < k_r < q_{n+1}$ be the critical times of type 1 for Δ. Then, for any interval $I \subset \Delta$,*

(a) *$I, I^1, \dots, I^{k_1}$ are pairwise ω-comparable*;
(b) *For $1 \leq j < r$, $I^{k_j+1}, I^{k_j+2}, \dots, I^{k_{j+1}}$ are pairwise ω-comparable*;
(c) *$I^{k_r+1}, \dots, I^{q_{n+1}}$ are pairwise ω-comparable*,

all with constant $K(f, s) = B_0^s$.

To proceed, we need to study the behaviour of ω close to the critical set of f. For this purpose, recall that $d > 1$ denotes the maximum of the criticalities of the critical points of f.

Lemma 5.10 *Let $n \geq n_0$ (from Remark 5.7), $\Delta \in \mathcal{P}_n$, and let $I \subset \Delta$ be an interval.*

(a) *If* $0 \leq j < q_{n+1}$ *is a critical time of type* 1 *for* Δ, $I^j \succcurlyeq I^{j+1}$ *with constant* $K(f, s) = (3d)^s$;
(b) *if* $0 \leq \ell_1 < \ell_2 \leq q_{n+1}$, $I^{\ell_1} \succcurlyeq I^{\ell_2}$, *with constant* $K(f, s) = (3dB_0)^{4Ns}$.

Proof Observe that part (a) and Corollary 5.9 together imply (b), since we can join I^{ℓ_1} and I^{ℓ_2} by a ω-domination chain as follows: let $\ell_1 \leq k_i < \cdots < k_m < \ell_2$ be the critical times of type 1 for Δ between ℓ_1 and ℓ_2. Then

$$I^{\ell_1} \sim I^{k_i} \succcurlyeq I^{k_i+1} \sim I^{k_{i+1}} \succcurlyeq \cdots \sim I^{k_m} \succcurlyeq I^{k_m+1} \sim I^{\ell_2}.$$

Since there are at most $6N + 2$ atoms in this chain, it follows that $I^{\ell_1} \succcurlyeq I^{\ell_2}$. To determine the constant of ω-domination, we start with $K(f, s) = 1$ and move along this chain, multiplying by B_0^s for every $\sim$ and by $(3d)^s$ for every $\succcurlyeq$. There are $m - i + 2$ $\sim$'s and $m - i + 1$ $\succcurlyeq$'s, so (since $m - i + 1 \leq 3N$) we can take

$$K(f, s) = (3dB_0)^{4Ns} \geq (3d)^{(m-i+1)s} B_0^{(m-i+2)s}.$$

Thus, we only need to prove (a).

Observe that, from Eq. (3.3),

$$\omega(I^{j+1}) = \frac{\int_{I^j \cap E} (Df)^s \, d\nu}{\left|I^{j+1}\right|^s} = \frac{\left|I^j\right|^s}{\left|I^{j+1}\right|^s} \frac{\int_{I^j \cap E} (Df)^s \, d\nu}{\left|I^j\right|^s}. \tag{5.12}$$

Now, Proposition 2.2 implies that

$$(Df(x))^s \leq (3d)^s \frac{\left|I^{j+1}\right|^s}{\left|I^j\right|^s} \quad \text{for all } x \in I^j. \tag{5.13}$$

Combining (5.12) and (5.13), we get

$$\omega(I^{j+1}) \leq (3d)^s \omega(I^j), \tag{5.14}$$

which proves (a). □

Before moving forward to the proof of Theorem 5.1, we will first prove a lemma which states, essentially, that *long atoms* ω*-dominate short atoms*. To simplify the proofs of this lemma and of Theorem 5.1 below, we will denote all constants of ω-domination generically by $K = K(f, s)$. It is worth noting that one could, in principle, keep track of all the constants appearing of ω-domination in the following proofs and write them down explicitly.

Lemma 5.11 *Let* $n \geq n_0$, $\Delta_1, \Delta_2 \in \mathcal{P}_n \cup \left\{I_n^{q_{n+1}}, I_{n+1}^{q_n}\right\}$. *If* Δ_1 *is a long atom (or* $I_n^{q_{n+1}}$*) and* Δ_2*, a short atom (or* $I_{n+1}^{q_n}$*), of* $\mathcal{P}_n$*, then* $\Delta_1 \succcurlyeq \Delta_2$. *Furthermore, if* $a_{n+1} \geq 2$ *or* $a_{n+1} = a_{n+2} = 1$, *then* $\Delta_1^{k_1} \succcurlyeq \Delta_2^{k_2}$ *for any* $0 \leq k_1, k_2 \leq q_{n+1}$.

Proof We split the proof in two parts: (i) that $\Delta_1 \succcurlyeq \Delta_2$; and (ii) that $\Delta_1^{k_1} \succcurlyeq \Delta_2^{k_2}$ if $a_{n+1} \geq 2$ or $a_{n+1} = a_{n+2} = 1$.

To prove (i), observe that, from Lemma 5.10, we have $\Delta_1 \succcurlyeq I_n^{q_{n+1}}$ and $I_{n+1} \succcurlyeq \Delta_2$. Therefore, (i) will follow if we prove that $I_n^{q_{n+1}} \succcurlyeq I_{n+1}$. Since $I_n^{q_{n+1}} \supset I_{n+1}$ (so $\nu(I_{n+1} \cap E) \leq \nu(I_n^{q_{n+1}} \cap E)$), this is a consequence of the fact that these two intervals have comparable lengths (see [8, Prop. 6.1]).

We now turn to the proof of (ii). From Lemma 5.10, we get $\Delta_1^{k_1} \succcurlyeq I_n^{2q_{n+1}-1}$ and $I_{n+1} \succcurlyeq \Delta_2^{k_2}$. By applying either Lemma 5.8 or Lemma 5.10, depending on whether $I_n^{2q_{n+1}-1} \cap$

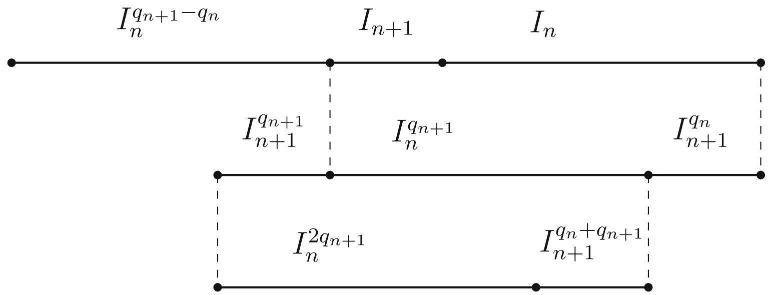

Fig. 3 Relative positions of the intervals I_n, I_{n+1}, $I_n^{q_{n+1}-q_n}$, $I_n^{q_{n+1}}$, $I_{n+1}^{q_n}$, $I_{n+1}^{q_{n+1}}$, $I_n^{2q_{n+1}}$, $I_{n+1}^{q_n+q_{n+1}}$ when $a_{n+1} \geq 2$

Crit $(f) = \varnothing$ or not, we get $I_n^{2q_{n+1}-1} \succcurlyeq I_n^{2q_{n+1}}$, from which it follows that $\Delta_1^{k_1} \succcurlyeq I_n^{2q_{n+1}}$. Thus, it suffices to show that $I_n^{2q_{n+1}} \succcurlyeq I_{n+1}$.

We first consider the case $a_{n+1} \geq 2$. For this end, observe from Fig. 3 that

$$I_n^{2q_{n+1}} = I_{n+1}^{q_{n+1}} \cup I_{n+1} \cup [I_n \setminus (I_{n+1}^{q_n} \cup I_{n+1}^{q_n+q_{n+1}})] \tag{5.15}$$

with the unions disjoint *modulo* endpoints. Thus, $I_n^{2q_{n+1}} \supset I_{n+1}$, which implies that

$$\omega(I_{n+1}) \leq \left(\frac{\left|I_n^{2q_{n+1}}\right|}{|I_{n+1}|}\right)^s \omega(I_n^{2q_{n+1}}). \tag{5.16}$$

Now, we use the Real Bounds to bound $\frac{\left|I_n^{2q_{n+1}}\right|}{|I_{n+1}|}$. From (5.15), we have:

$$\begin{aligned}
\frac{\left|I_n^{2q_{n+1}}\right|}{|I_{n+1}|} &= \frac{\left|I_{n+1}^{q_{n+1}}\right|}{|I_{n+1}|} + \frac{|I_{n+1}|}{|I_{n+1}|} + \frac{|I_n|}{|I_{n+1}|} - \frac{\left|I_{n+1}^{q_n}\right|}{|I_{n+1}|} - \frac{\left|I_{n+1}^{q_n+q_{n+1}}\right|}{|I_{n+1}|} \\
&< \frac{\left|I_{n+1}^{q_{n+1}}\right|}{|I_{n+1}|} + 1 + \frac{|I_n|}{|I_{n+1}|} \\
&\leq C + 1 + C = 1 + 2C < 3C
\end{aligned} \tag{5.17}$$

since $I_{n+1}^{q_{n+1}}$, I_{n+1} are adjacent atoms of $\mathcal{P}_{n+1}$, I_n, I_{n+1} are adjacent atoms of $\mathcal{P}_n$ and $C > 1$.

