The fundamental theorem of calculus for regular multidimensional Banach space valued Kurzweil-Henstock integrals

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Abstract

In [Ho,1] we gave a characterization of the functions $F : [a,b] \to X$ that are Kurzweil-Henstock integrals. In this paper we extend this result to dimension $n > 1$. A theorem of Saks, [S], proves that this is impossible with the usual definition of derivative for dimension $n > 1$. Hence we use the more general ***regular derivative*** and in order to keep the correspondence with the integration we use the more general ***regular integral***.

§1 - **BASIC DEFINITIONS** (In order to simplify the exposition we will consider dimension $n = 2$)

We work in a bidimensional interval $R = [a,b] = [a_1, b_1] \times [a_2, b_2]$. All intervals we consider have their sides parallel to the coordinate axis. For $x, y \in \mathbb{R}^2$ we write $x \leq y$ if $x_1 \leq y_1$ and $x_2 \leq y_2$. By $[x,y]$ we denote the corresponding closed interval; analogously for $]x,y[$ etc. J, J_i etc. always denote closed intervals; $\overset{\circ}{J}$ denotes the interior of J, $|J|$ denotes the area of J; m denotes the two-dimensional Lebesgue measure.

For $\xi \in R$ and $\sigma > 0$ we define $B_\sigma(\xi) = \{x \in R \mid \|x - \xi\| < \sigma\}$.

If $(J_i)_{i=1,2,\dots,n}$ is a division of R in nonoverlapping intervals and given points $\xi_i \in J_i$, *tags*, we say that $(\xi_i, J_i)_{i=1,2,\dots,n}$, or simply, (ξ_i, J_i), is a *tagged division* (TD) of R. If the union of the J_i is not all R we say that we have a *tagged partial division* (TPR) of R.

Given $c \in]0,1]$ and an interval J whose sides have length h and k we say that J is *c-regular* if $c \leq \frac{h}{k} \leq \frac{1}{c}$; we say that a TD or a TPD is *c-regular* if all its intervals are c-regular.

For $E \subset R$ a function $\delta : E \to [0, \infty[$ is called a *gauge* (defined on E). We say that a TD [TPD] $(\xi_i J_i)$ is *δ-fine*, where δ is a gauge on R, if for every i we have

$$\xi_i \in J_i \subset B_{\delta(\xi_i)}(\xi_i) \ .$$

If $c \in]0,1[$ it is immediate that we can divide R in a finite number of c-regular intervals; by further bisections it follows easily that for every gauge $\delta : R \to]0, \infty[$ there exists a c-regular, δ-fine TD of R: see [Ho, 2], Teor. 3.1.

Given a function $f : R \to X$, where X always denotes a Banach space, we say that f is *Kurzweil-Henstock integrable*, we write $f \in \mathcal{K}(R, X)$, or simply $f \in \mathcal{K}$, if there exists $I \in X$ such that for every $\varepsilon > 0$ there exists a gauge δ on R such that for every δ-fine TD (ξ_i, J_i) of R we have

$$\|\sum_i (f(\xi_i)|J_i| - I\| < \varepsilon \ ;$$

we write ${}^K\!\!\int_R f(t)\,dt = I$ or simply ${}^K\!\!\int_R f = I$.

If $f \in \mathcal{K}(R, I)$, for every $t \in R$ there exists $\tilde{f}(t) = {}^K\!\!\int_{[a,t]} f$.

We denote by $\mathcal{J}_R$ the set of all closed intervals $J \subset R$: we say that a function $F : \mathcal{J}_R \to X$ is *additive* if we have $F(J) = F(J_1) + F(J_2)$ whenever $J = J_1 \cup J_2$, with J_1, J_2 nonoverlapping; we denote by $\mathcal{A}(\mathcal{J}_R, X)$, or simply by $\mathcal{A}$, the set of all additive functions $F : \mathcal{J}_R \to X$.

