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EXTENSION THEORY APPROACH IN THE STABILITY OF THE STANDING WAVES FOR THE NLS EQUATION WITH POINT INTERACTIONS ON A STAR GRAPH

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Abstract. The aim of this work is to demonstrate the effectiveness of the extension theory of symmetric operators in the investigation of the stability of standing waves for the nonlinear Schrödinger equations with two types of non-linearities (power and logarithmic) and two types of point interactions (δ- and δ'-) on a star graph. Our approach allows us to overcome the use of variational techniques in the investigation of the Morse index for self-adjoint operators with non-standard boundary conditions which appear in the stability study. We also demonstrate how our method simplifies the proof of the stability results known for the NLS equation with point interactions on the line.

1. Introduction

In the last two decades the study of nonlinear dispersive models with point interactions has attracted a lot of attention of mathematicians and physicists. In particular, such models appear in nonlinear optics, Bose-Einstein condensates (BEC), and quantum graphs (or networks) (see [3, 17, 21–23, 28, 40, 42, 44] and references therein). The prototype equation for description of these models is the nonlinear Schrödinger equation (NLS henceforth)

$$i\partial_t u(t,x) + \partial_x^2 u(t,x) + |u(t,x)|^{p-1} u(t,x) = 0, \quad x \neq 0, \tag{1.1}$$

$(t,x) \in \mathbb{R}\times\mathbb{R}$, $p > 1$, with specific boundary conditions at $x = 0$ induced by a certain impurity or point interaction. The most studied are the models with so-called δ- and δ'-interaction (see Section 5 for details). Indeed, the Dirac distribution models an impurity or defect localized at the origin. Moreover, the NLS-δ equation on the line can be viewed as a prototype model for the

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interaction of a wide soliton with a highly localized potential. In nonlinear optics it models a soliton propagating in a medium with a point defect, or interaction of a wide soliton with a much narrower one in a bimodal fiber (see [36]). Recently, numerous results on the local well-posedness of initial value problem and periodic boundary value problem, the long time behavior of solutions, the existence of stationary states, blow up and scattering results (see [3,4,8,9,12,23,24,28,32,36] and references therein) have been obtained.

In this paper, we study the existence and the orbital stability of standing waves of the model (1.1) being extended to a star graph $\mathcal{G}$, i.e., N half-lines attached to the common vertex $\nu = 0$. Namely, we consider the following nonlinear Schrödinger equations on the star graph $\mathcal{G}$

$$i\partial_t \mathbf{U}(t,x) + \partial_x^2 \mathbf{U}(t,x) + |\mathbf{U}(t,x)|^{p-1}\mathbf{U}(t,x) = 0, \tag{1.2}$$

where $\mathbf{U}(t,x) = (u_j(t,x))_{j=1}^N : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}^N$, and $p > 1$. The nonlinearity acts componentwise, i.e., $(|\mathbf{U}|^{p-1}\mathbf{U})_j = |u_j|^{p-1}u_j$, and the function $\mathbf{U}$ is assumed to satisfy specific boundary δ- and δ'-interaction at the vertex $\nu = 0$ to be defined below (see Subsection 2.1).

Equation (1.2) models propagation through junctions in networks. The analysis of the behavior of NLS equation on networks is not yet fully developed, but it is currently growing (see [1, 2, 10, 11, 15, 16, 44] and references therein). In particular, models of BEC on graphs/networks is a topic of active research (see [22, 29]).

We recall that the quantum graphs (metric graphs equipped with a linear Hamiltonian $\mathbf{H}$) have been a very developed subject in the last couple of decades. They give simplified models in mathematics, physics, chemistry, and engineering, when one considers propagation of waves of various type through a quasi one-dimensional (e.g. meso- or nanoscale) system that looks like a thin neighborhood of a graph (see [17, 20, 22, 40, 42] for details and references).

Various recent analytical works (see [1,2,10,11,44] and references therein) deal with special solutions of (1.2) called *standing wave solutions*, i.e., the solutions of the form $\mathbf{U}(t,x) = e^{i\omega t}\mathbf{\Phi}(x)$, with the profile $\mathbf{\Phi}$ satisfying δ-interaction conditions defined by (2.3) below. In [2] it was established a complete description of the profiles $\mathbf{\Phi}$ for any $\alpha \in \mathbb{R}$, and the stability investigation for the N-tail profile (see (2.9)) under the restriction $\alpha < \alpha^* < 0$ which comes from the associated variational problem. In [1] the restriction $\alpha < \alpha^*$ was removed. It is worth noting that the problems of the existence and the stability/instability of standing waves are far richer and more complicated in the case of the NLS models with point interactions on star graphs than in the case of the NLS equation with point interactions on the line. We propose a novel short proof of the orbital stability of the N-tail profile for any $\alpha < 0$ in the framework of the extension theory approach (see Remark 3.20). Moreover, we prove the following new result on the orbital instability of N-bump profile $\mathbf{\Phi}$ in the case $\alpha > 0$.

Theorem 1.1. *Let* $\alpha > 0$, $1 < p < 5$, *and* $\omega > \frac{\alpha^2}{N^2}$. *Let also* $\mathbf{\Phi}_{\alpha,\delta}$ *be defined by* (2.9), *and the space* $\mathcal{E}(\mathcal{G})$ *be defined in notation section. Then the following assertions hold.*

(*i*) *If* $1 < p \leq 3$, *then* $e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}$ *is orbitally unstable in* $\mathcal{E}(\mathcal{G})$.

(*ii*) *If* $3 < p < 5$, *then there exists* $\omega_2 > \frac{\alpha^2}{N^2}$ *such that* $e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}$ *is orbitally unstable in* $\mathcal{E}(\mathcal{G})$ *for* $\omega > \omega_2$.

In the case $p \geq 5$ our method does not provide any information about orbital stability of $e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}$ (see Remark 3.20-(*i*)). Mention also that in the case $N = 2$ the above result coincides with [28, Theorem 4].

In Subsection 3.2, we prove the following novel stability theorem for the standing waves of the NLS-δ' equation on the star graph with a specific N-tail profile $\mathbf{\Phi}_{\lambda,\delta'}$ satisfying δ'-interaction conditions (2.5).

Theorem 1.2. *Let* $\lambda < 0$, *and* $\omega > \frac{N^2}{\lambda^2}$. *Let also* $\mathbf{\Phi}_{\lambda,\delta'}$ *be defined by* (2.11), *and the space* $H^1_{\mathrm{eq}}(\mathcal{G})$ *be defined by*

$$H^1_{\mathrm{eq}}(\mathcal{G}) = \{(v_j)_{j=1}^N \in H^1(\mathcal{G}) : v_1(x) = \cdots = v_N(x),\, x > 0\}.$$

Then the following assertions hold.

(*i*) *Let* $1 < p \leq 5$.

1) *If* $\omega < \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, *then* $e^{it\omega}\mathbf{\Phi}_{\lambda,\delta'}$ *is orbitally stable in* $H^1(\mathcal{G})$.

2) *If* $\omega > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ *and* N *is even, then* $e^{it\omega}\mathbf{\Phi}_{\lambda,\delta'}$ *is orbitally unstable in* $H^1(\mathcal{G})$.

(*ii*) *Let* $p > 5$ *and* $\omega \neq \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$. *Then there exists* $\omega^* > \frac{N^2}{\lambda^2}$ *such that* $e^{it\omega}\mathbf{\Phi}_{\lambda,\delta'}$ *is orbitally unstable in* $H^1(\mathcal{G})$ *for* $\omega > \omega^*$, *and* $e^{it\omega}\mathbf{\Phi}_{\lambda,\delta'}$ *is orbitally stable in* $H^1_{\mathrm{eq}}(\mathcal{G})$ *for* $\omega < \omega^*$.

The relative position of ω^* and $\frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ is discussed in Remark 3.27. In the case $N = 2$ the above result coincides with Proposition 6.9(1) (partially) and Theorem 6.11 in [3]. To our knowledge, the NLS-δ' equation on the star graph has never been studied before, and complete description of the standing waves for such model is unknown (see Remark 2.5).

In Section 4, we study the following NLS equation with logarithmic nonlinearity on the star graph $\mathcal{G}$ (NLS-log equation)

$$i\partial_t \mathbf{U}(t,x) + \partial_x^2 \mathbf{U}(t,x) + \mathbf{U}(t,x)\,\mathrm{Log}\,|\mathbf{U}(t,x)|^2 = 0, \tag{1.3}$$

where $\mathbf{U}(t,x) = (u_j(t,x))_{j=1}^N : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{C}^N$. The nonlinearity acts componentwise, i.e., $(\mathbf{U}\,\mathrm{Log}\,|\mathbf{U}|^2)_j = u_j\,\mathrm{Log}\,|u_j|^2$. Note that by $\mathrm{Log}\,|\mathbf{U}(t,x)|^2$, we mean $\mathrm{Log}(|\mathbf{U}(t,x)|^2)$.

For the NLS-log equation with δ-interaction, we extend the result from [15] (for any $\alpha < 0$) on the orbital stability of the Gaussian N-tail profile $\mathbf{\Psi}_{\alpha,\delta} = (\psi_{\alpha,\delta})_{j=1}^N$ defined by (2.14) below. In particular, we prove

Theorem 1.3. *Let $\omega \in \mathbb{R}$, and $\mathbf{\Psi}_{\alpha,\delta}$ be defined by* (2.14)*. Then the standing wave $e^{i\omega t}\mathbf{\Psi}_{\alpha,\delta}$ is orbitally stable in $W^1_{\mathcal{E}}(\mathcal{G})$ for any $\alpha < 0$, and $e^{i\omega t}\mathbf{\Psi}_{\alpha,\delta}$ is spectrally unstable for any $\alpha > 0$.*

We also show the result analogous to Theorem 1.2 for the NLS-log equation with δ'-interaction on $\mathcal{G}$.

Theorem 1.4. *Let $\lambda < 0$, and $\omega \in \mathbb{R}$. Let also $\mathbf{\Psi}_{\lambda,\delta'}$ be defined by* (2.16)*. Then the following assertions hold.*

(*i*) *If $-N < \lambda < 0$, then $e^{it\omega}\mathbf{\Psi}_{\lambda,\delta'}$ is orbitally stable in $W^1(\mathcal{G})$.*
(*ii*) *If $\lambda < -N$, then $e^{it\omega}\mathbf{\Psi}_{\lambda,\delta'}$ is spectrally unstable.*

The spaces $W^1_{\mathcal{E}}(\mathcal{G})$ and $W^1(\mathcal{G})$ are defined in notation section.

In Section 5, we propose a new approach to prove some known results on the orbital stability of standing waves for NLS equation (1.1) with δ- and δ'-interaction on the line. It should be noted that the most of previous results (for NLS on $\mathcal{G}$ and on the line) are based on either variational methods or the abstract stability theory by Grillakis, Shatah and Strauss [33, 34] which requires spectral analysis of certain self-adjoint Schrödinger operators. In particular, investigation of the spectrum of these operators is based on the analytic perturbations theory and the variational methods.

Our approach relies on the theory of extensions of symmetric operators, the spectral theory of self-adjoint Schrödinger operators and the analytic perturbations theory. In particular, the extension theory gives the advantage to estimate the number of negative eigenvalues (Morse index) of the linear Schrödinger operator associated with the NLS equation. We emphasize that we do not need to study any variational problem associated with the equation, and our method does not use any minimization properties of the standing waves studied. We would like to mention the papers [37, 38] where the non-variational methods were used for the investigation of the

Morse index in the case of the NLS equation on the star graph with classical and generalized Kirchhoff conditions at the vertex. In particular, the authors elaborated a kind of extension of the Sturm theory for the Schrödinger operators on the star graph.

The paper is organized as follows. In the Preliminaries (Section 2), we give some brief description of all the point interactions on the star graph and explain the origin of δ- and δ'-interaction. We also review previous results on the orbital stability. In Section 3, we discuss NLS equation (1.2) with δ- and δ'-interaction on the star graph $\mathcal{G}$. In Section 4, we study NLS-log equation (1.3) with δ- and δ'-interaction on $\mathcal{G}$. In Section 5, we briefly discuss how the tools of the extension theory can be applied to the stability study of the NLS equations with point interactions on the line.

Notation. By $H^1(\mathbb{R})$, $H^2(\mathbb{R}\setminus\{0\}) = H^2(\mathbb{R}_-)\oplus H^2(\mathbb{R}_+)$, we denote the Sobolev spaces. Denote by $\mathcal{G}$ the star graph constituted by N half-lines attached to a common vertex $\nu = 0$. On the graph, we define the spaces

$$L^p(\mathcal{G}) = \bigoplus_{j=1}^N L^p(\mathbb{R}_+),\ H^1(\mathcal{G}) = \bigoplus_{j=1}^N H^1(\mathbb{R}_+),\ H^2(\mathcal{G}) = \bigoplus_{j=1}^N H^2(\mathbb{R}_+),$$

$p > 1$. For instance, the norm of $\mathbf{V} = (v_j)_{j=1}^N$ in $L^p(\mathcal{G})$ is defined by $||\mathbf{V}||^p_{L^p(\mathcal{G})} = \sum\limits_{j=1}^N ||v_j||^p_{L^p(\mathbb{R}_+)}$. Depending on the context, we will use the following notations for different objects: by $||\cdot||$, we denote the norm in $L^2(\mathbb{R})$ or in $L^2(\mathcal{G})$ (accordingly $(\cdot,\cdot)$ denotes the scalar product in $L^2(\mathbb{R})$ or in $L^2(\mathcal{G})$); by $||\cdot||_p$, we denote the norm in $L^p(\mathbb{R})$ or in $L^p(\mathcal{G})$.
Denote $\mathcal{E}(\mathcal{G}) = \{(v_j)_{j=1}^N \in H^1(\mathcal{G}) : v_1(0) = \cdots = v_N(0)\}$, and

$$L^2_k(\mathcal{G}) = \{(v_j)_{j=1}^N \in L^2(\mathcal{G}) : v_1(x) = \cdots = v_k(x),\ v_{k+1}(x) = \cdots = v_N(x)\}.$$

In particular, $\mathcal{E}_k(\mathcal{G}) = \mathcal{E}(\mathcal{G})\cap L^2_k(\mathcal{G})$, and $H^1_k(\mathcal{G}) = H^1(\mathcal{G})\cap L^2_k(\mathcal{G})$. On $\mathcal{G}$, we define the following weighted Hilbert spaces

$$W^j(\mathcal{G}) = \bigoplus_{j=1}^N W^j(\mathbb{R}_+),\quad W^j(\mathbb{R}_+) = \{f \in H^j(\mathbb{R}_+) : x^j f \in L^2(\mathbb{R}_+)\},$$

$W^j_k(\mathcal{G}) = W^j(\mathcal{G})\cap L^2_k(\mathcal{G})$, $j\in\{1,2\}$, and the Banach space

$$W(\mathcal{G}) = \bigoplus_{j=1}^N W(\mathbb{R}_+),\ \ W(\mathbb{R}_+) = \{f\in H^1(\mathbb{R}_+) : |f|^2 \operatorname{Log}|f|^2 \in L^1(\mathbb{R}_+)\}.$$

In particular, $W_{\mathcal{E}}(\mathcal{G}) = W(\mathcal{G})\cap\mathcal{E}(\mathcal{G})$, $W^1_{\mathcal{E}}(\mathcal{G}) = W^1(\mathcal{G})\cap\mathcal{E}(\mathcal{G})$, and $W^1_{\mathcal{E},k}(\mathcal{G}) = W^1_{\mathcal{E}}(\mathcal{G}) \cap L^2_k(\mathcal{G})$.

Let A be a densely defined symmetric operator in the Hilbert space $\mathcal{H}$. The domain of A is denoted by $\mathrm{dom}(A)$. The *deficiency subspaces* and *deficiency numbers* of A are defined by $\mathcal{N}_\pm(A) := \ker(A^* \mp iI)$ and $n_\pm(A) := \dim\ker(A^* \mp iI)$ respectively. The number of negative eigenvalues counting multiplicities (or *the Morse index*) is denoted by $n(A)$. The spectrum and the resolvent set of A are denoted by $\sigma(A)$ and $\rho(A)$ respectively. In particular, $\sigma_p(A)$ and $\sigma_c(A)$ denote the point and the continuous spectrum of A. Let X be an arbitrary Banach space, then its dual is denoted by X'.

2. Preliminaries

In this Section, we provide a brief description of point interactions on the star graph and also discuss previous results on the orbital stability.

2.1. **The NLS equation with point interactions on a star graph.**

The family of self-adjoint conditions naturally arising at the vertex $\nu = 0$ of the star graph $\mathcal{G}$ has the following description

$$(U - I)\mathbf{U}(t,0) + i(U + I)\mathbf{U}'(t,0) = 0, \tag{2.1}$$

where $\mathbf{U}(t,0) = (u_j(t,0))_{j=1}^N$, $\mathbf{U}'(t,0) = (u_j'(t,0))_{j=1}^N$, U is an arbitrary unitary $N\times N$ matrix, and I is the $N\times N$ identity matrix. The conditions (2.1) at $\nu = 0$ define the N^2-parametric family of self-adjoint extensions of the closable symmetric operator (see [20, Chapter 17])

$$\mathbf{H}_0 = \bigoplus_{j=1}^N \frac{-d^2}{dx^2}, \quad \mathrm{dom}(\mathbf{H}_0) = \bigoplus_{j=1}^N C_0^\infty(\mathbb{R}_+).$$

We consider two choices of matrix U which correspond to so-called δ- and δ'- interactions on the star graph $\mathcal{G}$. More precisely, the matrix $U = \frac{2}{N+i\alpha}\mathcal{I} - I$, $\alpha \in \mathbb{R}\setminus\{0\}$, where $\mathcal{I}$ is the $N\times N$ matrix whose all entries equal one, induces the following nonlinear Schrödinger equation with δ-interaction on the star graph $\mathcal{G}$ (NLS-δ equation)

$$i\partial_t\mathbf{U} - \mathbf{H}^\delta_\alpha\mathbf{U} + |\mathbf{U}|^{p-1}\mathbf{U} = 0, \tag{2.2}$$

where $\mathbf{H}^\delta_\alpha$ is the self-adjoint operator on $L^2(\mathcal{G})$ acting as $(\mathbf{H}^\delta_\alpha\mathbf{V})(x) = (-v_j''(x))_{j=1}^N$, $x > 0$, on the domain $\mathrm{dom}(\mathbf{H}^\delta_\alpha) = \mathbf{D}_{\alpha,\delta}$, where

$$\mathbf{D}_{\alpha,\delta} := \Big\{\mathbf{V} \in H^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0),\ \sum_{j=1}^N v_j'(0) = \alpha v_1(0)\Big\}. \tag{2.3}$$

Model (2.2)-(2.3) has been extensively studied in [1, 2]. In particular, the authors showed well-posedness of the corresponding Cauchy problem. Moreover, they investigated the existence and the particular form of standing waves, as well as their variational and stability properties (see Theorems 2.2 and 2.4 below).

The second model we are interested in corresponds to $U = I - \frac{2}{N-i\lambda}\mathcal{I}$, $\lambda \in \mathbb{R} \setminus \{0\}$, which induces the nonlinear Schrödinger equation with δ'-interaction on the graph $\mathcal{G}$ (NLS-δ' equation)

$$i\partial_t \mathbf{U} - \mathbf{H}_\lambda^{\delta'}\mathbf{U} + |\mathbf{U}|^{p-1}\mathbf{U} = 0, \tag{2.4}$$

where $\mathbf{H}_\lambda^{\delta'}$ is the self-adjoint operator on $L^2(\mathcal{G})$ acting as $(\mathbf{H}_\lambda^{\delta'}\mathbf{V})(x) = (-v_j''(x))_{j=1}^N$, $x > 0$, on the domain $\mathrm{dom}(\mathbf{H}_\lambda^{\delta'}) = \mathbf{D}_{\lambda,\delta'}$, where

$$\mathbf{D}_{\lambda,\delta'} := \Big\{\mathbf{V} \in H^2(\mathcal{G}) : v_1'(0) = \cdots = v_N'(0),\ \sum_{j=1}^N v_j(0) = \lambda v_1'(0)\Big\}. \tag{2.5}$$

To our knowledge such type of interaction has never been studied for the NLS equation on the star graph. In this connection, one of the principal aims of this paper is to establish some results on the existence and the orbital stability of standing wave solutions to (2.4).

In Section 4, we consider the following NLS equations with logarithmic nonlinearity on the star graph (NLS-log-δ and NLS-log-δ' equation):

$$i\partial_t \mathbf{U} - \mathbf{H}_\alpha^{\delta}\mathbf{U} + \mathbf{U}\,\mathrm{Log}\,|\mathbf{U}|^2 = 0, \tag{2.6}$$

$$i\partial_t \mathbf{U} - \mathbf{H}_\lambda^{\delta'}\mathbf{U} + \mathbf{U}\,\mathrm{Log}\,|\mathbf{U}|^2 = 0. \tag{2.7}$$

Model (2.6) has been studied in [15]. In particular, the author showed well-posedness of the Cauchy problem in the Banach space $W_{\mathcal{E}}(\mathcal{G})$ (see Theorem 4.1), and studied stability properties of the ground state for the corresponding stationary equation.

2.2. **Review of the results on the orbital stability for the NLS equation with point interactions on a star graph.** Crucial role in the orbital stability analysis of standing waves is played by the symmetries of NLS equation (1.2) (and (1.3)) The basic symmetry associated to the mentioned equation is phase invariance, namely, if $\mathbf{U}$ is a solution of (1.2) then $e^{i\theta}\mathbf{U}$ is also a solution for any $\theta \in [0, 2\pi)$. Thus, it is reasonable to define orbital stability as follows (for the models (1.2) and (1.3)).

Definition 2.1. *The standing wave $\mathbf{U}(t,x) = e^{i\omega t}\mathbf{\Phi}(x)$ is said to be orbitally stable in a Banach space X if for any $\varepsilon > 0$ there exists $\eta > 0$ with the* *following property: if* $\mathbf{U}_0 \in X$ *satisfies* $||\mathbf{U}_0 - \mathbf{\Phi}||_X < \eta$, *then the solution* $\mathbf{U}(t)$ *of (1.2) (resp. (1.3)) with* $\mathbf{U}(0) = \mathbf{U}_0$ *exists for any* $t \in \mathbb{R}$ *and*

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} ||\mathbf{U}(t) - e^{i\theta}\mathbf{\Phi}||_X < \varepsilon.$$

Otherwise, the standing wave $\mathbf{U}(t,x) = e^{i\omega t}\mathbf{\Phi}(x)$ *is said to be orbitally unstable in* X.