Plugging (5.17) into (5.16), we get

$$\omega(I_{n+1}) \leq (3C)^s \omega(I_n^{2q_{n+1}}) \tag{5.18}$$

which proves that $I_n^{2q_{n+1}} \succcurlyeq I_{n+1}$.

Finally, we address the case $a_{n+1} = a_{n+2} = 1$. Observe from Fig. 4 that $I_n^{2q_{n+1}} \supset I_{n+2}^{q_{n+1}}$ and $I_{n+1} \subset I_{n+2}^{q_{n+1}} \cup I_{n+2}^{q_{n+1}+q_{n+2}}$; from Lemma 5.10, $I_{n+2}^{q_{n+1}} \succcurlyeq I_{n+2}^{q_{n+1}+q_{n+2}}$. By applying the Real Bounds and Lemma 5.3, we obtain

$$I_n^{2q_{n+1}} \succcurlyeq I_{n+2}^{q_{n+1}} \succcurlyeq I_{n+2}^{q_{n+1}} \cup I_{n+2}^{q_{n+1}+q_{n+2}} \succcurlyeq I_{n+1}$$

which finishes the proof. □

Though we expect Theorem 5.1 will prove more useful in future works, for our purposes we will require the following stronger result, which clearly implies Theorem 5.1.

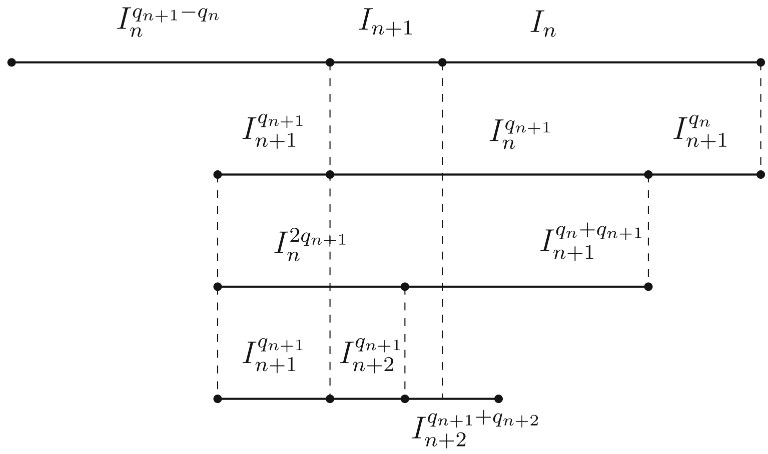

Fig. 4 Relative positions of the intervals I_n, I_{n+1}, $I_n^{q_{n+1}-q_n}$, $I_n^{q_{n+1}}$, $I_{n+1}^{q_n}$, $I_{n+1}^{q_{n+1}}$, $I_n^{2q_{n+1}}$, $I_{n+1}^{q_n+q_{n+1}}$, I_{n+2}, $I_{n+2}^{q_{n+1}}$, $I_{n+2}^{q_{n+1}+q_{n+2}}$ when $a_{n+1} = a_{n+2} = 1$. By applying the real bounds and Lemma 5.3, one can see all these intervals have comparable lengths

Theorem 5.12 *There exists a constant $B_2 = B_2(f, s) > 1$ with the following property. For any $n \geq n_0$ and $\Delta_1, \Delta_2 \in \mathcal{P}_n$, we have*:

(a) *If Δ_1, Δ_2 are both long atoms or both short atoms of $\mathcal{P}_n$, then*

$$B_2^{-1}\,\omega(\Delta_2) \leq \omega(\Delta_1) \leq B_2\,\omega(\Delta_2). \tag{5.19}$$

(b) *If Δ_1 is a short atom and Δ_2 is a long atom of $\mathcal{P}_n$, then*

$$\omega(\Delta_1) \leq B_2\,\omega(\Delta_2). \tag{5.20}$$

Furthermore, if $a_{n+1} \geq 2$ or $a_{n+1} = a_{n+2} = 1$, then Δ_1, Δ_2 can be respectively replaced in the above inequalities by images $\Delta_1^{k_1}, \Delta_2^{k_2}$, $0 \leq k_1, k_2 \leq q_{n+1} + 1$.

Proof Observe that (b) is precisely the content of Lemma 5.11. Moreover, since all short atoms of $\mathcal{P}_n$ become long atoms in $\mathcal{P}_{n+1}$, it suffices to prove (a) for long Δ_1, Δ_2.

Assume that Δ_1, Δ_2 are both long atoms of $\mathcal{P}_n$. We split the proof in two parts: (i) that $\Delta_1 \succcurlyeq \Delta_2$; and (ii) that $\Delta_1^{k_1} \succcurlyeq \Delta_2^{k_2}$ if $a_{n+1} \geq 2$ or $a_{n+1} = a_{n+2} = 1$.

We first prove (i). It suffices to show that Δ_1 ω-dominates Δ_2, since then the ω-comparability of the two follows by simply interchanging Δ_1 and Δ_2. Once more, from Lemma 5.10, $\Delta_1 \succcurlyeq I_n^{q_{n+1}}$ and $I_n \succcurlyeq \Delta_2$. Since $I_n \subset I_n^{q_{n+1}} \cup I_{n+1}^{q_n}$, $I_n^{q_{n+1}} \succcurlyeq I_{n+1}^{q_n}$ (by Lemma 5.11) and these intervals have pairwise comparable lengths, we conclude that $I_n^{q_{n+1}} \succcurlyeq I_n$. This finishes the proof of (i).

The proof of (ii) is essentially the same: we have $\Delta_1^{k_1} \succcurlyeq I_n^{2q_{n+1}}$ and $I_n \succcurlyeq \Delta_2^{k_2}$, so we need only show that $I_n^{2q_{n+1}} \succcurlyeq I_n$. But $I_n \subset I_n^{2q_{n+1}} \cup I_{n+1}^{q_n+q_{n+1}} \cup I_{n+1}^{q_n}$ and (from Lemma 5.11)

$$I_n^{2q_{n+1}} \succcurlyeq I_{n+1}^{q_n} \succcurlyeq I_{n+1}^{q_n+q_{n+1}} \tag{5.21}$$

so, since these intervals have pairwise comparable lengths, we conclude that $I_n^{2q_{n+1}} \succcurlyeq I_n$. This finishes the proof of (ii). □

6 Ergodicity and uniqueness

In this section, we prove that automorphic measures for multicritical circle maps with irrational rotation number are ergodic (Theorem 6.8 below). As a consequence, we will obtain the uniqueness part of Theorem A (we would like to remark that the non-flatness condition on each critical point of f is crucial in order to have uniqueness, see Sect. 7.2 below). In particular, Lebesgue is the unique f-automorphic measure of exponent 1 (Corollary 6.10).

We further show that this uniqueness remains true (up to a scalar multiple) in the context of continuous linear functionals on $C^0(S^1)$ (Corollary 6.11). As we will see in Sect. 7, Corollary 6.11, applied to Lebesgue measure ($s = 1$), is the core step towards proving Theorem B.

6.1 The Γ ratio

We first introduce a bit of notation. Fix some $s > 0$ and $\nu \in \mathcal{A}_s$. For an interval $I \subset S^1$ and a Borel f-invariant set $E \subset S^1$, we will denote by $\Gamma(I; E)$ the ratio

$$\Gamma(I; E) := \frac{\nu(I \cap E)}{\nu(I)}. \tag{6.1}$$

Observe that $\Gamma(I; E)$ can be expressed as the quotient of two ω-ratios with respect to I and different invariant sets. Indeed, in the numerator we take E as the f-invariant set, while in the denominator we just take the whole circle.