We say that a function $f : R \to X$ is *Henstock* or *variationally integrable*, we write $f \in \mathcal{H}(R, X)$, or simply $f \in \mathcal{H}$ if there exists a $F \in \mathcal{A}(\mathcal{J}_R, X)$ such that for every $\varepsilon > 0$ there exists a gauge δ on R such that for every δ-fine TD (ξ_i, J_i) we have

$$\sum_i \|f(\xi_i)|J_i| - F(J_i)\| < \varepsilon ;$$

it is immediate that $\mathcal{H} \subset \mathcal{K}$.

If $F \in \mathcal{A}(\mathcal{J}_R, X)$ and $\xi \in R$ we write $(DF)(\xi) = f(\xi)$ if there exists

(1) $$\lim_{J \ni \xi, |J| \to 0} \frac{1}{|J|} F(J) = f(\xi) .$$

One would expect that if $f \in \mathcal{L}_1(R)$ then for almost every (ae) $\xi \in R$ we have

(2) $$\lim_{J \ni \xi, |J| \to 0} \frac{1}{J} \int_J f(t)\, dt = f(\xi) ;$$

however Saks in [S] proves that with the exception of the functions belonging to a subset of first category of $\mathcal{L}_1(R)$, for all other functions f we do not have (2) at any point $\xi \in R$!

This fact suggests the necessity of generalizing the notion of derivative in dimension $n \geq 2$.

For $F \in \mathcal{A}(\mathcal{J}_R, X)$ and $c \in]0, 1]$ we say that F is *c-differentiable* at $\xi \in R$, and that $f(\xi)$ is its *c-derivative*, we write $(D^c F)(\xi) = f(\xi)$, if we have (1) when we take there only c-regular intervals. We say that F is *regularly differentiable* at $\xi \in R$, and that $f(\xi)$ is its *regular derivative*, we write $({}^r DF)(\xi) = f(\xi)$, if we have $(D^c F)(\xi) = f(\xi)$ for every $c \in]0, 1]$.

We say that $f : R \to X$ is a *regularly Kurzweil integrable*, we write $f \in {}^r\mathcal{K}(R, X)$ or simply $f \in {}^r\mathcal{K}$, if there exists $I \in X$ such that for every $\varepsilon > 0$ and every $c \in]0, 1[$ there exists a gauge δ on R such that for every c-regular, δ-fine TD (ξ, J_i) of R we have

$$\|\sum_i f(\xi_i)|J_i| - I\| < \varepsilon ;$$

then we write ${}^{r}\!\int_R f(t)\,dt = I$, or simply ${}^{r}\!\int_R f = I$, and for every $t \in R$ there exists ${}^{r}\tilde{f}(t) = {}^{r}\!\int_{[a,t]} f(s)\,ds$.

$\mathcal{K}(R,X) \subset {}^{r}\mathcal{K}(R,X)$ and for $f \in \mathcal{K}$ we have ${}^{r}\tilde{f} = \tilde{f}$.

We say that $f : R \to X$ is *regularly Henstock integrable*, we write $f \in {}^{r}\mathcal{H}(R,X)$ or simply $f \in {}^{r}\mathcal{H}$, if there exists $F \in \mathcal{A}(\mathcal{J}_R, X)$ such that for every $\varepsilon > 0$ and every $c \in]0,1[$ there exists a gauge δ on R such that for every c-regular, δ-fine TD (ξ_i, J_i) on R we have

(3) $$\sum_i \|f(\xi_i)|J_i| - F(J_i)\| < \varepsilon \,.$$

It is immediate that $\mathcal{H} \subset {}^{r}\mathcal{H}$ and ${}^{r}\mathcal{H} \subset {}^{r}\mathcal{K}$.

1.1 (Lemma of Saks-Henstock) - For $f \in {}^{r}\mathcal{H}(R,X)$, $\varepsilon > 0$ and $c \in]0,1[$, let δ be a gauge on R corresponding to the definition of $f \in {}^{r}\mathcal{H}$. Then for any c-regular, δ-fine TPD (ξ_i, J_i) of R we have (3).

