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Abstract

A weaker form of Zassenhaus' conjecture is proved for some infinite groups.

Introduction

Let G be a group and let $V\mathbb{Z}G$ be the group of units of augmentation one of the integral group ring $\mathbb{Z}G$. Given elements $\alpha = \sum \alpha(g)g \in \mathbb{Z}G$ and $g \in G$, we denote by C_g the conjugacy class of g and set $\tilde{\alpha}(g) = \sum_{h \in C_g} \alpha(g)$. If G is a finite group a well-known Conjecture of Zassenhaus states:

ZC1: *Let G be a finite group and $\alpha \in V\mathbb{Z}G$ a torsion unit; then there exist $\beta \in \mathbf{Q}G$ such that $\beta^{-1}\alpha\beta \in G$.*

For finite groups it is proved in [4] that the following is an equivalent form of ZC1.[1]

Lemma 1: *Let G be a finite group and $\alpha \in V\mathbb{Z}G$ a torsion unit. Then there exist $\beta \in \mathbf{Q}G$ such that $\beta^{-1}\alpha\beta \in G$ if and only if for every element γ*

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of the subgroup generated by α there exist an element $g_0 \in G$, unique up to conjugacy, such that $\tilde{\gamma}(g_0) \neq 0$.

As in [1] we say that a group G has the *unique trace property* (**UT**-*property*) if given an element $\alpha \in V\mathbb{Z}G$ of finite order, there exists an element $g \in G$, unique up to conjugacy, such that $\tilde{\alpha}(g) \neq 0$.

It is known that nilpotent groups are UT-groups,(see [1]). We find some other classes of UT-groups. The dificulty arrise exactly when we have to decide whether $\tilde{\alpha}(g) = 0$ for elements of infinite order. The first results in this direction are from [1]. It seems essencial to varify this case.

The Results

We begin by extending a result of [1] supplying at the same time a shorter proof.

Propositon 2 : *Let G be a group containing a normal subgroup H, which is locally noetherian, and such that G/H is a torsion group. If $\alpha \in V\mathbb{Z}G$ is a torsion unit and $g \in G$ is an element of infinite order then $\tilde{\alpha}(g) = 0$.*

Proof: Suppose that $\tilde{\alpha}(g) \neq 0$. Then, by [1, Prop.2] there exist an integer $k > 1$ and an element $x \in G$ such that $x^{-1}gx = g^k$. If x is of finite order, set $m = o(x)$. Then $g = x^{-m}gx^m = g^{k^m}$ and hence we have a contradiction. If x is of infinite order, since G/H is a torsion group, there exist an integer $m > 0$ such that x^m *and* g^m are in H. Set $t = x^m$, $h = g^m$, $n = k^m$. Then $t^{-1}ht = h^n$. Let $H_0 = \langle t, h \rangle$. Then, by our hypotesis, H_0 is noetherian hence, by [7, 1.2.7], $n = 1$ and consequently $k = 1$, a contradiction. □

Note that to prove the proposition we do not need H to be normal. Let $\alpha \in V\mathbb{Z}G$ be a torsion unit. The proof of our next result shows why it is necessary to decide whether $\tilde{\alpha}(g) = 0$ for elements of infinite order.

Propositon 3 : *Let G and H be as in the previous Proposition. Suppose further that H is torsion free, that the torsion elements of G form a subgroup $T(G)$ and that G/H is a UT-group. Then, G is a UT-group.*

Proof: Let $\alpha \in V\mathbb{Z}G$ be a torsion unit and $g \in G$ an element. If g is of infinite order then, by the previous Proposition, we have that $\tilde{\alpha}(g) = 0$.

If g has finite order, denote by β and $\overline{g}$ the projections of α and g in $V\mathbb{Z}(G/H)$. Let $C_{\overline{g}}$ be conjugacy class of β. Then, it is easy to see that $C_{\overline{g}}$ is the projection of the subset
$S = \{k \in G : k = t^{-1}gth,\ h \in H\ t \in G\}$. Now since $T(G)$ is a normal subgroup and H is normal and torsion free we see that $S \cap T(G) = C_g$. Furthermore, if we write $S = S_1 \cup C_g$, where S_1 are the elements of infinite order of S, then S_1 is a normal subset of G. Writing S_1 as a disjoint union of conjugacy classes and applying the previous Proposition, we see that $\sum_{h \in S_1} \alpha(h) = 0$ and hence we have that $\tilde{\beta}(\overline{g}) = \sum_{h \in S} \alpha(h) = \tilde{\alpha}(g)$. Since, by our assumption, G/H is a UT-group the result follows. □

As a consequence we obtain in a similar but shorter way, the following result of [1].