By direct analogy with Lemma 5.8, we thus obtain the following result.

Lemma 6.1 *Let $n \geq 0$ and $\Delta \in \mathcal{P}_n$. Then, for any interval $I \subset \Delta^*$ and $0 \leq j < k \leq q_{n+1}$ such that the intervals $f^j(\widehat{\Delta}), \ldots, f^{k-1}(\widehat{\Delta})$ do not contain any critical point of f, the following holds for all Borel f-invariant sets $E \subset S^1$:*

$$B_1^{-2s}\,\Gamma(I^j; E) \leq \Gamma(I^k; E) \leq B_1^{2s}\,\Gamma(I^j; E). \tag{6.2}$$

Definition 6.2 Let $\Delta \in \mathcal{P}_n$, $0 \leq k < q_{n+1}$. We will say that k is a *critical time of type 2* for Δ if $f^k(\widehat{\Delta}) \cap \mathrm{Crit}(f) \neq \varnothing$.

Since f has N critical points $c_1, \ldots, c_N$ and the collection $\{f^k(\widehat{\Delta})\}_{k=0}^{q_{n+1}-1}$ has intersection multiplicity at most 8 (Lemma 5.2), it follows that, for any $n \geq 0$ and $\Delta \in \mathcal{P}_n$, there are at most $8N$ critical times of type 2 for Δ.

Remark 6.3 By a direct analogy with Remark 5.7, there is some level $n_1 = n_1(f) \geq n_0(f) \in \mathbb{N}$, depending only on f, such that, for all $n \geq n_1$, $\Delta \in \mathcal{P}_n$ and $0 \leq k < q_{n+1}$ a critical time of type 2 for Δ, we have that: (i) $f^k(\widehat{\Delta})$ contains a *single* critical point of f; and (ii) $f^k(\widehat{\Delta}) \subset U$, where U is the interval about the critical point of f in $f^k(\widehat{\Delta})$ from Proposition 2.2.

With this terminology in mind, the following corollary is now an immediate consequence of Lemma 6.1:

Corollary 6.4 *Let $n \geq 0$, $\Delta \in \mathcal{P}_n$, and let $0 \leq k_1 < k_2 < \cdots < k_r < q_{n+1}$ be the critical times of type 2 for Δ. Then, for any interval $I \subset \Delta$, the following holds for all Borel f-invariant sets $E \subset S^1$:*

(a) *For any $0 \leq j, \ell \leq k_1$,*

$$B_1^{-2s}\,\Gamma(I^j; E) \leq \Gamma(I^\ell; E) \leq B_1^{2s}\,\Gamma(I^j; E); \tag{6.3}$$

(b) (6.3) *also holds for any* $k_i + 1 \le j,\ \ell \le k_{i+1},\ 1 \le i < r$;
(c) (6.3) *also holds for* $k_r + 1 \le j,\ \ell \le q_{n+1}$.

As we did in Lemma 5.10 with the ω-ratio, we now turn to the problem of understanding the behavior of Γ close to a critical point of f.

Lemma 6.5 *There exists a constant* $B_3 = B_3(f, s) > 1$ *with the following property. Let* $n \ge n_1$ *and* $\Delta \in \mathcal{P}_n$*. Assume that* n *is such that either* $a_{n+1} \ge 2$ *or* $a_{n+1} = a_{n+2} = 1$*. Then the following holds for all Borel* f*-invariant sets* $E \subset S^1$*: if* $0 \le j < q_{n+1}$ *is a critical time of type* 2 *for* Δ,

$$\Gamma(f^{j+1}(\Delta^*); E) \le B_3\, \Gamma(f^j(\Delta^*); E). \tag{6.4}$$

Remark 6.6 Note the hypotheses on the combinatorics of f at level n: $a_{n+1} \ge 2$ or $a_{n+1} = a_{n+2} = 1$. These are necessary to allow for use of the sharpened version of Theorem 5.12.

Proof We begin with the following observation (see Lemma 5.3): for any one of the atoms I of $\mathcal{P}_n$ that compose $\widehat{\Delta}$ and $0 \le k \le q_{n+1}$, we have $I^k \asymp f^k(\Delta^*)$. Further note that

$$((L)_3 \cup (L)_2 \cup \widetilde{L}) \cap (\widetilde{R} \cup (R)_2 \cup (R)_3) = \varnothing$$

so (from Remark 6.3) it cannot be that both $f^j((L)_3 \cup (L)_2 \cup \widetilde{L})$ and $f^j(\widetilde{R} \cup (R)_2 \cup (R)_3)$ contain a critical point of f. Since the other case is identical, we assume, without loss of generality, that

$$f^j((L)_3 \cup (L)_2 \cup \widetilde{L}) \cap \mathrm{Crit}\,(f) = \varnothing.$$

Now, since no two short atoms of $\mathcal{P}_n$ are adjacent, one of L, $(L)_2$ is a long atom; once more, since the other case is nearly identical, we assume, without loss of generality, that $(L)_2$ is a long atom of $\mathcal{P}_n$. Similarly, one of Δ, R is a long atom of $\mathcal{P}_n$, so we may assume, without loss of generality, that R is also a long atom of $\mathcal{P}_n$.

It thus follows as a straightforward consequence of Theorem 5.12 that $(L)_2^j \sim f^j(\Delta^*)$ and $(L)_2^{j+1} \sim f^{j+1}(\Delta^*)$. That is, there exists a constant $K_0 = K_0(f, s)$ such that

$$\begin{aligned}
&K_0^{-1}\, \frac{\nu((L)_2^j \cap E)}{\left|(L)_2^j\right|^s} \le \frac{\nu(f^j(\Delta^*) \cap E)}{\left|f^j(\Delta^*)\right|^s} \le K_0\, \frac{\nu((L)_2^j \cap E)}{\left|(L)_2^j\right|^s},\\
&K_0^{-1}\, \frac{\nu((L)_2^j)}{\left|(L)_2^j\right|^s} \le \frac{\nu(f^j(\Delta^*))}{\left|f^j(\Delta^*)\right|^s} \le K_0\, \frac{\nu((L)_2^j)}{\left|(L)_2^j\right|^s},\\
&K_0^{-1}\, \frac{\nu((L)_2^{j+1} \cap E)}{\left|(L)_2^{j+1}\right|^s} \le \frac{\nu(f^{j+1}(\Delta^*) \cap E)}{\left|f^{j+1}(\Delta^*)\right|^s} \le K_0\, \frac{\nu((L)_2^{j+1} \cap E)}{\left|(L)_2^{j+1}\right|^s},\\
&K_0^{-1}\, \frac{\nu((L)_2^{j+1})}{\left|(L)_2^{j+1}\right|^s} \le \frac{\nu(f^{j+1}(\Delta^*))}{\left|f^{j+1}(\Delta^*)\right|^s} \le K_0\, \frac{\nu((L)_2^{j+1})}{\left|(L)_2^{j+1}\right|^s}.
\end{aligned} \tag{6.5}$$

Therefore,

$$\begin{aligned}
&K_0^{-2}\, \Gamma((L)_2^j; E) \le \Gamma(f^j(\Delta^*); E) \le K_0^2\, \Gamma((L)_2^j; E),\\
&K_0^{-2}\, \Gamma((L)_2^{j+1}; E) \le \Gamma(f^{j+1}(\Delta^*); E) \le K_0^2\, \Gamma((L)_2^{j+1}; E).
\end{aligned} \tag{6.6}$$

Now, since $f^j(\widetilde{(L)_2})$ contains no critical point of f, from Lemma 5.4 and the mean value theorem,

$$B_0^{-1}\frac{\left|(L)_2^{j+1}\right|}{\left|(L)_2^{j}\right|} \leq Df(y) \leq B_0\frac{\left|(L)_2^{j+1}\right|}{\left|(L)_2^{j}\right|} \quad \text{for all } y \in (L)_2^j. \tag{6.7}$$

Thus,

$$\Gamma((L)_2^{j+1}; E) = \frac{\int_{(L)_2^j \cap E}(Df)^s\, d\nu}{\int_{(L)_2^j}(Df)^s\, d\nu} \leq \frac{B_0^s\left(\frac{\left|(L)_2^{j+1}\right|}{\left|(L)_2^{j}\right|}\right)^s \int_{(L)_2^j\cap E} d\nu}{B_0^{-s}\left(\frac{\left|(L)_2^{j+1}\right|}{\left|(L)_2^{j}\right|}\right)^s \int_{(L)_2^j} d\nu} = B_0^{2s}\,\Gamma((L)_2^j; E). \tag{6.8}$$

Combining Eqs. (6.6) and (6.8), we get:

$$\Gamma(f^{j+1}(\Delta^*); E) \leq K_0^2\,\Gamma((L)_2^{j+1}; E) \leq K_0^2\, B_0^{2s}\,\Gamma((L)_2^j; E) \leq B_0^{2s}\, K_0^4\,\Gamma(f^j(\Delta^*); E) \tag{6.9}$$

which is (6.4), with $B_3 := B_0^{2s}\, K_0^4$. □

Combining Corollary 6.4 and Lemma 6.5, we get the following.

