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## Research Paper

Characteristic subgroups and the  $R_\infty$ -property for virtual braid groupsKarel Dekimpe<sup>a</sup>, Daciberg Lima Gonçalves<sup>b</sup>, Oscar Ocampo<sup>c,\*</sup><sup>a</sup> KU Leuven Campus Kulak Kortrijk, Etienne Sabbelaan 53, 8500 Kortrijk, Belgium<sup>b</sup> Departamento de Matemática - IME-USP, Rua do Matão 1010 CEP: 05508-090 - São Paulo - SP - Brazil<sup>c</sup> Universidade Federal da Bahia, Departamento de Matemática - IME, Av. Milton Santos S/N CEP: 40170-110 - Salvador - BA - Brazil

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## ABSTRACT

Let  $n \geq 2$ . Let  $VB_n$  (resp.  $VP_n$ ) denote the virtual braid group (resp. virtual pure braid group), let  $WB_n$  (resp.  $WP_n$ ) denote the welded braid group (resp. welded pure braid group) and let  $UVB_n$  (resp.  $UVP_n$ ) denote the unrestricted virtual braid group (resp. unrestricted virtual pure braid group). In the first part of this paper we prove that, for  $n \geq 4$ , the group  $VP_n$  and for  $n \geq 3$  the groups  $WP_n$  and  $UVP_n$  are characteristic subgroups of  $VB_n$ ,  $WB_n$  and  $UVB_n$ , respectively. In the second part of the paper we show that, for  $n \geq 2$ , the virtual braid group  $VB_n$ , the unrestricted virtual pure braid group  $UVP_n$ , and the unrestricted virtual braid group  $UVB_n$  have the  $R_\infty$ -property. As a consequence of the technique used for few strings we also prove that, for  $n = 2, 3, 4$ , the welded braid group  $WB_n$  has the  $R_\infty$ -property and that for  $n = 2$  the corresponding pure braid groups have the  $R_\infty$ -property. On the other hand for  $n \geq 3$  it is unknown if the  $R_\infty$ -property holds or not for the virtual pure braid group  $VP_n$  and the welded pure braid group  $WP_n$ .

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## 1. Introduction

In this paper we are interested in characteristic subgroups and the  $R_\infty$ -property of the virtual braid groups.

The virtual braid group  $VB_n$  is the natural companion to the category of virtual knots, just as the Artin braid group is to usual knots and links. We note that a virtual knot diagram is like a classical knot diagram with one extra type of crossing, called a virtual crossing. The virtual braid groups have interpretations in terms of diagrams, see [22], [23] and [36]. The notion of virtual knots and links was introduced by Kauffman together with virtual braids in [23], and since then it has drawn the attention of several researchers. Virtual braid groups have interesting quotients, among them we are interested here in the welded braid group  $WB_n$  and the unrestricted virtual braid group  $UVB_n$ . As in the classical case for Artin braid groups, notable subgroups of  $VB_n$ ,  $WB_n$  and  $UVB_n$  are the respective pure subgroups  $VP_n$ ,  $WP_n$  and  $UVP_n$ . For  $VB_n$  there is also another notable subgroup that is an Artin group and will be denoted by  $KB_n$ . For the definitions of all these groups see Subsection 2.1.

Let  $K$  be a subgroup of a group  $G$ . Recall that  $K$  is said to be a characteristic subgroup of  $G$  if  $\varphi(K) = K$  for every automorphism  $\varphi$  of  $G$ . The existence of characteristic subgroups of a given group is in many cases useful. E.g. in this paper we will use characteristic subgroups to study the  $R_\infty$ -property (see the definition below) for the virtual braid groups as well for some quotients of it.

Let  $G, H$  be two groups. For every  $h \in H$  we have the inner automorphism  $c_h: H \rightarrow H$ , defined by  $c_h(x) = h x h^{-1}$ . We say that two homomorphisms  $\psi_1, \psi_2: G \rightarrow H$  are *conjugate*, and we denote it by  $\psi_1 \sim_c \psi_2$ , if there exists an element  $h \in H$  such that  $\psi_2 = c_h \circ \psi_1$ , which means that  $\psi_2(g) = h \psi_1(g) h^{-1}$ , for every  $g \in G$ . We note that  $\sim_c$  is an equivalence relation. Our first result is the following theorem which gives a condition on the kernel of a homomorphism to be a characteristic subgroup.

**Theorem 1.** *Let  $G$  and  $Q$  be two groups. Let  $\Sigma$  be the set of all surjective homomorphisms from  $G$  onto  $Q$ , let  $\mathcal{T} = \Sigma / \sim_c$  be the set of equivalence classes of  $\Sigma$  by  $\sim_c$  and let  $\Lambda$  be a set of representatives of  $\mathcal{T}$ . Let  $\lambda \in \Lambda$  be such that for all  $\omega \in \Lambda$  such that  $\text{Ker}(\omega)$  is isomorphic to  $\text{Ker}(\lambda)$  it actually holds that  $\text{Ker}(\lambda) = \text{Ker}(\omega)$ .*

*Then,  $\text{Ker}(\lambda)$  is a characteristic subgroup of  $G$ .*

Our main result about characteristic subgroups of virtual braid groups and some of its quotients is the following result.

**Theorem 2.** *Let  $n \geq 2$ .*

- (a) *The virtual pure braid group  $VP_n$  is a characteristic subgroup of  $VB_n$  if and only if  $n \geq 4$  and the group  $KB_n$  is a characteristic subgroup of the virtual braid group  $VB_n$  if and only if  $n \geq 3$ .*

- (b) *The welded pure braid group  $WP_n$  is a characteristic subgroup of the welded braid group  $WB_n$  if and only if  $n \geq 3$ .*
- (c) *The unrestricted virtual pure braid group  $UVP_n$  is a characteristic subgroup of the unrestricted virtual braid group  $UVB_n$  if and only if  $n \geq 3$ .*

We note that Theorem 2 is known for  $n \geq 5$ , but was not known for the case of few strings, so in this work we complete the knowledge about pure subgroups being characteristic in their respective virtual braid groups. Theorem 2 (for  $n \geq 5$ ) was proved for  $VB_n$  in [3] using the explicit description of the automorphism group of  $VB_n$ , see [3, Corollary 2.7]; for  $WB_n$  (see [26, Remark 2.17]) and  $UVB_n$  (see [26, Proposition 2.15]) the description of the automorphism group was not used. However, we highlight that our approach is different by using the general result in Theorem 1 for any number of strings: instead of using explicitly the automorphism group of each of the groups involved we use, up to conjugation, the set of surjective homomorphisms onto the symmetric group. The description of these sets for small values of  $n$  is given in Section 3. It is worth to notice that Theorem 2 will be used in the proof of Theorem 3 below.

Consider a group  $G$  and an endomorphism  $\alpha$  of  $G$ . We say that two elements  $x$  and  $y$  of  $G$  are twisted conjugate (via  $\alpha$ ) if and only if there exists a  $z \in G$  such that  $x = zy\alpha(z)^{-1}$ . It is easy to see that the relation of being twisted conjugate is an equivalence relation and the number of equivalence classes (also referred to as Reidemeister classes) is called the Reidemeister number  $R(\alpha)$  of  $\alpha$ . This Reidemeister number is either a positive integer or  $\infty$ .

Reidemeister numbers find their origins in algebraic topology and to be more precise in Nielsen–Reidemeister fixed point theory. Here one is interested in the study of the fixed point classes of a selfmap  $f$  of a space  $X$ . The number of fixed point classes of  $f$  is called the Reidemeister number of  $f$  and is denoted by  $R(f)$ . It is known that  $R(f) = R(f_*)$ , where  $f_*: \pi_1(X) \rightarrow \pi_1(X)$  is the induced endomorphism on the fundamental group  $\pi_1(X)$  of  $X$ .

There is currently a growing interest in the study of groups  $G$  having the  $R_\infty$ -property, these are groups for which  $R(\alpha) = \infty$  for any automorphism  $\alpha \in \text{Aut}(G)$ . The study of groups with that property was initiated by Fel'shtyn and Hill [15].

Since the beginning of this century many authors have been studying this property and for several families of groups it is known whether or not they have the  $R_\infty$ -property. Here are some families of groups with the  $R_\infty$ -property: the non-elementary Gromov hyperbolic groups [12,24], most of the Baumslag–Solitar groups [13] and groups quasi-isometric to Baumslag–Solitar groups [34], generalized Baumslag–Solitar groups [25], many linear groups [16,31] and also several families of lamplighter groups [19,35]. The study of the  $R_\infty$ -property for braid groups and braid-type groups has been increasing during the last years. For instance, in [14] it was shown that the Artin braid groups  $B_n$  and the mapping class groups of closed orientable surfaces different from the sphere have property  $R_\infty$ . In [8] the case of the pure Artin braid groups  $P_n$  was considered,

and they also share this property. More recently, the  $R_\infty$ -property was studied for some right angled Artin groups in [10] and for some Artin groups in [4].

In this paper we study the  $R_\infty$ -property for virtual braid groups and unrestricted virtual braid groups. Since these groups are trivial when  $n = 1$ , we shall consider, in general,  $n \geq 2$ . More precisely, the statement below summarises the main results in this work about this property.

**Theorem 3.** *Let  $n \geq 2$ . The following groups have the  $R_\infty$ -property:*

- (a) *The unrestricted virtual pure braid group  $UVP_n$ ,*
- (b) *the unrestricted virtual braid group  $UVB_n$ ,*
- (c) *the virtual braid group  $VB_n$ .*

*Further, if  $n = 2, 3$  and  $4$  also the welded braid group  $WB_n$  has the  $R_\infty$ -property, and for  $n = 2$  also the virtual  $(VP_2)$ , welded  $(WP_2)$  and unrestricted  $(UVP_2)$  pure braid groups have the  $R_\infty$ -property.*

For  $n \geq 5$ , part (c) of Theorem 3 above was simultaneously obtained by N. Nanda ([30]), using a different approach.

This paper is organized as follows. In the first subsection of Section 2 we will give the main definitions about the virtual braid groups that will be used in the text and in the second subsection we prove Theorem 1. In Section 3 we prove Theorem 2. To do that, we first treat the case  $n = 2$  and thereafter we describe the set of homomorphisms, up to conjugation, from  $VB_n$  (and also from  $WB_n$  and  $UVB_n$ ) to  $S_n$ , for  $n = 3, 4$ . Then, in Subsection 3.4 we use this information and Theorem 1 to prove Theorem 2. We prove Theorem 3 in Section 4, its proof is given in several different steps and using different techniques. For  $n \geq 2$ , the group  $UVP_n$  is isomorphic to a direct product of free groups and from Theorem 2 it is a characteristic subgroup of  $UVB_n$ . Using this, in Proposition 33, we prove item (a) and item (b) of Theorem 3. To prove Theorem 3 (c), for  $n \geq 5$ , we show that the kernel of the natural projection of  $VB_n$  onto  $UVB_n$  is a characteristic subgroup and then we use item (b) of Theorem 3, this will be done in Theorem 36. For  $n = 3$  or  $n = 4$ , we first show that the quotient  $VB_n/[VP_n, VP_n]$  has the  $R_\infty$ -property, where  $[VP_n, VP_n]$  is the commutator subgroup of the virtual pure braid group  $VP_n$ , see Theorems 28 and 31, respectively. Then the desired result for Theorem 3 (c) for the cases  $n = 3$  and  $n = 4$  is obtained in Corollaries 29 and 32, respectively.

We end this paper with an appendix explaining how the techniques of this paper can also be used to treat other braid-like groups such as the virtual twin groups.

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## 2. Preliminaries

In this section we give the definitions of virtual braid groups that we use in the text and we prove Theorem 1.

### 2.1. Virtual braid groups and forbidden relations

In this subsection, we recall the basic definitions of virtual braid groups. First, we write a presentation of the virtual braid group  $VB_n$  that will be very useful in this work. This presentation appears in [2] and it is a reformulation of the one given in [36, p.798].

**Definition 4** ([2, Theorem 4]). Let  $n \geq 2$ . The *virtual braid group on  $n$  strings*, denoted by  $VB_n$ , is the abstract group generated by  $\sigma_i$  (classical generators) and  $v_i$  (virtual generators), for  $i = 1, 2, \dots, n-1$ , with relations:

- (AR1)  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ ,  $i = 1, 2, \dots, n-2$ ;
- (AR2)  $\sigma_i \sigma_j = \sigma_j \sigma_i$ ,  $|i - j| \geq 2$ ;
- (PR1)  $v_i v_{i+1} v_i = v_{i+1} v_i v_{i+1}$ ,  $i = 1, 2, \dots, n-2$ ;
- (PR2)  $v_i v_j = v_j v_i$ ,  $|i - j| \geq 2$ ;
- (PR3)  $v_i^2 = 1$ ,  $i = 1, 2, \dots, n-1$ ;
- (MR1)  $\sigma_i v_j = v_j \sigma_i$ ,  $|i - j| \geq 2$ ;
- (MR2)  $v_i v_{i+1} \sigma_i = \sigma_{i+1} v_i v_{i+1}$ ,  $i = 1, 2, \dots, n-2$ .