Corollary 4 :*Let G be a nilpotent group. Then G is a UT-group.*

Proof: We may suppose that G is a finitely generated and hence we have, by [5, 5.4.6], that G is a polyciclyc group. So G is noetherian and, by [5, 5.4.15], has a normal torsion free subgroup H of finite index. By a result of Weiss (see [9]), G/H is a UT-group. Also, since G is nilpotent, we have that $T(G)$ is a subgroup of G and consequetly is in the centalizer of H. It follows by the previous Proposition that G is a UT-group. □

If G is a group and $g \in G$ is an element we denote by $K(g) = [g, G]$. Let now G be a group generated by an element t and an abelian normal subgroup A such that $t^{-1}at = a^{-1}$ for any $a \in A$ and $t^2 \in A$. Let $\alpha \in V\mathbb{Z}G$ be a torsion unit. By Lemma 2 we have that $\tilde{\alpha}(g) = 0$ for every element of infinite order. Now let $g = ta \in G$ be an element which is not in A. We compute $K(g)$. If $b \in A$ then $[g, b] = [t, b] = b^{-2}$. If $h = tb$ then $[g, h] = [ta, tb] = [tb, t][a, tb] = [b, t][a, t] = (ba)^{-2}$. Hence $K(g) = \{a^2 : a \in A\}$. So we have the following result:

Lemma 5 : *Let G be a group generated by an abelian subgroup A and an element $t \in G$, such that $t^{-1}at = a^{-1}$ for any $a \in A$ and $t^2 \in A$. Then*

1. *For every $g \notin A$ we have that $K(g) = \{a^2 : a \in A\}$*

2. *If $g \notin A$ then $gK(g) = C_g$.*

Proof: The considerations above show that (1) holds. So, let $g\theta \in gK(g)$. Since $g \notin A$ we have that conjugation by g inverts the elements of A. By (1) we have that $\theta = \varphi^2$ for some $\varphi \in A$. Setting $t = g\varphi$ we see easily that $t^{-1}g\theta t = g$. □

Remark : Note that item (2) of the previous Lemma holds whenever the elements of $K(g)$ are squares and are inverted by g.

We can now prove:

Theorem 6 : *Let $G = \langle t, A : t^2 \in A, t^{-1}at = a^{-1}, \forall a \in A\rangle$ where A is an abelian normal subgroup of G. Then G is an UT-group.*

Proof: Let $\alpha \in V\mathbb{Z}G$ be a torsion unit and $g \in G$ an element of the support of α. Suppose first that $o(g) = \infty$. Note that $[G : A] = 2$ and hence, by Proposition 2 , $\tilde{\alpha}(g) = 0$. Secondly, suppose that $g \notin A$. By Lemma 5 we have that $gK(g) = C_g$. Now the element $gK(g)$ is central in the quocient group $G/K(g)$ and hence, by [1, Prop. 4], we have that $\tilde{\alpha}(g) = \sum_{h \in gK(g)} \alpha(g) = 0$ *or* 1. Finally if $g \in T(A)$, since the support of α is finite and $t^2 \in A$, we may suppose that A is finitely generated. In particular A is a policyclic group and hence, by [5, 5.4.15], we have that there exist $H \lhd A$, which is torsion free and of finite index. Note that, since A is abelian and conjugating by t inverts the elements of A, H will also be normal in G. Consider the quocient group $\overline{G} = G/H$. This is a metabelian group which contains a normal abelian subgroup of index 2. Hence, by [3, Theorem 4.1], we have that $\overline{G}$ is a UT-group. Let $g \in A$ be an element and let $\overline{g}$ be its projection in $\overline{G}$. Then it is easily seen that $C_{\overline{g}}$ is the projection of the subset $S = \{b \in A : b = x^{-1}axh,\ h \in H\ x \in G\}$. Note that we may write S as a disjoint union $S = C_g \cup S_1$ where $S_1 = \{b \in S : h \neq 1\}$. Note that S_1 is a normal subset of G and its elements are all of infinite order. Hence, writing S_1 as a disjoint union of conjugacy classes we conclude, by Proposition 2, that $\sum_{h \in S} \alpha(h) = \tilde{\alpha}(g)$. Now consider the projection $\Psi : \mathbb{Z}G :\longrightarrow \mathbb{Z}\overline{G}$ and let $\beta = \Psi(\alpha)$. Then, since $\overline{G}$ is a UT-group, we have that $\tilde{\beta}(\overline{g}) = \sum_{\overline{h} \in C_{\overline{g}}} \beta(\overline{h}) \in \{0, 1\}$. Hence $\tilde{\alpha}(g) \in \{0, 1\}$ and so also in this case we have that $\tilde{\alpha}(g) \in \{0, 1\}$. Since α has augmentation 1 the result follows. □

A group G is called a T-group if normality is transitve in G. Let G be a solvable T-group and set $A = C_G(G')$. If A is not a torsion group then, by a result of [5, page 394], we have that G satisfies the condition of Theorem 6 and hence G is a UT-group.