In particular, for the NLS-δ and NLS-δ' equations on the star graph $\mathcal{G}$ defined by (2.2) and (2.4), the space X coincides with $\mathcal{E}(\mathcal{G})$ and $H^1(\mathcal{G})$, respectively.

In the first part of the paper, we study the orbital stability of the standing wave solutions $\mathbf{U}(t,x) = e^{i\omega t}\mathbf{\Phi}(x) = \left(e^{i\omega t}\varphi_j(x)\right)_{j=1}^N$ for the NLS-δ equation (2.2) on $\mathcal{G}$. It is easily seen that amplitude $\mathbf{\Phi} \in \mathbf{D}_{\alpha,\delta}$ satisfies the following stationary equation

$$\mathbf{H}_\alpha^\delta \mathbf{\Phi} + \omega\mathbf{\Phi} - |\mathbf{\Phi}|^{p-1}\mathbf{\Phi} = 0. \tag{2.8}$$

In [2], authors obtained the following description of all solutions to equation (2.8).

Theorem 2.2. *Let* $[s]$ *denote the integer part of* $s \in \mathbb{R}$, *and* $\alpha \neq 0$. *Then equation* (2.8) *has* $\left[\frac{N-1}{2}\right] + 1$ *(up to permutations of the edges of* $\mathcal{G}$*) vector solutions* $\mathbf{\Phi}_m^\alpha = (\varphi_{m,j}^\alpha)_{j=1}^N$, $m = 0, ..., \left[\frac{N-1}{2}\right]$, *which are given by*

$$\varphi_{m,j}^\alpha(x) = \begin{cases} \left[\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}x - a_m\right)\right]^{\frac{1}{p-1}}, & j = 1,..,m; \\ \left[\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}x + a_m\right)\right]^{\frac{1}{p-1}}, & j = m+1,...,N, \end{cases}$$

where $a_m = \tanh^{-1}\left(\frac{\alpha}{(2m-N)\sqrt{\omega}}\right)$, *and* $\omega > \frac{\alpha^2}{(N-2m)^2}$.

Remark 2.3. (i) Note that in the case $\alpha < 0$ vector $\mathbf{\Phi}_m^\alpha = (\varphi_{m,j}^\alpha)_{j=1}^N$ has m *bumps* and $N-m$ *tails*. It is easily seen that $\mathbf{\Phi}_0^\alpha$ is the N*-tail profile*. Moreover, the N-tail profile is the only symmetric (i.e., invariant under permutations of the edges) solution of equation (2.8). In the case $N = 5$, we have three types of profiles: *5-tail profile*, *4-tail/1-bump profile* and *3-tail/2-bump profile*. They are demonstrated on Figure 1 (from the left to the right).

(ii) In the case $\alpha > 0$ vector $\mathbf{\Phi}_m^\alpha = (\varphi_{m,j}^\alpha)_{j=1}^N$ has m *tails* and $N-m$ *bumps* respectively. For $N = 5$, we have: *5-bump profile, 4-bump/1-tail profile, 3-bump/ 2-tail profile.* They are demonstrated on Figure 2.

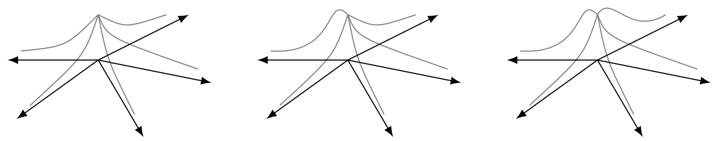

Figure 1

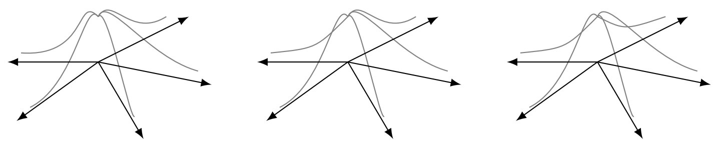

Figure 2

It was shown in [2] that for $-N\sqrt{\omega} < \alpha < \alpha^* < 0$, the vector solution $\mathbf{\Phi}_{\alpha,\delta} = (\varphi_{\alpha,\delta})_{j=1}^N := \mathbf{\Phi}_0^\alpha$,

$$\varphi_{\alpha,\delta} := \varphi_{0,j}^\alpha(x) = \left[\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}x + \tanh^{-1}\left(\frac{-\alpha}{N\sqrt{\omega}}\right)\right)\right]^{\frac{1}{p-1}} \tag{2.9}$$

is the ground state. The parameter α^* above originates from the variational problem associated with equation (2.8), and it guarantees constrained minimality of the action functional

$$\mathbf{S}_\alpha(\mathbf{V}) = \tfrac{1}{2}||\mathbf{V}'||^2 + \tfrac{\omega}{2}||\mathbf{V}||^2 - \tfrac{1}{p+1}||\mathbf{V}||_{p+1}^{p+1} + \tfrac{\alpha}{2}|v_1(0)|^2, \quad \mathbf{V} = (v_j)_{j=1}^N \in \mathcal{E}(\mathcal{G}). \tag{2.10}$$

Namely, the vector solution $\mathbf{\Phi}_{\alpha,\delta}$ is the ground state in the sense of the minimality of $\mathbf{S}_\alpha(\mathbf{V})$ at $\mathbf{\Phi}_{\alpha,\delta}$ with the constraint given by the Nehari manifold

$$\mathcal{N} = \{\mathbf{V} \in \mathcal{E}(\mathcal{G}) \setminus \{0\} : ||\mathbf{V}'||^2 + \omega||\mathbf{V}||^2 - ||\mathbf{V}||_{p+1}^{p+1} + \alpha|v_1(0)|^2 = 0\}.$$

For $\alpha > 0$ the N-bump profile $\mathbf{\Phi}_{\alpha,\delta}$ does not have the variational characterization (see [30, Remark 14]). In [2] the following orbital stability result has been shown.

Theorem 2.4. [2, Theorem 2] *Let $1 < p \le 5$, $\alpha < \alpha^* < 0$, and $\omega > \frac{\alpha^2}{N^2}$. Then the standing wave $e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}$ is orbitally stable in $\mathcal{E}(\mathcal{G})$.*

Authors in [2] showed also that for $p > 5$ there exists $\omega^* > \frac{\alpha^2}{N^2}$ such that $e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}$ is stable in $\mathcal{E}(\mathcal{G})$ for any $\omega \in (\frac{\alpha^2}{N^2}, \omega^*)$ and unstable for any $\omega > \omega^*$. Stronger version of the above theorem was proved in [1, Theorem 1]. In particular, they proved orbital stability of $e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}$ for $\alpha < 0$ without restriction

$\alpha < \alpha^* < 0$. For $m \neq 0$, $\alpha < 0$ in Theorem 2.2, we have $S(\mathbf{\Phi}_m^\alpha) > S(\mathbf{\Phi}_0^\alpha)$ which means that $\mathbf{\Phi}_m^\alpha$ for $m \neq 0$ is *an excited state*. Stability properties of the excited states as well as of $\mathbf{\Phi}_m^\alpha$ for $\alpha > 0$ were studied in [10].

To our knowledge, the problem of orbital stability of standing waves $\mathbf{U}(t,x) = e^{i\omega t}\mathbf{\Phi}(x)$ has never been considered for NLS-δ' equation (2.4) on the star graph. In the present paper, we study the orbital stability of the standing waves $\mathbf{U}(t,x) = e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ with N-tail profile $\mathbf{\Phi}_{\lambda,\delta'} = (\varphi_{\lambda,\delta'})_{j=1}^N$, where

$$\varphi_{\lambda,\delta'}(x) = \Big[\frac{(p+1)\omega}{2}\operatorname{sech}^2\Big(\frac{(p-1)\sqrt{\omega}}{2}x + \tanh^{-1}\Big(\frac{-N}{\lambda\sqrt{\omega}}\Big)\Big)\Big]^{\frac{1}{p-1}}, \quad (2.11)$$

with $\omega > \frac{N^2}{\lambda^2}$ and $\lambda < 0$. In Section 4, we prove new result on stability of $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ (see Theorem 1.2).

Remark 2.5. The description of the set of all solutions to the stationary equation

$$\mathbf{H}_\lambda^{\delta'}\mathbf{\Phi} + \omega\mathbf{\Phi} - |\mathbf{\Phi}|^{p-1}\mathbf{\Phi} = 0, \quad (2.12)$$

is unknown. We note that any L^2-solution to (2.12) has the form

$$\mathbf{\Phi}(x) = (\varphi_j(x))_{j=1}^N = (\sigma_j\varphi_0(x+x_j))_{j=1}^N,$$

where $\sigma_j \in \mathbb{C}, |\sigma_j| = 1$, $x_j \in \mathbb{R}$, and $\varphi_0(x) = [\frac{(p+1)\omega}{2}\operatorname{sech}^2(\frac{(p-1)\sqrt{\omega}}{2}x)]^{\frac{1}{p-1}}$. Hence, denoting $t_j = \tanh(x_j)$, from (2.5), we get the relations

$$\begin{cases} \sigma_1(1-t_1)^{\frac{1}{p-1}}t_1 = \dots = \sigma_N(1-t_N)^{\frac{1}{p-1}}t_N, \\ \sum_{j=1}^N \sigma_j(1-t_j)^{\frac{1}{p-1}} = -\lambda\sqrt{\omega}\sigma_1(1-t_1)^{\frac{1}{p-1}}t_1. \end{cases}$$

In [3], for the case of $\mathcal{G} = \mathbb{R}$ (δ'- interaction on the line), the authors established the existence of two families (odd and asymmetric) of solutions to (2.12). For $N \geq 3$, it seems to be very nontrivial problem to determine a complete description of the solutions to (2.12). Observe that in the case of NLS-δ equation the situation is easier since the continuity condition $\varphi_1(0) = \dots = \varphi_N(0)$ implies $|\varphi_1'(0)| = \dots = |\varphi_N'(0)|$, therefore, $\sigma_1 = \dots = \sigma_N$ and $x_j = \pm a$, $a > 0$.

In the case of NLS-log-δ equation the profile of the standing wave $e^{i\omega t}\mathbf{\Psi}$ satisfies the equality

$$\mathbf{H}_\alpha^\delta\mathbf{\Psi} + \omega\mathbf{\Psi} - \mathbf{\Psi}\operatorname{Log}|\mathbf{\Psi}|^2 = 0. \quad (2.13)$$

From [15] it follows that solutions to (2.13) have the following description.

Theorem 2.6. *Let $\alpha \neq 0$. Then equation* (2.13) *has* $\left[\frac{N-1}{2}\right]+1$ *vector solutions* $\mathbf{\Psi}_m^\alpha = (\psi_{m,j}^\alpha)_{j=1}^N$, $m = 0, ..., \left[\frac{N-1}{2}\right]$, *given by*

$$\psi_{m,j}^\alpha(x) = \begin{cases} e^{\frac{\omega+1}{2}} e^{-\frac{(x-a_m)^2}{2}}, & j = 1, .., m; \\ e^{\frac{\omega+1}{2}} e^{-\frac{(x+a_m)^2}{2}}, & j = m+1, ..., N, \end{cases} \qquad \text{where } a_m = \frac{\alpha}{2m-N}.$$

We should note that the structure of the profiles that solve (2.13) is similar to the one in the case of NLS-δ equation (see Remark 2.3). It was proved in [15] that for $\alpha < \alpha^*_{\mathrm{Log}} < 0$, the vector solution $\mathbf{\Psi}_{\alpha,\delta} = (\psi_{\alpha,\delta})_{j=1}^N$ defined by

$$\psi_{\alpha,\delta} = \psi_{0,j}^\alpha(x) = e^{\frac{\omega+1}{2}} e^{-\frac{(x-\frac{\alpha}{N})^2}{2}} \tag{2.14}$$

is the ground state. The condition $\alpha < \alpha^*_{\mathrm{Log}}$ guarantees constrained minimality of the following action functional for $\mathbf{V} \in W_{\mathcal{E}}(\mathcal{G})$,

$$\mathbf{S}_{\alpha,\mathrm{Log}}(\mathbf{V}) = \tfrac{1}{2}||\mathbf{V}'||^2 + \tfrac{(\omega+1)}{2}||\mathbf{V}||^2 - \tfrac{1}{2}\sum_{j=1}^N \int_0^\infty |v_j|^2 \,\mathrm{Log}\, |v_j|^2 dx + \tfrac{\alpha}{2}|v_1(0)|^2. \tag{2.15}$$

Namely, the vector solution $\mathbf{\Psi}_{\alpha,\delta}$ is the ground state in the sense of the minimality of $\mathbf{S}_{\alpha,\mathrm{Log}}(\mathbf{V})$ at $\mathbf{\Psi}_{\alpha,\delta}$ with the constraint given by the Nehari manifold $\mathcal{N}$, namely, $\mathbf{V} \in \mathcal{N}$ if and only if $\mathbf{V} \in W_{\mathcal{E}}(\mathcal{G}) \setminus \{0\}$ and

$$||\mathbf{V}'||^2 + \omega||\mathbf{V}||^2 - \sum_{j=1}^N \int_0^\infty |v_j|^2 \,\mathrm{Log}\, |v_j|^2 dx + \alpha|v_1(0)|^2 = 0.$$

In [15] the author proved that the standing wave $e^{i\omega t}\mathbf{\Psi}_{\alpha,\delta}$ is orbitally stable in $W_{\mathcal{E}}(\mathcal{G})$ for $\alpha < \alpha^*_{\mathrm{Log}} < 0$ and $\omega \in \mathbb{R}$. Below, we will overcome the restriction $\alpha < \alpha^*_{\mathrm{Log}}$ in the space $W^1_{\mathcal{E}}(\mathcal{G})$ (see Theorem 1.3), moreover, we will show spectral instability of the standing wave $e^{i\omega t}\mathbf{\Psi}_{\alpha,\delta}$ for any $\alpha > 0$ ($\mathbf{\Psi}_{\alpha,\delta}$ is the N-bump profile in this case).

Similarly, to the previous case, we show that the N-tail standing wave $e^{\omega i t}\mathbf{\Psi}_{\lambda,\delta'}$ for the NLS-log-δ' equation, where

$$\mathbf{\Psi}_{\lambda,\delta'} = (\psi_{\lambda,\delta'})_{j=1}^N, \quad \psi_{\lambda,\delta'} = e^{\frac{\omega+1}{2}} e^{-\frac{(x-\frac{N}{\lambda})^2}{2}}, \tag{2.16}$$

is orbitally stable in $W^1(\mathcal{G})$ for $-N < \lambda < 0$, and spectrally unstable for $\lambda < -N$ (see Theorem 1.4). Note that we do not need to assume that N is even to show the instability (compare with Theorem 1.2).

3. Stability theory of standing wave solutions for the NLS-δ and the NLS-δ' equations on a star graph

3.1. **The NLS-δ equation on a star graph.** In this Subsection, we study the orbital stability of the standing wave $\mathbf{U}(t,x) = e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}(x)$ of NLS-δ equation (2.2) with the particular N-bump profile $\mathbf{\Phi}_{\alpha,\delta} = (\varphi_{\alpha,\delta})_{j=1}^N$ defined by (2.9). As we are investigating orbital stability in $\mathcal{E}(\mathcal{G})$, we need to use the well-posedness of the initial value problem for equation (2.2) in this space. In [2] the authors established the results on local and global well-posedness of (2.2) in $\mathcal{E}(\mathcal{G})$. Below, we complete and extend these results, aiming to use them in the sequel for our instability analysis.

First, we establish the following property for the unitary group associated to (2.2).

Lemma 3.1. *Let $\{e^{-it\mathbf{H}_\delta^\alpha}\}_{t\in\mathbb{R}}$ be the family of unitary operators associated to NLS-δ model (2.2). Then, for every $\mathbf{V} = (v_j)_{i=1}^N \in \mathcal{E}(\mathcal{G})$, we have*

$$\partial_x(e^{-it\mathbf{H}_\delta^\alpha}\mathbf{V}) = -e^{-it\mathbf{H}_\delta^\alpha}\mathbf{V}' + \mathcal{B}(\mathbf{V}'), \tag{3.1}$$

where $\mathcal{B}(\mathbf{V}') = (2e^{it\partial_x^2}\tilde{v}_j)_{j=1}^N$, with $\tilde{v}_j(x) = \begin{cases} v_j'(x), & x \geq 0, \\ 0, & x < 0 \end{cases}$, and $e^{it\partial_x^2}$ is the unitary group associated with the free Schrödinger operator on $\mathbb{R}$.

Proof. Without loss of generality, we assume that $\alpha > 0$. Using functional calculus for unbounded self-adjoint operators and the classical expression for the resolvent of $-\frac{d^2}{dx^2}$ on the positive half-line, we get the formulas

$$e^{-it\mathbf{H}_\delta^\alpha}\mathbf{V}(x) = \tfrac{i}{\pi}\int_{-\infty}^{\infty} e^{-it\tau^2}\tau\mathbf{R}_{i\tau}\mathbf{V}(x)d\tau, \tag{3.2}$$

where $\mathbf{R}_\mu\mathbf{V} = (\mathbf{H}_\delta^\alpha + \mu^2 I)^{-1}\mathbf{V}$ has the components

$$(\mathbf{R}_\mu\mathbf{V})_j(x) = \tilde{c}_j e^{-\mu x} + \frac{1}{2\mu}\int_0^\infty v_j(y)e^{-|x-y|\mu}dy. \tag{3.3}$$

The coefficients $\tilde{c}_j$ are determined by the condition $\mathbf{R}_\mu\mathbf{V} \in \mathbf{D}_{\alpha,\delta}$. It is easily seen (e.g. [16, Appendix-6]) that $\mathbf{V} \in \mathbf{D}_{\alpha,\delta}$ iff $A\mathbf{V}(0) + B\mathbf{V}'(0) = \mathbf{0}$, where

$$A = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ \frac{\alpha}{N} & \frac{\alpha}{N} & \frac{\alpha}{N} & \dots & \frac{\alpha}{N} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \dots & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ & & \\ -1 & \dots & -1 \end{pmatrix}.$$

Let

$$t_j(\mu) = \frac{1}{2}\int_0^\infty v_j(y)e^{-\mu y}dy,$$

then from (3.3), we get $(\mathbf{R}_\mu \mathbf{V})_j(0) = \tilde{c}_j + \frac{1}{\mu}t_j(\mu)$ and $\partial_x[(\mathbf{R}_\mu\mathbf{V})_j](0) = -\mu\tilde{c}_j + t_j(\mu)$. Therefore, $(\tilde{c}_j)_{j=1}^N$ is the unique solution to the system

$$\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ \frac{\alpha}{N}+\mu & \frac{\alpha}{N}+\mu & \frac{\alpha}{N}+\mu & \dots & \frac{\alpha}{N}+\mu \end{pmatrix} \begin{pmatrix} \tilde{c}_1 \\ \\ \vdots \\ \\ \tilde{c}_N \end{pmatrix} = -\frac{1}{\mu} \begin{pmatrix} t_1(\mu)-t_2(\mu) \\ \vdots \\ \\ t_{N-1}(\mu)-t_N(\mu) \\ (\frac{\alpha}{N}-\mu)\sum_{j=1}^N t_j(\mu) \end{pmatrix}. \tag{3.4}$$

Below, we find $\mathbf{R}_\mu\mathbf{V}'$. Suppose initially that $v_j \in C_0^\infty(\mathbb{R}_+)$, $1 \le j \le N$, then there are coefficients $\tilde{d}_j$ such that

$$\begin{aligned} (\mathbf{R}_\mu\mathbf{V}')_j(x) &= \tilde{d}_j e^{-\mu x} + \frac{1}{2\mu}\int_0^\infty v_j'(y)e^{-\mu|x-y|}dy \\ &= \tilde{d}_j e^{-\mu x} - \frac{1}{2}\int_0^\infty v_j(y)\,\mathrm{sign}(x-y)e^{-\mu|x-y|}dy, \end{aligned} \tag{3.5}$$

where in the last equality, we have used integration by parts. Thus, we obtain $(\mathbf{R}_\mu\mathbf{V}')_j(0) = \tilde{d}_j + t_j(\mu)$. Moreover, since

$$\partial_x(\mathbf{R}_\mu\mathbf{V}')_j(x) = -\mu\tilde{d}_j e^{-\mu x} - \frac{1}{2}\int_0^\infty v_j'(y)\,\mathrm{sign}(x-y)e^{-\mu|x-y|}dy,$$

it follows from integration by parts $\partial_x(\mathbf{R}_\mu\mathbf{V}')_j(0) = -\mu\tilde{d}_j + \mu t_j(\mu)$. Hence, from the uniqueness of solution to system (3.4) it follows that $\mathbf{R}_\mu\mathbf{V}' \in \mathbf{D}_{\alpha,\delta}$ iff $\tilde{d}_j = \mu\tilde{c}_j$. Therefore, we obtain from (3.3) and the second equality in (3.5)

$$\begin{aligned} \partial_x(\mathbf{R}_\mu\mathbf{V})_j(x) &= -(\mathbf{R}_\mu\mathbf{V}')_j(x) - \int_0^\infty v_j(y)\,\mathrm{sign}(x-y)e^{-\mu|x-y|}dy \\ &= -(\mathbf{R}_\mu\mathbf{V}')_j(x) + \frac{1}{\mu}\int_0^\infty v_j'(y)e^{-\mu|x-y|}dy. \end{aligned}$$

Thus, from representation (3.2), we get

$$\partial_x(e^{-it\mathbf{H}_\lambda^{\delta'}}\mathbf{V}) = -e^{-it\mathbf{H}_\lambda^{\delta'}}\mathbf{V}' + \mathcal{B}(\mathbf{V}'),$$

where

$$(\mathcal{B}(\mathbf{V}'))_j(x) = \frac{1}{\pi}\int_{-\infty}^\infty e^{-it\tau^2}\int_0^\infty v_j'(y)e^{-i\tau|x-y|}dyd\tau.$$