The proof follows the usual ones for the Saks-Henstock Lemma; see for instance [Ho. 2], Teor. 3.7.

If $f \in {}^{r}\mathcal{K}$, for every $x \in R$ we have

(4) $${}^{r}\tilde{f}(x) = F([a,x]) - F([(a_1,a_2),(a_1,x_2)]) - F([(a_1,a_2),(x_1,a_2)]) + F([a,a]) \,.$$

We may associate to ${}^{r}\tilde{f}$ an additive function of intervals, which we still denote by ${}^{r}\tilde{f}$, ${}^{r}\tilde{f} : \mathcal{J}_X \to X$, defined by

(5) $${}^{r}\tilde{f}([x,y]) = {}^{r}\tilde{f}(y) - {}^{r}\tilde{f}(x_1,y_2) - {}^{r}\tilde{f}(x_2,y_1) + {}^{r}\tilde{f}(x) \,.$$

Reciprocally to $F \in \mathcal{A}(\mathcal{J}_R, X)$ we may associate a function $R \to X$, that we still denote by F:

(6) $$F(x) = F([a,x]) - F([(a_1,a_2),(a_1,x_2)]) - F([(a_1,a_2),(x_1,a_2)]) + F([a,a]) \,.$$

This function satisfies

(7) $$F(a_1,x_2) = F(x_1,a_2) = 0 \quad \text{for every } x \in R\ .$$

If $F \in \mathcal{A}(\mathcal{J}_R, X)$ satisfies (7) we write $F \in \mathcal{A}_0(\mathcal{J}_R, X)$ or simply $F \in \mathcal{A}_0$: to every $F \in \mathcal{A}(\mathcal{J}_R, X)$ we can associate a $F_0 \in \mathcal{A}$ that also satisfies (7), with $F_0(J) = F(J)$ for every $J \in \mathcal{J}_R$: $F_0([a,x])$ is given by the second member of (4) or (6).

1.2 - For $f : R \to X$ with $f = 0$ ae we have $f \in \mathcal{H}(R,X)$ and $\tilde{f} = 0$.

The proof is routine; see for instance the proof of 3.2 in [Ho, 2].

1.3 - For $f \in {}^r\mathcal{H}(R,X)$ and $g : R \to X$ with $g = f$ ae we have $g \in {}^r\mathcal{H}$ and ${}^r\tilde{g} = {}^r\tilde{f}$.

§2 - MAIN RESULTS

The lemma that follows is the bidimensional analogue of the Lemma of Austin. see [Ho. 2].

Lemma 2.1 - Let $(J_i)_{i\in A}$, A finite. be a covering of R or. more generally. of a compact $K \subset R$, by c-regular intervals. $0 < c \le 1$. Then there exists $B \subset A$ such that the intervals J_i. $i \in B$. are disjoint with

$$\sum_{i\in B} |J_i| \ge D_c m(K) \quad \text{where} \quad D_c = \frac{c^2}{7c^2+2}\ .$$

Proof. Let us denote by J^1 an interval J_i, $i \in A$. with maximal area. If $|J^1| \ge D_c m(K)$ the proof is finished. If not we define $A_1 = \{i \in A \mid J_i \cap J^1 \ne \emptyset\}$ and $K_1 = \bigcup_{i\in A_1} J_i$: from the maximality of $|J^1|$ it follows that

$$D_c|K_1| \le |J^1|\ .$$

Let us denote by J^2 an interval J_i, $i \in A\backslash A_1$. with maximal area. We have $|J^2| \le |J^1|$ and $J^2 \cap J^1 = \emptyset$. If $|J^2| + |J^1| \ge D_c m(K)$ the proof is finished. If not we define