**Remark 5.** The letters AR, PR and MR that appear in Definition 4 are used to indicate the type of relations in the given presentation of  $VB_n$ : *Artin Relations*, *Permutation Relations* and *Mixed Relations*.

Let  $n \geq 2$ . As in [3, Section 2], from the presentation of  $VB_n$  one can see that there are surjective homomorphisms  $\pi_P: VB_n \rightarrow S_n$  and  $\pi_K: VB_n \rightarrow S_n$  defined by  $\pi_P(\sigma_i) = \pi_P(v_i) = \tau_i = (i, i+1)$  for all  $1 \leq i \leq n-1$  and by  $\pi_K(\sigma_i) = 1$  and  $\pi_K(v_i) = \tau_i = (i, i+1)$  for all  $1 \leq i \leq n-1$ , respectively. The kernel of  $\pi_P$  is called the *virtual pure braid group* and it is denoted by  $VP_n$ . A presentation of this group can be found in [1, Theorem 1]. The kernel of  $\pi_K$ , denoted by  $KB_n$ , is known to be an Artin

group (this follows from the presentation of  $KB_n$  given in [2, Proposition 17]). This fact has been useful to obtain properties of the virtual braid group itself, for instance it was essential in determining the set of endomorphisms of  $VB_n$  in [3]. As mentioned in the first paragraph of Section 3 of [1] (resp. in [2, Section 6]) the virtual braid group admits a decomposition as semi-direct product  $VB_n = VP_n \rtimes S_n$  (resp.  $VB_n = KB_n \rtimes S_n$ ), with  $\iota: S_n \rightarrow VB_n$  given by  $\iota(\tau_i) = v_i$ , for  $i = 1, \dots, n-1$ , being a section for  $\pi_P$  (resp. for  $\pi_K$ ).

**Definition 6.** Consider, for  $i = 1, \dots, n-2$ , the following so-called forbidden relations in the virtual braid group:

- (a)  $v_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i v_{i+1}$ ,
- (b)  $v_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} v_i$ .

The *welded braid group*, denoted by  $WB_n$ , is the quotient of  $VB_n$  by the normal closure of the relations (a). The *unrestricted virtual braid group*, denoted by  $UVB_n$ , is the quotient of  $VB_n$  by the normal closure of the relations (a) and (b).

**Remark 7.** We note that the welded braid group  $WB_n$  appears with other names in the literature, for example as *the loop braid group*, see [6]. Also in [6] one can find an extensive exposition of it.

Since the forbidden relations are preserved by  $\pi_P: VB_n \rightarrow S_n$ , we may define the homomorphisms  $\overline{\pi_P}: WB_n \rightarrow S_n$  and  $\overline{\overline{\pi_P}}: UVB_n \rightarrow S_n$  by  $\overline{\pi_P}(\sigma_i) = \overline{\pi_P}(v_i) = (i, i+1)$  for all  $1 \leq i \leq n-1$  and  $\overline{\overline{\pi_P}}(\sigma_i) = \overline{\overline{\pi_P}}(v_i) = (i, i+1)$  for all  $1 \leq i \leq n-1$ , respectively. The kernel of  $\overline{\pi_P}$  is called the *welded pure braid group* and it is denoted by  $WP_n$ . The kernel of  $\overline{\overline{\pi_P}}$  is called the *unrestricted virtual pure braid group* and it is denoted by  $UVP_n$ .

We note that, for  $n \geq 3$ , is not possible to define a similar homomorphism  $\pi_K: VB_n \rightarrow S_n$  for the groups  $WB_n$  and  $UVB_n$  since the forbidden relations are not preserved by  $\pi_K$ .

## 2.2. On characteristic subgroups

In this subsection we prove a general result about characteristic subgroups of a group. Recall that two homomorphisms of groups  $\psi_1, \psi_2: G \rightarrow H$  are conjugate, denoted by  $\psi_1 \sim_c \psi_2$ , if there exists  $\eta \in \text{Inn}(H)$  such that  $\psi_2 = \eta \circ \psi_1$ . The following lemma is an easy observation.

**Lemma 8.** Let  $\Sigma$  be the set of all homomorphisms from  $G$  to  $H$ . Then  $\sim_c$  is an equivalence relation on  $\Sigma$ .

We note that Lemma 8 also holds if we restrict  $\Sigma$  to the set of all surjective homomorphisms from  $G$  to  $H$ .

**Proof of Theorem 1.** Let  $\lambda \in \Lambda$  such that for all  $\omega \in \Lambda$  it holds that  $\text{Ker}(\lambda) = \text{Ker}(\omega)$  as soon as they are isomorphic. Let  $\varphi: G \rightarrow G$  be any automorphism of  $G$ . Since  $\lambda \circ \varphi \in \Sigma$  then there is  $\zeta \in \Lambda$  such that  $\lambda \circ \varphi \sim_c \zeta$ , i.e. there exists an inner automorphism of  $Q$ , say  $\eta$ , such that  $\lambda \circ \varphi = \eta \circ \zeta$  and the following square is commutative

$$\begin{array}{ccc} G & \xrightarrow{\zeta} & Q \\ \varphi \downarrow \cong & & \eta \downarrow \cong \\ G & \xrightarrow{\lambda} & Q \end{array}$$

It is easy to see that  $\varphi(\text{Ker}(\zeta)) = \text{Ker}(\lambda)$  and so  $\text{Ker}(\zeta)$  is isomorphic to  $\text{Ker}(\lambda)$ . By the hypothesis, it follows that  $\text{Ker}(\zeta) = \text{Ker}(\lambda)$ , hence  $\varphi(\text{Ker}(\lambda)) = \varphi(\text{Ker}(\zeta)) = \text{Ker}(\lambda)$ .

So we proved that  $\varphi(\text{Ker}(\lambda)) = \text{Ker}(\lambda)$  for any automorphism of  $G$ , showing that  $\text{Ker}(\lambda)$  is a characteristic subgroup of  $G$ .  $\square$

**Remark 9.** The proof of Theorem 1 was in part motivated from the one given in [26] to prove that, for  $n \geq 5$ ,  $UVP_n$  is a characteristic subgroup of  $UVB_n$ , see [26, Proposition 2.15].

We finish this subsection with the following property about homomorphisms being conjugate.

**Proposition 10.** Let  $\overline{G}$  be a quotient of a group  $G$  and  $p: G \rightarrow \overline{G}$  the natural projection. Consider two homomorphisms  $\overline{\zeta}_1, \overline{\zeta}_2: \overline{G} \rightarrow H$  and define  $\zeta_1, \zeta_2: G \rightarrow H$  by  $\zeta_i = \overline{\zeta}_i \circ p$ , for  $i = 1, 2$ . Then,  $\zeta_1 \sim_c \zeta_2$  if and only if  $\overline{\zeta}_1 \sim_c \overline{\zeta}_2$ .

**Proof.** Suppose that  $\zeta_1 \sim_c \zeta_2$ . By definition there is  $h \in H$  such that  $\zeta_1(g) = h\zeta_2(g)h^{-1}$ , for all  $g \in G$ . So,  $\overline{\zeta}_1 \circ p(g) = h\overline{\zeta}_2 \circ p(g)h^{-1}$ , for all  $g \in G$ , that is equivalent to  $\overline{\zeta}_1(\overline{g}) = h\overline{\zeta}_2(\overline{g})h^{-1}$ , for all  $\overline{g} \in \overline{G}$ . Hence,  $\overline{\zeta}_1 \sim_c \overline{\zeta}_2$ .

The proof of the converse is similar.  $\square$

### 3. Characteristic subgroups of virtual braid groups

The main objective of this section is to prove that the pure virtual braid groups  $VP_n$  are characteristic subgroups of  $VB_n$  for  $n \geq 4$  and that the groups  $KB_n$  are characteristic subgroups of  $VB_n$  for  $n \geq 3$ . We recall that a group homomorphism  $\psi: G \rightarrow H$  is said to be *abelian* if its image  $\psi(G)$  is an abelian subgroup of  $H$ . The homomorphisms  $\pi_P$  and  $\pi_K$  defined in Subsection 2.1 will be mentioned several times in this section.

### 3.1. Two strings case

From the presentation of the virtual braid group  $VB_n$  and the definition of the groups  $WB_n$  and  $UVB_n$ , see Section 2, it follows that

$$VB_2 = WB_2 = UVB_2 = \langle \sigma_1, v_1 \mid v_1^2 = 1 \rangle \cong \mathbb{Z} * \mathbb{Z}_2.$$

**Remark 11.** In [26, Remark 2.6] the author constructed an automorphism  $\alpha: VB_2 \rightarrow VB_2$  defined by  $\alpha(\sigma_1) = \sigma_1^{-1}v_1$  and  $\alpha(v_1) = v_1$ . Using this automorphism she proved that  $VP_2 = WP_2 = UVP_2$  is not characteristic in  $VB_2 = WB_2 = UVB_2$ .

We note that the same automorphism may be used to verify that  $KB_2$  is not characteristic in  $VB_2$ .

### 3.2. Homomorphisms from virtual braid groups to the symmetric group: three strings case

In this subsection we will use the following presentation of the virtual braid group with 3 strings (see Section 2 for a presentation of  $VB_n$ ):

$$VB_3 = \langle \sigma_1, \sigma_2, v_1, v_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, v_1v_2v_1 = v_2v_1v_2, v_1v_2\sigma_1 = \sigma_2v_1v_2, v_1^2 = 1, v_2^2 = 1 \rangle \quad (1)$$

and the presentation

$$S_3 = \langle \tau_1, \tau_2 \mid \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2, \tau_1^2 = 1, \tau_2^2 = 1 \rangle \quad (2)$$

of the symmetric group  $S_3$ .

Define, for  $1 \leq i \leq 8$ , the following homomorphisms  $\psi_i: VB_3 \rightarrow S_3$ :

- (a)  $\psi_1(v_1) = \tau_1, \psi_1(v_2) = \tau_1, \psi_1(\sigma_1) = \tau_2, \psi_1(\sigma_2) = \tau_2$ ;
- (b)  $\psi_2(v_1) = \tau_1, \psi_2(v_2) = \tau_2, \psi_2(\sigma_1) = \tau_1, \psi_2(\sigma_2) = \tau_2$ . In this case  $\psi_2$  is equal to the homomorphism  $\pi_P$ ;
- (c)  $\psi_3(v_1) = \tau_1, \psi_3(v_2) = \tau_2, \psi_3(\sigma_1) = \tau_2, \psi_3(\sigma_2) = \tau_1\tau_2\tau_1$ ;
- (d)  $\psi_4(v_1) = \tau_1, \psi_4(v_2) = \tau_2, \psi_4(\sigma_1) = \tau_1\tau_2\tau_1, \psi_4(\sigma_2) = \tau_1$ ;
- (e)  $\psi_5(v_1) = \tau_1, \psi_5(v_2) = \tau_2, \psi_5(\sigma_1) = \tau_1\tau_2, \psi_5(\sigma_2) = \tau_1\tau_2$ ;
- (f)  $\psi_6(v_1) = \tau_1, \psi_6(v_2) = \tau_2, \psi_6(\sigma_1) = \tau_2\tau_1, \psi_6(\sigma_2) = \tau_2\tau_1$ ;
- (g)  $\psi_7(v_1) = \tau_1, \psi_7(v_2) = \tau_2, \psi_7(\sigma_1) = 1, \psi_7(\sigma_2) = 1$ . In this case  $\psi_7$  is equal to the homomorphism  $\pi_K$ ;
- (h)  $\psi_8(v_1) = \tau_1, \psi_8(v_2) = \tau_1, \psi_8(\sigma_1) = \tau_1\tau_2, \psi_8(\sigma_2) = \tau_1\tau_2$ .

**Remark 12.** We note that the homomorphisms  $\{\psi_i \mid 1 \leq i \leq 8\}$  are pairwise non conjugate. Indeed, first we can see that the homomorphisms  $\psi_1$  and  $\psi_8$  are not conjugate to  $\psi_i$  for  $i = 2, 3, \dots, 7$  because  $\psi_j(v_1) = \psi_j(v_2)$  for  $j = 1, 8$  and this is not the case for



the  $\psi_i$ 's with  $i = 2, 3, \dots, 7$ . Moreover,  $\psi_1$  is not conjugate to  $\psi_8$  because  $\psi_1(\sigma_1)$  is an element of order 2, while  $\psi_8(\sigma_1)$  has order 3.