Let G be a group such that the derived subgroup of G is cyclic of infinite order, say $G' = \langle \rho \rangle$. We shall use this notation in the following results.

Lemma 7: *Let G be a group with cyclic derived subgroup; then*

1. *If $g \in T(G)$ centralizes ρ then g is central.*
2. *Elements of odd order are central.*
3. $\{g^2 : g \in T(G)\} \subseteq \mathcal{Z}(G)$.
4. *If $g \in G$ has infinite order and $\alpha \in V\mathbb{Z}G$ is an element of finite order then $\tilde{\alpha}(g) = 0$.*

Proof: (1) Let $g \in T(G)$ and $x \in G$ then, since $\langle \rho \rangle$ is normal in G, we have that $g^{-1}xg = x\rho^k$ for some integer k. Let $m = o(g)$ then we have that $x = g^{-m}xg^m = x\rho^{km}$. Since ρ has infinite order we must have that $k = 0$.
(2) If $g \in G$ then g^2 centralizes ρ and hence is central. Since g has odd order we have that also g is central.
(3) The proof of (2) applies.
(4) Suppose that this is false; then, by [1, Prop.2], there exist $k > 1$, $x \in G$ such that $x^{-1}gx = g^k$. This implies that $g^{k-1} = [g,x] \in G'$. Set $n = k - 1$ *and* $h = g^n$; then the subgroup $\langle h \rangle$ is normal in G. Hence $x^{-1}hx \in \{h, h^{-1}\}$. But on the other hand $x^{-1}hx = h^k$ and hence we must have that $k = 1$, a contradiction. □

Lemma 8: *Let G be a group such that G' is infinite cyclic. Then, for any torsion element $g \in G$, we have that $gK(g) = C_g$.*

Proof: Let $G' = \langle \rho \rangle$. Then, since G' is a normal subgroup, we have that $g^{-1}\rho g \in \{\rho, \rho^{-1}\}$. If $g^{-1}\rho g = \rho$ then, by Lemma 7, g is central. So we may suppose that $g^{-1}\rho g = \rho^{-1}$. In this case also $g\rho g^{-1} = \rho^{-1}$. Hence we have that $g^{-1}g\rho^{-1}g = g\rho$ i.e. $g\rho^{-1}$is conjugated to $g\rho$. We now separate the proof in two cases.

Case 1: $K(g) \neq G'$.
Since $g^{-1}\rho g = \rho^{-1}$ and $K(g)$ is cyclic we must have that $K(g) = \langle \rho^2 \rangle$. Hence, by the Remark following Lemma 5, we have that $gK(g) = C_g$.

Case 2: $K(g) = G'$.
In this case, since G' is cyclic and ρ is inverted by elements not in its centralizer, we see easily that there is an element $t \in G$ such that $K(g) = \langle [g,t] \rangle$. In particular, we have that $[g,t] \in \{\rho, \rho^{-1}\}$. Hence g is conjugated either to $g\rho$ *or to* $g\rho^{-1}$. Since we have already proved that $g\rho$ is conjugated to $g\rho^{-1}$, we only have to prove that an element of $gK(g)$ is either conjugate to g or to $g\rho$. In fact, set $h = g\theta$ with $\theta \in K(g)$. If θ is a square then, by the Remark following Lemma 5, h is conjugated to g. If θ is not a square, we may write $h = g\rho\varphi$ where φ is a square. Hence, again by the same Remark, we have that h is conjugated to $g\rho$ which in turn is conjugated to g. □

We can now prove:

Theorem 9: *Let G be a group such that the derived subgroup of G is infinite cyclic. Then G is a UT-group.*

Proof: Let $\alpha \in V\mathbb{Z}G$ be a torsion unit and $g \in G$ an element. If g is of infinite order then, by lemma 7, we have that $\tilde{\alpha}(g) = 0$. If g is a torsion element then, by Lemma 8, we have that $\tilde{\alpha}(g) = \sum_{h \in gK(g)} \alpha(g)$. Since the element $gK(g)$ is central in the quocient group $G/K(g)$ we have, by, that $\sum_{h \in gK(g)} \tilde{\alpha}(g) \in \{0,1\}$. Since α has augmentation 1, the result is proved. □

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