Below, we find $\mathcal{B}(\mathbf{V}')$. It is well-known that $e^{it\partial_x^2}$ can be represented as $e^{it\partial_x^2}\phi = S_t * \phi$, where $\widehat{S}_t(\xi) = e^{-it\xi^2}$. Since for $t \neq 0$ and $x \in \mathbb{R}$

$$S_t(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\tau^2} e^{i\tau x} d\tau = \frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{-t}} e^{i\pi/4} e^{i\frac{x^2}{4t}} = \left(\frac{1}{4\pi i t}\right)^{1/2} e^{i\frac{x^2}{4t}},$$

it follows for $\phi(x) = \begin{cases} v_j'(x), & x \geq 0, \\ 0, & x < 0 \end{cases}$

$$\begin{aligned} I =& \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-it\tau^2}\int_{-\infty}^{\infty}\phi(y)\chi_{[0,x]}(y) e^{i\tau(y-x)} dy d\tau \\ =& 2\int_{-\infty}^{\infty}\phi(y)\chi_{[0,+\infty)}(x-y) S_t(x-y) dy = 2(\chi_{[0,+\infty)} S_t) * \phi(x). \end{aligned} \tag{3.6}$$

Similarly,

$$II = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-it\tau^2}\int_{-\infty}^{\infty}\phi(y)\chi_{[x,+\infty)}(y) e^{i\tau(x-y)} dy d\tau = 2(\chi_{(-\infty,0])} S_t) * \phi(x). \tag{3.7}$$

Thus, from (3.6)-(3.7), we have $(\mathcal{B}(\mathbf{V}'))_j(x) = I + II = 2S_t * \phi(x) = 2e^{it\partial_x^2}\phi(x)$. Hence relation (3.1) follows provided that each component of $\mathbf{V}$ has compact support. The general case follows from a density argument. $\square$

Remark 3.2. Observe that $e^{-it\mathbf{H}_\delta^\alpha}\mathbf{V} = e^{-it\mathbf{H}_\delta^\alpha}\mathbf{P}_c\mathbf{V} + e^{-it\mathbf{H}_\delta^\alpha}\mathbf{P}_p\mathbf{V}$, where $\mathbf{P}_c$ and $\mathbf{P}_p$ are L^2-orthogonal projections onto the subspaces corresponding to the continuous and the discrete spectral part of $\mathbf{H}_\delta^\alpha$. For $\alpha > 0$, we have $\sigma_c(\mathbf{H}_\delta^\alpha) = [0,\infty)$ and $\sigma_p(\mathbf{H}_\delta^\alpha) = \emptyset$, therefore, $\mathbf{P}_p = \mathbf{0}$. For $\alpha < 0$, $\sigma_c(\mathbf{H}_\delta^\alpha) = [0,\infty)$ and $\sigma_p(\mathbf{H}_\delta^\alpha) = \{-z_0^2\} = \{-\frac{\alpha^2}{N^2}\}$, where the corresponding eigenfunction is $\mathbf{V}_{z_0}(x) = (e^{\frac{\alpha}{N}x})_{j=1}^N$, and therefore $e^{-it\mathbf{H}_\delta^\alpha}\mathbf{P}_p\mathbf{V} = e^{itz_0^2}(\mathbf{V},\mathbf{V}_{z_0})\mathbf{V}_{z_0}$. In this case the formula (3.2) takes the form

$$e^{-it\mathbf{H}_\delta^\alpha}\mathbf{V}(x) = \tfrac{i}{\pi}\int_{-\infty}^{\infty} e^{-it\tau^2}\tau\mathbf{R}_{i\tau}\mathbf{V}(x) d\tau + e^{itz_0^2}(\mathbf{V},\mathbf{V}_{z_0})\mathbf{V}_{z_0}(x),$$

which however does not affect the proof of the well-posedness result. The proof of the spectral properties of $\mathbf{H}_\delta^\alpha$ repeats the one of [5, Theorem 3.1.4] for the case of the Schrödinger operator with the δ-interaction on the line. In particular, to describe the point spectrum for $\alpha < 0$ one needs to consider $\mathbf{H}_\delta^\alpha$ as the self-adjoint extension of the symmetric non-negative operator $\mathbf{L}$ defined by (3.17) with deficiency indices $n_\pm(\mathbf{L}) = 1$ and then to apply Proposition 3.9.

Lemma 3.3. *The family of unitary operators* $\{e^{-it\mathbf{H}_\delta^\alpha}\}_{t\in\mathbb{R}}$ *on* $L^2(\mathcal{G})$ *preserves the space* $\mathcal{E}(\mathcal{G})$*, i.e., for* $\mathbf{V} \in \mathcal{E}(\mathcal{G})$*, we have* $e^{-it\mathbf{H}_\delta^\alpha}\mathbf{V} \in \mathcal{E}(\mathcal{G})$.

Proof. Assume $\alpha > 0$. Let $\mathbf{V} \in \mathcal{E}(\mathcal{G})$, then it follows from (3.1) that $e^{-it\mathbf{H}_\delta^\alpha}\mathbf{V} \in H^1(\mathcal{G})$. Further, since $\mathbf{R}_\mu \mathbf{V} \in \mathbf{D}_{\alpha,\delta}$, we get from (3.2) the equality $(e^{-it\mathbf{H}_\delta^\alpha}\mathbf{V})_1(0) = \cdots = (e^{-it\mathbf{H}_\delta^\alpha}\mathbf{V})_N(0)$. □

Theorem 3.4. *Let $p > 1$. Then for any $\mathbf{U}_0 \in \mathcal{E}(\mathcal{G})$ there exists $T > 0$ such that equation (2.2) has a unique solution $\mathbf{U} \in C([-T,T], \mathcal{E}(\mathcal{G})) \cap C^1([-T,T], \mathcal{E}'(\mathcal{G}))$ satisfying $\mathbf{U}(0) = \mathbf{U}_0$. For each $T_0 \in (0,T)$ the mapping $\mathbf{U}_0 \in \mathcal{E}(\mathcal{G}) \to \mathbf{U} \in C([-T_0,T_0], \mathcal{E}(\mathcal{G}))$, is continuous. In particular, for $p > 2$ this mapping is at least of class C^2. Moreover, if $\mathbf{U}_0 \in \mathcal{E}_k(\mathcal{G})$, then $\mathbf{U}(t) \in \mathcal{E}_k(\mathcal{G})$ for all $t \in [-T,T]$.*

Proof. The local well-posedness result in $\mathcal{E}(\mathcal{G})$ follows from standard arguments of the Banach fixed point theorem applied to non-linear Schrödinger equations (see [27]). We will give the sketch of the proof for the case $\alpha > 0$. Consider the mapping $J_{\mathbf{U}_0} : C([-T,T], \mathcal{E}(\mathcal{G})) \longrightarrow C([-T,T], \mathcal{E}(\mathcal{G}))$ given by

$$J_{\mathbf{U}_0}[\mathbf{U}](t) = e^{-it\mathbf{H}_\delta^\alpha}\mathbf{U}_0 + i\int_0^t e^{-i(t-s)\mathbf{H}_\delta^\alpha}|\mathbf{U}(s)|^{p-1}\mathbf{U}(s)ds,$$

where $e^{-it\mathbf{H}_\delta^\alpha}$ is the unitary group given by (3.2). One needs to show that the mapping $J_{\mathbf{U}_0}$ is well-defined. To do this it is necessary to estimate initially the nonlinear term $|\mathbf{U}(s)|^{p-1}\mathbf{U}(s)$. Using the one-dimensional Gagliardo-Nirenberg inequality one may show (see formula (2.3) in [2])

$$\|\mathbf{U}\|_q \leq C\|\mathbf{U}'\|^{\frac{1}{2}-\frac{1}{q}}\|\mathbf{U}\|^{\frac{1}{2}+\frac{1}{q}}, \quad q > 2,\ C > 0. \tag{3.8}$$

Using (3.8), the relation $|(|f|^{p-1}f)'| \leq C_0|f|^{p-1}|f'|$ and Hölder's inequality, we obtain for $\mathbf{U} \in H^1(\mathcal{G})$

$$|||\mathbf{U}|^{p-1}\mathbf{U}||_{H^1(\mathcal{G})} \leq C_1||\mathbf{U}||^p_{H^1(\mathcal{G})}. \tag{3.9}$$

Let $\mathbf{U}_0, \mathbf{U} \in \mathcal{E}(\mathcal{G})$, then from Lemmas 3.1-3.3 and (3.9) it follows that $J_{\mathbf{U}_0}[\mathbf{U}](t) \in \mathcal{E}(\mathcal{G})$. Moreover, using (3.1), (3.9), L^2-unitarity of $e^{-it\mathbf{H}_\delta^\alpha}$ and $e^{it\partial_x^2}$, we get

$$||J_{\mathbf{U}_0}[\mathbf{U}](t)||_{H^1(\mathcal{G})} \leq C_2||\mathbf{U}_0||_{H^1(\mathcal{G})} + C_3T\sup_{s\in[0,T]}||\mathbf{U}(s)||^p_{H^1(\mathcal{G})},$$

where the positive constants C_2, C_3 do not depend on $\mathbf{U}_0$. The continuity and contraction property of $J_{\mathbf{U}_0}$ are proved in a standard way. Therefore, we obtain the existence of a unique solution to the Cauchy problem associated to (2.2) on $\mathcal{E}(\mathcal{G})$.

Next, we recall that the argument based on the contraction mapping principle above has the advantage that if the nonlinearity $F(\mathbf{U}, \overline{\mathbf{U}}) = |\mathbf{U}|^{p-1}\mathbf{U}$ has a specific regularity, then it is inherited by the mapping data-solution. In particular, following the ideas in the proof of [41, Corollary 5.6], we consider for $(\mathbf{V}_0, \mathbf{V}) \in B(\mathbf{U}_0; \epsilon) \times C([-T, T], \mathcal{E}(\mathcal{G}))$ the mapping

$$\Gamma(\mathbf{V}_0, \mathbf{V})(t) = \mathbf{V}(t) - J_{\mathbf{V}_0}[\mathbf{V}](t), \qquad t \in [-T, T].$$

Then $\Gamma(\mathbf{U}_0, \mathbf{U})(t) = 0$ for all $t \in [-T, T]$. For $p - 1$ being an even integer, $F(\mathbf{U}, \overline{\mathbf{U}})$ is smooth, and therefore Γ is smooth. Hence, using the arguments applied for obtaining the local well-posedness in $\mathcal{E}(\mathcal{G})$ above, we can show that the operator $\partial_{\mathbf{V}}\Gamma(\mathbf{U}_0, \mathbf{U})$ is one-to-one and onto. Thus, by the Implicit Function Theorem there exists a smooth mapping $\mathbf{\Lambda} : B(\mathbf{U}_0; \delta) \to C([-T, T], \mathcal{E}(\mathcal{G}))$ such that $\Gamma(\mathbf{V}_0, \mathbf{\Lambda}(\mathbf{V}_0)) = 0$ for all $\mathbf{V}_0 \in B(\mathbf{U}_0; \delta)$. This argument establishes the smoothness property of the mapping data-solution associated to equation (2.4) when $p - 1$ is an even integer.

If $p - 1$ is not an even integer and $p > 2$, then $F(\mathbf{U}, \overline{\mathbf{U}})$ is $C^{[p]}$-function, and consequently the mapping data-solution is of class $C^{[p]}$ (see [41, Remark 5.7]). Therefore, for $p > 2$, we conclude that the mapping data-solution is at least of class C^2.

Next, we show that the unitary group $e^{-it\mathbf{H}_\delta^\alpha}$ preserves the subspace $\mathcal{E}_k(\mathcal{G})$. Indeed, let $\mathbf{V} = (v_j) \in \mathcal{E}_k(\mathcal{G})$, then we obtain $t_1(\mu) = \cdots = t_k(\mu)$ and $t_{k+1}(\mu) = \cdots = t_N(\mu)$, where $t_j(\mu) = \frac{1}{2}\int_0^\infty v_j(y)e^{-\mu y}dy$. Hence, from (3.4) it follows $\tilde{c}_1 = \cdots = \tilde{c}_k$ and $\tilde{c}_{k+1} = \cdots = \tilde{c}_N$. Thus, by (3.2), we get $e^{-it\mathbf{H}_\delta^\alpha}\mathbf{V} \in \mathcal{E}_k(\mathcal{G})$. Lastly, the well-posedness in $\mathcal{E}_k(\mathcal{G})$ follows from the uniqueness of the solution to the Cauchy problem in $\mathcal{E}(\mathcal{G})$ and the invariance of the space $\mathcal{E}_k(\mathcal{G})$ for the unitary group $e^{-it\mathbf{H}_\delta^\alpha}$ shown above. □

Remark 3.5. (i) In [2, Proposition 2.2] the authors proved that for any solution to Cauchy problem associated with (2.2), the conservation of charge and energy hold, i.e.,

$$Q(\mathbf{U}(t)) = ||\mathbf{U}(t)||^2 = ||\mathbf{U}_0||^2, \text{ and } E_\alpha(\mathbf{U}(t)) = E_\alpha(\mathbf{U}_0),\ t \in [-T, T],$$

where E_α is defined for $\mathbf{V} = (v_j)_{j=1}^N \in \mathcal{E}(\mathcal{G})$ by

$$E_\alpha(\mathbf{V}) = \tfrac{1}{2}||\mathbf{V}'||^2 - \tfrac{1}{p+1}||\mathbf{V}||_{p+1}^{p+1} + \tfrac{\alpha}{2}\left|v_1(0)\right|^2.$$

Using the Sobolev embedding theorem, Gagliardo-Nirenberg inequality (3.8), the above conservation laws, one can induce global well-posedness of (2.2) for $1 < p < 5$ (i.e., we can choose $T = +\infty$).

(ii) Observe that $E_\alpha \in C^2(\mathcal{E}(\mathcal{G}), \mathbb{R})$ since $p > 1$. This fact allows us to apply the results by Ohta [45] in our instability analysis.

Next, we introduce the basic objects of the classical theory by Grillakis, Shatah and Strauss. Consider the following two self-adjoint matrix operators associated with $\mathbf{\Phi}_{\alpha,\delta} = (\varphi_{\alpha,\delta})_{j=1}^N$

$$\mathbf{L}_{1,\alpha} = \Big(\Big(-\frac{d^2}{dx^2} + \omega - p(\varphi_{\alpha,\delta})^{p-1}\Big)\delta_{k,j}\Big),$$
$$\mathbf{L}_{2,\alpha} = \Big(\Big(-\frac{d^2}{dx^2} + \omega - (\varphi_{\alpha,\delta})^{p-1}\Big)\delta_{k,j}\Big),$$
$$\mathrm{dom}(\mathbf{L}_{1,\alpha}) = \mathrm{dom}(\mathbf{L}_{2,\alpha}) = \mathbf{D}_{\alpha,\delta},$$

where $\delta_{k,j}$ is the Kronecker symbol, $\mathbf{D}_{\alpha,\delta}$ and $\varphi_{\alpha,\delta}$ are defined by (2.3) and (2.9). The operators $\mathbf{L}_{1,\alpha}$ and $\mathbf{L}_{2,\alpha}$ are associated with the functional $\mathbf{S}_\alpha$ defined by (2.10) via the following equality

$$(\mathbf{S}_\alpha)''(\mathbf{\Phi}_{\alpha,\delta})(\mathbf{U},\mathbf{V}) = (\mathbf{L}_{1,\alpha}\mathbf{U}_1, \mathbf{V}_1) + (\mathbf{L}_{2,\alpha}\mathbf{U}_2, \mathbf{V}_2),$$

where $\mathbf{U} = \mathbf{U}_1 + i\mathbf{U}_2$ and $\mathbf{V} = \mathbf{V}_1 + i\mathbf{V}_2$. The vector functions $\mathbf{U}_j, \mathbf{V}_j$, $j \in \{1,2\}$, are assumed to be real valued.

Formally $(\mathbf{S}_\alpha)''(\mathbf{\Phi}_{\alpha,\delta})$ can be considered as a self-adjoint $2N \times 2N$ matrix operator (see [33, 34] for the details) $\mathbf{H}_\alpha = \begin{pmatrix} \mathbf{L}_{1,\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{2,\alpha} \end{pmatrix}$. Define

$$p(\omega_0) = \begin{cases} 1 & \text{if } \partial_\omega ||\mathbf{\Phi}_{\alpha,\delta}||^2 > 0 \text{ at } \omega = \omega_0, \\ 0 & \text{if } \partial_\omega ||\mathbf{\Phi}_{\alpha,\delta}||^2 < 0 \text{ at } \omega = \omega_0. \end{cases}$$

Having established *Assumptions 1, 2* in [33], i.e., well-posedness of the associated Cauchy problem (see Theorem 3.4) and the existence of C^1 in ω standing wave, the next stability/instability result follows from [33, Theorem 3] and [45, Corollary 3 and 4].

Theorem 3.6. *Let $\alpha \neq 0$, $\omega > \frac{\alpha^2}{N^2}$, and $n(\mathbf{H}_\alpha)$ be the number of negative eigenvalues of $\mathbf{H}_\alpha$. Suppose also that*

1) $\ker(\mathbf{L}_{2,\alpha}) = \mathrm{span}\{\mathbf{\Phi}_{\alpha,\delta}\}$,
2) $\ker(\mathbf{L}_{1,\alpha}) = \{\mathbf{0}\}$,
3) *the negative spectrum of $\mathbf{L}_{1,\alpha}$ and $\mathbf{L}_{2,\alpha}$ consists of a finite number of negative eigenvalues (counting multiplicities),*
4) *the rest of the spectrum of $\mathbf{L}_{1,\alpha}$ and $\mathbf{L}_{2,\alpha}$ is positive and bounded away from zero. Then the following assertions hold.*

(*i*) *If $n(\mathbf{H}_\alpha) = p(\omega) = 1$, then the standing wave $e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}$ is orbitally stable in $\mathcal{E}(\mathcal{G})$.*
(*ii*) *If $n(\mathbf{H}_\alpha) - p(\omega) = 1$ in $L^2_k(\mathcal{G})$, then the standing wave $e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}$ is orbitally unstable in $\mathcal{E}_k(\mathcal{G})$ and, consequently, in $\mathcal{E}(\mathcal{G})$.*

Remark 3.7. The instability part of the above theorem needs some additional comments.

(i) It is known from [34] that when $n(\mathbf{H}_\alpha) - p(\omega)$ is odd, we obtain only spectral instability of $e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}$. To obtain orbital instability due to [34, Theorem 6.1], it is sufficient to show estimate (6.2) in [34] for the semigroup $e^{t\mathbf{A}_\alpha}$ generated by $\mathbf{A}_\alpha = \begin{pmatrix} \mathbf{0} & \mathbf{L}_{2,\alpha} \\ -\mathbf{L}_{1,\alpha} & \mathbf{0} \end{pmatrix}$. In our particular case it is not clear how to prove estimate (6.2).

(ii) When $n(\mathbf{H}_\alpha) = 2$ (which happens for $\alpha > 0$), we can apply the results by Ohta [45, Corollary 3 and 4] to get the instability part of the above Theorem. We note that in this case the orbital instability follows without using spectral instability.

(iii) Generally, to imply the orbital instability from the spectral one, the approach by [35] can be used (see Theorem 2). The key point of this method is to use the fact that the mapping data-solution associated to the model is of class C^2. In particular, for the NLS-δ and NLS-δ' models the mapping data-solution is of class C^2 as $p > 2$ (see Theorem 3.4 and 3.22). The approach by [35] have been applied successfully in [13] and [14] for the models of KdV-type.

Below, we describe the spectrum of the operators $\mathbf{L}_{1,\alpha}$ and $\mathbf{L}_{2,\alpha}$ which will help us to verify the conditions of Theorem 3.6. Our ideas are based on the extension theory of symmetric operators and the perturbation theory. For convenience of the reader and for the future references, we formulate the following extension theory results (see [43, Chapter IV, §14]).

Proposition 3.8. (von Neumann decomposition) *Let A be a closed densely defined symmetric operator. Then the following decomposition holds*

$$\mathrm{dom}(A^*) = \mathrm{dom}(A) \oplus \mathcal{N}_+(A) \oplus \mathcal{N}_-(A). \tag{3.10}$$

Therefore, for $u \in \mathrm{dom}(A^*)$ such that $u = f + f_i + f_{-i}$, with $f \in \mathrm{dom}(A)$, $f_{\pm i} \in \mathcal{N}_\pm(A)$, we get $A^*u = Af + if_i - if_{-i}$.

Proposition 3.9. *Let A be a densely defined lower semi-bounded symmetric operator (that is, $A \geq mI$) with finite deficiency indices $n_\pm(A) = k < \infty$ in the Hilbert space $\mathcal{H}$, and let $\widetilde{A}$ be a self-adjoint extension of A. Then the spectrum of $\widetilde{A}$ in $(-\infty, m)$ is discrete and consists of at most k eigenvalues counting multiplicities.*

Remark 3.10. When $m = 0$, Proposition 3.9 provides an estimate for $n(\widetilde{A})$.

Below, using the perturbation theory, we show the equality $n(\mathbf{L}_{1,\alpha}) = 2$ in the space $L^2_k(\mathcal{G})$ for any $k \in \{1, ..., N-1\}$, i.e., $n(\mathbf{L}_{1,\alpha}|_{L^2_k(\mathcal{G})}) = 2$. For this purpose let us define the following self-adjoint matrix Schrödinger operator on $L^2(\mathcal{G})$ with Kirchhoff condition at $\nu = 0$

$$\mathbf{L}_{1,0} = \Big(\Big(-\frac{d^2}{dx^2} + \omega - p\varphi_0^{p-1}\Big)\delta_{i,j}\Big), \tag{3.11}$$
$$\mathrm{dom}(\mathbf{L}_{1,0}) = \Big\{\mathbf{V} \in H^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0),\ \sum_{j=1}^{N} v_j'(0) = 0\Big\},$$

where $\varphi_0 = \left[\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}x\right)\right]^{\frac{1}{p-1}}$, $x > 0$, is the half-soliton for the classical NLS model (1.1).

Let $\mathbf{\Phi}_0 = (\varphi_0)_{j=1}^N$, then it is not difficult to see that $\mathbf{\Phi}_{\alpha,\delta} \to \mathbf{\Phi}_0$, as $\alpha \to 0$, in $H^1(\mathcal{G})$. The following lemma states the analyticity of the family of operators $(\mathbf{L}_{1,\alpha})$.