Now assume that  $\psi_i$  is conjugate to  $\psi_j$  with  $3 \leq i, j \leq 7$  and let  $\lambda \in S_3$  be such that  $\psi_i = c_\lambda \circ \psi_j$ . As  $\psi_i(v_1) = \psi_j(v_1) = \tau_1$ , it follows that  $\lambda\tau_1\lambda^{-1} = \tau_1$  and analogously from  $\psi_i(v_2) = \psi_j(v_2) = \tau_2$  we find that  $\lambda\tau_2\lambda^{-1} = \tau_2$ . Hence  $\lambda$  centralizes both  $\tau_1$  and  $\tau_2$  and so  $\lambda$  belongs to the centre of  $S_3$  (because  $S_3$  is generated by  $\tau_1$  and  $\tau_2$ ). But the centre of  $S_3$  is trivial, so  $\lambda = 1$  from which we get that  $\psi_i = \psi_j$  and so  $i = j$ .

**Theorem 13.** *Let  $\psi: VB_3 \rightarrow S_3$  be a homomorphism. Then, up to conjugation, one of the following possibilities holds*

- (a)  $\psi$  is abelian;
- (b)  $\psi \in \{\psi_i \mid 1 \leq i \leq 8\}$ .

**Proof.** Let  $\psi: VB_3 \rightarrow S_3$  be a homomorphism and let  $\iota: S_3 \rightarrow VB_3$  be the natural inclusion defined by  $\iota(\tau_1) = v_1$  and  $\iota(\tau_2) = v_2$ . Then,  $\psi \circ \iota$  is an endomorphism of  $S_3$ . It is straightforward to verify that, up to conjugation,  $\psi \circ \iota$  is either the identity homomorphism or  $\text{im}(\psi \circ \iota) = \langle \tau_1 \rangle$  or  $\text{im}(\psi \circ \iota) = \{1\}$  (the trivial subgroup). To say that  $\psi \circ \iota$  is abelian is equivalent to saying that  $\psi \circ \iota$  is, up to equivalence, not the identity.

First suppose that  $\psi \circ \iota$  is not the identity. Hence, from the relations  $v_1^2 = 1$ ,  $v_2^2 = 1$  and  $v_1v_2v_1 = v_2v_1v_2$  we have  $\psi(v_1) = \psi(v_2) = w_1 \in S_3$  with  $w_1^2 = 1$ . From the relation  $v_1v_2\sigma_1 = \sigma_2v_1v_2$  we get  $\psi(\sigma_1) = \psi(\sigma_2) = w_2 \in S_3$ . Notice that, up to conjugacy,  $w_1$  is either 1 or  $\tau_1$ . Also if  $w_1 = 1$  or  $w_2 = 1$  then  $\psi$  is abelian.

So, we suppose that  $w_1 = \tau_1$  and that  $w_2$  is a non trivial element in  $S_3$ . Now we analyse the possible values of  $w_2 \in S_3$  such that  $\psi$  is a homomorphism.

- If  $\psi(v_1) = \tau_1$ ,  $\psi(v_2) = \tau_1$ ,  $\psi(\sigma_1) = \tau_1$ ,  $\psi(\sigma_2) = \tau_1$  then  $\psi$  is abelian.
- Suppose that  $\psi(v_1) = \tau_1$ ,  $\psi(v_2) = \tau_1$ ,  $\psi(\sigma_1) = \tau_2$ ,  $\psi(\sigma_2) = \tau_2$ . This homomorphism is  $\psi_1$ .
- If  $\psi(v_1) = \tau_1$ ,  $\psi(v_2) = \tau_1$ ,  $\psi(\sigma_1) = \tau_1\tau_2$ ,  $\psi(\sigma_2) = \tau_1\tau_2$  then  $\psi$  is  $\psi_8$ .
- Suppose that  $\psi(v_1) = \tau_1$ ,  $\psi(v_2) = \tau_1$ ,  $\psi(\sigma_1) = \tau_1\tau_2\tau_1$ ,  $\psi(\sigma_2) = \tau_1\tau_2\tau_1$ . Then  $\psi$  is conjugate to  $\psi_1$ .
- Let  $\psi(v_1) = \tau_1$ ,  $\psi(v_2) = \tau_1$ ,  $\psi(\sigma_1) = \tau_2\tau_1$ ,  $\psi(\sigma_2) = \tau_2\tau_1$ . Then  $\psi$  is conjugate to  $\psi_8$ .

For any choice of  $w_2 = \psi(\sigma_1) = \psi(\sigma_2)$  in  $S_3$  we obtain an abelian homomorphism or a homomorphism that is conjugate to  $\psi_1$  or  $\psi_8$ .

Now, suppose that  $\psi \circ \iota$  is the identity homomorphism. This implies that

$$\psi(v_1) = \tau_1 \text{ and } \psi(v_2) = \tau_2.$$

From the mixed relation  $v_1v_2\sigma_1v_2v_1 = \sigma_2$  it follows that if we know  $\psi(\sigma_1)$  then  $\psi(\sigma_2)$  is completely determined. We analyse the possible values of  $\psi(\sigma_1)$ .

- Suppose that  $\psi(\sigma_1) = 1$ ,  $\psi(\sigma_2) = 1$ . This homomorphism is  $\psi_7 = \pi_K$ .
- Suppose that  $\psi(\sigma_1) = \tau_1$ . Then  $\psi(\sigma_2) = \tau_1\tau_2\tau_1\tau_2\tau_1 = \tau_2$ . This homomorphism is  $\psi_2 = \pi_P$ .
- Let  $\psi(\sigma_1) = \tau_2$ , then  $\psi(\sigma_2) = \tau_1\tau_2\tau_2\tau_2\tau_1 = \tau_1\tau_2\tau_1$ . This homomorphism is  $\psi_3$ .
- If  $\psi(\sigma_1) = \tau_1\tau_2\tau_1$ ,  $\psi(\sigma_2) = \tau_1\tau_2\tau_2\tau_1\tau_2\tau_2\tau_1 = \tau_1$ . This homomorphism is  $\psi_4$ .
- Let  $\psi(\sigma_1) = \tau_1\tau_2$ . So,  $\psi(\sigma_2) = \tau_1\tau_2\tau_1\tau_2\tau_2\tau_1 = \tau_1\tau_2$ . This homomorphism is  $\psi_5$ .
- Suppose that  $\psi(\sigma_1) = \tau_2\tau_1$ . Then  $\psi(\sigma_2) = \tau_1\tau_2\tau_2\tau_1\tau_2\tau_1 = \tau_2\tau_1$ . This homomorphism is  $\psi_6$ .

From the computations above we proved that, up to conjugation,  $\psi$  is abelian or  $\psi \in \{\psi_i \mid 1 \leq i \leq 8\}$ .  $\square$

Now we study the homomorphisms from the welded braid group  $WB_3$  and the unrestricted braid group  $UVB_3$  to the symmetric group  $S_3$ . The study of these homomorphisms follows the same lines as the proof of Theorem 13. Instead of repeating this method, we may verify which homomorphisms given in Theorem 13 respect the forbidden relations given in Definition 6.

**Theorem 14.** *Let  $\omega: WB_3 \rightarrow S_3$  be a homomorphism. Then, up to conjugation, one of the following possibilities holds*

- $\omega$  is abelian;
- $\omega = \bar{\psi}$ , where  $\psi \in \{\psi_i \mid 1 \leq i \leq 5\}$  as in Theorem 13 and  $\bar{\psi}: WB_3 \rightarrow S_3$  is the induced homomorphism in the quotient of  $VB_3$  by adding the forbidden relation (a) of Definition 6.

**Proof.** Let  $\omega: WB_3 \rightarrow S_3$  be a homomorphism. If  $\omega$  is abelian there is nothing to prove. Suppose that  $\omega$  is non-abelian. Let  $\psi: VB_3 \rightarrow S_3$  be a homomorphism that belongs to  $\{\psi_i \mid 1 \leq i \leq 8\}$  as in Theorem 13. We verify which of these homomorphisms satisfy the forbidden relation (a) of Definition 6.

Let  $\psi = \psi_1$ . Then  $\psi_1(v_1\sigma_2\sigma_1) = \tau_1\tau_2\tau_2 = \tau_1$  and  $\psi_1(\sigma_2\sigma_1v_2) = \tau_2\tau_2\tau_1 = \tau_1$ . Hence,  $\bar{\psi}_1: WB_3 \rightarrow S_3$  is a homomorphism. Now, let  $\psi = \psi_7$ . Recall that  $\psi_7(v_1) = \tau_1$ ,  $\psi_7(v_2) = \tau_2$ ,  $\psi_7(\sigma_1) = 1$  and  $\psi_7(\sigma_2) = 1$ . Then  $\psi_7(v_1\sigma_2\sigma_1) = \tau_1$  and  $\psi_7(\sigma_2\sigma_1v_2) = \tau_2$ . Hence,  $\psi_7$  does not induce a homomorphism in the quotient group  $WB_3$ . The verification of the other homomorphisms  $\psi \in \{\psi_i \mid 1 \leq i \leq 8\}$  is similar and with this computation we obtain the result of this theorem.  $\square$

Using the same idea of the proof of the last theorem we get the following result about unrestricted virtual braid groups.

**Theorem 15.** *Let  $\mu: UVB_3 \rightarrow S_3$  be a homomorphism. Then, up to conjugation, one of the following possibilities holds*

- (a)  $\mu$  is abelian;  
 (b)  $\mu = \overline{\overline{\psi}}$ , where  $\psi \in \{\psi_i \mid 1 \leq i \leq 4\}$  as in Theorem 13 and  $\overline{\overline{\psi}}: UVB_3 \rightarrow S_3$  is the induced homomorphism in the quotient of  $VB_3$  by adding the forbidden relations (a) and (b) of Definition 6.

**Proof.** The proof of this theorem is similar to the one of Theorem 14.  $\square$

**Remark 16.** It follows from Proposition 10 and Theorem 13 that the homomorphisms  $\{\overline{\psi_i} \mid 1 \leq i \leq 5\}$  of Theorem 14(b) are pairwise non conjugate. Similarly for the homomorphisms of Theorem 15(b).

### 3.3. Homomorphisms from virtual braid groups to the symmetric group: four strings case

In this subsection we will use the following presentation of the symmetric group

$$S_4 = \langle \tau_1, \tau_2, \tau_3 \mid \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2, \tau_2\tau_3\tau_2 = \tau_3\tau_2\tau_3, \tau_1\tau_3 = \tau_3\tau_1, \tau_1^2 = \tau_2^2 = \tau_3^2 = 1 \rangle \quad (3)$$

and the presentation of the virtual braid group with 4 strings (see Section 2 for a presentation of  $VB_n$ ) with generators  $\sigma_1, \sigma_2, \sigma_3, v_1, v_2, v_3$  and defining relations:

- (AR)  $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \sigma_1\sigma_3 = \sigma_3\sigma_1,$   
 (PR)  $v_1v_2v_1 = v_2v_1v_2, v_2v_3v_2 = v_3v_2v_3, v_1v_3 = v_3v_1, v_1^2 = 1, v_2^2 = 1, v_3^2 = 1,$   
 (MR)  $\sigma_1v_3 = v_3\sigma_1, \sigma_3v_1 = v_1\sigma_3, v_1v_2\sigma_1 = \sigma_2v_1v_2, v_2v_3\sigma_2 = \sigma_3v_2v_3.$

Let  $\eta: S_4 \rightarrow S_4$  be the homomorphism defined by  $\eta(\tau_1) = \eta(\tau_3) = \tau_1$  and  $\eta(\tau_2) = \tau_2$ .

**Lemma 17.** Let  $\varphi: S_4 \rightarrow S_4$  be any endomorphism of  $S_4$ . Then, up to conjugation, one of the following possibilities holds.

- (a)  $\varphi$  is abelian,  
 (b)  $\varphi$  is the identity homomorphism,  
 (c)  $\varphi = \eta$ .

**Proof.** This follows from examining case by case all the possible images of the transpositions. We note that if, for some  $i = 1, 2, 3$ ,  $\varphi(\tau_i) = 1$  then  $\varphi$  is the trivial homomorphism. Hence, we do not consider this case. Also, we notice that if, for some  $i = 1, 2, 3$ ,  $\varphi(\tau_i)$  is a product of different transpositions then  $\varphi$  is abelian with image the cyclic group of order 2.