Corollary 6.7 *There exists a constant $B_4 = B_4(f, s) > 1$ with the following property. Let $n \geq n_1$ and $\Delta \in \mathcal{P}_n$. Assume that n is such that $a_{n+1} \geq 2$ or $a_{n+1} = a_{n+2} = 1$. Then, for any Borel f-invariant set $E \subset S^1$ and $0 \leq \ell_1 < \ell_2 \leq q_{n+1}$,*

$$\Gamma(f^{\ell_2}(\Delta^*); E) \leq B_4\,\Gamma(f^{\ell_1}(\Delta^*); E). \tag{6.10}$$

6.2 Ergodicity

We are now ready to prove the ergodicity of ν with respect to f.

Theorem 6.8 (Ergodicity) *Let $s > 0$ and let $\nu \in \mathcal{A}_s$ be an automorphic measure of exponent s for f. Then ν is ergodic with respect to f.*

Proof Let $E \subset S^1$ be a Borel f-invariant set such that $\nu(E) < 1$. Our aim is to show that, in fact, $\nu(E) = 0$. For $x \in S^1$, consider the sequence

$$\mathcal{V}(x) = \left\{ \Delta^* \,\middle|\, n \geq 0, \Delta \in \mathcal{P}_n, x \in \Delta \right\}$$

of triples of adjacent atoms from the dynamical partitions $\mathcal{P}_n$ such that x is contained in the central atom of the triple. As n increases, the triples of level n in this family shrink to x while maintaining definite space on both sides (by the Real Bounds). If $x \in \Lambda$, then there is a unique atom $\Delta_n(x) \in \mathcal{P}_n$ that contains x, and x is contained in its interior. Therefore, for $x \in \Lambda$, $\mathcal{V}(x)$ contains precisely one triple $(\Delta_n^*(x))$ of each level $n \geq 0$.

Since $\Lambda \setminus E$ has positive ν-measure, we claim that for any given $\epsilon > 0$ there exist $x \in \Lambda \setminus E$ and $n_2 \geq n_1(f) \geq n_0(f)$ such that, for all $n \geq n_2$,

$$\Gamma(\Delta_n^*(x); E) = \frac{\nu(\Delta_n^*(x) \cap E)}{\nu(\Delta_n^*(x))} < \epsilon. \tag{6.11}$$

Indeed, note that since ν has no atoms and is supported on the whole circle (recall Sect. 3), the map $h\colon S^1 \to S^1$ given by $h(x) = \int_{[0,x]} d\nu$ is a circle homeomorphism, which identifies the measure ν with Lebesgue measure on S^1. Thus, the existence of a point x satisfying (6.11) follows from the standard Lebesgue Density Theorem.

Now, some necessary distinctions depending on the combinatorics of f must be made. If ρ is the rotation number of f, $\rho = [a_0, a_1, \cdots]$, then either we have $a_n = 1$ for every sufficiently large n, or $a_n \geq 2$ occurs infinitely often. In any case, we can choose $n \geq n_2$ such that either $a_{n+1} = a_{n+2} = 1$ (in the first case) or $a_{n+1} \geq 2$ (in the latter case). We thus fix some level $n \geq n_2$ that satisfies one of these conditions, so that both (6.11) and Corollary 6.7 hold simultaneously.

Observe that the collection $\left\{f^i(\Delta_n^*(x))\right\}_{i=0}^{q_{n+1}}$ covers the circle and has intersection multiplicity at most 4 (see Lemma 5.2), so

$$\begin{aligned}\nu(E) &= \nu\left(\bigcup_{i=0}^{q_{n+1}} f^i(\Delta_n^*(x)) \cap E\right) \leq \sum_{i=0}^{q_{n+1}} \nu(f^i(\Delta_n^*(x)) \cap E)\\ &= \sum_{i=0}^{q_{n+1}} \nu(f^i(\Delta_n^*(x)))\, \Gamma(f^i(\Delta_n^*(x)); E)\end{aligned} \tag{6.12}$$

and furthermore,

$$\sum_{i=0}^{q_{n+1}} \nu(f^i(\Delta_n^*(x))) \leq 4 \tag{6.13}$$

(see Remark 2.6).

From Corollary 6.7,

$$\Gamma(f^i(\Delta_n^*(x)); E) \leq B_4\, \Gamma(\Delta_n^*(x)); E) \tag{6.14}$$

so, plugging (6.14), (6.13) and (6.11) into (6.12):

$$\begin{aligned}\nu(E) &\leq \sum_{i=0}^{q_{n+1}} \nu(f^i(\Delta_n^*(x)))\, \Gamma(f^i(\Delta_n^*(x)); E) \leq B_4\, \Gamma(\Delta_n^*(x); E) \sum_{i=0}^{q_{n+1}} \nu(f^i(\Delta_n^*(x)))\\ &\leq 4\, B_4\, \frac{\nu(\Delta_n^*(x) \cap E)}{\nu(\Delta_n^*(x))} < 4\, B_4\, \epsilon.\end{aligned} \tag{6.15}$$

By letting $\epsilon \to 0$, we get $\nu(E) = 0$.

Thus, there is no Borel f-invariant set $E \subset S^1$ such that $0 < \nu(E) < 1$, which proves that ν is ergodic. □

For future reference (see Sect. 7.3 below) let us point out the following particular case of Theorem 6.8, which is important in its own right.

Theorem 6.9 *Any given multicritical circle map with irrational rotation number is ergodic with respect to Lebesgue measure.*

6.3 Uniqueness

We now show that ergodicity of f-automorphic measures of positive exponent implies that there is a unique f-automorphic measure of exponent s for each $s > 0$. We further extend this uniqueness statement to finite signed measures, i.e., continuous linear functionals on $C^0(S^1)$.

Proof of Theorem A, uniqueness part Arguing by contradiction, suppose there is some $s > 0$ such that $\mathcal{A}_s$ contains two distinct measures, say ν_1, ν_2. First, suppose $\nu_1 \ll \nu_2$, and let $\psi \in L^1(\nu_2)$ be the Radon–Nikodym derivative

$$\psi = \frac{d\nu_1}{d\nu_2}.$$

As a simple calculation shows, $\psi \circ f = \psi$ ν_2-almost everywhere, i.e., ψ is f-invariant ν_2-a.e. But, from Theorem 6.8, ν_2 is ergodic for f, so ψ must be constant ν_2-a.e. Since $\int_{S^1} \psi \, d\nu_2 = 1$, we conclude that $\psi = 1$ ν_2-a.e. But then $\nu_1 = \nu_2$, contradicting our assumption that ν_1, ν_2 are distinct.

Finally, if ν_1 is not absolutely continuous with respect to ν_2, let $\nu_3 = \frac{1}{2}(\nu_1 + \nu_2)$. Then $\nu_3 \in \mathcal{A}_s$ (since $\mathcal{A}_s$ can be easily verified to be convex) and $\nu_2 \ll \nu_3$, $\nu_1 \ll \nu_3$. Thus, from the previous case, we must have $\nu_1 = \nu_2 = \nu_3$, once again in contradiction to our assumption that ν_1, ν_2 are distinct. □

Thus, we have now given a complete proof of Theorem A. Correspondingly, for $s > 0$, we will denote the unique f-automorphic measure of exponent s by μ_s.