Lemma 3.11. *As a function of α, $(\mathbf{L}_{1,\alpha})$ is real-analytic family of self-adjoint operators of type (B) in the sense of Kato.*

Proof. By [39, Theorem VII-4.2], it suffices to note that the family of bilinear forms $(B_{1,\alpha})$ defined for $\mathbf{U} = (u_j)_{j=1}^N, \mathbf{V} = (v_j)_{j=1}^N \in \mathcal{E}(\mathcal{G})$ by

$$B_{1,\alpha}(\mathbf{U},\mathbf{V}) = \sum_{j=1}^{N}\int_0^\infty (u_j'v_j' + \omega u_j v_j - p(\varphi_{\alpha,\delta})^{p-1}u_j v_j)dx + \alpha u_1(0)v_1(0),$$

is real-analytic of type (B). $\square$

As we intend to study the negative spectrum of $\mathbf{L}_{1,\alpha}$ using perturbation theory, we need to describe spectral properties of $\mathbf{L}_{1,0}$ (which is a "limit value" of $\mathbf{L}_{1,\alpha}$ as $\alpha \to 0$).

Theorem 3.12. *Let $\mathbf{L}_{1,0}$ be defined by* (3.11) *and $k \in \{1, ..., N-1\}$. Then the following assertions hold.*

(*i*) $\ker(\mathbf{L}_{1,0}) = \mathrm{span}\{\hat{\mathbf{\Phi}}_{0,1}, ..., \hat{\mathbf{\Phi}}_{0,N-1}\}$, *where*

$$\hat{\mathbf{\Phi}}_{0,j} = (0, ..., 0, \underset{\mathbf{j}}{\varphi_0'}, \underset{\mathbf{j+1}}{-\varphi_0'}, 0, ..., 0).$$

(*ii*) *In the space $L_k^2(\mathcal{G})$, we have* $\ker(\mathbf{L}_{1,0}) = \mathrm{span}\{\widetilde{\mathbf{\Phi}}_{0,k}\}$, *where*

$$\widetilde{\mathbf{\Phi}}_{0,k} = \Big(\underset{\mathbf{1}}{\tfrac{N-k}{k}\varphi_0'}, ..., \underset{\mathbf{k}}{\tfrac{N-k}{k}\varphi_0'}, \underset{\mathbf{k+1}}{-\varphi_0'}, ..., \underset{\mathbf{N}}{-\varphi_0'}\Big), \tag{3.12}$$

i.e., $\ker(\mathbf{L}_{1,0}|_{L_k^2(\mathcal{G})}) = \mathrm{span}\{\widetilde{\mathbf{\Phi}}_{0,k}\}$.

(*iii*) *The operator* $\mathbf{L}_{1,0}$ *has one simple negative eigenvalue in* $L^2(\mathcal{G})$*, i.e.,* $n(\mathbf{L}_{1,0}) = 1$*. Moreover,* $\mathbf{L}_{1,0}$ *has one simple negative eigenvalue in* $L^2_k(\mathcal{G})$ *for any* k*, i.e.,* $n(\mathbf{L}_{1,0}|_{L^2_k(\mathcal{G})}) = 1$*.*

(*iv*) *The rest of the spectrum of* $\mathbf{L}_{1,0}$ *is positive and bounded away from zero.*

Proof. (*i*) Recall that the only $L^2(\mathbb{R}_+)$-solution to the equation

$$-v_j'' + \omega v_j - p\varphi_0^{p-1} v_j = 0$$

is $v_j = \varphi_0'$ (up to a factor). Thus, any element of $\ker(\mathbf{L}_{1,0})$ has the form $\mathbf{V} = (v_j)_{j=1}^N = (c_j \varphi_0')_{j=1}^N$, $c_j \in \mathbb{R}$. The continuity condition is satisfied since $\varphi_0'(0) = 0$. Condition $\sum_{j=1}^N v_j'(0) = 0$ gives rise to $(N-1)$-dimensional kernel of $\mathbf{L}_{1,0}$. It is easily seen that the functions $\hat{\mathbf{\Phi}}_{0,j}$, $j = 1, ..., N-1$, form basis there.

(*ii*) Arguing as in the previous item, we can see that $\ker(\mathbf{L}_{1,0})$ is one-dimensional in $L^2_k(\mathcal{G})$, and it is spanned on $\widetilde{\mathbf{\Phi}}_{0,k}$.

(*iii*) The main idea of the proof is to apply Proposition 3.9. In what follows, we use the notation $\boldsymbol{l}_0 = \Big(\Big(-\frac{d^2}{dx^2} + \omega - p\varphi_0^{p-1}\Big)\delta_{k,j}\Big)$. First, note that $\mathbf{L}_{1,0}$ is the self-adjoint extension of the following symmetric operator (see Remark 3.14)

$$\mathbf{L}_0 = \boldsymbol{l}_0,$$
$$\operatorname{dom}(\mathbf{L}_0) = \Big\{\mathbf{V} \in H^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0) = 0, \sum_{j=1}^N v_j'(0) = 0\Big\}. \tag{3.13}$$

Below, we show that the operator $\mathbf{L}_0$ is non-negative, and $n_\pm(\mathbf{L}_0) = 1$. First, let us show that the adjoint operator of $\mathbf{L}_0$ is given by

$$\mathbf{L}_0^* = \boldsymbol{l}_0, \quad \operatorname{dom}(\mathbf{L}_0^*) = \left\{\mathbf{V} \in H^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0)\right\}. \tag{3.14}$$

Using standard arguments one can prove that $\operatorname{dom}(\mathbf{L}_0^*) \subset H^2(\mathcal{G})$ and $\mathbf{L}_0^* = \boldsymbol{l}_0$ (see [43, Chapter V,§17]). Denoting

$$D_0^* := \{\mathbf{V} \in H^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0)\},$$

we easily arrive at $D_0^* \subseteq \operatorname{dom}(\mathbf{L}_0^*)$. Indeed, for any $\mathbf{U} = (u_j)_{j=1}^N \in D_0^*$ and $\mathbf{V} = (v_j)_{j=1}^N \in \operatorname{dom}(\mathbf{L}_0)$ denoting $\mathbf{U}^* = \boldsymbol{l}_0(\mathbf{U}) \in L^2(\mathcal{G})$, we get

$$(\mathbf{L}_0\mathbf{V}, \mathbf{U}) = (\mathbf{V}, \boldsymbol{l}_0(\mathbf{U})) + \sum_{j=1}^N \left[-v_j' u_j + v_j u_j'\right]_0^\infty = (\mathbf{V}, \boldsymbol{l}_0(\mathbf{U})) = (\mathbf{V}, \mathbf{U}^*),$$

which, by definition of the adjoint operator, means that $\mathbf{U} \in \mathrm{dom}(\mathbf{L}_0^*)$ or $D_0^* \subseteq \mathrm{dom}(\mathbf{L}_0^*)$.

Let us show the inverse inclusion $D_0^* \supseteq \mathrm{dom}(\mathbf{L}_0^*)$. Take $\mathbf{U} \in \mathrm{dom}(\mathbf{L}_0^*)$, then for any $\mathbf{V} \in \mathrm{dom}(\mathbf{L}_0)$, we have

$$(\mathbf{L}_0\mathbf{V}, \mathbf{U}) = (\mathbf{V}, \boldsymbol{l}_0(\mathbf{U})) + \sum_{j=1}^{N} \left[-v_j' u_j + v_j u_j'\right]_0^\infty = (\mathbf{V}, \mathbf{L}_0^*\mathbf{U}) = (\mathbf{V}, \boldsymbol{l}_0(\mathbf{U})).$$

Thus, we arrive at the equality

$$\sum_{j=1}^{N} \left[-v_j' u_j + v_j u_j'\right]_0^\infty = \sum_{j=1}^{N} v_j'(0) u_j(0) = 0 \tag{3.15}$$

for any $\mathbf{V} \in \mathrm{dom}(\mathbf{L}_0)$. Let $\mathbf{W} = (w_j)_{j=1}^N \in \mathrm{dom}(\mathbf{L}_0)$ such that $w_3'(0) = w_4'(0) = \cdots = w_N'(0) = 0$. Then for $\mathbf{U} \in \mathrm{dom}(\mathbf{L}_0^*)$ from (3.15) it follows that

$$\sum_{j=1}^{N} w_j'(0) u_j(0) = w_1'(0) u_1(0) + w_2'(0) u_2(0) = 0. \tag{3.16}$$

Recalling that $\sum_{j=1}^N w_j'(0) = w_1'(0) + w_2'(0) = 0$ and assuming $w_2'(0) \neq 0$, we obtain from (3.16) the equality $u_1(0) = u_2(0)$. Repeating the similar arguments for $\mathbf{W} = (w_j)_{j=1}^N \in \mathrm{dom}(\mathbf{L}_0)$ such that $w_4'(0) = w_5'(0) = \cdots = w_N'(0) = 0$, we get $u_1(0) = u_2(0) = u_3(0)$ and so on. Finally taking $\mathbf{W} = (w_j)_{j=1}^N \in \mathrm{dom}(\mathbf{L}_0)$ such that $w_N'(0) = 0$, we arrive at $u_1(0) = u_2(0) = \cdots = u_{N-1}(0)$, and consequently $u_1(0) = u_2(0) = \cdots = u_N(0)$. Thus, $\mathbf{U} \in D_0^*$ or $D_0^* \supseteq \mathrm{dom}(\mathbf{L}_0^*)$, and (3.14) holds.

Let us show that the operator $\mathbf{L}_0$ is non-negative. First, note that every component of the vector $\mathbf{V} = (v_j)_{j=1}^N \in H^2(\mathcal{G})$ satisfies the following identity

$$-v_j'' + \omega v_j - p\varphi_0^{p-1} v_j = \frac{-1}{\varphi_0'} \frac{d}{dx}\Big[(\varphi_0')^2 \frac{d}{dx}\Big(\frac{v_j}{\varphi_0'}\Big)\Big], \quad x > 0.$$

Using the above equality and integrating by parts, we get for $\mathbf{V} \in \mathrm{dom}(\mathbf{L}_0)$

$$\begin{aligned}(\mathbf{L}_0\mathbf{V}, \mathbf{V}) &= \sum_{j=1}^{N} \int_0^\infty (\varphi_0')^2 \Big[\frac{d}{dx}\Big(\frac{v_j}{\varphi_0'}\Big)\Big]^2 dx + \sum_{j=1}^{N} \Big[-v_j' v_j + v_j^2 \frac{\varphi_0''}{\varphi_0'}\Big]_0^\infty \\ &= \sum_{j=1}^{N} \int_0^\infty (\varphi_0')^2 \Big[\frac{d}{dx}\Big(\frac{v_j}{\varphi_0'}\Big)\Big]^2 dx \geq 0.\end{aligned}$$

Note that the equality

$$\sum_{j=1}^{N}\Big[-v_j'v_j+v_j^2\frac{\varphi_0''}{\varphi_0'}\Big]_0^\infty=0$$

follows from the condition $v_j(0)=0$ and the fact that $x=0$ is the first-order zero for $\varphi_0'(x)$ (i.e., $\varphi_0''(0)\neq 0$).

Due to the von Neumann decomposition (3.10),

$$\begin{aligned}\mathrm{dom}(\mathbf{L}_0^*)&=\big\{\mathbf{V}\in H^2(\mathcal{G}): v_1(0)=\cdots=v_N(0)\big\}\\ &=\mathrm{dom}(\mathbf{L}_0)\oplus\mathrm{span}\{\mathbf{V}_i\}\oplus\mathrm{span}\{\mathbf{V}_{-i}\},\end{aligned}$$

where $\mathbf{V}_{\pm i}=(e^{i\sqrt{\pm i}x})_{j=1}^N$, $\Im(\sqrt{\pm i})>0$. Indeed, since $\varphi_0\in L^\infty(\mathbb{R}_+)$, it follows $\mathrm{dom}(\mathbf{L}_0^*)=\mathrm{dom}(\mathbf{L}^*)=\mathrm{dom}(\mathbf{L})\oplus\mathrm{span}\{\mathbf{V}_i\}\oplus\mathrm{span}\{\mathbf{V}_{-i}\}$, where

$$\mathbf{L}=\Big(\Big(-\frac{d^2}{dx^2}\Big)\delta_{k,j}\Big),\quad \mathrm{dom}(\mathbf{L})=\mathrm{dom}(\mathbf{L}_0),\quad \mathcal{N}_\pm(\mathbf{L})=\mathrm{span}\{\mathbf{V}_{\pm i}\}. \tag{3.17}$$

Since $n_\pm(\mathbf{L})=1$, by [43, Chapter IV, Theorem 6], it follows that $n_\pm(\mathbf{L}_0)=1$.

Due to Proposition 3.9, $n(\mathbf{L}_{1,0})\leq 1$. For $\mathbf{\Phi}_0=(\varphi_0)_{j=1}^N$, we obviously have $(\mathbf{L}_{1,0}\mathbf{\Phi}_0,\mathbf{\Phi}_0)=-(p-1)||\mathbf{\Phi}_0||_{p+1}^{p+1}<0$. By minimax principle, we arrive at $n(\mathbf{L}_{1,0})=1$. Noting that $\mathbf{\Phi}_0\in L^2_k(\mathcal{G})$ for any k, we get $n_\pm(\mathbf{L}_0|_{L^2_k(\mathcal{G})})=1$.

(iv) By Weyl's theorem (see [46, Theorem XIII.14]), the essential spectrum of $\mathbf{L}_{1,0}$ coincides with $[\omega,\infty)$. Since $\mathbf{\Phi}_0\in L^\infty(\mathcal{G})$, there can be only finitely many isolated eigenvalues in $(-\infty,\omega')$ for any $\omega'<\omega$. Then (iv) follows easily. □

Remark 3.13. Observe that, when we deal with deficiency indices, the operator $\mathbf{L}_0$ is assumed to act on complex-valued functions which however does not affect the analysis of negative spectrum of $\mathbf{L}_{1,0}$ acting on real-valued functions.

Remark 3.14. Let us show that the domain of any self-adjoint extension $\widehat{\mathbf{L}}$ of the operator $\mathbf{L}_0$ defined by (3.13)(and acting on complex-valued functions) is given by

$$\mathrm{dom}(\widehat{\mathbf{L}})=\Big\{\mathbf{V}\in H^2(\mathcal{G}): v_1(0)=\cdots=v_N(0),\sum_{j=1}^N v_j'(0)=zv_1(0),\ z\in\mathbb{R}\Big\}.$$

Indeed, due to [5, Theorem A.1],

$$\mathrm{dom}(\widehat{\mathbf{L}})=\Big\{\mathbf{F}=\mathbf{F}_0+c\mathbf{F}_i+ce^{i\theta}\mathbf{F}_{-i}:\ \mathbf{F}_0\in\mathrm{dom}(\mathbf{L}_0),c\in\mathbb{C},\theta\in[0,2\pi)\Big\},$$

where $\mathbf{F}_{\pm i} = (\frac{i}{\sqrt{\pm i}} e^{i\sqrt{\pm i}x})_{j=1}^N$, $\Im(\sqrt{\pm i}) > 0$. It is easily seen that for $\mathbf{F} \in \mathrm{dom}(\widehat{\mathbf{L}})$, we have

$$\sum_{j=1}^N (\mathbf{F})_j'(0) = -Nc(1+e^{i\theta}), \quad (\mathbf{F})_j(0) = c\left(e^{i\pi/4} + e^{i(\theta-\pi/4)}\right).$$

From the last equalities it follows that

$$\sum_{j=1}^N (\mathbf{F})_j'(0) = z(\mathbf{F})_1(0), \text{ where } z = \frac{-N(1+e^{i\theta})}{\left(e^{i\pi/4} + e^{i(\theta-\pi/4)}\right)} \in \mathbb{R}.$$

Combining Lemma 3.11 and Theorem 3.12, in the framework of the perturbation theory, we obtain the following proposition.

Proposition 3.15. *Let $k \in \{1, ..., N-1\}$. Then there exist $\alpha_0 > 0$ and two analytic functions $\mu : (-\alpha_0, \alpha_0) \to \mathbb{R}$ and $\mathbf{F} : (-\alpha_0, \alpha_0) \to L_k^2(\mathcal{G})$ such that*

(*i*) *$\mu(0) = 0$ and $\mathbf{F}(0) = \widetilde{\mathbf{\Phi}}_{0,k}$, where $\widetilde{\mathbf{\Phi}}_{0,k}$ is defined by (3.12).*
(*ii*) *For all $\alpha \in (-\alpha_0, \alpha_0)$, $\mu(\alpha)$ is the simple isolated second eigenvalue of $\mathbf{L}_{1,\alpha}$ in $L_k^2(\mathcal{G})$, and $\mathbf{F}(\alpha)$ is the associated eigenvector for $\mu(\alpha)$.*
(*iii*) *α_0 can be chosen small enough to ensure that for $\alpha \in (-\alpha_0, \alpha_0)$ the spectrum of $\mathbf{L}_{1,\alpha}$ in $L_k^2(\mathcal{G})$ is positive, except at most the first two eigenvalues.*

Proof. Using the spectral structure of the operator $\mathbf{L}_{1,0}$ (see Theorem 3.12), we can separate the spectrum $\sigma(\mathbf{L}_{1,0})$ into two parts $\sigma_0 = \{\mu_{1,0}^0, 0\}$ and σ_1 by a closed curve Γ (for example, a circle), such that σ_0 belongs to the inner domain of Γ and σ_1 to the outer domain of Γ (note that $\sigma_1 \subset (\epsilon, +\infty)$ for $\epsilon > 0$). Next, Lemma 3.11 and the analytic perturbations theory imply that $\Gamma \subset \rho(\mathbf{L}_{1,\alpha})$ for sufficiently small $|\alpha|$, and $\sigma(\mathbf{L}_{1,\alpha})$ is likewise separated by Γ into two parts, such that the part of $\sigma(\mathbf{L}_{1,\alpha})$ inside Γ consists of a finite number of eigenvalues with total multiplicity (algebraic) two. Therefore, we obtain from the Kato-Rellich Theorem (see [46, Theorem XII.8]) the existence of two analytic functions $\mu, \mathbf{F}$ defined in a neighborhood of zero such that items (*i*), (*ii*), and (*iii*) hold. □

Below, we investigate how the perturbed second eigenvalue moves depending on the sign of α.

Proposition 3.16. *There exists $0 < \alpha_1 < \alpha_0$ such that $\mu(\alpha) > 0$ for any $\alpha \in (-\alpha_1, 0)$, and $\mu(\alpha) < 0$ for any $\alpha \in (0, \alpha_1)$. Thus, in the space $L_k^2(\mathcal{G})$ for α small, we have $n(\mathbf{L}_{1,\alpha}) = 1$ as $\alpha < 0$, and $n(\mathbf{L}_{1,\alpha}) = 2$ as $\alpha > 0$.*

Proof. From Taylor's theorem, we have the following expansions

$$\mu(\alpha) = \mu_0 \alpha + O(\alpha^2) \quad \text{and} \quad \mathbf{F}(\alpha) = \widetilde{\mathbf{\Phi}}_{0,k} + \alpha \mathbf{F}_0 + \mathbf{O}(\alpha^2), \tag{3.18}$$

where $\mu_0 = \mu'(0) \in \mathbb{R}$, $\mathbf{F}_0 = \partial_\alpha \mathbf{F}(\alpha)|_{\alpha=0} \in L^2_k(\mathcal{G})$, and $\widetilde{\mathbf{\Phi}}_{0,k}$ is defined by (3.12). The desired result will follow if we show that $\mu_0 < 0$. We compute $(\mathbf{L}_{1,\alpha}\mathbf{F}(\alpha), \widetilde{\mathbf{\Phi}}_{0,k})$ in two different ways.

In what follows, we will use the following decomposition for $\mathbf{\Phi}_{\alpha,\delta}$ defined by (2.9)

$$\mathbf{\Phi}_{\alpha,\delta}(\alpha) = \mathbf{\Phi}_0 + \alpha \mathbf{G}_0 + \mathbf{O}(\alpha^2), \; \mathbf{G}_0 = \partial_\alpha(\mathbf{\Phi}_{\alpha,\delta})|_{\alpha=0} = \tfrac{-2}{(p-1)N\omega} \left(\varphi_0'\right)_{j=1}^N. \tag{3.19}$$

From (3.18), we obtain

$$(\mathbf{L}_{1,\alpha}\mathbf{F}(\alpha), \widetilde{\mathbf{\Phi}}_{0,k}) = \mu_0 \alpha ||\widetilde{\mathbf{\Phi}}_{0,k}||^2 + O(\alpha^2). \tag{3.20}$$

By $\mathbf{L}_{1,0}\widetilde{\mathbf{\Phi}}_{0,k} = \mathbf{0}$ and (3.18), we get

$$\begin{aligned} \mathbf{L}_{1,\alpha}\widetilde{\mathbf{\Phi}}_{0,k} &= p\left((\mathbf{\Phi}_0)^{p-1} - (\mathbf{\Phi}_{\alpha,\delta})^{p-1}\right)\widetilde{\mathbf{\Phi}}_{0,k} \\ &= -\alpha p(p-1)(\mathbf{\Phi}_0)^{p-2}\mathbf{G}_0 \widetilde{\mathbf{\Phi}}_{0,k} + \mathbf{O}(\alpha^2). \end{aligned} \tag{3.21}$$

The operations in the last equality are componentwise. Equations (3.21) and (3.19) induce

$$\begin{aligned} (\mathbf{L}_{1,\alpha}\mathbf{F}(\alpha), \widetilde{\mathbf{\Phi}}_{0,k}) &= -\left(\widetilde{\mathbf{\Phi}}_{0,k}, \alpha p(p-1)(\mathbf{\Phi}_0)^{p-2}\mathbf{G}_0\widetilde{\mathbf{\Phi}}_{0,k}\right) + O(\alpha^2) \\ &= \tfrac{2\alpha p(N-k)}{k\omega} \int_0^\infty (\varphi_0')^3 \varphi_0^{p-2} dx + O(\alpha^2). \end{aligned} \tag{3.22}$$

Finally, combining (3.22) and (3.20), we obtain for $k \in \{1, ..., N-1\}$

$$\mu_0 = \frac{2p(N-k)}{k\omega ||\widetilde{\mathbf{\Phi}}_{0,k}||^2} \int_0^\infty (\varphi_0')^3 \varphi_0^{p-2} dx + O(\alpha).$$

It follows that μ_0 is negative for sufficiently small $|\alpha|$ (due to the negativity of φ_0' on $\mathbb{R}_+$), which in view of (3.18) ends the proof. □

Now, we can count the number of negative eigenvalues of $\mathbf{L}_{1,\alpha}$ for any α using the classical continuation argument based on the Riesz-projection (see [28]) and the extension theory.