Suppose that  $\varphi(\tau_1) = \tau_1$ . It follows from the relation  $\tau_1\tau_3 = \tau_3\tau_1$  that  $\varphi(\tau_3)$  lies in  $\{1, \tau_1, \tau_3, \tau_1\tau_3\}$ , the centralizer of  $\langle \tau_1 \rangle$  in  $S_4$ . By examining the possible images of  $\tau_3$  we obtain, up to conjugation, that either  $\varphi$  is abelian, or  $\varphi$  is the identity homomorphism or  $\varphi = \eta$ .  $\square$

Define, for  $1 \leq i \leq 6$ , the following homomorphisms  $\delta_i: VB_4 \longrightarrow S_4$ :

- (a)  $\delta_1(v_1) = \tau_1, \delta_1(v_2) = \tau_2, \delta_1(v_3) = \tau_1, \delta_1(\sigma_1) = \tau_1, \delta_1(\sigma_2) = \tau_2, \delta_1(\sigma_3) = \tau_1$ ;
- (b)  $\delta_2(v_1) = \tau_1, \delta_2(v_2) = \tau_2, \delta_2(v_3) = \tau_1, \delta_2(\sigma_1) = \tau_3, \delta_2(\sigma_2) = \tau_3\tau_2\tau_1\tau_2\tau_3, \delta_2(\sigma_3) = \tau_3$ ;
- (c)  $\delta_3(v_1) = \tau_1, \delta_3(v_2) = \tau_2, \delta_3(v_3) = \tau_3, \delta_3(\sigma_1) = \tau_1, \delta_3(\sigma_2) = \tau_2, \delta_3(\sigma_3) = \tau_3$ . This  $\delta_3$  is equal to the homomorphism  $\pi_P$ ;
- (d)  $\delta_4(v_1) = \tau_1, \delta_4(v_2) = \tau_2, \delta_4(v_3) = \tau_3, \delta_4(\sigma_1) = \tau_3, \delta_4(\sigma_2) = \tau_3\tau_2\tau_1\tau_2\tau_3, \delta_4(\sigma_3) = \tau_1$ ;
- (e)  $\delta_5(v_1) = \tau_1, \delta_5(v_2) = \tau_2, \delta_5(v_3) = \tau_3, \delta_5(\sigma_1) = 1, \delta_5(\sigma_2) = 1, \delta_5(\sigma_3) = 1$ . This  $\delta_5$  is equal to the homomorphism  $\pi_K$ ;
- (f)  $\delta_6(v_1) = \tau_1, \delta_6(v_2) = \tau_2, \delta_6(v_3) = \tau_1, \delta_6(\sigma_1) = 1, \delta_6(\sigma_2) = 1, \delta_6(\sigma_3) = 1$ .

**Remark 18.** We note that the homomorphisms  $\{\delta_i \mid 1 \leq i \leq 6\}$  are pairwise non conjugate.

**Theorem 19.** Let  $\delta: VB_4 \longrightarrow S_4$  be a homomorphism. Then, up to conjugation, one of the following possibilities holds

- (a)  $\delta$  is abelian;
- (b)  $\delta \in \{\delta_i \mid 1 \leq i \leq 6\}$ .

**Proof.** Let  $\delta: VB_4 \longrightarrow S_4$  be a homomorphism and let  $\iota: S_4 \longrightarrow VB_4$  be the natural inclusion defined by  $\iota(\tau_1) = v_1, \iota(\tau_2) = v_2$  and  $\iota(\tau_3) = v_3$ . Then,  $\delta \circ \iota$  is an endomorphism of  $S_4$ . From Lemma 17, up to conjugation,  $\delta \circ \iota$  is abelian or it is  $\eta$  or it is the identity homomorphism.

We claim that when  $\delta \circ \iota$  is abelian then  $\delta$  is abelian. The proof given for [3, Theorem 2.1] in the case in which the composition is abelian works for  $n = 4$ , proving our claim.

Suppose that  $\delta \circ \iota = \eta$ . Therefore

$$\delta(v_1) = \tau_1, \delta(v_2) = \tau_2 \text{ and } \delta(v_3) = \tau_1.$$

From the mixed relations  $v_1v_2\sigma_1v_2v_1 = \sigma_2$  and  $v_2v_3\sigma_2v_3v_2 = \sigma_3$  we see that  $\delta(\sigma_2)$  and  $\delta(\sigma_3)$  are completely determined by  $\delta(\sigma_1)$ . We analyse the possible values of  $\delta(\sigma_1)$ . From the relation  $\sigma_1v_3 = v_3\sigma_1$  and  $\delta(v_3) = \tau_1$  it follows that  $\delta(\sigma_1)$  lies in the centralizer of  $\langle \tau_1 \rangle$  in  $S_4$ , i.e.  $\delta(\sigma_1) \in \{1, \tau_1, \tau_3, \tau_1\tau_3\}$ .

- Suppose that  $\delta(\sigma_1) = 1$ , then  $\delta(\sigma_2) = 1$  and  $\delta(\sigma_3) = 1$ . This homomorphism is  $\delta_6$ .
- Suppose that  $\delta(\sigma_1) = \tau_1$ . Then  $\delta(\sigma_2) = \tau_1\tau_2\tau_1\tau_2\tau_1 = \tau_2$  and  $\delta(\sigma_3) = \tau_2\tau_1\tau_2\tau_1\tau_2 = \tau_1$ . This homomorphism is  $\delta_1$ .
- Suppose that  $\delta(\sigma_1) = \tau_3$ . Then  $\delta(\sigma_2) = \tau_1\tau_2\tau_3\tau_2\tau_1 = \tau_3\tau_2\tau_1\tau_2\tau_3$  and  $\delta(\sigma_3) = \tau_2\tau_1\tau_1\tau_2\tau_3\tau_2\tau_1\tau_1\tau_2 = \tau_3$ . This homomorphism is  $\delta_2$ .

- Finally, if  $\delta(\sigma_1) = \tau_1\tau_3$  we do not get a homomorphism since the relation  $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$  is not preserved.

Now, suppose that  $\delta \circ \iota$  is the identity homomorphism. This implies that

$$\delta(v_1) = \tau_1, \delta(v_2) = \tau_2 \text{ and } \delta(v_3) = \tau_3.$$

As before, from the mixed relations  $v_1v_2\sigma_1v_2v_1 = \sigma_2$  and  $v_2v_3\sigma_2v_3v_2 = \sigma_3$  it follows that  $\delta(\sigma_1)$  determines  $\delta(\sigma_2)$  and  $\delta(\sigma_3)$  completely. Moreover the relation  $\sigma_1v_3 = v_3\sigma_1$  implies that  $\delta(\sigma_1) \in \{1, \tau_1, \tau_3, \tau_1\tau_3\}$  the centralizer of  $\langle \tau_3 \rangle$  in  $S_4$ . We analyse the possible values of  $\delta(\sigma_1)$ .

- Suppose that  $\delta(\sigma_1) = 1$ , then  $\delta(\sigma_2) = 1$  and  $\delta(\sigma_3) = 1$ . This homomorphism is  $\delta_5 = \pi_K$ .
- Suppose that  $\delta(\sigma_1) = \tau_1$ . Then  $\delta(\sigma_2) = \tau_1\tau_2\tau_1\tau_2\tau_1 = \tau_2$  and  $\delta(\sigma_3) = \tau_2\tau_3\tau_2\tau_3\tau_2 = \tau_3$ . This homomorphism is  $\delta_3 = \pi_P$ .
- Suppose that  $\delta(\sigma_1) = \tau_3$ . Then  $\delta(\sigma_2) = \tau_1\tau_2\tau_3\tau_2\tau_1 = \tau_3\tau_2\tau_1\tau_2\tau_3$  and  $\delta(\sigma_3) = \tau_2\tau_3\tau_3\tau_2\tau_1\tau_2\tau_3\tau_3\tau_2 = \tau_1$ . This homomorphism is  $\delta_4$ .
- Finally, if  $\delta(\sigma_1) = \tau_1\tau_3$  we do not get a homomorphism since the relation  $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$  is not preserved.

From the computations above we proved that, up to conjugation,  $\delta$  is abelian or  $\delta \in \{\delta_i \mid 1 \leq i \leq 6\}$ .  $\square$

Similar to the case  $n = 3$ , we verify which homomorphisms given in Theorem 19 respect the forbidden relations given in Definition 6.

**Theorem 20.** *Let  $\omega: WB_4 \longrightarrow S_4$  be a homomorphism. Then, up to conjugation, one of the following possibilities holds*

- $\omega$  is abelian;
- $\omega = \bar{\delta}$ , where  $\delta \in \{\delta_i \mid 1 \leq i \leq 4\}$  as in Theorem 19 and  $\bar{\delta}: WB_4 \longrightarrow S_4$  is the induced homomorphism in the quotient of  $VB_4$  by adding the forbidden relation (a) of Definition 6.

**Proof.** The proof is completely similar to the one given for Theorem 14.  $\square$

Finally we get the result for unrestricted virtual braid groups.

**Theorem 21.** *Let  $\mu: UVB_4 \longrightarrow S_4$  be a homomorphism. Then, up to conjugation, one of the following possibilities holds*

- $\mu$  is abelian;

- (b)  $\mu = \bar{\bar{\delta}}$ , where  $\delta \in \{\delta_i \mid 1 \leq i \leq 4\}$  as in Theorem 19 and  $\bar{\bar{\delta}}: UVB_4 \rightarrow S_4$  is the induced homomorphism in the quotient of  $VB_4$  by adding the forbidden relations (a) and (b) of Definition 6.

**Proof.** The proof is completely similar to the one given for Theorem 15.  $\square$

**Remark 22.** It follows from Proposition 10 and Theorem 19 that the homomorphisms  $\{\bar{\delta}_i \mid 1 \leq i \leq 4\}$  of Theorem 20(b) are pairwise non conjugate. Similarly for the homomorphisms of Theorem 21(b).

### 3.4. Proof of Theorem 2

We will use Theorem 1 to prove that for  $n \geq 4$  (resp.  $n \geq 3$ ) the pure subgroup of the virtual braid group (resp. the welded braid group and the unrestricted virtual braid group) is characteristic in the virtual braid group (resp. the welded braid group and the unrestricted virtual braid group). In the previous subsections we computed, up to conjugation, all surjective homomorphisms from  $VB_n$  to  $S_n$  (and also for  $WB_n$  and  $UVB_n$ ), for  $n = 3, 4$ . In the next two lemmas we will compare some kernels of these maps.

We will use the following notation. Let  $G$  be a group, the abelianization of  $G$  will be denoted by  $G^{Ab}$ , i.e.  $G^{Ab} = G/[G, G]$ .

**Lemma 23.** Let  $n \geq 3$ .

- (a) The groups  $VP_n$  and  $KB_n$  are not isomorphic.
- (b) Let  $n = 3$  and let  $\psi \in \{\psi_i \mid 1 \leq i \leq 8\}$  as in Theorem 13.
- The group  $KB_3 = \text{Ker}(\psi_7)$  is not isomorphic to  $\text{Ker}(\psi_i)$  for  $1 \leq i \leq 8$  with  $i \neq 7$ .
- (c) Let  $n = 4$  and let  $\delta \in \{\delta_i \mid 1 \leq i \leq 6\}$  as in Theorem 19.
- The group  $VP_4 = \text{Ker}(\delta_3)$  is not isomorphic to  $\text{Ker}(\delta_i)$  for  $1 \leq i \leq 6$  with  $i \neq 3$ .
  - The group  $KB_4 = \text{Ker}(\delta_5)$  is not isomorphic to  $\text{Ker}(\delta_i)$  for  $1 \leq i \leq 6$  with  $i \neq 5$ .