$A_2 = \{i \in A \backslash A_1 \mid J_i \cap J^2 = \emptyset\}$ and $K_2 = \bigcup_{i \in A_2} J_i$. We have $D_c|K_2| \leq |J^2|$; let us denote by J^3 an interval J_i, $i \in A \backslash (A_1 \cup A_2)$, with $|J^3|$ maximal. We have $|J^3| \leq |J^2|$ and J^1, J^2, J^3 are disjoint. If $|J^1| + |J^2| + |J^3| \geq D_c m(K)$ the proof is finished. If not, we proceed; the process must finish after a finite number of choices $J^1, J^2, \ldots, J^s$ since A is finite. Then we will have

$$\frac{1}{D_c}[|J^1| + |J^2| + \cdots + |J^s|] \geq |K_1| + |K_2| + \cdots + |K_s| \geq m(K)$$

since $\bigcup_{1 \leq i \leq s} K_i \supset \bigcup_{i \in A} J_i \supset K$ and $D_c|K_i| \leq |J^i|$ for $i = 1, 2, \ldots, s$.

Theorem 2.2 - If $f \in {}^r\mathcal{H}(R, X)$ then there exists ${}^rD\, {}^r\tilde{f} = f$ ae.

Proof. Let us denote by E the set of all $\xi \in R$ such that there does not exist $({}^rD\, {}^r\tilde{f})(\xi)$ or such that $({}^rD\, {}^r\tilde{f})(\xi) \neq f(\xi)$. We will prove that $mE = 0$. Let us take $\xi \in E$; then there exists $c = \frac{1}{k}$ ($k \in N$; hence any $\frac{1}{n} < \frac{1}{k}$ will also do) such that there does not exist $(D^c\, {}^r\tilde{f})(\xi)$ or $(D^c\, {}^r\tilde{f})(\xi) \neq f(\xi)$. We write then $\xi \in E^k$ and we have $E = \bigcup_{k \in N} E^k$. We denote by E_n^k the set of all $\xi \in E^k$ that are such that any neighbourhood of ξ contains a c-regular interval $J_\xi \ni \xi$ such that

$$(a) \qquad \| {}^r\tilde{f}(J_\xi) - f(\xi)|J_\xi| \| > \frac{1}{n}|J_\xi| .$$

Since $f \in {}^r\mathcal{K}$, given $\varepsilon > 0$ there exists a gauge δ on R such that for any c-regular, δ-fine TPD (ξ_i, J_i) we have

$$(b) \qquad \sum_i \| {}^r\tilde{f}(J_i) - f(\xi)|J_i| \| < \varepsilon .$$

If $\xi \in E_n^k$ we take J_ξ such that we have (a) with

$$\xi \in \overset{\circ}{J}_\xi \subset J_\xi \subset B_{\delta(\xi)}(\xi) .$$

It is enough to prove that $mU < 2D_c n\varepsilon$ where $U = \bigcup_{\xi \in E_n^k} \overset{\circ}{J}_\xi$. It is immediate that there exists a compact $K \subset U$ with $mK > \frac{1}{2}mU$ and the open covering $(\overset{\circ}{J}_\xi)_{\xi \in E_n^k}$ of K contains a finite subcovering $(\overset{\circ}{J}_{\xi_i})_{i\in A}$, A finite. We write $J_i = J_{\xi_i}$ if $i \in A$. By Lemma 2.1 there exist $B \subset A$ such that the J_i, $i \in B$, are disjoint and

$$(c) \qquad m\left(\bigcup_{i\in B} J_i\right) \geq \frac{1}{D_c} m\left(\bigcup_{i\in A} J_i\right) \geq \frac{1}{D_c} mK > \frac{1}{2D_c} mU .$$

Since $(\xi_i, J_i)_{i\in B}$ is a c-regular, δ-fine TPD, by (c), (a) and (b) we have

$$\frac{1}{2D_c} mU < m\left(\bigcup_{i\in B} J_i\right) < n \sum_{i\in B} \| {}^r\tilde{f}(J_i) - f(\xi_i)|J_i| \, \| < n\varepsilon . \qquad \blacksquare$$