Since Lebesgue measure is always f-automorphic of exponent 1, we have the following immediate consequence of Theorem A:

Corollary 6.10 *Lebesgue measure is the unique automorphic measure of exponent* 1 *for* f.

We now show that the uniqueness statement of Theorem A (and in particular, Corollary 6.10) remains true (up to a scalar multiple) in the context of continuous linear functionals on $C^0(S^1)$:

Corollary 6.11 *Let* $s > 0$. *If* $L \in C^0(S^1)^*$ *is such that*

$$\langle L, \phi \rangle = \left\langle L, (\phi \circ f)(Df)^s \right\rangle \tag{6.16}$$

for all $\phi \in C^0(S^1)$, *then*

$$L = \langle L, 1 \rangle \; \mu_s. \tag{6.17}$$

Proof The proof is reproduced almost verbatim from [2, Remarque 1]. From the Riesz Representation Theorem, there is a unique signed finite Radon measure ν on S^1 such that

$$\langle L, \phi \rangle = \int_{S^1} \phi \, d\nu$$

for all $\phi \in C^0(S^1)$. Therefore, it suffices to show that

$$\nu = \nu(S^1) \, \mu_s. \tag{6.18}$$

First, suppose ν is positive. Then $\tilde{\nu} := \frac{\nu}{\nu(S^1)} \in \mathcal{A}_s$, so

$$\nu = \nu(S^1) \, \tilde{\nu} = \nu(S^1) \, \mu_s.$$

Now, for the general case, let $\nu = \nu_+ - \nu_-$ be the Jordan decomposition of ν. We wish to show that

$$\int_{S^1} \phi \, d\nu_+ = \int_{S^1} (\phi \circ f)(Df)^s \, d\nu_+,$$
$$\int_{S^1} \phi \, d\nu_- = \int_{S^1} (\phi \circ f)(Df)^s \, d\nu_- \tag{6.19}$$

for all $\phi \in C^0(S^1)$. Since the linear operator $U_s \colon C^0(S^1) \to C^0(S^1)$ given by

$$U_s(\psi) = (\psi \circ f)(Df)^s$$

is positive, for all $\phi \in C^0(S^1)$ with $\phi \geq 0$, we have

$$\begin{aligned} \int_{S^1} \phi \, d\nu_+ &= \sup_{0 \leq \psi \leq \phi} \int_{S^1} \psi \, d\nu = \sup_{0 \leq \psi \leq \phi} \int_{S^1} (\psi \circ f)(Df)^s \, d\nu \\ &= \sup_{0 \leq \psi \leq U_s(\phi)} \int_{S^1} \psi \, d\nu = \int_{S^1} U_s(\phi) \, d\nu_+ \end{aligned} \tag{6.20}$$

and similarly for ν_-. It follows that (6.19) holds for all continuous $\phi \geq 0$; by linearity, it must hold for all $\phi \in C^0(S^1)$. Thus, applying the first case to ν_+, ν_-, we have

$$\nu = \nu_+ - \nu_- = \nu_+(S^1)\,\mu_s - \nu_-(S^1)\,\mu_s = \nu(S^1)\,\mu_s$$

which finishes the proof. □

Remark 6.12 It follows from (6.17) that all linear functionals $L \in C^0(S^1)^*$ that satisfy (6.16) have a *definite sign*: if $L = L_+ - L_-$ is its Jordan decomposition, then $L_+ = 0$ or $L_- = 0$, depending on whether $\langle L, 1 \rangle$ is negative, positive or zero (and in this last case, $L = 0$).

Remark 6.13 With uniqueness at hand, it is not difficult to prove *continuity* of automorphic measures in the weak* topology. To be more precise, let $\{f_n\}$ be a sequence of C^3 multicritical circle maps converging, in the C^1 topology, to a C^3 multicritical circle map f. If $s_n \to s$ in $[0, +\infty)$, then $\mu_{s_n}(f_n)$ converges weakly to $\mu_s(f)$, the unique f-automorphic measure of exponent s.

7 Applications

In this final section we prove Theorem B (Sect. 7.1), Theorem C (Sect. 7.2) and Theorem D (Sect. 7.3). With these purposes, we first review some basic results regarding invariant distributions for dynamical systems on compact manifolds.

7.1 Cohomological equations and invariant distributions

Let M be a compact smooth manifold. For integer $0 \leq r \leq \infty$, let $C^r(M)$ be the space of C^r functions $u \colon M \to \mathbb{R}$, equipped with its C^r topology. Recall that the C^r topology turns $C^r(M)$ into a Banach space for finite r and $C^\infty(M)$ into a Fréchet space, and that a distribution on M is simply an element of the continuous dual space $C^\infty(M)^*$; we will denote the space of distributions on M by $\mathcal{D}'(M)$, and the value of a distribution $T \in \mathcal{D}'(M)$ acting on a function $u \in C^\infty(M)$ by $\langle T, u \rangle$.

Suppose $T \in \mathcal{D}'(M)$ and $0 \leq k < \infty$ are such that there exists $C > 0$ with

$$|\langle T, u \rangle| \leq C \, \|u\|_k \quad \forall\, u \in C^\infty(M).$$

In this case, T has a unique continuous extension $\tilde{T} \in C^k(M)^*$; we say that T has *order at most* k. In fact, every $T \in C^k(M)^*$ is the unique continuous extension of a distribution on M. Denoting $C^k(M)^*$ by $\mathcal{D}'_k(M)$, we have the following chain of inclusions *modulo* unique extensions:

$$\mathcal{D}'_0(M) \hookrightarrow \mathcal{D}'_1(M) \hookrightarrow \cdots \hookrightarrow \mathcal{D}'(M).$$

Observe that the Riesz Representation Theorem naturally identifies $\mathcal{D}'_0(M)$ with the space $\mathcal{M}(M)$ of signed finite Radon measures on M.

If $T \in \mathcal{D}'(M)$ belongs to $\mathcal{D}'_k(M)$ for some finite k, we say that T has finite order, and we define its order as the least such k. A noteworthy consequence of the compactness of M is that all distributions on M have finite order, i.e.,

$$\mathcal{D}'(M) = \bigcup_{k=0}^{\infty} \mathcal{D}'_k(M).$$

Now, let $f: M \to M$ be a C^r endomorphism of M, $0 \le r \le \infty$.

Definition 7.1 (C^ℓ*-coboundary*) Let $0 \le \ell \le \infty$ and $\phi \in C^\ell(M)$. We say that ϕ is a C^ℓ*-coboundary* for f if the *cohomological equation*

$$u \circ f - u = \phi \tag{7.1}$$

has a solution $u \in C^\ell(M)$.

For integer $0 \le \ell \le \infty$, the set of C^ℓ-coboundaries for f forms a vector subspace of $C^\ell(M)$, which we will denote by $\mathrm{B}\left(f, C^\ell(M)\right)$.

Definition 7.2 (*Invariant distribution*) We say that $T \in \mathcal{D}'_r(M)$ is f*-invariant* if

$$\langle T, u \circ f \rangle = \langle T, u \rangle \tag{7.2}$$

for all $u \in C^\infty(M)$.

Remark 7.3 Let the manifold M be the unit circle S^1. An f-automorphic measure ν of exponent 1 naturally induces an f-invariant distribution $T \in \mathcal{D}'_1\left(S^1\right)$ by letting

$$\langle T, u \rangle = \int_{S^1} u' \, d\nu$$

for all $u \in C^1(S^1)$. Indeed, note that

$$\langle T, u \circ f \rangle = \int_{S^1} (u \circ f)' \, d\nu = \int_{S^1} (u' \circ f)\, Df \, d\nu = \int_{S^1} u' \, d\nu = \langle T, u \rangle.$$

Of course, Lebesgue measure (which is automorphic of exponent 1 for any C^1 circle homeomorphism) induces the null distribution $\langle T, u \rangle = 0$. However, automorphic measures of exponent 1 different from Lebesgue provide non-trivial invariant distributions, see Sect. 7.2 below.

For all integer $0 \le k \le r$, the set $\mathcal{D}'_k(f)$ of f-invariant distributions of order at most k forms a vector subspace of $\mathcal{D}'_k(M)$. In fact, Eqs. (7.1) and (7.2) (by unique extension to C^k functions) identify $\mathcal{D}'_k(f)$ with the (continuous) annihilator of $\mathrm{B}\left(f, C^k(M)\right)$.