**Proof.** (a) This item is the same as [2, Proposition 21].

- (b) Let  $n = 3$  and let  $\psi \in \{\psi_i \mid 1 \leq i \leq 8\}$  as in Theorem 13. We used the GAP System [17] to compute the abelianization of the groups involved. We elucidate the routine used in the computations for the case  $n = 3$ :

```
f4:=FreeGroup("x","y","a","b");;
AssignGeneratorVariables(f4);;
r:=ParseRelators([x,y,a,b],"xyx=yxy,aba=bab,a^2=1,b^2=1,bayab=x");;
g:= f4/r; # g is the virtual braid group on 3 strings
p1:=GroupHomomorphismByImages(g, SymmetricGroup(3), [g.1,g.2, g.3, g.4],
[(2,3), (2,3), (1,2), (1,2)]); AbelianInvariants(Kernel(p1));
```

```

[ f1, f2, f3, f4 ] -> [ (2,3), (2,3), (1,2), (1,2) ]
[ 0, 0, 0, 0, 3, 3, 3 ]
p2:=GroupHomomorphismByImages(g, SymmetricGroup(3), [g.1,g.2, g.3, g.4],
[(1,2), (2,3), (1,2), (2,3)]); AbelianInvariants(Kernel(p2));
[ f1, f2, f3, f4 ] -> [ (1,2), (2,3), (1,2), (2,3) ]
[ 0, 0, 0, 0, 0, 0, 0 ]
p3:=GroupHomomorphismByImages(g, SymmetricGroup(3), [g.1,g.2, g.3, g.4],
[(2,3), (1,2)*(2,3)*(1,2), (1,2), (2,3)]); AbelianInvariants(Kernel(p3));
[ f1, f2, f3, f4 ] -> [ (2,3), (1,3), (1,2), (2,3) ]
[ 0, 0, 0, 0, 0, 0, 0 ]
p4:=GroupHomomorphismByImages(g, SymmetricGroup(3), [g.1,g.2, g.3, g.4],
[(1,2)*(2,3)*(1,2), (1,2), (1,2), (2,3)]); AbelianInvariants(Kernel(p4));
[ f1, f2, f3, f4 ] -> [ (1,3), (1,2), (1,2), (2,3) ]
[ 0, 0, 0, 0, 0, 0, 0 ]
p5:=GroupHomomorphismByImages(g, SymmetricGroup(3), [g.1,g.2, g.3, g.4],
[(1,2)*(2,3), (1,2)*(2,3), (1,2), (2,3)]); AbelianInvariants(Kernel(p5));
[ f1, f2, f3, f4 ] -> [ (1,3,2), (1,3,2), (1,2), (2,3) ]
[ 0, 0, 2, 2, 2, 2, 2 ]
p6:=GroupHomomorphismByImages(g, SymmetricGroup(3), [g.1,g.2, g.3, g.4],
[(2,3)*(1,2), (2,3)*(1,2), (1,2), (2,3)]); AbelianInvariants(Kernel(p6));
[ f1, f2, f3, f4 ] -> [ (1,2,3), (1,2,3), (1,2), (2,3) ]
[ 0, 0, 2, 2, 2, 2, 2 ]
p7:=GroupHomomorphismByImages(g, SymmetricGroup(3), [g.1,g.2, g.3, g.4],
[(), (), (1,2), (2,3)]); AbelianInvariants(Kernel(p7));
[ f1, f2, f3, f4 ] -> [ (), (), (1,2), (2,3) ]
[ 0, 0 ]
p8:=GroupHomomorphismByImages(g, SymmetricGroup(3), [g.1,g.2, g.3, g.4],
[(1,2)*(2,3), (1,2)*(2,3), (1,2), (1,2)]); AbelianInvariants(Kernel(p8));
[ f1, f2, f3, f4 ] -> [ (1,3,2), (1,3,2), (1,2), (1,2) ]
[ 0, 0, 0, 0, 0, 3 ]

```

Summarising, we get

- $(Ker(\psi_1))^{Ab} \cong \mathbb{Z}^4 \oplus (\mathbb{Z}_3)^3$
- $(Ker(\psi_2))^{Ab} = (VP_3)^{Ab} \cong \mathbb{Z}^6$
- $(Ker(\psi_3))^{Ab} \cong \mathbb{Z}^6$
- $(Ker(\psi_4))^{Ab} \cong \mathbb{Z}^6$
- $(Ker(\psi_5))^{Ab} \cong \mathbb{Z}^2 \oplus (\mathbb{Z}_2)^4$
- $(Ker(\psi_6))^{Ab} \cong \mathbb{Z}^2 \oplus (\mathbb{Z}_2)^4$
- $(Ker(\psi_7))^{Ab} = (KB_3)^{Ab} \cong \mathbb{Z}^2$
- $(Ker(\psi_8))^{Ab} \cong \mathbb{Z}^4 \oplus \mathbb{Z}_3$

From this we conclude that the group  $KB_3 = Ker(\psi_7)$  is not isomorphic to  $Ker(\psi_i)$  for  $1 \leq i \leq 8$  with  $i \neq 7$ .

(c) Let  $n = 4$  and let  $\delta \in \{\delta_i \mid 1 \leq i \leq 6\}$  as in Theorem 19. We use the same idea of the previous item. From the computations using GAP we get

- $(\text{Ker}(\delta_1))^{\text{Ab}} \cong \mathbb{Z}^3 \oplus (\mathbb{Z}_2)^2$
- $(\text{Ker}(\delta_2))^{\text{Ab}} \cong \mathbb{Z}^6 \oplus (\mathbb{Z}_2)^8$
- $(\text{Ker}(\delta_3))^{\text{Ab}} = (VP_4)^{\text{Ab}} \cong \mathbb{Z}^{12}$
- $(\text{Ker}(\delta_4))^{\text{Ab}} \cong \mathbb{Z}^6 \oplus (\mathbb{Z}_2)^2$
- $(\text{Ker}(\delta_5))^{\text{Ab}} = (KB_4)^{\text{Ab}} \cong \mathbb{Z}$
- $(\text{Ker}(\delta_6))^{\text{Ab}} \cong \mathbb{Z} \oplus (\mathbb{Z}_2)^2$

From these computations we conclude that the group  $VP_4 = \text{Ker}(\delta_3)$  is not isomorphic to  $\text{Ker}(\delta_i)$  for  $1 \leq i \leq 6$  with  $i \neq 3$  and also that the group  $KB_4 = \text{Ker}(\delta_5)$  is not isomorphic to  $\text{Ker}(\delta_i)$  for  $1 \leq i \leq 6$  with  $i \neq 5$ .  $\square$

In the next result we consider the cases of welded and unrestricted virtual braid groups with few strings.

**Lemma 24.**

- (a) Let  $n = 3$
- Let  $\overline{\psi}_i: WB_3 \rightarrow S_3$  as in Theorem 14, where  $\{\psi_i \mid 1 \leq i \leq 5\}$  are the homomorphisms given in Theorem 13. The pure welded braid subgroup  $WP_3 = \text{Ker}(\overline{\psi}_2)$  is not isomorphic to  $\text{Ker}(\overline{\psi}_i)$ , for any  $1 \leq i \leq 5$  with  $i \neq 2$ .
  - Let  $\overline{\overline{\psi}}_i: UVB_3 \rightarrow S_3$  as in Theorem 15, where  $\{\psi_i \mid 1 \leq i \leq 4\}$  are the homomorphisms given in Theorem 13. The pure unrestricted virtual braid subgroup  $UVP_3 = \text{Ker}(\overline{\overline{\psi}}_2)$  is not isomorphic to  $\text{Ker}(\overline{\overline{\psi}}_i)$ , for any  $1 \leq i \leq 4$  with  $i \neq 2$ .
- (b) Let  $n = 4$ .
- Let  $\overline{\delta}_i: WB_4 \rightarrow S_4$  as in Theorem 20, where  $\{\delta_i \mid 1 \leq i \leq 4\}$  are the homomorphisms given in Theorem 19. The pure welded braid subgroup  $WP_4 = \text{Ker}(\overline{\delta}_3)$  is not isomorphic to  $\text{Ker}(\overline{\delta}_i)$ , for any  $1 \leq i \leq 4$  with  $i \neq 3$ .
  - Let  $\overline{\overline{\delta}}_i: UVB_4 \rightarrow S_4$  as in Theorem 21, where  $\{\delta_i \mid 1 \leq i \leq 4\}$  are the homomorphisms given in Theorem 19. The pure unrestricted virtual braid subgroup  $UVP_4 = \text{Ker}(\overline{\overline{\delta}}_3)$  is not isomorphic to  $\text{Ker}(\overline{\overline{\delta}}_i)$ , for any  $1 \leq i \leq 4$  with  $i \neq 3$ .

**Proof.** The proof of this result is similar to the previous one in which we use GAP [17] to compute the abelianization of the kernel of each homomorphism. We just list below the abelianizations of the groups involved from which we conclude this result.

- (a) Let  $n = 3$

- $(\text{Ker}(\overline{\psi}_1))^{\text{Ab}} \cong \mathbb{Z}^2 \oplus (\mathbb{Z}_3)^3$
- $(\text{Ker}(\overline{\psi}_2))^{\text{Ab}} \cong \mathbb{Z}^6$
- $(\text{Ker}(\overline{\psi}_3))^{\text{Ab}} \cong \mathbb{Z}^4$
- $(\text{Ker}(\overline{\psi}_4))^{\text{Ab}} \cong \mathbb{Z}^4$
- $(\text{Ker}(\overline{\psi}_5))^{\text{Ab}} \cong \mathbb{Z} \oplus (\mathbb{Z}_3)^5$
- $(\text{Ker}(\overline{\overline{\psi}}_1))^{\text{Ab}} \cong \mathbb{Z}^2 \oplus (\mathbb{Z}_3)^2$
- $(\text{Ker}(\overline{\overline{\psi}}_2))^{\text{Ab}} \cong \mathbb{Z}^6$
- $(\text{Ker}(\overline{\overline{\psi}}_3))^{\text{Ab}} \cong \mathbb{Z}^2 \oplus \mathbb{Z}_3$
- $(\text{Ker}(\overline{\overline{\psi}}_4))^{\text{Ab}} \cong \mathbb{Z}^2 \oplus \mathbb{Z}_3$



(b) Let  $n = 4$

- $(\text{Ker}(\overline{\delta_1}))^{Ab} \cong \mathbb{Z}^3 \oplus (\mathbb{Z}_2)^2$
- $(\text{Ker}(\overline{\delta_2}))^{Ab} \cong \mathbb{Z}^3 \oplus (\mathbb{Z}_2)^8$
- $(\text{Ker}(\overline{\delta_3}))^{Ab} \cong \mathbb{Z}^{12}$
- $(\text{Ker}(\overline{\delta_4}))^{Ab} \cong \mathbb{Z}^3 \oplus (\mathbb{Z}_2)^3$
- $(\text{Ker}(\overline{\overline{\delta_1}}))^{Ab} \cong \mathbb{Z}^3 \oplus (\mathbb{Z}_2)^2$
- $(\text{Ker}(\overline{\overline{\delta_2}}))^{Ab} \cong \mathbb{Z}^3 \oplus (\mathbb{Z}_2)^6$
- $(\text{Ker}(\overline{\overline{\delta_3}}))^{Ab} \cong \mathbb{Z}^{12}$
- $(\text{Ker}(\overline{\overline{\delta_4}}))^{Ab} \cong \mathbb{Z}^3 \oplus (\mathbb{Z}_2)^3 \quad \square$

**Remark 25.** Given a group homomorphism  $\xi: G \longrightarrow H$  and  $\gamma \in \text{Aut}(H)$ , then  $\text{Ker}(\xi) = \text{Ker}(\gamma \circ \xi)$ .

With the above information we can now determine exactly when the virtual pure braid group is characteristic in the virtual braid group.

**Proof of Theorem 2.** The case  $n = 2$  follows from Remark 11.

Let  $n \geq 3$ . We first show that  $VP_3$  is not characteristic in  $VB_3$ . Let  $\alpha: VB_3 \longrightarrow VB_3$  be the homomorphism determined by

$$\alpha(v_1) = v_1, \alpha(v_2) = v_2, \alpha(\sigma_1) = v_1 v_2 \sigma_1 v_2 v_1 = \sigma_2 \text{ and } \alpha(\sigma_2) = v_1 v_2 \sigma_2 v_2 v_1.$$

We leave it to the reader to check that this  $\alpha$  preserves the relations of the presentation (1) of  $VB_3$  and so indeed determines a homomorphism. Moreover, as  $v_1 v_2$  is an element of order 3, we have that

$$\alpha^3(v_i) = v_i \text{ and } \alpha^3(\sigma_i) = v_1 v_2 v_1 v_2 v_1 v_2 \sigma_i v_2 v_1 v_2 v_1 v_2 v_1 = \sigma_i, \quad i = 1, 2,$$

hence  $\alpha^3$  is the identity on  $VB_3$  from which we conclude that  $\alpha$  is an automorphism of  $VB_3$ . Note that  $\pi_P(v_1 \sigma_1) = 1$  so  $v_1 \sigma_1 \in VP_3$ , but  $\pi_P(\alpha(v_1 \sigma_1)) = \pi_P(v_2 \sigma_1 v_2 v_1) = \tau_2 \tau_1 \tau_2 \tau_1 = \tau_1 \tau_2 \neq 1$  showing that  $\alpha(VP_3) \neq VP_3$  and so  $VP_3$  is not a characteristic subgroup of  $VB_3$ .

We will apply Theorem 1 to show that in the other cases we do obtain characteristic subgroups. Recall that  $\text{Out}(S_n)$  is trivial for  $n \neq 6$  and that  $\text{Out}(S_6)$  is a cyclic group of order 2. In [3] the authors used the notation  $\nu_6$  for the automorphism such that its class generates  $\text{Out}(S_6)$ , see [3, Introduction] for an explicit definition of this outer automorphism.

(a) Let  $\Sigma_n$  be the set of all surjective homomorphisms from  $VB_n$  onto  $S_n$ , let  $\mathcal{T}_n = \Sigma_n / \sim_c$  be the set of equivalence classes of  $\Sigma_n$  by  $\sim_c$ . We choose the following set of representatives  $\Lambda_n$  of  $\mathcal{T}_n$ :

- $\Lambda_3 = \{\psi_i \mid 1 \leq i \leq 8\}$ , from Theorem 13,
- $\Lambda_4 = \{\delta_i \mid 1 \leq i \leq 6\}$ , from Theorem 19,
- $\Lambda_6 = \{\pi_K, \pi_P, \nu_6 \circ \pi_K, \nu_6 \circ \pi_P\}$ , from [3, Theorem 2.1], and
- $\Lambda_n = \{\pi_K, \pi_P\}$ , for  $n \geq 5$  and  $n \neq 6$  from [3, Theorem 2.1].