For $F \in \mathcal{A}(\mathcal{J}_R, X)$ and $B \subset R$ we say that F satisfies the ***regular strong Lusin condition*** on B, we write $F \in {}^rSL(B, X)$, if for every $\varepsilon > 0$, every $c \in]0,1[$ and every $A \subset B$ with $mA = 0$, there exists a gauge δ on A such that for every c-regular, δ-fine TPD (ξ_i, J_i) with $\xi_i \in A$ we have

$$\sum_i \|F(J_i)\| < \varepsilon .$$

If $F \in {}^rSL(R, X)$ and if we have f furthermore (7) we write $F \in {}^rSL_a(R, X)$ or simply $F \in {}^rSL_a$. If there exists also ${}^rDF$ ae we write $F \in {}^rSL_a{}^{rD}(R, X)$ or simply $F \in {}^rSL_a{}^{rD}$.

Theorem 2.3 - For $f \in {}^r\mathcal{H}(R, X)$ we have ${}^r\tilde{f} \in {}^rSL_a{}^{rD}(R, X)$ and ${}^rD\,{}^r\tilde{f} = f$ ae.

Proof. Take $A \subset R$ with $mA = 0$ and define $f_{\complement A} = f\chi_{\complement A}$. By 1.3 we have $f_{\complement A} \in {}^r\mathcal{H}$ with ${}^r\tilde{f}_{\complement A} = {}^r\tilde{f}$. Given $\varepsilon > 0$ and $c \in]0,1[$ let δ be a gauge on R corresponding to the definition of ${}^r\!\int_R f$. By 1.1, for any c-regular, δ-fine TPD (ξ_i, J_i) with $\xi_i \in A$ we have $\sum_i \| {}^r\tilde{f}_{\complement A}(J_i) - f_{\complement A}(\xi_i)|J_i| \, \| = \sum_i \| {}^r\tilde{f}(J_i)\| < \varepsilon$ and by Theorem 2.2 there exists ${}^rD\,{}^r\tilde{f} = f$ ae. $\blacksquare$

Reciprocally:

Theorem 2.4 - If $F \in {}^rSL_a{}^{rD}(R, X)$ and $f : R \to X$ are such that ${}^rDF = f$ ae then $f \in {}^r\mathcal{H}(R, X)$ and $F = {}^r\tilde{f}$.

Proof. We define $E = \{\xi \in R \mid \exists({}^{r}DF)(\xi) = f(\xi)\}$. Hence for $A = \complement E$ we have $mA = 0$. By 1.3 we may suppose that $f(\xi) = 0$ if $\xi \in A$. It is enough to prove that for every $\varepsilon > 0$ and every $c \in]0,1[$ there exists a gauge δ on R such that for any c-regular δ-fine TD (ξ_i, J_i) we have $\sum_i \|F(J_i) - f(\xi_i)|J_i|\,\| \leq 2\varepsilon$.

Definition of δ: 1) if $\xi \in E$, i.e., if there exists $({}^{r}DF)(\xi) = f(\xi)$ then by definition of ${}^{r}DF$ there exists $\delta(\xi) > 0$ such that if $\xi \in J \subset B_{\delta(\xi)}(\xi)$, with J c-regular, we have

$$\|F(J) - f(\xi)|J|\,\| < \frac{\varepsilon}{|R|}|J|\ ;$$

2) Since $mA = 0$ and $F \in {}^{r}SL$ there exists a gauge δ on A such that for any c-regular, δ-fine TPD (ξ_k, J_k) with $\xi_k \in A$ we have

$$\sum_k \|F(J_k)\| < \varepsilon\ .$$

Now, for any c-regular, δ-fine TD (ξ_i, J_i) of R we have

$$\begin{aligned} &\sum_i \|F(J_i) - f(\xi_i)|J_i|\,\| \\ \leq\ &\sum_{\xi_i \in E} \|F(J_i) - f(\xi_i)|J_i|\,\| + \sum_{\xi_i \in A} \|F(J_i)\| + \sum_{\xi_i \in A} \|f(\xi_i)\|\,|J_i| \\ \leq\ &\sum_{\xi_i \in E} \frac{\varepsilon}{|R|}|J_i| + \varepsilon + 0 \leq 2\varepsilon\ . \end{aligned}$$