Proposition 3.17. *Let* $k \in \{1, ..., N-1\}$ *and* $\alpha \neq 0$. *Then*

(*i*) $\ker(\mathbf{L}_{2,\alpha}) = \text{span}\{\mathbf{\Phi}_{\alpha,\delta}\}$ *and* $\mathbf{L}_{2,\alpha} \geq 0$,

(*ii*) $\ker(\mathbf{L}_{1,\alpha}) = \{\mathbf{0}\}$,

(*iii*) *for* $\alpha > 0$, $n(\mathbf{L}_{1,\alpha}) = 2$ *in* $L^2_k(\mathcal{G})$, *i.e.*, $n(\mathbf{L}_{1,\alpha}|_{L^2_k(\mathcal{G})}) = 2$, (*iv*) *for* $\alpha < 0$, $n(\mathbf{L}_{1,\alpha}) = 1$ *in* $L^2_k(\mathcal{G})$, *i.e.*, $n(\mathbf{L}_{1,\alpha}|_{L^2_k(\mathcal{G})}) = 1$, *moreover,* $n(\mathbf{L}_{1,\alpha}) = 1$ *in* $L^2(\mathcal{G})$.

Proof. Assertions (*i*)-(*ii*) were proved in [2, Proposition 6.1].

(*iii*) Recall that $\ker(\mathbf{L}_{1,\alpha}) = \{\mathbf{0}\}$ for $\alpha \neq 0$. Define α_∞ by

$$\alpha_\infty = \inf\{\tilde{\alpha} > 0 : \mathbf{L}_{1,\alpha} \text{ has two negative eigenvalues for all } \alpha \in (0, \tilde{\alpha})\}.$$

Proposition 3.16 implies that α_∞ is well defined and $\alpha_\infty \in (0, \infty]$. We claim that $\alpha_\infty = \infty$. Suppose that $\alpha_\infty < \infty$. Let $M = n(\mathbf{L}_{1,\alpha_\infty})$ and Γ be a closed curve (for example, a circle or a rectangle) such that $0 \in \Gamma \subset \rho(\mathbf{L}_{1,\alpha_\infty})$, and all the negative eigenvalues of $\mathbf{L}_{1,\alpha_\infty}$ belong to the inner domain of Γ. The existence of such Γ can be deduced from the lower semi-boundedness of the quadratic form associated to $\mathbf{L}_{1,\alpha_\infty}$.

Next, from Lemma 3.11 it follows that there is $\epsilon > 0$ such that for $\alpha \in [\alpha_\infty - \epsilon, \alpha_\infty + \epsilon]$, we have $\Gamma \subset \rho(\mathbf{L}_{1,\alpha})$ and for $\xi \in \Gamma$, $\alpha \to (\mathbf{L}_{1,\alpha} - \xi)^{-1}$ is analytic. Therefore, the existence of an analytic family of Riesz-projections $\alpha \to P(\alpha)$ given by

$$P(\alpha) = -\frac{1}{2\pi i}\int_\Gamma (\mathbf{L}_{1,\alpha} - \xi)^{-1} d\xi$$

implies that $\dim(\operatorname{Ran} P(\alpha)) = \dim(\operatorname{Ran} P(\alpha_\infty)) = M$ for all $\alpha \in [\alpha_\infty - \epsilon, \alpha_\infty + \epsilon]$. Next, by definition of α_∞, $\mathbf{L}_{1,\alpha_\infty - \epsilon}$ has two negative eigenvalues, and $M = 2$, hence $\mathbf{L}_{1,\alpha}$ has two negative eigenvalues for $\alpha \in (0, \alpha_\infty + \epsilon]$, which contradicts with the definition of α_∞. Therefore, $\alpha_\infty = \infty$.

(*iv*) Analogously, we can prove that $n(\mathbf{L}_{1,\alpha}) = 1$ in $L^2_k(\mathcal{G})$ in the case $\alpha < 0$. To show the equality in the whole space $L^2(\mathcal{G})$, we need to repeat the arguments of the proof of Theorem 3.12-(*iii*) (i.e., $\mathbf{L}_{1,0}$ has to be replaced by $\mathbf{L}_{1,\alpha}$, and $\mathbf{\Phi}_0$ by $\mathbf{\Phi}_{\alpha,\delta}$). Namely, $\mathbf{L}_{1,\alpha}$ has to be considered as the self-adjoint extension of the non-negative symmetric operator

$$\begin{aligned}\mathbf{L}_\alpha &= \Big(\Big(-\tfrac{d^2}{dx^2} + \omega - p(\varphi_{\alpha,\delta})^{p-1}\Big)\delta_{k,j}\Big),\\ \operatorname{dom}(\mathbf{L}_\alpha) &= \Big\{\mathbf{V} \in H^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0) = 0, \sum_{j=1}^N v_j'(0) = 0\Big\},\end{aligned}$$

with deficiency indices $n_\pm(\mathbf{L}_\alpha) = 1$. Note that since $\alpha < 0$, we have $\varphi'_{\alpha,\delta}(x) < 0$, $x \geq 0$. $\square$

Remark 3.18. (*i*) Using instruments of the extension theory, it can be shown that $n(\mathbf{L}_{1,\alpha}) \leq N$ in $L^2(\mathcal{G})$.

(ii) Note that by Weyl's theorem (see [46, Theorem XIII.14]) the rest of the spectrum of $\mathbf{L}_{1,\alpha}$ and $\mathbf{L}_{2,\alpha}$ in $L^2(\mathcal{G})$ is positive and bounded away from zero.

To apply Theorem 3.6, we need to study the sign of $\partial_\omega ||\mathbf{\Phi}_{\alpha,\delta}||^2$.

Proposition 3.19. *Let $\omega > \frac{\alpha^2}{N^2}$ and $J(\omega) = \partial_\omega ||\mathbf{\Phi}_{\alpha,\delta}||^2$. Then the following assertions hold.*

(i) *Let $\alpha < 0$, then*

1) *for $1 < p \leq 5$, we have $J(\omega) > 0$;*

2) *for $p > 5$, there exists ω_1 such that $J(\omega_1) = 0$, and $J(\omega) > 0$ for $\omega \in (\frac{\alpha^2}{N^2}, \omega_1)$, while $J(\omega) < 0$ for $\omega \in (\omega_1, \infty)$.*

(ii) *Let $\alpha > 0$, then*

1) *for $1 < p \leq 3$, we have $J(\omega) > 0$;*

2) *for $3 < p < 5$, there exists ω_2 such that $J(\omega_2) = 0$, and $J(\omega) < 0$ for $\omega \in (\frac{\alpha^2}{N^2}, \omega_2)$, while $J(\omega) > 0$ for $\omega \in (\omega_2, \infty)$;*

3) *for $p \geq 5$, we have $J(\omega) < 0$.*

Proof. To prove all the assertions, we will use the equality (see [2])

$$J(\omega) = C\omega^{\frac{7-3p}{2(p-1)}} J_1(\omega),$$

where $C = \frac{N}{p-1}(\frac{p+1}{2})^{\frac{2}{p-1}} > 0$ and

$$J_1(\omega) = \tfrac{5-p}{p-1} \int_{\frac{-\alpha}{N\sqrt{\omega}}}^{1} (1-t^2)^{\frac{3-p}{p-1}} dt + \tfrac{-\alpha}{N\sqrt{\omega}}(1 - \tfrac{\alpha^2}{N^2\omega})^{\frac{3-p}{p-1}}.$$

Thus,

$$J_1'(\omega) = \tfrac{-\alpha}{N\omega^{3/2}} \tfrac{3-p}{p-1} \Big[\Big(1 - \tfrac{\alpha^2}{N^2\omega}\Big)^{\frac{3-p}{p-1}} + \tfrac{\alpha^2}{N^2\omega} \Big(1 - \tfrac{\alpha^2}{N^2\omega}\Big)^{-\frac{2(p-2)}{p-1}} \Big].$$

Item (i) was proved in [2].

Let us prove the assertion (ii). Item 3) is immediate. Consider $p \in (1,5)$. It is easily seen that

$$a_0 = \lim_{\omega \to +\infty} J_1(\omega) = \frac{5-p}{p-1} \int_0^1 (1-t^2)^{\frac{3-p}{p-1}} dt > 0, \tag{3.23}$$

and

$$\lim_{\omega \to \frac{\alpha^2}{N^2}} J_1(\omega) = \begin{cases} 2a_0, & p \in (1,3], \\ -\infty, & p \in (3,5). \end{cases} \tag{3.24}$$

Observing that $J_1'(\omega) \le 0$ for $p \in (1, 3]$ ($J_1'(\omega) \equiv 0$ as $p = 3$) and using (3.23)-(3.24), we get $J(\omega) > 0$. Let $p \in (3, 5)$, then $J_1'(\omega) > 0$. Thus, from (3.23)-(3.24) it follows that there exists unique $\omega_2 > \frac{\alpha^2}{N^2}$ such that $J_1(\omega_2) = J(\omega_2) = 0$, and $J(\omega) < 0$ for $\omega \in (\frac{\alpha^2}{N^2}, \omega_2)$, while $J(\omega) > 0$ for $\omega \in (\omega_2, \infty)$. □

Proof of Theorem 1.1. From Theorem 3.4, we obtain well-posedness of (2.2) in $\mathcal{E}_k(\mathcal{G})$ for any $k \in \{1, ..., N-1\}$. For $\alpha > 0$, from Proposition 3.17-(*iii*) and Proposition 3.19 -(*ii*), we obtain $n(\mathbf{H}_\alpha|_{L_k^2(\mathcal{G})}) - p(\omega) = 1$ as $p \in (1, 3]$, $\omega > \frac{\alpha^2}{N^2}$, and $p \in (3, 5)$, $\omega > \omega_2$. Thus, from Theorem 3.6, we get orbital instability of $e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}$ in $\mathcal{E}_k(\mathcal{G})$ and consequently in $\mathcal{E}(\mathcal{G})$. □

Remark 3.20. (*i*) Let $p \ge 5$ and $\alpha > 0$, then by Proposition 3.19-(*ii*) and Proposition 3.17-(*iii*), we get $n(\mathbf{H}_\alpha|_{L_k^2(\mathcal{G})}) - p(\omega) = 2$. This means that Theorem 3.6 does not provide any information about stability properties of $e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}$ in $\mathcal{E}_k(\mathcal{G})$.

(*ii*) Since the mapping data-solution is of class C^2 for $p > 2$, we can apply the approach by [35], to imply the orbital instability from the spectral one for $p \in (2, 5)$.

(*iii*) Theorem 2.4 above initially established in [2] easily follows for any $\alpha < 0$ from our approach. Indeed, combining Theorem 3.4, Proposition 3.17-(*i*)-(*ii*)-(*iv*), Proposition 3.19-(*i*) and Theorem 3.6, we get the orbital stability of $e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}$ in $\mathcal{E}(\mathcal{G})$ for $1 < p \le 5$. Moreover, applying the approach by [35], we may deduce the orbital instability of $e^{i\omega t}\mathbf{\Phi}_{\alpha,\delta}$ from the spectral one for $p > 5$ and $\omega > \omega^*$ (see [2, Remark 6.1]).

3.2. **The NLS-δ' equation on a star graph.** As it was announced in the Introduction, in this Subsection, we discuss a new problem. In particular, we study the orbital stability of the standing wave $\mathbf{U}(t, x) = e^{i\omega t}\mathbf{\Phi}(x)$ of NLS-δ' equation (2.4) with the particular N-tail profile $\mathbf{\Phi}_{\lambda,\delta'} = (\varphi_{\lambda,j})_{j=1}^N$ satisfying the stationary equation

$$\mathbf{H}_\lambda^{\delta'}\mathbf{\Phi} + \omega\mathbf{\Phi} - |\mathbf{\Phi}|^{p-1}\mathbf{\Phi} = 0$$

under the conditions $\varphi_{\lambda,1} = \cdots = \varphi_{\lambda,N} =: \varphi_{\lambda,\delta'}$ and $N\varphi_{\lambda,j}(0) = \lambda\varphi'_{\lambda,j}(0)$. It is easily seen that $\mathbf{\Phi}_{\lambda,\delta'}$ is defined by (2.11) for $\lambda < 0$ and $\omega > \frac{N^2}{\lambda^2}$.

As we are investigating orbital stability in $H^1(\mathcal{G})$ we need to show the well-posedness of the initial value problem for equation (2.4) in this space (*Assumption 2* in [34]). First, we establish the following property for the unitary group associated to (2.4).

Lemma 3.21. *Let $\{e^{-it\mathbf{H}_\lambda^{\delta'}}\}_{t\in\mathbb{R}}$ be the family of unitary operators associated to NLS-δ' model (2.4). Then for every $\mathbf{V}\in H^1(\mathcal{G})$ we have the relation $\partial_x(e^{-it\mathbf{H}_\lambda^{\delta'}}\mathbf{V}) = -e^{-it\mathbf{H}_\lambda^{\delta'}}\mathbf{V}' + \mathcal{B}(\mathbf{V}')$, where $\mathcal{B}(\mathbf{V}') = (2e^{it\partial_x^2}\tilde{v}_j)_{j=1}^N$, with $\tilde{v}_j(x) = \begin{cases} v_j'(x), & x\geq 0,\\ 0, & x<0\end{cases}$, and $e^{it\partial_x^2}$ is the unitary group associated with the free Schrödinger operator on $\mathbb{R}$.*

Proof. The proof repeats the one of Lemma 3.1. The only difference is that δ'-interaction on $\mathcal{G}$ is induced by the following condition

$$\mathbf{V}\in \mathbf{D}_{\lambda,\delta'} \quad \text{iff} \quad \tilde{A}\mathbf{V}(0)+\tilde{B}\mathbf{V}'(0)=\mathbf{0}, \text{ where}$$

$$\tilde{A} = \begin{pmatrix} 0 & ... & 0\\ 0 & & 0\\ \vdots & & \vdots\\ & & \\ -1 & ... & -1\end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 & -1 & 0 & ... & 0\\ 0 & 1 & -1 & ... & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & ... & -1\\ \frac{\lambda}{N} & \frac{\lambda}{N} & \frac{\lambda}{N} & ... & \frac{\lambda}{N}\end{pmatrix}.$$

□

Theorem 3.22. *Let $p>1$. Then for any $\mathbf{U}_0\in H^1(\mathcal{G})$ there exists $T>0$ such that equation (2.2) has a unique solution $\mathbf{U}\in C([-T,T],H^1(\mathcal{G}))\cap C^1([-T,T],[H^1(\mathcal{G})]')$ satisfying $\mathbf{U}(0)=\mathbf{U}_0$. For each $T_0\in(0,T)$ the mapping $\mathbf{U}_0\in H^1(\mathcal{G})\to\mathbf{U}\in C([-T_0,T_0],H^1(\mathcal{G}))$, is continuous. In particular, for $p>2$ this mapping is at least of class C^2.*

Moreover, the conservation of energy and charge holds:

$E_\lambda(\mathbf{U}(t)) = E_\lambda(\mathbf{U}_0)$, and $Q(\mathbf{U}(t)) = ||\mathbf{U}(t)||^2 = ||\mathbf{U}_0||^2$, $t\in[-T,T]$, where the energy E_λ is defined for $\mathbf{V}=(v_j)_{j=1}^N\in H^1(\mathcal{G})$ by

$$E_\lambda(\mathbf{V}) = \tfrac{1}{2}||\mathbf{V}'||^2 - \tfrac{1}{p+1}||\mathbf{V}||_{p+1}^{p+1} + \tfrac{1}{2\lambda}\Big|\sum_{j=1}^N v_j(0)\Big|^2.$$

Consequently, for $1<p<5$, we can choose $T=+\infty$.

Proof. The prove repeats the one of Theorem 3.4. In particular, it essentially uses Lemma 3.21 and the Banach contraction theorem. □

Remark 3.23. Analogously to the case of NLS-δ equation the following equality holds $e^{-it\mathbf{H}_{\delta'}^\lambda}\mathbf{V} = e^{-it\mathbf{H}_{\delta'}^\lambda}\mathbf{P}_c\mathbf{V} + e^{-it\mathbf{H}_{\delta'}^\lambda}\mathbf{P}_p\mathbf{V}$. Similarly, for $\lambda>0$, we have $\sigma_c(\mathbf{H}_{\delta'}^\lambda) = [0,\infty)$ and $\sigma_p(\mathbf{H}_{\delta'}^\lambda)=\emptyset$, therefore, $\mathbf{P}_p=\mathbf{0}$. For

$\lambda < 0$, $\sigma_c(\mathbf{H}^\lambda_{\delta'}) = [0,\infty)$ and $\sigma_p(\mathbf{H}^\lambda_{\delta'}) = \{-z_0^2\} = \{-\frac{N^2}{\lambda^2}\}$, where the corresponding eigenfunction is $\mathbf{V}_{z_0}(x) = (e^{\frac{N}{\lambda}x})_{j=1}^N$, and therefore, $e^{-it\mathbf{H}^\lambda_{\delta'}}\mathbf{P}_p\mathbf{V} = e^{itz_0^2}(\mathbf{V},\mathbf{V}_{z_0})\mathbf{V}_{z_0}$.

The proof of the spectral properties of $\mathbf{H}^\lambda_{\delta'}$ repeats the one of [5, Theorem 4.3] for the case of the Schrödinger operator with δ'-interaction on the line. In particular, to describe the point spectrum for $\lambda < 0$ one needs to consider $\mathbf{H}^\lambda_{\delta'}$ as the self-adjoint extension of the symmetric non-negative operator $\mathbf{L}'$ defined by (3.30) with deficiency indices $n_\pm(\mathbf{L}') = 1$ and then to apply Proposition 3.9.

Consider the following two self-adjoint matrix operators

$$\mathbf{L}_{1,\lambda} = \Big(\Big(-\frac{d^2}{dx^2} + \omega - p(\varphi_{\lambda,\delta'})^{p-1}\Big)\delta_{k,j}\Big),$$
$$\mathbf{L}_{2,\lambda} = \Big(\Big(-\frac{d^2}{dx^2} + \omega - (\varphi_{\lambda,\delta'})^{p-1}\Big)\delta_{k,j}\Big),$$

with $\mathrm{dom}(\mathbf{L}_{1,\lambda}) = \mathrm{dom}(\mathbf{L}_{2,\lambda}) = \mathbf{D}_{\lambda,\delta'}$. Here, $\delta_{k,j}$ is the Kronecker symbol. These operators are associated in a standard way with the second derivative of the following action functional

$$\mathbf{S}_\lambda(\mathbf{V}) = \tfrac{1}{2}||\mathbf{V}'||^2 - \tfrac{1}{p+1}||\mathbf{V}||_{p+1}^{p+1} + \tfrac{1}{2\lambda}\Big|\sum_{j=1}^N v_j(0)\Big|^2 + \tfrac{\omega}{2}||\mathbf{V}||^2,$$

where $\mathbf{V} = (v_j)_{j=1}^N \in H^1(\mathcal{G})$. Namely, $(\mathbf{S}_\lambda)''(\mathbf{\Phi}_{\lambda,\delta'})(\mathbf{U},\mathbf{V}) = (\mathbf{L}_{1,\lambda}\mathbf{U}_1,\mathbf{V}_1) + (\mathbf{L}_{2,\lambda}\mathbf{U}_2,\mathbf{V}_2)$ with $\mathbf{U} = \mathbf{U}_1 + i\mathbf{U}_2$ and $\mathbf{V} = \mathbf{V}_1 + i\mathbf{V}_2$. As in the previous paragraph, we consider the form $(\mathbf{S}_\lambda)''(\mathbf{\Phi}_{\lambda,\delta'})$ as a linear operator

$$\mathbf{H}_\lambda = \begin{pmatrix} \mathbf{L}_{1,\lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{2,\lambda} \end{pmatrix}. \tag{3.25}$$

The energy functional E_λ defined by (3.22) belongs to $C^2(H^1(\mathcal{G}),\mathbb{R})$ and *Assumptions 1,2* in [33] are satisfied. Thus, the analog of stability/instability Theorem 3.6 is true for $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$.

Below, we give the description of the spectrum of the operators $\mathbf{L}_{1,\lambda}$ and $\mathbf{L}_{2,\lambda}$, which due to formula (3.25), will help us to verify the conditions of mentioned stability/instability result.

Proposition 3.24. *Let $\lambda < 0$ and $\omega > \frac{N^2}{\lambda^2}$, then the following results hold.*

(i) $\ker(\mathbf{L}_{2,\lambda}) = \mathrm{span}\{\mathbf{\Phi}_{\lambda,\delta'}\}$, *and* $\mathbf{L}_{2,\lambda} \geq 0$.

(ii) *If* $\omega < \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, *then* $\ker(\mathbf{L}_{1,\lambda}) = \{\mathbf{0}\}$, *and* $n(\mathbf{L}_{1,\lambda}) = 1$.

(*iii*) *If* $\omega = \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, *then* $n(\mathbf{L}_{1,\lambda}) = 1$, *and the kernel of* $\mathbf{L}_{1,\lambda}$ *is given by* $\ker(\mathbf{L}_{1,\lambda}) = \mathrm{span}\{\hat{\mathbf{\Phi}}_{\lambda,1}, .., \hat{\mathbf{\Phi}}_{\lambda,N-1}\}$, *where*

$$\hat{\mathbf{\Phi}}_{\lambda,j} = (0, .., 0, \underset{\mathbf{j}}{\varphi'_{\lambda,\delta'}}, \underset{\mathbf{j+1}}{-\varphi'_{\lambda,\delta'}}, 0, .., 0). \tag{3.26}$$

(*iv*) *If* $\omega > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, *then* $\ker(\mathbf{L}_{1,\lambda}) = \{\mathbf{0}\}$, *and* $n(\mathbf{L}_{1,\lambda}) \le N$. *Moreover, for* N *even in the space* $L^2_{\frac{N}{2}}(\mathcal{G})$, *we have* $n(\mathbf{L}_{1,\lambda}|_{L^2_{\frac{N}{2}}(\mathcal{G})}) = 2$.