Thus, by the Hahn–Banach separation theorem,

$$\mathrm{cl}_k \; \mathrm{B}\left(f, C^k(M)\right) = \bigcap_{T \in \mathcal{D}'_k(f)} \ker T \tag{7.3}$$

where cl_k denotes closure in the C^k topology.

Furthermore, we have the chain of inclusions

$$\mathcal{D}'_0(f) \hookrightarrow \mathcal{D}'_1(f) \hookrightarrow \cdots \hookrightarrow \mathcal{D}'_r(f)$$

and also

$$\mathcal{D}'_r(f) = \bigcup_{k=0}^{r} \mathcal{D}'_k(f). \tag{7.4}$$

The following proposition is a simple but crucial consequence of (7.3).

Proposition 7.4 *Let $f : M \to M$ be a C^r endomorphism of a compact smooth manifold M, $0 \le r \le \infty$, and let μ be an f-invariant Radon probability measure on M. Let $0 \le k \le r$ be an integer. Then*

$$\mathcal{D}'_k(f) = \mathbb{R}\mu$$

if, and only if, the following holds. For any $\phi \in C^k(M)$ with $\int_M \phi \, d\mu = 0$, there is a sequence $\{\phi_n = u_n \circ f - u_n\}_{n \ge 1} \subset \mathrm{B}\left(f, C^k(M)\right)$ of C^k-coboundaries for f converging to ϕ in the C^k topology.

With the above criterion at hand, we are ready to prove that Theorem A implies Theorem B. The proof given below is taken almost verbatim from [28]. We reproduce it here for the sake of completeness as well as to indicate the points of the proof in which estimates depending on the bounded variation of $\log Df$ for $C^{1+\mathrm{bv}}$-diffeomorphisms must be replaced by estimates suitable for multicritical circle maps and where results from [2] must be replaced by consequences of Theorem A. From Proposition 7.4, it suffices to show that Theorem A implies the following lemma.

Lemma 7.5 *Let $u \in C^1(S^1)$ have zero μ-mean, that is, $\int_{S^1} u \, d\mu = 0$. Then there is a sequence v_n of C^1 functions $S^1 \to \mathbb{R}$ such that*

$$v_n \circ f - v_n \longrightarrow u \tag{7.5}$$

and

$$(v'_n \circ f)Df - v'_n \longrightarrow u' \tag{7.6}$$

uniformly.

Let $u \in C^1(S^1)$, $\int_{S^1} u \, d\mu = 0$, be fixed. The construction of the sequence v_n from Lemma 7.5 will be derived as a consequence of the following fact.

Proposition 7.6 *There exists a sequence $\{w_n\}_{n \ge 1} \subset C^0(S^1)$ such that*

$$(w_n \circ f)Df - w_n \longrightarrow u' \tag{7.7}$$

uniformly, and such that $\int_{S^1} w_n \, dm = 0$ for all $n \ge 1$, where m denotes the Lebesgue measure in the unit circle.

Indeed, assume that Proposition 7.6 is true, and for each $n \ge 1$, let $v_n : S^1 \to \mathbb{R}$ be defined by

$$v_n(x) = \int_{[0,x]} w_n(y) \, dy,$$

where $[0, x]$ is the positively oriented closed circle interval with endpoints $0, x$.

Observe that, since $\int_{S^1} w_n \, dm = 0$, v_n is well-defined as a $\mathbb{Z}$-periodic function from $\mathbb{R}$ to $\mathbb{R}$ (*i.e.*, $v_n \in C^0(S^1)$). Furthermore, $v_n \in C^1(S^1)$ and $v'_n = w_n$, so

$$(v'_n \circ f)Df - v'_n = (w_n \circ f)Df - w_n.$$

Thus, Proposition 7.6 implies that the sequence v_n we have just defined satisfies (7.6). It remains to show that the $v_n \circ f - v_n$ also converge uniformly to u.

Well, for any $x \in S^1$, we have

$$\begin{aligned} v_n(f(x)) - v_n(x) - u(x) &= \int_{[0,f(x)]} w_n(y)\,dy - \int_{[0,x]} w_n(y)\,dy \\ &\quad - \left(u(0) + \int_{[0,x]} u'(y)\,dy\right) \\ &= \int_{[0,x]} \left[(w_n(f(y))Df(y) - w_n(y)) - u'(y)\right] dy - c_n \end{aligned} \tag{7.8}$$

where $c_n{:=}u(0) - \int_{[0,f(0)]} w_n(y)\,dy$. Thus,

$$\|(v_n \circ f - v_n + c_n) - u\|_{C^0} \le \left\|[(w_n \circ f)Df - w_n] - u'\right\|_{C^0}. \tag{7.9}$$

Now, from Proposition 7.6, the right-hand side in (7.9) converges to 0, so the sequence $\{v_n \circ f - v_n + c_n\}_{n\ge1}$ converges uniformly to u. Consequently, since μ is f-invariant,

$$c_n = \int_{S^1} (v_n \circ f - v_n + c_n)\,d\mu \longrightarrow \int_{S^1} u\,d\mu = 0 \tag{7.10}$$

so we conclude that (7.5) holds for the sequence $\{v_n\}$ as well. This finishes the proof of Lemma 7.5, assuming Proposition 7.6.

With the knowledge that Proposition 7.6 implies Lemma 7.5 (which in turn implies Theorem B), we now show that Theorem A implies this proposition. First, a technical lemma, which consists of Proposition 7.6 in the special case $u = f - \mathrm{Id}$.

Lemma 7.7 *There exists a sequence* $\left\{\hat{w}_k\right\}_{k\ge1} \subset C^0(S^1)$, *with* $(\hat{w}_k \circ f)Df - \hat{w}_k$ *converging uniformly to* $Df - 1$, *such that, for all* $k \ge 1$, $\int_{S^1} \hat{w}_k\,dm = 0$.

Proof For $k \ge 1$, let $\hat{w}_k \in C^0(S^1)$ be defined by

$$\hat{w}_k = 1 - \frac{1}{q_k}\sum_{i=0}^{q_k-1} Df^i. \tag{7.11}$$

Observe that

$$\int_{S^1} Df^i\,dm = 1,$$

so

$$\int_{S^1} \hat{w}_k\,dm = 1 - \frac{1}{q_k}\sum_{i=0}^{q_k-1}\int_{S^1} Df^i\,dm = 1 - 1 = 0. \tag{7.12}$$

Furthermore,

$$\begin{aligned} &\left\|\left[(\hat{w}_k \circ f)Df - \hat{w}_k\right] - (Df - 1)\right\|_{C^0} \\ &= \left\|\left[\left(Df - \frac{1}{q_k}\sum_{i=0}^{q_k-1}(Df^i \circ f)Df\right) - 1 + \frac{1}{q_k}\sum_{i=0}^{q_k-1} Df^i\right] - Df + 1\right\|_{C^0} \\ &= \frac{1}{q_k}\left\|1 - Df^{q_k}\right\|_{C^0} \le \frac{1 + \|Df^{q_k}\|_{C^0}}{q_k}. \end{aligned} \tag{7.13}$$

Now, from Lemma 2.9,

$$\left\| \left[(\hat{w}_k \circ f)Df - \hat{w}_k \right] - (Df - 1) \right\|_{C^0} \leq \frac{1 + \|Df^{q_k}\|_{C^0}}{q_k} \leq \frac{1 + C_1}{q_k} \longrightarrow 0 \quad (7.14)$$

since $q_k \longrightarrow \infty$. Equations (7.14) and (7.12) prove the lemma. □

Remark 7.8 The main difference between the proofs of Lemma 7.7 above (in the critical case) and the corresponding lemma in [28, p. 317] (in the diffeomorphism case) is the use of Lemma 2.9 to bound $\|Df^{q_n}\|_{C^0}$, instead of the standard Denjoy inequality (see [8, Section 3.2]). Furthermore, in our case the $\hat{w}_k$ are not only continuous, but in fact C^2, since we require f to be at least C^3.

The proof of Proposition 7.6 given below depends essentially on the crucial fact that, if $L \in C^0(S^1)^*$ satisfies

$$\langle L, (\phi \circ f)Df - \phi \rangle = 0 \quad (7.15)$$

for all $\phi \in C^0(S^1)$, then L is a scalar multiple of Lebesgue measure. In the diffeomorphism case, this fact is a consequence of [2, Théorème 1], while in the critical case, it follows from Theorem A (recall Corollary 6.11).