■

§3 - COMMENTS

For $f \in \mathcal{H}([a,b], X)$ the function $\tilde{f}$ is continuous but even in the numerical case, $X = R$, for $f \in {}^{r}\mathcal{H}([a,b])$, ${}^{r}\tilde{f}$ may be discontinuous at the border of $[a,b]$, see [J-K-S].

If $J = J_1 \cup J_2$, with nonoverlapping J_1 and J_2, for $f : J \to \mathbb{R}$ we may have $f \in {}^{r}\mathcal{H}(J_i)$, $i = 1, 2$, but $f \notin {}^{r}\mathcal{H}(J)$: see [J-K-S].

The idea of regular intervals goes back to at least 60 years.

Results related to multidimensional regular derivation and integration can be found, among others, in the following recent papers: [J-K-S], [K-J, 1], [J-K, 1], [K-J, 2], [J-K, 2] and [K-J, 3]; their nomenclature however is different:

1) Their integrals are called: GP integral [J-K-S], weak Perron integral [J-K, 1], Z-integral [K-J, 1], Pf-integral [K-J,2], α-regular integral [K-J,3].

2) Their derivatives are called: weak derivative [J-K,1], Z-derivative [K-J, 1], α-regular derivative [K-J,2].

3) For the regular strong Lusin condition we have the following names: n_f^+ class [J-K, 1], well-behaved [K-J, 1], α-variationally normal [K-J, 2].

We took the expression "strong Lusin condition" from [L].

In another paper we will develop further the results presented here and compare them with those of the papers quoted above.

REFERENCES

[Ho, 1] Hönig, C.S. On a remarkable differential characterization of the functions that are Kurzweil-Henstock integrals. *33º Seminário Brasileiro de Análise* (1991), 331–341.

[Ho, 2] Hönig, C.S. As integrais de gauge. *37º Seminário Brasileiro de Análise* (1993), 1–60.

[J-K, 1] Jurkat, W.B.; Knizia, R.W. A characterization of multi-dimensional Perron integrals and the fundamental theorem. *Can. J. Math.*, **43** (1991), 526–539.

[J-K, 2] Jurkat, W.B.; Knizia, R.W. Generalized absolute continuous interval functions and multi-dimensional Perron integration. *Analysis*, **12** (1992), 303–313.

[J-K-S] Jarnik, J.; Kurzweil, J.; Schwabik, Št. On Mawhin's approach to multiple non-absolute convergent integral. *Časopis pro pest. mat.*, 108 (1983), 356–380.

[K-J, 1] Kurzweil, J.; Jarnik, J. Equi integrability and controlled convergence of Perron-type integrable function. *Real Analysis Exchange*, **17** (1991-92), 110–139.

[K-J, 2] Kurzweil, J.; Jarnik, J. Equivalent definitions of regular generalized Perron integral. *Czech. Math. Journal*, **42** (1992), 365–378.

[K-J, 3] Kurzweil, J.; Jarnik, J. Differentiability and integrability in n dimensions with respect to α-regular intervals. *Results in Math.*, **21** (1992), 138–151.

[L] Lee, P.Y. On ACG^* function. *Real Analysis Exchange*. **15** (1989-90), 754–756.

[S] Saks, S. Remark of the differentiability of the Lebesgue indefinite integral. *Fund. Math.*, **XXII** (1934), 257–261.