(*v*) *The rest of the spectrum of* $\mathbf{L}_{1,\lambda}$ *and* $\mathbf{L}_{2,\lambda}$ *is positive and bounded away from zero.*

Proof. (*i*) It is clear that $\mathbf{\Phi}_{\lambda,\delta'} \in \ker(\mathbf{L}_{2,\lambda})$. To show the equality $\ker(\mathbf{L}_{2,\lambda}) = \mathrm{span}\{\mathbf{\Phi}_{\lambda,\delta'}\}$ let us note that any $\mathbf{V} = (v_j)_{j=1}^N \in H^2(\mathcal{G})$ satisfies the following identity

$$-v_j'' + \omega v_j - (\varphi_{\lambda,\delta'})^{p-1} v_j = \frac{-1}{\varphi_{\lambda,\delta'}}\frac{d}{dx}\Big[\varphi^2_{\lambda,\delta'}\frac{d}{dx}\Big(\frac{v_j}{\varphi_{\lambda,\delta'}}\Big)\Big], \quad x > 0. \tag{3.27}$$

Thus, for $\mathbf{V} \in \mathbf{D}_{\lambda,\delta'}$, we obtain from (3.27), (2.5), and (2.11)

$$(\mathbf{L}_{2,\lambda}\mathbf{V}, \mathbf{V}) = \sum_{j=1}^N \int_0^\infty \varphi^2_{\lambda,\delta'}\Big[\frac{d}{dx}\Big(\frac{v_j}{\varphi_{\lambda,\delta'}}\Big)\Big]^2 dx + R_{\lambda,N}, \text{ where}$$

$$R_{\lambda,N} = \sum_{j=1}^N \Big[v_j'(0)v_j(0) - v_j^2(0)\frac{\varphi'_{\lambda,\delta'}(0)}{\varphi_{\lambda,\delta'}(0)}\Big] = \frac{1}{\lambda}\Big[\sum_{j=1}^N v_j(0)\Big]^2 - \frac{N}{\lambda}\sum_{j=1}^N v_j^2(0).$$

The term $R_{\lambda,N}$ if positive for $\lambda < 0$ by Jensen's inequality applied to $f(x) = x^2$. Thus, $(\mathbf{L}_{2,\lambda}\mathbf{V}, \mathbf{V}) > 0$ for $\mathbf{V} \in \mathbf{D}_{\lambda,\delta'} \setminus \mathrm{span}\{\mathbf{\Phi}_{\lambda,\delta'}\}$ which proves (*i*).

(*ii*) Concerning the kernel of $\mathbf{L}_{1,\lambda}$, we recall that the only $L^2(\mathbb{R}_+)$-solution of the equation $-v_j'' + \omega v_j - p(\varphi_{\lambda,\delta'})^{p-1}v_j = 0$ is given by $v_j = \varphi'_{\lambda,\delta'}$ (up to a factor). Thus, any element of $\ker(\mathbf{L}_{1,\lambda})$ has the form $\mathbf{V} = (v_j)_{j=1}^N = (c_j\varphi'_{\lambda,\delta'})_{j=1}^N$, $c_j \in \mathbb{R}$. If $v_1'(0) = \cdots = v_N'(0) \neq 0$, then by (2.5), we get $c_1 = \cdots = c_N \neq 0$, and consequently $N\varphi'_{\lambda,\delta'}(0) = \lambda\varphi''_{\lambda,\delta'}(0)$. Therefore, $\omega = \frac{N^2}{\lambda^2}$, which is impossible. Otherwise, the condition $v_j'(0) = 0$ implies that $\varphi''_{\lambda,\delta'}(0) = 0$, which is equivalent to the identity $\omega = \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$. Thus, we get that $c_1 = \cdots = c_N = 0$ and $\mathbf{V} \equiv \mathbf{0}$ for $\omega \neq \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$.

The proof of the equality $n(\mathbf{L}_{1,\lambda}) = 1$ for $\omega < \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ is similar to the one in the case of the operator $\mathbf{L}_{1,0}$ defined by (3.11). Namely, denoting

$$\boldsymbol{l}_\lambda = \Big(\Big(- \frac{d^2}{dx^2} + \omega - p(\varphi_{\lambda,\delta'})^{p-1}\Big)\delta_{k,j}\Big), \tag{3.28}$$

we define the following symmetric operator $\mathbf{L}'_0 = \boldsymbol{l}_\lambda$ with

$$\mathrm{dom}(\mathbf{L}'_0) = \Big\{\mathbf{V} \in H^2(\mathcal{G}) : v'_1(0) = \cdots = v'_N(0) = 0,\ \sum_{j=1}^N v_j(0) = 0\Big\}.$$

It is easily seen that $\mathbf{L}_{1,\lambda}$ is the self-adjoint extension of $\mathbf{L}'_0$. Let us show that the operator $\mathbf{L}'_0$ is non-negative. First, note that any $\mathbf{V} = (v_j)_{j=1}^N \in H^2(\mathcal{G})$ satisfies the following identity

$$-v''_j + \omega v_j - p(\varphi_{\lambda,\delta'})^{p-1} v_j = \frac{-1}{\varphi'_{\lambda,\delta'}} \frac{d}{dx}\Big[(\varphi'_{\lambda,\delta'})^2 \frac{d}{dx}\Big(\frac{v_j}{\varphi'_{\lambda,\delta'}}\Big)\Big], \quad x > 0.$$

Using the above equality and integrating by parts, we get for $\mathbf{V} \in \mathrm{dom}(\mathbf{L}'_0)$

$$(\mathbf{L}'_0\mathbf{V}, \mathbf{V}) = \sum_{j=1}^N \int_0^\infty (\varphi'_{\lambda,\delta'})^2 \Big[\frac{d}{dx}\Big(\frac{v_j}{\varphi'_{\lambda,\delta'}}\Big)\Big]^2 dx - \sum_{j=1}^N v_j^2(0)\frac{\varphi''_{\lambda,\delta'}(0)}{\varphi'_{\lambda,\delta'}(0)}.$$

Taking into account that

$$-v_j^2(0)\frac{\varphi''_{\lambda,\delta'}(0)}{\varphi'_{\lambda,\delta'}(0)} = v_j^2(0)\frac{\lambda\omega}{2N}\Big(p - 1 - (p+1)\frac{N^2}{\lambda^2\omega}\Big), \tag{3.29}$$

we get non-negativity of $\mathbf{L}'_0$ for $\omega \le \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$.

Next, the adjoint operator of $\mathbf{L}'_0$ is given by

$$(\mathbf{L}'_0)^* = \boldsymbol{l}_\lambda, \quad \mathrm{dom}((\mathbf{L}'_0)^*) = \big\{\mathbf{V} \in H^2(\mathcal{G}) : v'_1(0) = \cdots = v'_N(0)\big\}.$$

The last formula can be shown analogously to (3.14). Due to the von Neumann decomposition (3.10), we get (assuming that $\mathbf{L}'_0$ acts on complex-valued functions)

$$\mathrm{dom}((\mathbf{L}'_0)^*) = \mathrm{dom}(\mathbf{L}'_0) \oplus \mathrm{span}\{\mathbf{V}_i\} \oplus \mathrm{span}\{\mathbf{V}_{-i}\},$$

where $\mathbf{V}_{\pm i} = (e^{i\sqrt{\pm i}x})_{j=1}^N$, $\Im(\sqrt{\pm i}) > 0$. Indeed, since $\varphi_{\lambda,\delta'} \in L^\infty(\mathbb{R}_+)$, we get $\mathrm{dom}((\mathbf{L}'_0)^*) = \mathrm{dom}((\mathbf{L}')^*)$, where

$$\mathbf{L}' = \Big(\Big(- \frac{d^2}{dx^2}\Big)\delta_{k,j}\Big), \quad \mathrm{dom}(\mathbf{L}') = \mathrm{dom}(\mathbf{L}'_0). \tag{3.30}$$

Finally, by [43, Chapter IV, Theorem 6], $n_\pm(\mathbf{L}_0') = n_\pm(\mathbf{L}') = 1$. By Proposition 3.9, $n(\mathbf{L}_{1,\lambda}) \leq 1$. Due to $(\mathbf{L}_{1,\lambda}\mathbf{\Phi}_{\lambda,\delta'}, \mathbf{\Phi}_{\lambda,\delta'}) = -(p-1)||\mathbf{\Phi}_{\lambda,\delta'}||_{p+1}^{p+1} < 0$, we finally arrive at $n(\mathbf{L}_{1,\lambda}) = 1$, and (ii) is proved.

(iii) From the proof of item (ii), we induce that $n(\mathbf{L}_{1,\lambda}) = 1$, and the kernel of $\mathbf{L}_{1,\lambda}$ is nonempty as $\omega = \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$. Moreover, we know that any element of the kernel has the form $\mathbf{V} = (v_j)_{j=1}^N = (c_j\varphi'_{\lambda,\delta'})_{j=1}^N$, $c_j \in \mathbb{R}$, and it is necessary that $v_1'(0) = \cdots = v_N'(0) = 0$. Hence, the condition

$$\lambda v_1'(0) = \sum_{j=1}^N v_j(0) = 0 \tag{3.31}$$

gives rise to $(N-1)$-dimensional kernel of $\mathbf{L}_{1,\lambda}$. Since the functions $\hat{\mathbf{\Phi}}_{\lambda,j}$, $1 \leq j \leq N-1$, defined in (3.26) are linearly independent and satisfy the condition (3.31), they form the basis in $\ker(\mathbf{L}_{1,\lambda})$, and (iii) is proved.

(iv) The identity $\ker(\mathbf{L}_{1,\lambda}) = \{\mathbf{0}\}$ was shown in (ii). To show the inequality $n(\mathbf{L}_{1,\lambda}) \leq N$, we introduce the following minimal symmetric operator $\mathbf{L}_{\min} = \boldsymbol{l}_\lambda$ with

$$\mathrm{dom}(\mathbf{L}_{\min}) = \left\{\mathbf{V} \in H^2(\mathcal{G}) : \begin{array}{c} v_1'(0) = \cdots = v_N'(0) = 0, \\ v_1(0) = \cdots = v_N(0) = 0 \end{array}\right\}, \tag{3.32}$$

where $\boldsymbol{l}_\lambda$ is defined in (3.28). The operator $\mathbf{L}_{1,\lambda}$ is self-adjoint extension of $\mathbf{L}_{\min}$. From the formula (3.29) it follows that $\mathbf{L}_{\min}$ is a non-negative operator. It is obvious that $\mathbf{L}_{\min}^* = \boldsymbol{l}_\lambda$, $\mathrm{dom}(\mathbf{L}_{\min}^*) = H^2(\mathcal{G})$. Then, due to the von Neumann formula (for $\mathbf{L}_{\min}$ acting on complex-valued functions)

$$\mathrm{dom}(\mathbf{L}_{\min}^*) = \mathrm{dom}(\mathbf{L}_{\min}) \oplus \mathrm{span}\{\mathbf{V}_i^1, .., \mathbf{V}_i^N\} \oplus \mathrm{span}\{\mathbf{V}_{-i}^1, .., \mathbf{V}_{-i}^N\},$$

where $\mathbf{V}_{\pm i}^j = (0, ..., \underset{\mathbf{j}}{e^{i\sqrt{\pm i}x}}, 0, ..., 0)$, $\Im(\sqrt{\pm i}) > 0$, and consequently $n_\pm(\mathbf{L}_{\min}) = N$. By Proposition 3.9, $n(\mathbf{L}_{1,\lambda}) \leq N$.

Let N be even. It is easily seen that $n_\pm(\mathbf{L}_{\min}) = 2$ in $L^2_{\frac{N}{2}}(\mathcal{G})$. Indeed, $\mathrm{dom}(\mathbf{L}_{\min}^*) = \mathrm{dom}(\mathbf{L}_{\min}) \oplus \mathrm{span}\{\widetilde{\mathbf{V}}_i^1, \widetilde{\mathbf{V}}_i^2\} \oplus \mathrm{span}\{\widetilde{\mathbf{V}}_{-i}^1, \widetilde{\mathbf{V}}_{-i}^2\}$, where $\widetilde{\mathbf{V}}_{\pm i}^1 = (\underset{\mathbf{1}}{e^{i\sqrt{\pm i}x}}, ..., \underset{\mathbf{N/2}}{e^{i\sqrt{\pm i}x}}, \underset{\mathbf{N/2+1}}{0}, ..., \underset{\mathbf{N}}{0})$, and $\widetilde{\mathbf{V}}_{\pm i}^2 = (\underset{\mathbf{1}}{0}, ..., \underset{\mathbf{N/2}}{0}, \underset{\mathbf{N/2+1}}{e^{i\sqrt{\pm i}x}}, ..., \underset{\mathbf{N}}{e^{i\sqrt{\pm i}x}})$. By Proposition 3.9, we get $n(\mathbf{L}_{1,\lambda}) \leq 2$ in $L^2_{\frac{N}{2}}(\mathcal{G})$.

Let us introduce the following quadratic form $\mathbf{F}_{1,\lambda}$ associated with the operator $\mathbf{L}_{1,\lambda}$

$$\mathbf{F}_{1,\lambda}(\mathbf{V}) = ||\mathbf{V}'||^2 + \omega||\mathbf{V}||^2 - p\sum_{j=1}^N \int_0^\infty (\varphi_{\lambda,\delta'})^{p-1}|v_j|^2 dx + \tfrac{1}{\lambda}\Big|\sum_{j=1}^N v_j(0)\Big|^2,$$

with $\mathrm{dom}(\mathbf{F}_{1,\lambda}) = H^1(\mathcal{G})$. Let $\mathbf{\Phi}_\lambda^- = (\underset{\mathbf{1}}{\varphi'_{\lambda,\delta'}}, ..., \underset{\mathbf{N/2}}{\varphi'_{\lambda,\delta'}}, \underset{\mathbf{N/2+1}}{-\varphi'_{\lambda,\delta'}}, ..., \underset{\mathbf{N}}{-\varphi'_{\lambda,\delta'}})$, then integrating by parts, we obtain

$$\mathbf{F}_{1,\lambda}(\mathbf{\Phi}_\lambda^-) = N\int_0^\infty \varphi'_{\lambda,\delta'}\left(-\varphi'''_{\lambda,\delta'} + \omega\varphi'_{\lambda,\delta'} - p(\varphi_{\lambda,\delta'})^{p-1}\varphi'_{\lambda,\delta'}\right)dx$$

$$-N\varphi'_{\lambda,\delta'}(0)\varphi''_{\lambda,\delta'}(0) = \tfrac{N^2}{2\lambda}\omega\left[\left(\tfrac{(p+1)\omega}{2}\right)\left(1-\tfrac{N^2}{\lambda^2\omega}\right)\right]^{\frac{2}{p-1}}\left(p-1-(p+1)\tfrac{N^2}{\lambda^2\omega}\right),$$

which is negative for $\omega > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$. Since $(\mathbf{L}_{1,\lambda}\mathbf{\Phi}_{\lambda,\delta'}, \mathbf{\Phi}_{\lambda,\delta'}) < 0$, we get by orthogonality of $\mathbf{\Phi}_\lambda^-$ and $\mathbf{\Phi}_{\lambda,\delta'}$

$$\mathbf{F}_{1,\lambda}(s\mathbf{\Phi}_{\lambda,\delta'} + r\mathbf{\Phi}_\lambda^-) = |s|^2F_{1,\omega}^\lambda(\mathbf{\Phi}_{\lambda,\delta'}) + |r|^2F_{1,\omega}^\lambda(\mathbf{\Phi}_\lambda^-) < 0.$$

Thus, we obtain that $\mathbf{F}_{1,\lambda}$ is negative on two-dimensional subspace $\mathcal{M} = \mathrm{span}\{\mathbf{\Phi}_{\lambda,\delta'}, \mathbf{\Phi}_\lambda^-\}$. Therefore, by minimax principle, we get $n(\mathbf{L}_{1,\lambda}) \geq 2$. The assertion (iv) is proved. The proof of item (v) is standard and relies on Weyl's theorem. This finishes the proof of the Proposition. □

Finally, we have to study the sign of $\partial_\omega||\mathbf{\Phi}_{\lambda,\delta'}||^2$.

Proposition 3.25. *Let $\omega > \frac{N^2}{\lambda^2}$, $\lambda < 0$, and $J(\omega) = \partial_\omega||\mathbf{\Phi}_{\lambda,\delta'}||^2$.*

(i) If $1 < p \leq 5$, then $J(\omega) > 0$.

(ii) If $p > 5$, then there exists ω^* such that $J(\omega^*) = 0$, and $J(\omega) > 0$ for $\omega \in (\frac{N^2}{\lambda^2}, \omega^*)$, while $J(\omega) < 0$ for $\omega \in (\omega^*, \infty)$.

Proof. Recall that $\mathbf{\Phi}_{\lambda,\delta'} = (\varphi_{\lambda,\delta'})_{j=1}^N$, where $\varphi_{\lambda,\delta'}$ is defined by (2.11), we have via change of variables

$$\int_0^\infty(\varphi_{\lambda,\delta'}(x))^2dx = \left(\frac{p+1}{2}\right)^{\frac{2}{p-1}}\frac{2\omega^{\frac{2}{p-1}-\frac{1}{2}}}{p-1}\int_{\frac{N}{|\lambda|\sqrt{\omega}}}^1(1-t^2)^{\frac{2}{p-1}-1}dt.$$

From the last equality, we get

$$J(\omega) = C\omega^{\frac{7-3p}{2(p-1)}}J_1(\omega), \qquad C = \tfrac{N}{p-1}\left(\frac{p+1}{2}\right)^{\frac{2}{p-1}}, \tag{3.33}$$

where

$$J_1(\omega) = \tfrac{5-p}{p-1}\int_{\frac{N}{|\lambda|\sqrt{\omega}}}^1(1-t^2)^{\frac{3-p}{p-1}}dt + \tfrac{N}{|\lambda|\sqrt{\omega}}(1-\tfrac{N^2}{\lambda^2\omega})^{\frac{3-p}{p-1}}.$$

Thus,

$$J_1'(\omega) = \tfrac{N}{|\lambda|\omega^{3/2}}\tfrac{3-p}{p-1}\left[\left(1-\tfrac{N^2}{\lambda^2\omega}\right)^{\frac{3-p}{p-1}} + \tfrac{N^2}{\lambda^2\omega}\left(1-\tfrac{N^2}{\lambda^2\omega}\right)^{-\frac{2(p-2)}{p-1}}\right]. \tag{3.34}$$

It is immediate that $J(\omega) > 0$ for $1 < p \leq 5$. Consider the case $p > 5$. It is easily seen

$$\lim_{\omega \to +\infty} J_1(\omega) = \frac{5-p}{p-1} \int_0^1 (1-t^2)^{\frac{3-p}{p-1}} dt < 0, \qquad \lim_{\omega \to \frac{N^2}{\lambda^2}} J_1(\omega) = \infty.$$

Moreover, from (3.34) it follows that $J_1'(\omega) < 0$ for $\omega > \frac{N^2}{\lambda^2}$, and consequently $J_1(\omega)$ is strictly decreasing. Therefore, there exists a unique $\omega^* > \frac{N^2}{\lambda^2}$ such that $J_1(\omega^*) = J(\omega^*) = 0$, consequently, $J(\omega) > 0$ for $\omega \in (\frac{N^2}{\lambda^2}, \omega^*)$, and $J(\omega) < 0$ for $\omega \in (\omega^*, \infty)$. $\square$

Proof of Theorem 1.2. (i) 1) Combining Theorem 3.22, Theorem 3.6 (adapted to the case of the NLS-δ' equation), Proposition 3.24 (items (i), (ii) and (v)), and Proposition 3.25-(i), we get stability of $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ in $H^1(\mathcal{G})$.

2) Combining Theorem 3.6, Proposition 3.24 (items (i), (iv) and (v)), and Proposition 3.25-(i), we get orbital instability of $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ in $H^1_{\frac{N}{2}}(\mathcal{G})$ (compare with Remark 3.7-(ii)). We note that well-posedness of the Cauchy problem associated with equation (2.4) in $H^1_{\frac{N}{2}}(\mathcal{G})$ follows from the uniqueness of the solution to the Cauchy problem in $H^1(\mathcal{G})$ and the fact that the group $e^{-it\mathbf{H}^{\delta'}_{\lambda}}$ preserves the space $H^1_{\frac{N}{2}}(\mathcal{G})$. Finally, instability in the smaller space $H^1_{\frac{N}{2}}(\mathcal{G})$ induces instability in all $H^1(\mathcal{G})$.

(ii) Relative position of ω^* and $\omega = \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ is not clear (see Remark 3.27), which complicates the analysis in the framework of Theorem 3.6. But we can overcome this difficulty restricting the operator $\mathbf{L}_{1,\lambda}$ onto the space $L^2_{\text{eq}}(\mathcal{G})$ defined by $L^2_{\text{eq}}(\mathcal{G}) = \{\mathbf{V} = (v_j)_{j=1}^N \in L^2(\mathcal{G}) : v_1(x) = \dots = v_N(x), \, x > 0\}$. Moreover, we introduce $H^1_{\text{eq}}(\mathcal{G}) = H^1(\mathcal{G}) \cap L^2_{\text{eq}}(\mathcal{G})$. We note that $H^1_{\text{eq}}(\mathcal{G})$ is also preserved by the group $e^{-it\mathbf{H}^{\delta'}_{\lambda}}$.

Recall that $\mathbf{L}_{1,\lambda}$ is the self-adjoint extension of the minimal symmetric operator $\mathbf{L}_{\min}$ defined by (3.32). It is easily seen that the operator $\mathbf{L}_{\min}|_{L^2_{\text{eq}}(\mathcal{G})}$ satisfies $\mathcal{N}_{\pm}(\mathbf{L}_{\min}|_{L^2_{\text{eq}}(\mathcal{G})}) = \text{span}\{(e^{i\sqrt{\pm i}x})_{j=1}^N\}$. The last equality, by Proposition 3.9, implies $n(\mathbf{L}_{1,\lambda}|_{L^2_{\text{eq}}(\mathcal{G})}) = 1$ since $(\mathbf{L}_{1,\lambda}\mathbf{\Phi}_{\lambda,\delta'}, \mathbf{\Phi}_{\lambda,\delta'}) < 0$ and $\mathbf{\Phi}_{\lambda,\delta'} \in L^2_{\text{eq}}(\mathcal{G})$.

Without loss of generality, we can assume that $\omega^* \neq \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$. All our forthcoming conclusions about orbital stability are based on Theorem 3.6 for the spaces $H^1(\mathcal{G})$ and $H^1_{\text{eq}}(\mathcal{G})$, Remark 3.7, Theorem 3.22, Proposition 3.24, and Proposition 3.25. Consider 2 cases.