If  $n = 6$ , from Remark 25,  $\text{Ker}(\nu_6 \circ \pi_K) = \text{Ker}(\pi_K)$  and  $\text{Ker}(\nu_6 \circ \pi_P) = \text{Ker}(\pi_P)$ . Then, from Lemma 23 and Theorem 1 we get that  $VP_n$  is characteristic in  $VB_n$  when  $n \geq 4$  and that  $KB_n$  is characteristic in  $VB_n$  when  $n \geq 3$ .

(b) Let  $\overline{\Sigma}_n$  be the set of all surjective homomorphisms from  $WB_n$  onto  $S_n$ , let  $\overline{\mathcal{T}}_n = \overline{\Sigma}_n / \sim_c$  be the set of equivalence classes of  $\overline{\Sigma}_n$  by  $\sim_c$ . We choose the following set of representatives  $\overline{\Lambda}_n$  of  $\overline{\mathcal{T}}_n$ :

- $\overline{\Lambda}_3 = \{\overline{\psi_i} \mid 1 \leq i \leq 5\}$ , from Theorem 14,
- $\overline{\Lambda}_4 = \{\overline{\delta_i} \mid 1 \leq i \leq 4\}$ , from Theorem 20,
- $\overline{\Lambda}_6 = \{\overline{\pi_P}, \nu_6 \circ \overline{\pi_P}\}$ , from [26, Remark 2.8], and
- $\overline{\Lambda}_n = \{\overline{\pi_P}\}$ , for  $n \geq 5$  and  $n \neq 6$  from [26, Remark 2.8].

If  $n = 6$ , from Remark 25,  $\text{Ker}(\nu_6 \circ \overline{\pi_P}) = \text{Ker}(\overline{\pi_P})$ . Then, from Lemma 24 and Theorem 1 we get the result of the second item.

(c) The proof of this item is similar to the last one. For the proof we use Theorem 15, Theorem 21, [26, Theorem 1], Lemma 24 and Theorem 1.  $\square$

#### 4. Virtual braid groups and the $R_\infty$ -property

In this section we prove Theorem 3. It will be solved case by case, in three subsections, and using slightly different approaches. We note that from the presentation of the virtual braid group it follows that

$$VB_2 = WB_2 = UVB_2 = \langle \sigma_1, v_1 \mid v_1^2 = 1 \rangle \cong \mathbb{Z} * \mathbb{Z}_2.$$

Hence,  $VB_2 = WB_2 = UVB_2$  has the  $R_\infty$ -property, see [18]. We start by recalling a result that we will use repeatedly to prove Theorem 3.

**Lemma 26** ([27, Lemma 6]). *Consider an exact sequence of groups*

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

*where  $K$  is a characteristic subgroup of  $G$ . Then,*

- If  $Q$  has the  $R_\infty$ -property, then so does  $G$ .*
- If  $Q$  is finite and  $K$  has the  $R_\infty$ -property, then  $G$  so does.*

In the sequel we will also make use of some facts about crystallographic groups (see e.g. [7] and [33] for more details on these groups). A  $n$ -dimensional crystallographic group  $\Gamma$  is a group which fits in a short exact sequence

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow F \longrightarrow 1$$

where  $F$  is a finite group and  $\mathbb{Z}^n$  is maximal abelian in  $\Gamma$ . Such a short exact sequence induces a representation  $\varphi: F \longrightarrow GL_n(\mathbb{Z})$  which is called the holonomy representation.

In fact, requiring that  $\mathbb{Z}^n$  is maximal abelian in  $\Gamma$  is equivalent to asking that  $\varphi$  is a faithful representation. It is known (see e.g. [32, Lemma 2.4]) that  $\mathbb{Z}^n$  is then the unique normal and maximal abelian subgroup of  $\Gamma$  and hence  $\mathbb{Z}^n$  is characteristic in  $\Gamma$ . For such a  $n$ -dimensional crystallographic group  $\Gamma$  there exists an embedding  $\rho: \Gamma \rightarrow \text{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL_n(\mathbb{R})$  with  $\rho(\Gamma) \cap \mathbb{R}^n = \rho(\mathbb{Z}^n) = \mathbb{Z}^n$ . After identifying  $\Gamma$  with its image  $\rho(\Gamma)$  in  $\text{Aff}(\mathbb{R}^n)$ , the second Bieberbach theorem implies that any automorphism  $\psi \in \text{Aut}(\Gamma)$  can be realised as an affine conjugation, i.e.  $\exists(d, D) \in \text{Aff}(\mathbb{R}^n)$  such that  $\psi(\gamma) = (d, D)\gamma(d, D)^{-1}$  for all  $\gamma \in \Gamma$  (see also [11, Remark 3.2]).

#### 4.1. The case of 3 strings

We use the presentation of the virtual braid group with 3 strings given in equation (1).

**Lemma 27.** *The normal closure of the coset of the element  $v_1v_2$  in  $VB_3/[VP_3, VP_3]$  (resp. in  $VB_3/[KB_3, KB_3]$ ) is a characteristic subgroup of  $VB_3/[VP_3, VP_3]$  (resp. of  $VB_3/[KB_3, KB_3]$ ).*

**Proof.** We recall from [5, Theorem 3.3 and equation (8)] that there is a decomposition  $VB_3/[VP_3, VP_3] \cong VP_3/[VP_3, VP_3] \rtimes S_3$  where  $VP_3/[VP_3, VP_3]$  is the free abelian group of rank 6 generated by  $\{\lambda_{i,j} \mid 1 \leq i \neq j \leq 3\}$ , the symmetric group is generated by two transpositions  $v_1, v_2$ , and such that the action is given by permutation of indices. So, we may write a presentation for  $VB_3/[VP_3, VP_3]$  with generators  $v_1, v_2$  and  $\lambda_{i,j}$  for  $1 \leq i \neq j \leq 3$  and defining relations given by

- $v_1v_2v_1 = v_2v_1v_2, v_1^2 = 1, v_2^2 = 1,$
- $[\lambda_{i,j}, \lambda_{k,l}] = 1$  for  $1 \leq i \neq j \leq 3$  and  $1 \leq k \neq l \leq 3,$
- $v_k \cdot \lambda_{i,j} \cdot v_k = \lambda_{v_k(i), v_k(j)},$  for  $1 \leq i \neq j \leq 3$  and  $k = 1, 2.$

Let  $\gamma = v_1v_2 \in VB_3/[VP_3, VP_3]$ . Hence  $\gamma$  has order 3 in  $VB_3/[VP_3, VP_3]$ . Let  $N$  be the normal closure of the element  $\gamma$  in  $VB_3/[VP_3, VP_3]$ . Since every element in  $VB_3/[VP_3, VP_3]$  of order 3 is conjugate to  $\gamma$  (see [5, Corollary 3.8]) we get that  $N$  is a characteristic subgroup of  $VB_3/[VP_3, VP_3]$ .

Now, we prove the result for  $VB_3/[KB_3, KB_3]$ . Proposition 17 and Corollary 18 of [2] show that  $VB_3$  can be seen as a semidirect product  $VB_3 = KB_3 \rtimes S_3$ , where  $KB_3$  can be viewed as a group generated by 6 generators  $x_{i,j}$  with  $1 \leq i \neq j \leq 3$  subject to 6 relations  $x_{i,k}x_{k,j}x_{i,k} = x_{k,j}x_{i,k}x_{k,j}$  (for  $\{i, j, k\} = \{1, 2, 3\}$ ) and where  $S_3$  acts on the generators by permuting the indices. In the proof of Proposition 19 of [2] it was shown that in the quotient  $VB_3/[KB_3, KB_3]$  these relations lead to an equality of cosets  $x_{1,2} = x_{2,3} = x_{3,1}$  and  $x_{1,3} = x_{3,2} = x_{2,1}$  (which we abusively also denote by the same symbols). Hence, from [2, Proposition 19], the group  $KB_3/[KB_3, KB_3]$  is a free abelian group of rank 2 generated by the cosets of the elements  $x_{1,2}$  and  $x_{1,3}$  and we obtain a split extension

$$1 \longrightarrow KB_3/[KB_3, KB_3] \cong \mathbb{Z}^2 \longrightarrow VB_3/[KB_3, KB_3] \xrightarrow{\overline{\pi_K}} S_3 \longrightarrow 1$$

where  $\overline{\pi_K}$  is the homomorphism induced from  $\pi_K: VB_3 \longrightarrow S_3$  (see Subsection 2.1). We consider now the following presentation of  $S_3$ ,  $S_3 = \langle a, b \mid a^3 = 1, b^2 = 1, (ba)^2 = 1 \rangle$ , where  $a = v_1 v_2$  and  $b = v_1$ . By using the method described in [21, Chapter 10] we find a presentation of the group  $VB_3/[KB_3, KB_3]$  with generators  $a, b, x_{1,2}, x_{1,3}$  and defining relations

- $a^3 = 1; b^2 = 1; (ba)^2 = 1;$
- $[x_{1,2}, x_{1,3}] = 1;$
- $bx_{1,2}b^{-1} = x_{1,3}; bx_{1,3}b^{-1} = x_{1,2};$
- $ax_{1,2}a^{-1} = x_{1,2}; ax_{1,3}a^{-1} = x_{1,3}.$

We consider now the following extension

$$1 \longrightarrow KB_3/[KB_3, KB_3] \longrightarrow \overline{\pi_K}^{-1}(\mathbb{Z}_3) \xrightarrow{\overline{\pi_K}} \mathbb{Z}_3 \longrightarrow 1$$

where  $\mathbb{Z}_3$  is the group generated by  $a = v_1 v_2$ . Notice that  $\overline{\pi_K}^{-1}(\mathbb{Z}_3)$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$  generated by the set  $\{x_{1,2}, x_{1,3}, a\}$ . From the above we obtain the extension

$$1 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3 \longrightarrow VB_3/[KB_3, KB_3] \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

where  $\mathbb{Z}_2$  is the group generated by  $b = v_1$ . From this extension we see that the torsion subgroup of  $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$  is the unique subgroup of order 3 in  $VB_3/[KB_3, KB_3]$ . So this subgroup, which is generated by  $v_1 v_2$  is a characteristic subgroup of  $VB_3/[KB_3, KB_3]$ .  $\square$

Lemma 27 is useful to prove the next result.

**Theorem 28.** *The quotient groups  $VB_3/[VP_3, VP_3]$  and  $VB_3/[KB_3, KB_3]$  have the  $R_\infty$ -property.*

**Proof.** Let  $N$  be the normal closure of the coset of  $v_1 v_2$  in  $VB_3/[VP_3, VP_3]$ . We consider the quotient  $G = (VB_3/[VP_3, VP_3])/N$  that has a presentation given by the one of  $VB_3/[VP_3, VP_3]$  (see the proof of Lemma 27) adding the relation  $v_1 v_2 = 1$ , which is equivalent to the relation  $v_1 = v_2$  since  $v_1$  and  $v_2$  are transpositions. From  $v_1 = v_2$  and the relations  $v_k \cdot \lambda_{i,j} \cdot v_k = \lambda_{v_k(i), v_k(j)}$ , for  $1 \leq i \neq j \leq 3$  and  $k = 1, 2$  we conclude that  $G \cong \mathbb{Z}^2 \rtimes \mathbb{Z}_2$  has a presentation with generators  $\lambda_{1,2}, \lambda_{2,1}$  and  $v_1$  and defining relations

- $v_1^2 = 1,$
- $[\lambda_{1,2}, \lambda_{2,1}] = 1,$
- $v_1 \lambda_{1,2} v_1 = \lambda_{2,1},$
- $v_1 \lambda_{2,1} v_1 = \lambda_{1,2}.$

Let  $M$  be the normal closure of the coset of  $v_1 v_2$  in  $VB_3/[KB_3, KB_3]$  (which is actually the group of order 3 generated by  $v_1 v_2$ ) and let  $H = (VB_3/[KB_3, KB_3])/M$ .

**Table 1**The character table of  $S_4$ .

Representation / Conjugacy class representative and size	( ) Identity element (Size 1)	(1, 2)(3, 4) (Size 3)	(1, 2) (Size 6)	(1, 2, 3, 4) (Size 6)	(1, 2, 3) (Size 8)
Trivial representation $\chi_1$	1	1	1	1	1
Sign representation $\chi_2$	1	1	-1	-1	1
Irreducible representation of degree two with kernel of order four $\chi_3$	2	2	0	0	-1
Standard representation $\chi_4$	3	-1	1	-1	0
Product of standard and sign representation $\chi_5$	3	-1	-1	1	0

From the proof of Lemma 27 is clear that  $H$  is isomorphic to the group  $G$  above in this proof.