Proof of Proposition 7.6 The proof will result from two claims.
Claim #1: There is a sequence $\{\bar{w}_n\}_{n\geq 1} \subset C^0(S^1)$ such that

$$(\bar{w}_n \circ f)Df - \bar{w}_n \longrightarrow u' \quad (7.16)$$

uniformly.
Indeed, consider the continuous linear operator $U_1 : C^0(S^1) \to C^0(S^1)$ given by

$$U_1\, w = (w \circ f)Df - w \quad (7.17)$$

and let M be the image of U_1. If no sequence $\bar{w}_n$ satisfying (7.16) exists, then $u' \notin \mathrm{cl}_0\ M$, so the Hahn–Banach separation theorem implies the existence of a linear functional $L \in C^0(S^1)^*$ such that L is identically null on M and $\langle L, u' \rangle = 1$. But the fact that L is null on M is easily seen to be equivalent to (7.15), so L must be a multiple of Lebesgue measure; this contradicts the fact that $\int_{S^1} u'\, dm = 0$ (since u is $\mathbb{Z}$-periodic). Thus, a sequence $\{\bar{w}_n\}_{n\geq 1}$ satisfying (7.16) must exist. For each $n \geq 1$, let

$$c_n := \int_{S^1} \bar{w}_n\, dm, \quad \tilde{w}_n := \bar{w}_n - c_n.$$

and choose $k_n \in \mathbb{N}$ such that

$$|c_n| \left\| \left[(\hat{w}_{k_n} \circ f)Df - \hat{w}_{k_n} \right] - (Df - 1) \right\|_{C^0} \leq 2^{-n}$$

(Lemma 7.7 guarantees that this is possible). Finally, define

$$w_n := \tilde{w}_n + c_n \hat{w}_{k_n} = \bar{w}_n + c_n(\hat{w}_{k_n} - 1). \quad (7.18)$$

Observe that $\int_{S^1} w_n\, dm = 0$.
Claim # 2:

$$(w_n \circ f)Df - w_n \longrightarrow u' \quad (7.19)$$

uniformly. Observe that proving claim # 2 will finish the proof.

To prove the claim, let $\epsilon > 0$ be arbitrary, and choose $n_0 \in \mathbb{N}$ such that $2^{-n_0} < \frac{\epsilon}{2}$ and

$$\left\|(\bar{w}_n \circ f)Df - \bar{w}_n - u'\right\|_{C^0} < \frac{\epsilon}{2}$$

for all $n \geq n_0$.

Thus, for all $n \geq n_0$,

$$\begin{aligned}
&\left\|(w_n \circ f)Df - w_n - u'\right\|_{C^0} \\
&\quad \leq \left\|(\bar{w}_n \circ f)Df - \bar{w}_n - u'\right\|_{C^0} + |c_n| \left\|\left[(\hat{w}_{k_n} - 1) \circ f\right] Df - (\hat{w}_{k_n} - 1)\right\|_{C^0} \\
&\quad < \frac{\epsilon}{2} + |c_n| \left\|\left[(\hat{w}_{k_n} \circ f)Df - \hat{w}_{k_n}\right] - (Df - 1)\right\|_{C^0} \\
&\quad \leq \frac{\epsilon}{2} + 2^{-n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned} \tag{7.20}$$

which proves the claim. □

We have thus proved Theorem B.

7.2 Wandering intervals and invariant distributions

Before entering the proof of Theorem D, let us briefly explain why Theorem C holds.

For any given irrational number $\rho \in (0, 1)$, Hall was able to construct in [21] a C^∞ homeomorphism $f : S^1 \to S^1$, with rotation number $\rho(f) = \rho$, having a wandering interval I (i.e., I is an open interval such that $f^n(I)$ is disjoint from $f^m(I)$ whenever $n \neq m$ in $\mathbb{Z}$). These examples, so-called *Hall's examples*, present a single critical point c which is *flat*: the successive derivatives of f (of all orders) vanish at c. Note that this critical point necessarily belongs to the invariant Cantor set of f (otherwise, a C^∞ perturbation supported on the wandering interval containing c would produce a C^∞ diffeomorphism with irrational rotation number and wandering intervals). In particular, $f^n : I \to f^n(I)$ is a diffeomorphism for all $n \in \mathbb{Z}$.

To these Hall's examples, we will apply the following general remark.

Lemma 7.9 *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with a wandering interval $I \subset S^1$ such that $f^n : I \to f^n(I)$ is a C^1 diffeomorphism for all $n \in \mathbb{Z}$. Then, the series*

$$\sum_{n\in\mathbb{Z}} Df^n(x)$$

is finite for Lebesgue almost every $x \in I$.

Proof Since Df is non-negative on the whole circle, the function $x \mapsto \sum_{n\in\mathbb{Z}} Df^n(x)$ is the pointwise limit of a monotone sequence of measurable functions,[3] and therefore it is measurable. Hence, by the Monotone Convergence Theorem,

$$\int_I \sum_{n\in\mathbb{Z}} Df^n \, dm = \sum_{n\in\mathbb{Z}} \int_I Df^n \, dm.$$

But

$$\sum_{n\in\mathbb{Z}} \int_I Df^n \, dm = \sum_{n\in\mathbb{Z}} \left|f^n(I)\right| \leq 1,$$

[3] To wit, the functions $x \mapsto \sum_{|n|\leq N} Df^n(x)$ with $N = 1, 2, \ldots$.

since I is a wandering interval and $f^n\colon I \to f^n(I)$ is a C^1 diffeomorphism for all $n \in \mathbb{Z}$. Thus, we have

$$\int_I \sum_{n\in\mathbb{Z}} Df^n \, dm \leq 1,$$

and therefore the integrand has to be finite at Lebesgue almost every $x \in I$. □

Proof of Theorem C Let $f\colon S^1 \to S^1$ be a Hall's example as above, having a wandering interval $I \subset S^1$. Pick some $x \in I$ such that $S = \sum_{n\in\mathbb{Z}} Df^n(x)$ is finite (recall that, by Lemma 7.9, this series is finite for Lebesgue almost every x outside the f-invariant Cantor set). Following [2, Section 3.1], we consider the probability measure

$$\nu = \frac{1}{S} \sum_{n\in\mathbb{Z}} Df^n(x)\, \delta_{f^n(x)}.$$

We see at once that ν is f-automorphic of exponent 1 (note, in particular, that the *uniqueness* part of Theorem A breaks down if we remove the non-flatness condition on the critical points of f). Now we consider $T \in \mathcal{D}'_1\left(S^1\right)$ given by

$$\langle T, u\rangle = \int_{S^1} u' \, d\nu.$$

As explained in Remark 7.3 above, the distribution T is f-invariant. Finally, to prove that T is *not* a scalar multiple of the unique f-invariant probability measure μ is straightforward (compare [28, Section 3]). Indeed, let $u\colon\ S^1 \to \mathbb{R}$ be of class C^1, supported on the wandering interval I, and such that x is not a critical point of u. Then, on one hand, we have

$$\langle T, u\rangle = \int_{S^1} u' \, d\nu = \frac{1}{S}\, u'(x) \neq 0.$$

On the other hand, since the support of u is disjoint from the non-wandering set of f, we certainly have $\int_{S^1} u \, d\mu = 0$. This finishes the proof of Theorem C. □

7.3 Denjoy–Koksma inequality improved

We finish this paper by proving Theorem D. In fact, we will present two different proofs of Theorem D. The first proof works only when the observable ϕ is of class C^1, whereas the second works in the general case, i.e., when ϕ is absolutely continuous with respect to Lebesgue. The former follows [1, pages 513–514], and relies on the absence of invariant distributions of order 1 (obtained in Theorem B), while the latter follows [27, pages 379–381], and it only uses the ergodicity of the Lebesgue measure under a multicritical circle map (as established in Theorem 6.9).