1. Suppose that $\omega^* < \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$.

Let $\omega < \omega^* < \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, then $n(\mathbf{L}_{1,\lambda}) = 1$ in $L^2(\mathcal{G})$ and we have $\partial_\omega ||\mathbf{\Phi}_{\lambda,\delta'}||^2 > 0$. Therefore, $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ is orbitally stable in $H^1(\mathcal{G})$, and hence in $H^1_{\rm eq}(\mathcal{G})$.

If $\omega^* < \omega < \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, then $n(\mathbf{L}_{1,\lambda}) = 1$ in $L^2(\mathcal{G})$ and $\partial_\omega ||\mathbf{\Phi}_{\lambda,\delta'}||^2 < 0$, which induces orbital instability of $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ in $H^1(\mathcal{G})$.

Let $\omega > \frac{N^2}{\lambda^2}\frac{p+1}{p-1} > \omega^*$. Then $n(\mathbf{L}_{1,\lambda}|_{L^2_{\rm eq}(\mathcal{G})}) = 1$ and also $\partial_\omega ||\mathbf{\Phi}_{\lambda,\delta'}||^2 < 0$, which induces orbital instability of $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ in $H^1_{\rm eq}(\mathcal{G})$ and consequently in $H^1(\mathcal{G})$.

2. Suppose that $\omega^* > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$.

If $\omega < \frac{N^2}{\lambda^2}\frac{p+1}{p-1} < \omega^*$, then $n(\mathbf{L}_{1,\lambda}) = 1$ in $L^2(\mathcal{G})$ and $\partial_\omega ||\mathbf{\Phi}_{\lambda,\delta'}||^2 > 0$, consequently, $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ is orbitally stable in $H^1(\mathcal{G})$, and therefore in $H^1_{\rm eq}(\mathcal{G})$.

If $\frac{N^2}{\lambda^2}\frac{p+1}{p-1} < \omega < \omega^*$, then $n(\mathbf{L}_{1,\lambda}|_{L^2_{\rm eq}(\mathcal{G})}) = 1$ and $\partial_\omega ||\mathbf{\Phi}_{\lambda,\delta'}||^2 > 0$, which induces stability of $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ in $H^1_{\rm eq}(\mathcal{G})$.

Let $\omega > \omega^* > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, then $n(\mathbf{L}_{1,\lambda}|_{L^2_{\rm eq}(\mathcal{G})}) = 1$ and $\partial_\omega ||\mathbf{\Phi}_{\lambda,\delta'}||^2 < 0$, which induces orbital instability of $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ in $H^1_{\rm eq}(\mathcal{G})$ and consequently in $H^1(\mathcal{G})$.

Summarizing all the cases, we get for $\omega > \omega^*$ nonlinear instability of $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ in $H^1(\mathcal{G})$, and for $\omega < \omega^*$ stability of $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ at least in $H^1_{\rm eq}(\mathcal{G})$. This finishes the proof. □

Remark 3.26. (*i*) It is worth mentioning that the orbital instability result follows easily for $2 < p < 5$ from the spectral instability using the fact that the mapping data-solution for (2.4) is of class C^2 (see Theorem 3.22 and Remark 3.7-(*iii*)).

(*ii*) Observe that for $p > 5$ the orbital instability results are obtained via classical approach by [33] without using spectral instability. Otherwise, the orbital instability can be deduced from the spectral one since for $p > 5$ the mapping data-solution for (2.4) is of class C^2.

Remark 3.27. Note that the integral appearing in (3.33) (via change of variables) is related to the incomplete Beta function

$$B\Big(y; \frac{1}{2}, b\Big) = \int_0^y x^{-\frac{1}{2}}(1-x)^{b-1}dx,$$

with $b = \frac{2}{p-1}$. Using basic numerical simulations, one can show that for $p = 6, 7, ...$, relation $\omega^* > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ holds. By the continuity of the function J as a function of p, we get the relation $\omega^* > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ in the neighborhood of every integer $p > 5$.

We conjecture that $\omega^* > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ holds for any $p > 5$. This conjecture by Theorem 3.6 implies the following stability properties of $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ in the case $p > 5$:

(i) if $\omega \in (\frac{N^2}{\lambda^2}, \frac{N^2}{\lambda^2}\frac{p+1}{p-1})$, then $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ is stable in $H^1(\mathcal{G})$;

(ii) if $\omega \in (\frac{N^2}{\lambda^2}\frac{p+1}{p-1}, \omega^*)$ and N is even, then $e^{i\omega t}\mathbf{\Phi}_{\lambda,\delta'}$ is unstable in $H^1(\mathcal{G})$.

4. Stability theory of standing wave solutions for the NLS-log-δ and the NLS-log-δ' equation on a star graph

4.1. **The NLS-log-δ equation on a star graph.** In this subsection, we prove spectral instability of the N-bump stationary state solution $\mathbf{\Psi}_{\alpha,\delta} = (\psi_{\alpha,\delta})_{j=1}^N$ of Gaussian type, where $\psi_{\alpha,\delta}(x) = e^{\frac{\omega+1}{2}} e^{-\frac{(x-\frac{\alpha}{N})^2}{2}}$, $\alpha > 0$, $\omega \in \mathbb{R}$. We also extend the stability result in [15] for any $\alpha < 0$ (see Theorem 1.3).

Since well-posedness is a crucial assumption for stability theory, it is worth proving that equation (2.6) is well-posed in the space $W^1_{\mathcal{E}}(\mathcal{G})$. In [15] the following well-posedness result in $W_{\mathcal{E}}(\mathcal{G})$ was proved.

Proposition 4.1. *For any $\mathbf{U}_0 \in W_{\mathcal{E}}(\mathcal{G})$ there is a unique solution $\mathbf{U} \in C(\mathbb{R}, W_{\mathcal{E}}(\mathcal{G})) \cap C^1(\mathbb{R}, W'_{\mathcal{E}}(\mathcal{G}))$ of (2.6) such that*

$$\mathbf{U}(0) = \mathbf{U}_0 \quad \textit{and} \quad \sup_{t\in\mathbb{R}} ||\mathbf{U}(t)||_{W_{\mathcal{E}}(\mathcal{G})} < \infty.$$

Furthermore, the conservation of energy and charge holds, that is,

$$E_{\alpha,Log}(\mathbf{U}(t)) = E_{\alpha,Log}(\mathbf{U}_0), \quad \textit{and} \quad Q(\mathbf{U}(t)) = ||\mathbf{U}(t)||^2 = ||\mathbf{U}_0||^2,$$

where the energy $E_{\alpha,Log}$ is defined for $\mathbf{V} = (v_j)_{j=1}^N \in W_{\mathcal{E}}(\mathcal{G})$ by

$$E_{\alpha,Log}(\mathbf{V}) = \tfrac{1}{2}||\mathbf{V}'||^2 - \tfrac{1}{2}\sum_{j=1}^N \int_0^\infty |v_j|^2 \operatorname{Log}|v_j|^2 dx + \tfrac{\alpha}{2}|v_1(0)|^2.$$

Using the above result, we obtain well-posedness in $W^1_{\mathcal{E}}(\mathcal{G})$.

Theorem 4.2. *If $\mathbf{U}_0 \in W^1_{\mathcal{E}}(\mathcal{G})$, there is a unique solution $\mathbf{U}(t)$ of (2.6) such that $\mathbf{U}(t) \in C(\mathbb{R}, W^1_{\mathcal{E}}(\mathcal{G}))$ and $\mathbf{U}(0) = \mathbf{U}_0$.*

Proof. The proof can be found in [11]. Basically it follows from Proposition 4.1 and two additional facts. The first one is that $W^1_{\mathcal{E}}(\mathcal{G}) \subset W_{\mathcal{E}}(\mathcal{G})$ (see [9, Lemma 3.1]), and the second one is the continuity of the mapping $t \mapsto ||x\mathbf{U}(t)||^2$ on $\mathbb{R}$. □

The strategy of the proof of Theorem 1.3 is analogous to the one in the previous case of the NLS equation with power nonlinearity. In particular, we will use the adapted (weaker) version of the stability/instability Theorem 3.6 (to the specific Gaussian profile $\mathbf{\Psi}_{\alpha,\delta}$ and the space $W^1_{\mathcal{E}}(\mathcal{G})$).

Consider the following two harmonic oscillator self-adjoint matrix operators with domain $\mathrm{dom}(\mathbf{T}_{1,\alpha}) = \mathrm{dom}(\mathbf{T}_{2,\alpha}) = \mathbf{D}^{\mathrm{Log}}_{\alpha,\delta}$ defined by

$$\mathbf{T}_{1,\alpha} = \Big(\Big(-\frac{d^2}{dx^2} + (x - \tfrac{\alpha}{N})^2 - 3\Big)\delta_{k,j}\Big),$$
$$\mathbf{T}_{2,\alpha} = \Big(\Big(-\frac{d^2}{dx^2} + (x - \tfrac{\alpha}{N})^2 - 1\Big)\delta_{k,j}\Big),$$
$$\mathbf{D}^{\mathrm{Log}}_{\alpha,\delta} := \Big\{\mathbf{V} \in W^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0), \sum_{j=1}^{N} v'_j(0) = \alpha v_1(0)\Big\},$$

where $\delta_{k,j}$ is the Kronecker symbol. These operators are associated with $\mathbf{H}_{\alpha,\mathrm{Log}} := (\mathbf{S}_{\alpha,\mathrm{Log}})''(\mathbf{\Psi}_{\alpha,\delta})$ (where $\mathbf{S}_{\alpha,\mathrm{Log}}$ is defined by (2.15)) in a standard way, i.e.,

$$\mathbf{H}_{\alpha,\mathrm{Log}} = \begin{pmatrix} \mathbf{T}_{1,\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{2,\alpha} \end{pmatrix}.$$

Noting that $\partial_\omega ||\mathbf{\Psi}_{\alpha,\delta}||^2 > 0$, $E_{\alpha,\mathrm{Log}} \in C(W^1_{\mathcal{E}}(\mathcal{G}), \mathbb{R})$ (see [11, Proposition 2.3]), and combining [33, Theorem 3.5] with [34, Theorem 5.1], we can formulate the stability/instability theorem for the NLS-log-δ equation.

Theorem 4.3. *Let $\alpha \neq 0$, and $n(\mathbf{H}_{\alpha,Log})$ be the number of negative eigenvalues of $\mathbf{H}_{\alpha,Log}$. Suppose also that*

1) $\ker(\mathbf{T}_{2,\alpha}) = \mathrm{span}\{\mathbf{\Psi}_{\alpha,\delta}\}$,

2) $\ker(\mathbf{T}_{1,\alpha}) = \{\mathbf{0}\}$,

3) *the negative spectrum of $\mathbf{T}_{1,\alpha}$ and $\mathbf{T}_{2,\alpha}$ consists of a finite number of negative eigenvalues (counting multiplicities),*

4) *the rest of the spectrum of $\mathbf{T}_{2,\alpha}$ and $\mathbf{T}_{1,\alpha}$ is positive and bounded away from zero. Then the following assertions hold.*

(*i*) *If $n(\mathbf{H}_{\alpha,Log}) = 1$, then the standing wave $e^{i\omega t}\mathbf{\Psi}_{\alpha,\delta}$ is orbitally stable in $W^1_{\mathcal{E}}(\mathcal{G})$.*

(*ii*) *If $n(\mathbf{H}_{\alpha,Log}) = 2$ in $L^2_k(\mathcal{G})$, then the standing wave $e^{i\omega t}\mathbf{\Psi}_{\alpha,\delta}$ is spectrally unstable.*

Remark 4.4. (*i*) By saying $e^{i\omega t}\mathbf{\Psi}_{\alpha,\delta}$ "*is spectrally unstable*," we mean that the spectrum of the linear part $\mathbf{A}_{\alpha,\mathrm{Log}} = \begin{pmatrix} \mathbf{0} & \mathbf{T}_{2,\alpha} \\ -\mathbf{T}_{1,\alpha} & \mathbf{0} \end{pmatrix}$ of the linearization of the NLS-log-δ equation around $\mathbf{\Psi}_{\alpha,\delta}$ contains an eigenvalue with positive real part.

(ii) In item (ii), we affirm only spectral instability since we cannot apply neither [45, Corollary 3 and 4] (since we do not know if $E_{\alpha,\mathrm{Log}} \in C^2(W^1_{\mathcal{E}}(\mathcal{G}),\mathbb{R})$), nor [35, Theorem 2 Remark, Section 2] (since we do not know if the mapping data-solution associated to the NLS-log-δ equation is of class C^2 around $\mathbf{\Psi}_{\alpha,\delta}$) to prove orbital instability (see Remark 3.7 above).

Below, we study the spectral properties of $\mathbf{T}_{1,\alpha}$ and $\mathbf{T}_{2,\alpha}$. To investigate the spectrum of the operator $\mathbf{T}_{1,\alpha}$, we will use the perturbation theory analogously to the previous case of the NLS-δ equation with power nonlinearity. In particular, define the following self-adjoint Schrödinger operator on $L^2(\mathcal{G})$ with Kirchhoff condition at $\nu = 0$

$$\mathbf{T}_{1,0} = \Big(\Big(-\frac{d^2}{dx^2} + x^2 - 3\Big)\delta_{i,j}\Big), \tag{4.1}$$
$$\mathrm{dom}(\mathbf{T}_{1,0}) = \Big\{\mathbf{V} \in W^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0),\ \sum_{j=1}^N v_j'(0) = 0\Big\}.$$

As above $\mathbf{T}_{1,\alpha}$ "tends" to $\mathbf{T}_{1,0}$ for $\alpha \to 0$. In the next Theorem, we describe the spectral properties of $\mathbf{T}_{1,0}$.

Theorem 4.5. *Let $\mathbf{T}_{1,0}$ be defined by (4.1) and $k \in \{1,...,N-1\}$. Then the following assertions hold*

(i) $\ker(\mathbf{T}_{1,0}) = \mathrm{span}\{\hat{\mathbf{\Psi}}_{0,1},...,\hat{\mathbf{\Psi}}_{0,N-1}\}$, *where*

$$\hat{\mathbf{\Psi}}_{0,j} = (0,...,0,\underset{\mathbf{j}}{\psi_0'},\underset{\mathbf{j+1}}{-\psi_0'},0,...,0), \quad \psi_0(x) = e^{-\frac{x^2}{2}}.$$

(ii) *In the space $L^2_k(\mathcal{G})$, we have* $\ker(\mathbf{T}_{1,0}) = \mathrm{span}\{\widetilde{\mathbf{\Psi}}_{0,k}\}$, *where*

$$\widetilde{\mathbf{\Psi}}_{0,k} = \Big(\underset{\mathbf{1}}{\tfrac{N-k}{k}\psi_0'},...,\underset{\mathbf{k}}{\tfrac{N-k}{k}\psi_0'},\underset{\mathbf{k+1}}{-\psi_0'},...,\underset{\mathbf{N}}{-\psi_0'}\Big), \tag{4.2}$$

i.e., $\ker(\mathbf{T}_{1,0}|_{L^2_k(\mathcal{G})}) = \mathrm{span}\{\widetilde{\mathbf{\Psi}}_{0,k}\}$.

(iii) *The operator $\mathbf{T}_{1,0}$ has one simple negative eigenvalue, i.e., $n(\mathbf{T}_{1,0}) = 1$. Moreover, the operator $\mathbf{T}_{1,0}$ has one simple negative eigenvalue in $L^2_k(\mathcal{G})$, i.e., $n(\mathbf{T}_{1,0}|_{L^2_k(\mathcal{G})}) = 1$.*

(iv) *The spectrum of $\mathbf{T}_{1,0}$ is discrete.*

Proof. The proof of items (i)-(ii) repeats the one of Theorem 3.12 (i)-(ii).

(*iii*) We will follow the ideas of the proof of item (*iii*) of Theorem 3.12 and Lemma 4.11 in [9]. Denote $\boldsymbol{t}_0 = ((-\frac{d^2}{dx^2} + x^2 - 3)\delta_{k,j})$. First, one needs to show that the operator $\mathbf{T}_0$ acting as $\mathbf{T}_0 = \boldsymbol{t}_0$ on

$$\operatorname{dom}(\mathbf{T}_0) = \Big\{\mathbf{V} \in W^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0) = 0, \sum_{j=1}^{N} v_j'(0) = 0\Big\}.$$

is non-negative. The proof follows from the identity

$$-v_j'' + (x^2 - 3)v_j = \frac{-1}{\psi_0'}\frac{d}{dx}\Big[(\psi_0')^2 \frac{d}{dx}\Big(\frac{v_j}{\psi_0'}\Big)\Big], \quad x > 0,$$

for any $\mathbf{V} = (v_j)_{j=1}^N \in W^2(\mathcal{G})$.

Next, we need to prove that $n_\pm(\mathbf{T}_0) = 1$. We use the ideas of the proof of [9, Lemma 4.11]. First, we establish the scale of Hilbert spaces associated with the self-adjoint non-negative operator (see [6, Section I,§1.2.2]) $\mathbf{T} = \boldsymbol{t}_0 + 3I$ defined on

$$\operatorname{dom}(\mathbf{T}) = \Big\{\mathbf{V} \in W^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0), \sum_{j=1}^{N} v_j'(0) = 0\Big\}.$$

Define for $s \geq 0$ the space

$$\mathfrak{H}_s(\mathbf{T}) = \Big\{\mathbf{V} \in L^2(\mathcal{G}) : \|\mathbf{V}\|_{s,2} = \Big\|(\mathbf{T} + I)^{s/2}\mathbf{V}\Big\| < \infty\Big\}.$$

The space $\mathfrak{H}_s(\mathbf{T})$ with norm $\|\cdot\|_{s,2}$ is complete. The dual space of $\mathfrak{H}_s(\mathbf{T})$ is denoted by $\mathfrak{H}_{-s}(\mathbf{T}) = \mathfrak{H}_s(\mathbf{T})'$. The norm in the space $\mathfrak{H}_{-s}(\mathbf{T})$ is defined by the formula $\|\mathbf{V}\|_{-s,2} = \|(\mathbf{T}+I)^{-s/2}\mathbf{V}\|$. The spaces $\mathfrak{H}_s(\mathbf{T})$ form the following chain ... $\subset \mathfrak{H}_2(\mathbf{T}) \subset \mathfrak{H}_1(\mathbf{T}) \subset L^2(\mathcal{G}) = \mathfrak{H}_0(\mathbf{T}) \subset \mathfrak{H}_{-1}(\mathbf{T}) \subset \mathfrak{H}_{-2}(\mathbf{T}) \subset$ ...

The norm in the space $\mathfrak{H}_1(\mathbf{T})$ can be calculated as follows

$$\begin{aligned}\|\mathbf{V}\|_{1,2}^2 &= ((\mathbf{T} + I)^{1/2}\mathbf{V}, (\mathbf{T} + I)^{1/2}\mathbf{V}) \\ &= \sum_{j=1}^{N}\int_0^\infty \left(|v_j'(x)|^2 + |v_j(x)|^2 + x^2|v_j(x)|^2\right) dx.\end{aligned}$$

Therefore, we have the embedding $\mathfrak{H}_1(\mathbf{T}) \hookrightarrow H^1(\mathcal{G})$ and, by the Sobolev embedding, $\mathfrak{H}_1(\mathbf{T}) \hookrightarrow L^\infty(\mathcal{G})$. From the former remark, we obtain that the functional $\delta_1 : \mathfrak{H}_1(\mathbf{T}) \to \mathbb{C}$ acting as $\delta_1(\mathbf{V}) = v_1(0)$ belongs to $\mathfrak{H}_1(\mathbf{T})' = \mathfrak{H}_{-1}(\mathbf{T})$, and consequently $\delta_1 \in \mathfrak{H}_{-2}(\mathbf{T})$. Therefore, using [6, Lemma 1.2.3], it follows that the restriction $\hat{\mathbf{T}}_0$ of the operator $\mathbf{T}$ onto the domain $\operatorname{dom}(\hat{\mathbf{T}}_0) = \{\mathbf{V} \in \operatorname{dom}(\mathbf{T}) : \delta_1(\mathbf{V}) = v_1(0) = 0\} = \operatorname{dom}(\mathbf{T}_0)$ is a densely defined symmetric operator with equal deficiency indices $n_\pm(\hat{\mathbf{T}}_0) = 1$. By [43, Chapter

IV, Theorem 6], the operators $\hat{\mathbf{T}}_0$ and $\mathbf{T}_0$ have equal deficiency indices. Therefore, $n(\mathbf{T}_{1,0}) \leq 1$. Since $\mathbf{T}_{1,0}\mathbf{\Psi}_0 = -2\mathbf{\Psi}_0$, where $\mathbf{\Psi}_0 = (\psi_0)_{j=1}^N$, we get $n(\mathbf{T}_{1,0}) = 1$. As $\mathbf{\Psi}_0 \in L^2_k(\mathcal{G})$ for any k, we get $n(\mathbf{T}_{1,0}|_{L^2_k(\mathcal{G})}) = 1$.

(iv) With slight modifications, we can repeat the proof of [19, Theorem 3.1, Chapter II] to show that the spectrum of $\mathbf{T}_{1,0}$ is discrete by $\lim\limits_{x\to+\infty}(x^2 - 3) = +\infty$, i.e., $\sigma(\mathbf{T}_{1,0}) = \sigma_p(\mathbf{T}_{1,0}) = \{\mu_{0,j}\}_{j\in\mathbb{N}}$. In particular, we have the following distribution of the eigenvalues $\mu_{0,1} < \mu_{0,2} < \cdots < \mu_{0,j} < \cdots$, with $\mu_{0,j} \to +\infty$ as $j \to +\infty$. □

Proposition 4.6. *Let $k \in \{1, ..., N-1\}$, $\alpha \neq 0$, and $\mathbf{\Psi}_{\alpha,\delta}$ be defined by* (2.14)*. Then*

(i) $\ker(\mathbf{T}_{2,\alpha}) = \mathrm{span}\{\mathbf{\Psi}_{\alpha,\delta}\}$ *and* $\mathbf{T}_{2,\alpha} \geq 0$*,*
(ii) $\ker(\mathbf{T}_{1,\alpha}) = \{\mathbf{0}\}$*,*
(iii) *for $\alpha > 0$, $n(\mathbf{T}_{1,\alpha}) = 2$ in $L^2_k(\mathcal{G})$, i.e., $n(\mathbf{T}_{1,\alpha}|_{L^2_k(\mathcal{G})}) = 2$,*
(iv) *for $\alpha < 0$, $n(\mathbf{T}_{1,\alpha}) = 1$ in $L^2(\mathcal{G})$,*
(v) *the spectrum of the operators $\mathbf{T}_{1,\alpha}$ and $\mathbf{T}_{2,\alpha}$ in $L^2(\mathcal{G})$ is discrete.*

Proof. (i) The proof repeats the one of [2, Proposition 6.1]. We only need to note that any $\mathbf{V} = (v_j)_{j=1}^N \in W^2(\mathcal{G})$ satisfies the following identity

$$-v_j'' + ((x - \tfrac{\alpha}{N})^2 - 1)v_j = \frac{-1}{\psi_{\alpha,\delta}}\frac{d}{dx}\Big[\psi^2_{\alpha,\delta}\frac{d}{dx}\Big(\frac{v_j}{\psi_{\alpha,\delta}}\Big)\Big], \quad x > 0.$$

(ii) The proof is standard. It is sufficient to note that any vector from the kernel of $\mathbf{T}_{1,\alpha}$ has the form $\mathbf{V} = (v_j)_{j=1}^N$, where $v_j = c_j\psi'_{\alpha,\delta}$ $c_j \in \mathbb{R}$.