We note that  $G$  and  $H$  are isomorphic to the crystallographic group of dimension 2 of Case 5 of the list of all 17 wallpaper groups given in [20, Section 3] (there it was denoted by  $G_1^2$ ). Hence, from [20, Section 3], it follows that  $G$  and  $H$  have the  $R_\infty$ -property. Therefore, from Lemma 26 and Lemma 27, we have that  $VB_3/[VP_3, VP_3]$  and  $VB_3/[KB_3, KB_3]$  also have the  $R_\infty$ -property.  $\square$

With the last result we may prove that  $VB_3$ ,  $WB_3$  and  $UVB_3$  have the  $R_\infty$ -property.

**Corollary 29.** *The virtual braid group  $VB_3$ , the welded braid group  $WB_3$  and the unrestricted virtual braid group  $UVB_3$  have the  $R_\infty$ -property.*

**Proof.** From Theorem 28 we know that the groups  $VB_3/[VP_3, VP_3]$  and  $VB_3/[KB_3, KB_3]$  have the  $R_\infty$ -property. From [5, Theorem 5.1] the group  $VB_3/[VP_3, VP_3]$  is isomorphic to  $WB_3/[WP_3, WP_3]$  as well as to  $UVB_3/[UVP_3, UVP_3]$ . Then, by applying Lemma 26 and Theorem 2 we get this result.  $\square$

#### 4.2. The case of 4 strings

Let  $\mathbb{Z}^{12} \rtimes S_4$  be a crystallographic group such that the generators of  $\mathbb{Z}^{12}$  are denoted by  $\lambda_{i,j}$  for  $1 \leq i \neq j \leq 4$  and such that the action of  $w \in S_4$  on  $\lambda_{i,j}$  is given by  $w \cdot \lambda_{i,j} = \lambda_{w^{-1}(i), w^{-1}(j)}$  for all  $1 \leq i \neq j \leq 4$  and where for us the operation  $\tau\mu$  in  $S_4$  means  $\mu \circ \tau$  (so first  $\tau$ , then  $\mu$ ). This is then a left action of  $S_4$  on  $\mathbb{Z}^{12}$ . Let  $\varphi: S_4 \rightarrow GL_{12}(\mathbb{Z})$  be the holonomy representation of  $\mathbb{Z}^{12} \rtimes S_4$ . From the natural homomorphism  $GL_{12}(\mathbb{Z}) \hookrightarrow GL_{12}(\mathbb{Q})$  we shall view the holonomy representation as  $\varphi: S_4 \hookrightarrow GL_{12}(\mathbb{Q})$ .

First, we describe the  $S_4$ -module structure of  $\mathbb{Q}^{12}$  using character theory. We record in Table 1 the character table of  $S_4$ .

Let  $\chi$  be the character of the representation  $\varphi: S_4 \hookrightarrow GL_{12}(\mathbb{Q})$ . Recall that, for an element  $\tau \in S_4$ , the number  $\chi(\tau) = \text{Tr}(\varphi(\tau))$  is equal to the number of generators  $\lambda_{i,j}$

**Table 2**The character  $\chi: S_4 \rightarrow GL_{12}(\mathbb{Q})$ .

( ) Identity element (Size 1)	$\tau = (1, 2)(3, 4)$ (Size 3)	$\tau = (1, 2)$ (Size 6)	$\tau = (1, 2, 3, 4)$ (Size 6)	$\tau = (1, 2, 3)$ (Size 8)
$\chi(1) = 12$	$\chi(\tau) = 0$	$\chi(\tau) = 2$	$\chi(\tau) = 0$	$\chi(\tau) = 0$

that are fixed by  $\tau$ . In Table 2 we show the character  $\chi$  evaluated in each of the five conjugacy classes of elements in  $S_4$ , given by representatives.

Now, we compute the components of the character  $\chi$ :

$$\begin{aligned}
 (\chi \mid \chi_1) &= \frac{1}{24}(12 \cdot 1 + 6 \cdot 2 \cdot 1) &= 1 \\
 (\chi \mid \chi_2) &= \frac{1}{24}(12 \cdot 1 + 6 \cdot 2 \cdot (-1)) &= 0 \\
 (\chi \mid \chi_3) &= \frac{1}{24}(12 \cdot 2 + 6 \cdot 2 \cdot 0) &= 1 \\
 (\chi \mid \chi_4) &= \frac{1}{24}(12 \cdot 3 + 6 \cdot 2 \cdot 1) &= 2 \\
 (\chi \mid \chi_5) &= \frac{1}{24}(12 \cdot 3 + 6 \cdot 2 \cdot (-1)) &= 1
 \end{aligned}$$

Hence, the character  $\chi$  has the decomposition

$$\chi = \chi_1 + \chi_3 + 2\chi_4 + \chi_5. \quad (4)$$

Let  $V \subseteq \mathbb{Q}^{12}$  be the submodule of  $\mathbb{Q}^{12}$  corresponding to  $\chi_1 + \chi_3 + 2\chi_4$ . Then,  $V' = V \cap \mathbb{Z}^{12}$  is a submodule of  $\mathbb{Z}^{12}$  and so a normal subgroup of  $\mathbb{Z}^{12} \rtimes S_4$  such that  $\mathbb{Z}^{12}/V'$  is torsion-free.

It follows that, as groups, we can write  $\mathbb{Z}^{12} = V' \oplus W'$  where both  $V' \cong \mathbb{Z}^9$  and  $W' \cong \mathbb{Z}^3$  are free abelian.

**Lemma 30.** *The group  $V'$  is a characteristic subgroup of  $\mathbb{Z}^{12} \rtimes S_4$ .*

**Proof.** Let  $\{e_1, e_2, \dots, e_{12}\}$  be a generating set of  $\mathbb{Z}^{12}$  such that  $\{e_1, e_2, \dots, e_9\}$  generates  $V'$ . Let  $w \in S_4$ . With respect to this generating set we can write  $\varphi(w)$  as a  $12 \times 12$  matrix and since  $V'$  is a submodule, we have that

$$\varphi(w) = \begin{pmatrix} \sigma_1(w) & \alpha(w) \\ 0 & \sigma_2(w) \end{pmatrix}$$

with  $\sigma_1: S_4 \rightarrow GL_9(\mathbb{Z})$  corresponding to  $\chi_1 + \chi_3 + 2\chi_4$  and  $\sigma_2: S_4 \rightarrow GL_3(\mathbb{Z})$  corresponding to  $\chi_5$ .

We can embed  $\mathbb{Z}^{12} \rtimes S_4$  into  $\text{Aff}(\mathbb{R}^{12}) = \mathbb{R}^{12} \rtimes GL_{12}(\mathbb{R})$  by mapping  $(z, w)$  to  $(z, \varphi(w))$ . Let  $\psi \in \text{Aut}(\mathbb{Z}^{12} \rtimes S_4)$ . Recall that  $\mathbb{Z}^{12}$  is characteristic (see page 38). So,  $\psi$  induces an automorphism  $\bar{\psi}$  on  $S_4$ . We know that  $\text{Aut}(S_4) = \text{Inn}(S_4)$ . Hence, there is an inner automorphism  $\mu \in \text{Inn}(\mathbb{Z}^{12} \rtimes S_4)$  such that  $\psi \circ \mu$  induces the identity on  $S_4$ .

As  $V' \triangleleft \mathbb{Z}^{12} \rtimes S_4$  we know that  $\mu(V') = V'$ . So, we may assume from now onwards that  $\psi$  induces the identity on  $S_4$ . As mentioned before (see page 38)  $\psi$  is realized by an affine conjugation. So, there exists an element  $(d, D) \in \text{Aff}(\mathbb{R}^{12})$  so

that  $\psi(z, \varphi(w)) = (d, D)(z, \varphi(w))(d, D)^{-1}$ . As  $\psi$  induces the identity on  $S_4$ , we must have that  $D\varphi(w)D^{-1} = \varphi(w)$ , for all  $w \in S_4$ .

Write  $D$  as  $\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$ . Therefore,

$$\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \begin{pmatrix} \sigma_1(w) & \alpha(w) \\ 0 & \sigma_2(w) \end{pmatrix} = \begin{pmatrix} \sigma_1(w) & \alpha(w) \\ 0 & \sigma_2(w) \end{pmatrix} \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

and so

$$D_3\sigma_1(w) = \sigma_2(w)D_3, \text{ for all } w \in S_4. \quad (5)$$

Notice that  $D_3$  is a  $3 \times 9$  matrix and can be viewed as a map  $D_3: \mathbb{Q}^9 \rightarrow \mathbb{Q}^3$ , with  $\mathbb{Q}^9$  an  $S_4$ -module via  $\sigma_1$  and  $\mathbb{Q}^3$  an  $S_4$ -module via  $\sigma_2$ . Equation (5) shows that  $D_3$  is an  $S_4$ -module map from  $\mathbb{Q}^9$  to  $\mathbb{Q}^3$ , where  $\mathbb{Q}^3$  is an irreducible module and  $\mathbb{Q}^9$  does not contain a submodule isomorphic to  $\mathbb{Q}^3$  and so  $D_3 = 0$ . Hence  $D = \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix}$ .

It now follows that for  $z \in V'$  we have that

$$\begin{aligned} \psi(z) &= (d, D)(z, 1)(d, D)^{-1} \\ &= (d + Dz, D)(-D^{-1}d, D^{-1}) \\ &= (d + Dz - d, 1) \\ &= (Dz, 1). \end{aligned}$$

But, since  $D = \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix}$  it follows that  $Dz \in V'$ .  $\square$

**Theorem 31.** *The group  $\mathbb{Z}^{12} \rtimes S_4$  has the  $R_\infty$ -property.*

**Proof.** The quotient of  $\mathbb{Z}^{12} \rtimes S_4$  by the characteristic subgroup  $V'$  of Lemma 30 satisfies

$$\mathbb{Z}^{12}/V' \rtimes S_4 \cong \mathbb{Z}^3 \rtimes S_4$$

where the action is faithful (it corresponds to  $\chi_5$ ). Then it is a 3-dimensional crystallographic group. From [9, Theorem 4.2] we know that this group has the  $R_\infty$ -property. Hence, from Lemma 26 the result follows.  $\square$

**Corollary 32.** *The virtual braid group  $VB_4$ , the welded braid group  $WB_4$  and the unrestricted virtual braid group  $UVB_4$  have the  $R_\infty$ -property.*

**Proof.** We note that the group  $VB_4/[VP_4, VP_4]$  is isomorphic to the group  $\mathbb{Z}^{12} \rtimes S_4$  of Theorem 31 (see [5, Theorem 3.3]), so by Theorem 31 it has the  $R_\infty$ -property. From [5, Theorem 5.1]  $VB_4/[VP_4, VP_4]$  is isomorphic to  $WB_4/[WP_4, WP_4]$  as well as to  $UVB_4/[UVP_4, UVP_4]$ . Then, from Lemma 26 and Theorem 2 we conclude the result for this corollary.  $\square$

### 4.3. General cases

In the next proposition we show that, for  $n \geq 2$ , the  $R_\infty$ -property holds for the unrestricted virtual braid group  $UVB_n$  and its pure subgroup  $UVP_n$ . Then we use it to prove the result for the virtual braid group  $VB_n$ , with  $n \geq 5$ .

**Proposition 33.** *Let  $n \geq 2$ . The unrestricted virtual pure braid group  $UVP_n$  and the unrestricted virtual braid group  $UVB_n$  have the  $R_\infty$ -property.*

**Proof.** The case  $n = 2$  for  $UVB_n$  was mentioned in the first paragraph of this section. From [26, Remark 2.10] it follows that, for  $n \geq 2$ ,  $UVP_n$  is isomorphic to the direct product of  $n(n-1)/2$  copies of the free group of rank 2. Hence, from [10, Example 5.1.8] we conclude that  $UVP_n$  has the  $R_\infty$ -property.

Now, let  $n \geq 3$ . From Theorem 2, the group  $UVP_n$  is a characteristic subgroup of  $UVB_n$  (see also [26, Proposition 2.15] for  $n \geq 5$ ). Then, from Lemma 26 applied to the short exact sequence  $1 \rightarrow UVP_n \rightarrow UVB_n \rightarrow S_n \rightarrow 1$  we obtain the result for  $UVB_n$ .  $\square$

**Remark 34.** We note that, for  $n = 3$  and 4, we also proved the  $R_\infty$ -property for  $UVB_n$  in Corollaries 29 and 32, respectively, but using different techniques.