Proof of Theorem D for C^1 observables For any given $\phi \in C^1(S^1)$, $\phi - \int_{S^1} \phi \, d\mu$ belongs to $\ker \mu$. Combining Theorem B with Proposition 7.4 we have that, for any given $\varepsilon > 0$, there exists $u \in C^1(S^1)$ such that

$$\left\| \left(u \circ f - u\right) - \left(\phi - \int_{S^1} \phi \, d\mu\right) \right\|_{C^1} \leq \frac{\varepsilon}{2}.$$

Let $\tilde{\phi} \in C^1(S^1)$ be given by $\tilde{\phi} = u \circ f - u + \int_{S^1} \phi \, d\mu$, so that $\|\tilde{\phi} - \phi\|_{C^1} \leq \varepsilon/2$. Since μ is f-invariant, we have $\int_{S^1} \tilde{\phi} \, d\mu = \int_{S^1} \phi \, d\mu$. Now for any given $x \in S^1$,

$$\left| \sum_{i=0}^{q_n - 1} \phi\big(f^i(x)\big) - q_n \int_{S^1} \phi \, d\mu \right| \leq \left| \sum_{i=0}^{q_n - 1} (\phi - \tilde{\phi})\big(f^i(x)\big) - q_n \int_{S^1} (\phi - \tilde{\phi}) \, d\mu \right| + \left| \sum_{i=0}^{q_n - 1} \tilde{\phi}\big(f^i(x)\big) - q_n \int_{S^1} \tilde{\phi} \, d\mu \right|.$$

Let us estimate both terms at the right side of this inequality. On one hand, by the standard Denjoy–Koksma inequality (see for instance [8, Thm. 3.3]),

$$\left| \sum_{i=0}^{q_n - 1} (\phi - \tilde{\phi})\big(f^i(x)\big) - q_n \int_{S^1} (\phi - \tilde{\phi}) \, d\mu \right| \leq \|\tilde{\phi} - \phi\|_{C^1} \leq \varepsilon/2.$$

On the other hand,

$$\begin{aligned} \sum_{i=0}^{q_n - 1} \tilde{\phi}\big(f^i(x)\big) - q_n \int_{S^1} \tilde{\phi} \, d\mu &= \sum_{i=0}^{q_n - 1} \left[u\big(f^{i+1}(x)\big) - u\big(f^i(x)\big) + \int_{S^1} \phi \, d\mu \right] - q_n \int_{S^1} \tilde{\phi} \, d\mu \\ &= u\big(f^{q_n}(x)\big) - u(x) + q_n \left(\int_{S^1} \phi \, d\mu - \int_{S^1} \tilde{\phi} \, d\mu \right) \\ &= u\big(f^{q_n}(x)\big) - u(x). \end{aligned}$$

In particular,

$$\left| \sum_{i=0}^{q_n - 1} \tilde{\phi}\big(f^i(x)\big) - q_n \int_{S^1} \tilde{\phi} \, d\mu \right| \leq \|u\|_{C^1} \, \|f^{q_n} - \mathrm{Id}\,\|_{C^0}.$$

By minimality of f, we can choose $n_0 \in \mathbb{N}$ such that $\|u\|_{C^1} \, \|f^{q_n} - \mathrm{Id}\,\|_{C^0} < \varepsilon/2$ for all $n \geq n_0$. Therefore, $\left| \sum_{i=0}^{q_n - 1} \phi\big(f^i(x)\big) - q_n \int_{S^1} \phi \, d\mu \right| < \varepsilon$ for all $x \in S^1$ and $n \geq n_0$. Since ε is arbitrary, this finishes the proof. □

Let us now give a proof of Theorem D that works in general, following [27, Section 2]. With this purpose, we will need the following lemma.

Lemma 7.10 *If $v \in L^1(m)$ is a Lebesgue-integrable function on the circle such that $\int_{S^1} v \, dm = 0$, then there exists a sequence v_n of Lebesgue-integrable functions on the circle such that $\int_{S^1} v_n \, dm = 0$ for all n and*

$$(v_n \circ f) Df - v_n \longrightarrow v$$

in the L^1 sense.

Proof Consider the continuous linear operator $U \colon L^1(m) \to L^1(m)$ given by $Uw = (w \circ f) Df - w$, and let M be the image of U. First, assume that $v \notin \mathrm{cl}\ M$; then, by the Hahn–Banach theorem, there exists $L \in L^1(m)^*$ such that L is identically null on M and $\langle L, v \rangle = 1$. By identification of $L^1(m)^*$ with $L^\infty(m)$, there exists an L^∞ function ϕ such that

$$\langle L, w \rangle = \int_{S^1} \phi \, w \, dm$$

for all $w \in L^1(m)$. But then, for all $w \in L^1(m)$,

$$0 = \int_{S^1} \phi \, [(w \circ f)Df - w] \, dm = \int_{S^1} (\phi \circ f^{-1}) \, w \, dm - \int_{S^1} \phi \, w \, dm.$$

Since the previous equality holds for all Lebesgue-integrable w, we conclude that ϕ is f-invariant m-almost everywhere. But, as proved in Sect. 6.2, f is ergodic with respect to Lebesgue (Theorem 6.9). Therefore, ϕ must be almost everywhere constant, i.e., there exists some constant β such that $\phi = \beta$ m-almost everywhere. But then

$$1 = \langle L, v \rangle = \int_{S^1} v \, \phi \, dm = \beta \int_{S^1} v \, dm,$$

contradicting the fact that $v \in \ker m$. Thus, $v \in \mathrm{cl}\ M$, and there is some sequence $\bar{w}_n$ of L^1 functions for which

$$(\bar{w}_n \circ f)Df - \bar{w}_n \longrightarrow v$$

in the L^1 sense.

Now, recall the functions $\hat{w}_k$ from Lemma 7.7: we have that

$$(\hat{w}_k \circ f)Df - \hat{w}_k \longrightarrow Df - 1$$

uniformly. If we define $c_n := \int_{S^1} \bar{w}_n \, dm$, then the desired sequence w_n is given by

$$w_n := \bar{w}_n - c_n + c_n \, \hat{w}_{k_n},$$

where the k_n are chosen as in the proof of Proposition 7.6. □

Proof of Theorem D For any given $\phi \in AC(S^1)$, we have that its derivative $v := \phi'$ exists m-almost everywhere, and furthermore, $\int_{S^1} v \, dm = 0$ (since ϕ is $\mathbb{Z}$-periodic). Let $\epsilon > 0$ be fixed. By Lemma 7.10, there exists $w \in L^1(m)$, $w \in \ker m$, such that

$$u := v - [(w \circ f)Df - w] \tag{7.21}$$

satisfies

$$\int_{S^1} |u| \, dm \leq \frac{\epsilon}{2}. \tag{7.22}$$

Now, let $\psi, \xi \in AC(S^1)$ be given by

$$\psi(x) = \int_{[0,x]} w \, dm, \quad \xi = \phi - \psi \circ f + \psi - \int_{S^1} \phi \, d\mu.$$

Then

$$\phi = \xi + \psi \circ f - \psi + \int_{S^1} \phi \, d\mu, \tag{7.23}$$

and it is immediate from the f-invariance of μ that $\int_{S^1} \xi \, d\mu = 0$. Furthermore,

$$\xi' = \phi' - (\psi' \circ f)Df + \psi' = v - (w \circ f)Df + w = u.$$

By a telescoping sum, we have from (7.23) that

$$\sum_{i=0}^{q_n - 1} \phi \circ f^i - q_n \int_{S^1} \phi \, d\mu = \sum_{i=0}^{q_n - 1} \xi \circ f^i + \psi \circ f^{q_n} - \psi. \tag{7.24}$$

We now proceed to estimate the right hand side of the above equation. On one hand, by the standard Denjoy–Koksma inequality and (7.22),

$$\left\| \sum_{i=0}^{q_n-1} \xi \circ f^i \right\|_{C^0} \leq \mathrm{var}(\xi) \leq \left\| \xi' \right\|_{L^1(m)} = \|u\|_{L^1(m)} \leq \frac{\epsilon}{2}. \tag{7.25}$$

On the other hand, by the minimality of f,

$$\left\| \psi \circ f^{q_n} - \psi \right\|_{C^0} \leq \frac{\epsilon}{2} \tag{7.26}$$

for sufficiently large n. By the triangle inequality, Eqs. (7.24)–(7.26) together show that, for sufficiently large n,

$$\left\| \sum_{i=0}^{q_n-1} \phi \circ f^i - q_n \int_{S^1} \phi \, d\mu \right\|_{C^0} \leq \epsilon. \tag{7.27}$$

Since ϵ is arbitrary, this concludes the proof of Theorem D. □

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