All possible homomorphisms from  $VB_n$  to  $VB_m$  were determined in [3, Theorem 2.3], for  $n \geq 5$ ,  $m \geq 2$  and  $n \geq m$ . In particular, for  $n \geq 5$ ,  $\text{Out}(VB_n)$ , the outer automorphism group of  $VB_n$ , is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  and is generated by the classes of  $\zeta_1$  and  $\zeta_2$  where

- $\zeta_1: VB_n \rightarrow VB_n$  is defined by  $\zeta_1(\sigma_i) = v_i \sigma_i v_i$  and  $\zeta_1(v_i) = v_i$ ;
- $\zeta_2: VB_n \rightarrow VB_n$  is defined by  $\zeta_2(\sigma_i) = \sigma_i^{-1}$  and  $\zeta_2(v_i) = v_i$ ;

for  $i = 1, \dots, n-1$ , see [3, Corollary 2.5].

**Lemma 35.** *Let  $n \geq 5$ . The normal closure  $K$  of the set*

$$\{v_i \sigma_{i+1} \sigma_i v_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1}; v_{i+1} \sigma_i \sigma_{i+1} v_i \sigma_{i+1}^{-1} \sigma_i^{-1} \mid i = 1, \dots, n-2\}$$

*is a characteristic subgroup of  $VB_n$ .*

**Proof.** We shall use the presentation of  $VB_n$  given in Definition 4. Recall that, for all  $i = 1, \dots, n-1$ ,  $v_i = v_i^{-1}$  in  $VB_n$ . In the following computations we use the mixed relation (MR2)  $v_i v_{i+1} \sigma_i = \sigma_{i+1} v_i v_{i+1}$  of  $VB_n$ , for  $i = 1, 2, \dots, n-2$ , which is equivalent to  $\sigma_i v_{i+1} v_i = v_{i+1} v_i \sigma_{i+1}$  or to  $v_i v_{i+1} \sigma_i^{-1} = \sigma_{i+1}^{-1} v_i v_{i+1}$  or to  $\sigma_i^{-1} v_{i+1} v_i = v_{i+1} v_i \sigma_{i+1}^{-1}$ .



$$\begin{aligned}
\zeta_1(v_i\sigma_{i+1}\sigma_i v_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}) &= v_i v_{i+1}\sigma_{i+1} v_{i+1} v_i \sigma_i v_i v_{i+1} v_i \sigma_i^{-1} v_i v_{i+1} \sigma_{i+1}^{-1} v_{i+1} \\
&= v_i v_{i+1}\sigma_{i+1} v_{i+1} v_i \sigma_i v_{i+1} v_i v_{i+1} \sigma_i^{-1} v_i v_{i+1} \sigma_{i+1}^{-1} v_{i+1} \\
&= v_i v_{i+1}\sigma_{i+1} v_{i+1} v_i v_{i+1} v_i \sigma_i v_{i+1} v_i \sigma_i^{-1} v_i v_{i+1} v_i v_{i+1} \sigma_{i+1}^{-1} v_{i+1} \\
&= v_i v_{i+1} \sigma_{i+1} v_i v_{i+1} \sigma_{i+1}^{-1} v_i v_{i+1} \sigma_{i+1}^{-1} v_{i+1} v_i v_{i+1} \\
&= \frac{v_i v_{i+1} v_i v_{i+1} \sigma_i \sigma_{i+1} v_i \sigma_{i+1}^{-1} \sigma_i^{-1} v_{i+1} v_i v_{i+1}}{v_{i+1} v_i v_{i+1} \cdot v_{i+1} \sigma_i \sigma_{i+1} v_i \sigma_{i+1}^{-1} \sigma_i^{-1} \cdot v_{i+1} v_i v_{i+1}} \\
\zeta_1(v_{i+1}\sigma_i \sigma_{i+1} v_i \sigma_{i+1}^{-1} \sigma_i^{-1}) &= v_{i+1} v_i \sigma_i v_i v_{i+1} \sigma_{i+1} v_{i+1} v_i v_{i+1} \sigma_{i+1}^{-1} v_{i+1} v_i \sigma_i^{-1} v_i \\
&= v_{i+1} v_i \sigma_i v_i v_{i+1} \sigma_{i+1} v_i v_{i+1} v_i \sigma_{i+1}^{-1} v_{i+1} v_i \sigma_i^{-1} v_i \\
&= v_{i+1} v_i \sigma_i v_i v_{i+1} v_i v_{i+1} \sigma_i v_{i+1} \sigma_i^{-1} v_{i+1} v_i v_{i+1} v_i \sigma_i^{-1} v_i \\
&= v_{i+1} v_i \sigma_i v_{i+1} v_i \sigma_i v_{i+1} \sigma_i^{-1} v_i v_{i+1} \sigma_{i+1}^{-1} v_i \\
&= \frac{v_{i+1} v_i v_{i+1} v_i \sigma_i \sigma_{i+1} \sigma_i v_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} v_i v_{i+1} v_i}{v_i v_{i+1} v_i \cdot v_i \sigma_i \sigma_{i+1} \sigma_i v_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} \cdot v_i v_{i+1} v_i} \\
\zeta_2(v_i \sigma_{i+1} \sigma_i v_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1}) &= v_i \sigma_{i+1}^{-1} \sigma_i^{-1} v_{i+1} \sigma_i \sigma_{i+1} \\
&= (v_{i+1} \sigma_i \sigma_{i+1})^{-1} v_{i+1} \sigma_i \sigma_{i+1} v_i \sigma_{i+1}^{-1} \sigma_i^{-1} (v_{i+1} \sigma_i \sigma_{i+1}) \\
\zeta_2(v_{i+1} \sigma_i \sigma_{i+1} v_i \sigma_{i+1}^{-1} \sigma_i^{-1}) &= v_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} v_i \sigma_{i+1} \sigma_i \\
&= (v_i \sigma_{i+1} \sigma_i)^{-1} v_i \sigma_{i+1} \sigma_i v_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} (v_i \sigma_{i+1} \sigma_i) \quad \square
\end{aligned}$$

We recall that the unrestricted virtual braid group  $UVB_n$  is the quotient group  $VB_n/K$  of the virtual braid group, see Definition 6.

**Theorem 36.** *Let  $n \geq 5$ . The virtual braid group  $VB_n$  has the  $R_\infty$ -property.*

**Proof.** From Lemma 35 we know that  $K$  is characteristic in  $VB_n$ . The quotient  $VB_n/K$  is the unrestricted virtual braid group  $UVB_n$  that, from Proposition 33, has the  $R_\infty$ -property. The desired result then follows by applying Lemma 26.  $\square$

**Remark 37.**

- (a) To the best of our knowledge, for  $n \geq 5$ , it is not known if the welded braid group  $WB_n$  has the  $R_\infty$ -property.
- (b) From the presentation of  $VP_n$  (see [1, Theorem 1]) we get  $VP_2 \cong \mathbb{Z} * \mathbb{Z}$ , which we know it has the  $R_\infty$ -property. Since  $VP_2 = WP_2 = UVP_2$ , these groups have the  $R_\infty$ -property. For  $n \geq 3$ , as far as we know it is unknown if the  $R_\infty$ -property holds or not for virtual pure braid groups and for the welded pure braid groups.

As explained at the end of the introduction, Theorem 3 is now proved by collecting all the results of this section.

## Data availability

No data was used for the research described in the article.

## Appendix A

We note that the technique used in this work to prove that some subgroups of virtual braid groups are characteristic may be applied to other braid-like groups. We exemplify it in this section by showing that some remarkable subgroups of virtual twin groups are characteristic. For more details about these groups see [28] and [29] and the references therein.

The virtual twin group  $VT_n$ ,  $n \geq 2$ , admits a presentation with generators  $\sigma_i, \rho_i$  for  $i = 1, \dots, n-1$  and defining relations:

- $\sigma_i^2 = 1$  for  $i = 1, 2, \dots, n-1$ .
- $\sigma_i \sigma_j = \sigma_j \sigma_i$  for  $|i-j| \geq 2$ .
- $\rho_i^2 = 1$  for  $i = 1, \dots, n-1$ .
- $\rho_i \rho_j = \rho_j \rho_i$  for  $|i-j| \geq 2$ .
- $\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}$ , for  $i = 1, 2, \dots, n-2$ .
- $\rho_i \sigma_j = \sigma_j \rho_i$ , for  $|i-j| \geq 2$ .
- $\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}$ , for  $i = 1, \dots, n-2$ .

Let  $n \geq 2$ . For  $1 \leq i \leq n-1$  let  $\tau_i = (i, i+1)$  as before. Let  $\pi_P: VT_n \rightarrow S_n$  be the homomorphism defined by  $\pi_P(\sigma_i) = \pi_P(\rho_i) = \tau_i$  for  $i = 1, \dots, n-1$ . The *pure virtual twin group*  $PVT_n$  is defined to be the kernel of  $\pi_P$ . Let  $\pi_K: VT_n \rightarrow S_n$  be the homomorphism defined by  $\pi_K(\sigma_i) = 1$  and  $\pi_K(\rho_i) = \tau_i$  for  $i = 1, \dots, n-1$ . The kernel of  $\pi_K$  will be denoted by  $KT_n$ .

**Theorem 38.** *Let  $n \geq 2$ .*

- (a) *The groups  $PVT_n$  and  $KT_n$  are not isomorphic.*
- (b) *The virtual pure twin group  $PVT_n$  is a characteristic subgroup of the virtual twin group  $VT_n$  if and only if  $n \neq 3$  and the group  $KT_n$  is a characteristic subgroup of  $VT_n$  if and only if  $n \geq 3$ .*

**Proof.** Let  $n \geq 2$ .

- (a) From [28] the pure virtual twin group  $PVT_n$  is a right-angled Artin group (hence it is torsion free) and from [29] the group  $KT_n$  is a right-angled Coxeter group (so it has finite order elements), hence they are not isomorphic.
- (b) The proof of this item follows the same lines as the proof of Theorem 2, so we just indicate some steps of the proof.

*Claim 1:* There are, up to conjugation,

- (i) 3 surjective homomorphisms from  $VT_2$  to  $S_2$ ;
- (ii) 5 surjective homomorphisms from  $VT_3$  to  $S_3$ ;

- (iii) 6 surjective homomorphisms from  $VT_4$  to  $S_4$ ;
- (iv) 4 surjective homomorphisms from  $VT_6$  to  $S_6$ ; and
- (v) 2 surjective homomorphisms from  $VT_n$  to  $S_n$ , for  $n \geq 5$  and  $n \neq 6$ .

The proof of *Claim 1* for  $n \geq 5$  may be found in [29]. For the cases  $n = 2, 3, 4$  the verification is done as in Section 3 for the virtual braid group.

*Claim 2:* It is clear that the image of  $KT_2$  by the automorphism  $\psi: VT_2 \rightarrow VT_2$  defined by  $\psi(\sigma_1) = \rho_1$  and  $\psi(\rho_1) = \sigma_1$  is not  $KT_2$ . Also, it is not difficult to verify that the image of  $PVT_3$  by the automorphism  $\varphi: VT_3 \rightarrow VT_3$  defined by  $\varphi(\sigma_1) = \sigma_2$ ,  $\varphi(\sigma_2) = \rho_1\rho_2\sigma_2\rho_2\rho_1$ ,  $\varphi(\rho_1) = \rho_1$  and  $\varphi(\rho_2) = \rho_2$  is not  $PVT_3$ . Therefore, the group  $KT_2$  (resp.  $PVT_3$ ) is not a characteristic subgroup of  $VT_2$  (resp.  $VT_3$ ).

*Claim 3:* The groups  $PVT_2$ ,  $PVT_4$ ,  $KT_3$  and  $KT_4$  are not isomorphic to the kernels of the other homomorphisms (for the same number of strings) from Claim 1. The verification of this claim can be done along the same lines as we did for  $VB_n$  in Section 3.

Then, applying Theorem 1, we get that for  $n \neq 3$  (resp.  $n \geq 3$ ) the groups  $PVT_n$  (resp.  $KT_n$ ) are characteristic subgroups of  $VT_n$ .  $\square$

An application of Theorem 38 is the following result.

**Corollary 39.** *Let  $n \geq 2$ . The virtual twin group  $VT_n$  has the  $R_\infty$ -property.*

**Proof.** Since  $VT_2$  is isomorphic to  $\mathbb{Z}_2 * \mathbb{Z}_2$  then from [20, Proposition 2.8] (see also [18, Lemma 2]) it has the  $R_\infty$ -property. From the presentation of  $KT_n$  given in [29, Theorem 3.3] we get the isomorphism  $KT_3 \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ . From [18, Lemma 2] the group  $KT_3$  has the  $R_\infty$ -property and since it is a characteristic subgroup of  $VT_3$  (Theorem 38) then from Lemma 26 the latter group also has the  $R_\infty$ -property. For  $n \geq 4$  the pure virtual twin group  $PVT_n$  has the  $R_\infty$ -property, see [28]. Then, from Lemma 26 and Theorem 38, the virtual twin group  $VT_n$  has the  $R_\infty$ -property.  $\square$

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