(iii) The proof of this item is analogous to the one of the item (iii) of Proposition 3.17. It suffices to note that for the operator $\mathbf{T}_{1,\alpha}$ the coefficient μ_0 in decomposition (3.18) is negative. Indeed, (see the proof of Proposition 4.17 in [9])

$$\mu_0 = -\frac{2(N-k)}{k||\widetilde{\mathbf{\Psi}}_{0,k}||^2}\int_0^\infty x(\psi_0')^2dx + O(\alpha),$$

where $\widetilde{\mathbf{\Psi}}_{0,k}$ is defined by (4.2).

(iv) To show the equality in the whole space $L^2(\mathcal{G})$, we need to repeat the arguments of the proof of Theorem 4.5-(iii) (i.e., $\mathbf{T}_{1,0}$ has to be replaced by $\mathbf{T}_{1,\alpha}$, and $\mathbf{\Psi}_0$ by $\mathbf{\Psi}_{\alpha,\delta}$).

(v) The proof follows from [19, Chapter II, Theorem 3.1]. □

Proof of Theorem 1.3. Combining Theorem 4.2, Theorem 4.3, Proposition 4.6, we get orbital stability of $e^{i\omega t}\mathbf{\Psi}_{\alpha,\delta}$ in $W^1_{\mathcal{E}}(\mathcal{G})$ for $\alpha < 0$ and spectral instability of $e^{i\omega t}\mathbf{\Psi}_{\alpha,\delta}$ for $\alpha > 0$. □

4.2. **The NLS-log-δ' equation on a star graph.** In this subsection, we study the stability properties for the N-tail profile $\mathbf{\Psi}_{\lambda,\delta'} = (\psi_{\lambda,\delta'})_{j=1}^N$, where $\psi_{\lambda,\delta'} = e^{\frac{\omega+1}{2}} e^{-\frac{(x-\frac{N}{\lambda})^2}{2}}$, $\lambda < 0,\ \omega \in \mathbb{R}$. Similarly to [15, Proposition 1.1], we get the well-posedness result in $W(\mathcal{G})$.

Proposition 4.7. *For any $\mathbf{U}_0 \in W(\mathcal{G})$ there is a unique solution $\mathbf{U} \in C(\mathbb{R}, W(\mathcal{G})) \cap C^1(\mathbb{R}, W'(\mathcal{G}))$ of (2.7) such that $\mathbf{U}(0) = \mathbf{U}_0$ and $\sup_{t\in\mathbb{R}} ||\mathbf{U}(t)||_{W(\mathcal{G})} < \infty$. Furthermore, the conservation of energy and charge holds, that is,*

$$E_{\lambda,Log}(\mathbf{U}(t)) = E_{\lambda,Log}(\mathbf{U}_0), \ \ \textit{and} \ Q(\mathbf{U}(t)) = ||\mathbf{U}(t)||^2 = ||\mathbf{U}_0||^2,$$

where the energy $E_{\lambda,Log}$ is defined for $\mathbf{V} = (v_j)_{j=1}^N \in W(\mathcal{G})$ by

$$E_{\lambda,Log}(\mathbf{V}) = \tfrac{1}{2}||\mathbf{V}'||^2 - \tfrac{1}{2}\sum_{j=1}^N \int_0^\infty |v_j|^2 \operatorname{Log}|v_j|^2 dx + \tfrac{1}{2\lambda}\Big|\sum_{j=1}^N v_j(0)\Big|^2.$$

Proof. The proof repeats the one of [15, Proposition 1.1]. One just needs to replace

$$\mathfrak{F}_\gamma[u] = \sum_{j=1}^N \int_{\mathbb{R}_+} |u_j'|^2 dx - \gamma|u_1(0)|^2$$

by

$$\sum_{j=1}^N \int_{\mathbb{R}_+} |u_j'|^2 dx + \tfrac{1}{\lambda}\Big|\sum_{j=1}^N u_j(0)\Big|^2.$$

We also refer the reader to [27, Section 9.2]. □

Using the above result, one may show the well-posedness in $W^1(\mathcal{G})$.

Theorem 4.8. *If $\mathbf{U}_0 \in W^1(\mathcal{G})$, there is a unique solution $\mathbf{U}(t)$ of (2.7) such that $\mathbf{U}(t) \in C(\mathbb{R}, W^1(\mathcal{G}))$ and $\mathbf{U}(0) = \mathbf{U}_0$.*

Proof. One should repeat the proof of [11, Theorem 2.2] substituting $W^1_{\mathcal{E}}(\mathcal{G})$ by $W^1(\mathcal{G})$. □

Consider the action functional associated with equation (2.7) for $\mathbf{V} \in W^1(\mathcal{G})$,

$$\mathbf{S}_{\lambda,\mathrm{Log}}(\mathbf{V}) = \tfrac{1}{2}||\mathbf{V}'||^2 + \tfrac{(\omega+1)}{2}||\mathbf{V}||^2 - \tfrac{1}{2}\sum_{j=1}^N \int_0^\infty |v_j|^2 \operatorname{Log}|v_j|^2 dx + \tfrac{1}{2\lambda}\Big|\sum_{j=1}^N v_j(0)\Big|^2.$$

As above our idea is to study the spectral properties of the self-adjoint operators associated with $(\mathbf{S}_{\lambda,\mathrm{Log}})''(\mathbf{\Phi}_{\lambda,\delta'})$

$$\mathbf{T}_{1,\lambda}=\Big(\Big(-\frac{d^2}{dx^2}+(x-\tfrac{N}{\lambda})^2-3\Big)\delta_{k,j}\Big),$$
$$\mathbf{T}_{2,\lambda}=\Big(\Big(-\frac{d^2}{dx^2}+(x-\tfrac{N}{\lambda})^2-1\Big)\delta_{k,j}\Big),$$

acting on $\mathrm{dom}(\mathbf{T}_{1,\lambda})=\mathrm{dom}(\mathbf{T}_{2,\lambda})=\mathbf{D}^{\mathrm{Log}}_{\lambda,\delta'}$, where

$$\mathbf{D}^{\mathrm{Log}}_{\lambda,\delta'}:=\Big\{\mathbf{V}\in W^2(\mathcal{G}):v_1'(0)=\cdots=v_N'(0),\quad \sum_{j=1}^N v_j(0)=\lambda v_1'(0)\Big\}.$$

Using arguments from the proof of Proposition 4.6 and Proposition 3.24, we can show the following result.

Proposition 4.9. *Let* $k\in\{1,...,N-1\}$, $\lambda<0$, *and* $\mathbf{\Psi}_{\lambda,\delta'}$ *be defined by* (2.16). *Then the following assertions hold.*

- (i) $\ker(\mathbf{T}_{2,\lambda})=\mathrm{span}\{\mathbf{\Psi}_{\lambda,\delta'}\}$, and $\mathbf{T}_{2,\lambda}\geq 0$.
- (ii) If $-N<\lambda<0$, then $\ker(\mathbf{T}_{1,\lambda})=\{\mathbf{0}\}$, and $n(\mathbf{T}_{1,\lambda})=1$ in $L^2(\mathcal{G})$.
- (iii) If $\lambda=-N$, then $n(\mathbf{T}_{1,\lambda})=1$, and the kernel of $\mathbf{T}_{1,\lambda}$ is given by $\ker(\mathbf{T}_{1,\lambda})=\mathrm{span}\{\hat{\mathbf{\Psi}}_{\lambda,1},..,\hat{\mathbf{\Psi}}_{\lambda,N-1}\}$, where
$$\hat{\mathbf{\Psi}}_{\lambda,j}=(0,..,0,\underset{\mathbf{j}}{\psi'_{-N,\delta'}},\underset{\mathbf{j+1}}{-\psi'_{-N,\delta'}},0,..,0).$$
In particular, in this case $n(\mathbf{T}_{1,\lambda}|_{L^2_k(\mathcal{G})})=1$, and $\ker(\mathbf{T}_{1,\lambda}|_{L^2_k(\mathcal{G})})=\mathrm{span}\{\widetilde{\mathbf{\Psi}}_{-N,k}\}$, where
$$\widetilde{\mathbf{\Psi}}_{-N,k}=\Big(\underset{\mathbf{1}}{\tfrac{N-k}{k}\psi'_{-N,\delta'}},...,\underset{\mathbf{k}}{\tfrac{N-k}{k}\psi'_{-N,\delta'}},\underset{\mathbf{k+1}}{-\psi'_{-N,\delta'}},...,\underset{\mathbf{N}}{-\psi'_{-N,\delta'}}\Big).$$
- (iv) If $\lambda<-N$, then $\ker(\mathbf{T}_{1,\lambda})=\{\mathbf{0}\}$, and $n(\mathbf{T}_{1,\lambda}|_{L^2_k(\mathcal{G})})=2$.
- (v) The spectrum of $\mathbf{T}_{1,\lambda}$ and $\mathbf{T}_{2,\lambda}$ is discrete.

Proof. (i) The proof is analogous to the one of item (i) of Proposition 3.24.

(ii) The proof repeats the one of item (ii) of Proposition 3.24. We only need to note that the non-negative (for $-N<\lambda<0$) symmetric operator

$$\mathbf{T}'_0=\Big(\Big(-\frac{d^2}{dx^2}+(x-\tfrac{N}{\lambda})^2-3\Big)\delta_{k,j}\Big),$$
$$\mathrm{dom}(\mathbf{T}'_0)=\Big\{\mathbf{V}\in W^2(\mathcal{G}):v_1'(0)=\cdots=v_N'(0)=0,\quad \sum_{j=1}^N v_j(0)=0\Big\}.$$

has deficiency indices equal one. It can be shown repeating the arguments of the proof of item (*iii*) of Theorem 4.5.

(*iii*) It suffices to repeat the arguments of the proof of item (*iii*) of Proposition 3.24.

(*iv*) By the analyticity of the family $(\mathbf{T}_{1,\lambda})$ as a function of $\lambda < 0$ and the spectral properties of $\mathbf{T}_{1,\lambda}$, for $\lambda = -N$, we obtain (via the Kato-Rellich Theorem):

1) There exist $\delta > 0$ small and two analytic functions $\mu(\lambda) : (-N - \delta, -N + \delta) \to \mathbb{R}$ and $\mathbf{F}(\lambda) : (-N - \delta, -N + \delta) \to L^2_k(\mathcal{G})$ such that $\mu(-N) = 0$ and $\mathbf{F}(-N) = \widetilde{\mathbf{\Psi}}_{-N,k}$.
2) $\mu(\lambda)$ is a simple isolated eigenvalue of $\mathbf{T}_{1,\lambda}$, and $\mathbf{F}(\lambda)$ is an associated eigenvector for $\mu(\lambda)$.
3) Except at most the first two eigenvalues, the spectrum of $\mathbf{T}_{1,\lambda}|_{L^2_k(\mathcal{G})}$ is positive.

Below, we show that $\mu(\lambda) < 0$ for $\lambda < -N$, and $\mu(\lambda) > 0$ for $\gamma > -N$. From Taylor's theorem, we have the following expansions

$$\begin{aligned} \mu(\lambda) &= \mu_{-N}(\lambda + N) + O((\lambda + N)^2), \text{ and} \\ \mathbf{F}(\lambda) &= \widetilde{\mathbf{\Psi}}_{-N,k} + (\lambda + N)\mathbf{G}_{-N} + \mathbf{O}((\lambda + N)^2), \end{aligned} \tag{4.3}$$

where $\mu_{-N} = \mu'(-N) \in \mathbb{R}$ and $\mathbf{G}_{-N} = \partial_\lambda \mathbf{F}(\lambda)|_{\lambda=-N} \in L^2_k(\mathcal{G})$.

Let us show that $\mu_{-N} > 0$. To show the positivity of μ_{-N}, we compute $(\mathbf{T}_{1,\lambda}\mathbf{F}(\lambda), \widetilde{\mathbf{\Psi}}_{-N,k})$ in two different ways. Since $\mathbf{T}_{1,\lambda}\mathbf{F}(\lambda) = \mu(\lambda)\mathbf{F}(\lambda)$, it follows from (4.3) that

$$(\mathbf{T}_{1,\lambda}\mathbf{F}(\lambda), \widetilde{\mathbf{\Psi}}_{-N,k}) = \mu_{-N}(\lambda + N)\|\widetilde{\mathbf{\Psi}}_{-N,k}\|^2 + O((\lambda + N)^2). \tag{4.4}$$

By $\mathbf{T}_{1,-N}\widetilde{\mathbf{\Psi}}_{-N,k} = \mathbf{0}$, we obtain

$$\mathbf{T}_{1,\lambda}\widetilde{\mathbf{\Psi}}_{-N,k} = \left(-2x\tfrac{N+\lambda}{\lambda} + \tfrac{N^2-\lambda^2}{\lambda^2}\right)\widetilde{\mathbf{\Psi}}_{-N,k}. \tag{4.5}$$

Since $\mathbf{T}_{1,\lambda}$ is self-adjoint, we obtain from (4.3) and (4.5)

$$\begin{aligned} &(\mathbf{T}_{1,\lambda}\mathbf{F}(\lambda), \widetilde{\mathbf{\Psi}}_{-N,k}) = (\mathbf{F}(\lambda), \mathbf{T}_{1,\lambda}\widetilde{\mathbf{\Psi}}_{-N,k}) \\ &= \left(\widetilde{\mathbf{\Psi}}_{-N,k}, \left[-2x\tfrac{N+\lambda}{\lambda} + \tfrac{N^2-\lambda^2}{\lambda^2}\right]\widetilde{\mathbf{\Psi}}_{-N,k}\right) + O((\lambda + N)^2). \end{aligned} \tag{4.6}$$

Combination of (4.4) and (4.6) leads to

$$\mu_{-N}\|\widetilde{\mathbf{\Psi}}_{-N,k}\|^2 = \left(\widetilde{\mathbf{\Psi}}_{-N,k}, \left[-\tfrac{2}{\lambda}x + \tfrac{N-\lambda}{\lambda^2}\right]\widetilde{\mathbf{\Psi}}_{-N,k}\right) + O(\lambda + N). \tag{4.7}$$

Define $g(\lambda) := \left(\widetilde{\mathbf{\Psi}}_{-N,k}, \left[-\frac{2}{\lambda}x + \frac{N-\lambda}{\lambda^2}\right] \widetilde{\mathbf{\Psi}}_{-N,k}\right)$, then

$$g(\lambda) = \tfrac{(N-k)N}{k} \int_0^\infty \left[-\tfrac{2}{\lambda}x + \tfrac{N-\lambda}{\lambda^2}\right] (\psi'_{-N,\delta'})^2 dx.$$

By Taylor's theorem, $g(\lambda) = g(-N) + g'(-N)(\lambda + N) + O((\lambda + N)^2)$. It is easily seen that

$$g(-N) = 2e^{\omega+1}\tfrac{N-k}{k} \int_0^\infty (x+1)^3 e^{-(x+1)^2} dx > 0.$$

From (4.7), we get

$$\mu_{-N} = \frac{g(\lambda)}{\|\widetilde{\mathbf{\Psi}}_{-N,k}\|^2} + O(\lambda + N) = \frac{g(-N)}{\|\widetilde{\mathbf{\Psi}}_{-N,k}\|^2} + O(\lambda + N),$$

and consequently $\mu_{-N} > 0$ for λ close to $-N$.

Let λ be close to $-N$ and $\lambda < -N$, then from item (*iii*) and the analysis above ($\mu(\lambda) < 0$) it follows that $n(\mathbf{T}_{1,\lambda}|_{L^2_k(\mathcal{G})}) = 2$. Finally, by the continuation argument (see item (*iii*) of Proposition 3.17), we extend the former property for all $\lambda < -N$.

(*iv*) To prove the last spectral property it is sufficient to note that the spectrum of $\mathbf{T}_{1,\lambda}$ and $\mathbf{T}_{2,\lambda}$ is discrete due to the growth of $q(x) = (x - \frac{N}{\lambda})^2$ as $x \to \infty$. □

Proof of Theorem 1.4. Combining Theorem 4.7, Proposition 4.9, Theorem 4.3 (adapted to the case of the NLS-log-δ' equation), we get orbital stability of $e^{i\omega t}\mathbf{\Psi}_{\lambda,\delta'}$ in $W^1(\mathcal{G})$ for $-N < \lambda < 0$. Spectral instability of $e^{i\omega t}\mathbf{\Psi}_{\lambda,\delta'}$ follows for $\lambda < -N$. □

5. Applications to other models

In the above sections the use of the extension theory of symmetric operators was essential for the estimates of the Morse index of the specific self-adjoint Schrödinger operators. In this section, we show how this approach can be applied to the case of the nonlinear Schrödinger equations with specific point interactions on the line. In particular, we reprove in concise form (avoiding the use of variational techniques) some stability results for these equations established recently by the other authors (see [3, 28, 31, 32]).

5.1. **The NLS with point interactions on the line.** In the scalar case the family of self-adjoint boundary conditions for (1.1) at $x = 0$ is formally defined by

$$\begin{pmatrix} \psi(0+) \\ \psi'(0+) \end{pmatrix} = \tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi(0-) \\ \psi'(0-) \end{pmatrix}, \tag{5.1}$$

with a, b, c, d and τ satisfying the conditions (see [6, Theorem 3.2.3] or formula (K.1.2) from [5, Appendix K])

$$\{a, b, c, d \in \mathbb{R},\ \tau \in \mathbb{C} : ad - bc = 1,\ |\tau| = 1\}. \tag{5.2}$$

The parameters (5.1) label the self-adjoint extensions of the closable symmetric operator $H_0 = -\frac{d^2}{dx^2}$ defined, for instance, on the space $C_0^\infty(\mathbb{R} \setminus \{0\})$.

We are interested in two specific choices of the parameters in (5.2), which are relevant in physical applications (see [3, 24]). The first choice $\tau = a = d = 1$, $b = 0$, $c = -\gamma, \gamma \in \mathbb{R}\setminus\{0\}$ corresponds to the δ-interaction of strength $-\gamma$ which gives rise to the following NLS-δ model

$$i\partial_t u - H_\gamma^\delta u + |u|^{p-1}u = 0, \tag{5.3}$$

where H_γ^δ is the self-adjoint operator on $L^2(\mathbb{R})$ acting as $(H_\gamma^\delta v)(x) = -v''(x)$, for $x \neq 0$, on the domain $\mathrm{dom}(H_\gamma^\delta) = D_{\gamma,\delta}$, where

$$D_{\gamma,\delta} := \big\{v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) : v'(0+) - v'(0-) = -\gamma v(0)\big\}.$$

The operator H_γ^δ is formally defined by the expression $l_\gamma^\delta = -\frac{d^2}{dx^2} - \gamma\delta(x)$, where $\delta(x)$ is the Dirac delta distribution.

The second choice of parameters $\tau = a = d = 1$, $c = 0$, $b = -\beta, \beta \in \mathbb{R}\setminus\{0\}$ corresponds to the case of so-called δ'-interaction of strength $-\beta$. It gives rise to the following model (NLS-δ' henceforth)

$$i\partial_t u - H_\beta^{\delta'} u + |u|^{p-1}u = 0, \tag{5.4}$$

in which $H_\beta^{\delta'}$ is the self-adjoint operator on $L^2(\mathbb{R})$ acting as $(H_\beta^{\delta'} v)(x) = -v''(x)$, for $x \neq 0$, on the domain $\mathrm{dom}(H_\beta^{\delta'}) = D_{\beta,\delta'}$, where

$$D_{\beta,\delta'} := \{v \in H^2(\mathbb{R} \setminus \{0\}) : v(0+) - v(0-) = -\beta v'(0),\ v'(0+) = v'(0-)\}.$$

Recall that $H_\beta^{\delta'}$ is formally defined by the expression $l_\beta^{\delta'} = -\frac{d^2}{dx^2} - \beta\langle\cdot, \delta'\rangle\delta'(x)$.

The NLS-δ model has been extensively studied in the last decade (see [8, 12, 23, 24, 28, 30–32, 36, 45] and reference therein). The NLS-δ' model is less studied, in [3, 4] the authors investigated variational properties and the orbital stability of the ground states of the NLS-δ' equation with the repulsive δ'-interaction ($\beta > 0$).

5.2. **The NLS-δ' equation on the line.** As above the existence of standing wave solutions $u(t,x) = e^{i\omega t}\varphi_(x)$ of equation (5.4) requires that the profile $\varphi \in D_{\beta,\delta'}$ satisfies the semi-linear elliptic equation

$$H_\beta^{\delta'}\varphi + \omega\varphi - |\varphi|^{p-1}\varphi = 0. \tag{5.5}$$

It was shown in [3] that for $\beta > 0$ equation (5.5) has two types of solutions (odd and asymmetric)

$$\varphi^{odd}_{\omega,\beta}(x) = \operatorname{sign}(x)\Big[\frac{(p+1)\omega}{2}\operatorname{sech}^2\Big(\frac{(p-1)\sqrt{\omega}}{2}(|x|+y)\Big)\Big]^{\frac{1}{p-1}}, \tag{5.6}$$

with $x \neq 0$ and $\omega > \frac{4}{\beta^2}$,

$$\varphi^{as}_{\omega,\beta}(x) = \begin{cases} \Big[\frac{(p+1)\omega}{2}\operatorname{sech}^2\Big(\frac{(p-1)\sqrt{\omega}}{2}(x+y_1)\Big)\Big]^{\frac{1}{p-1}}, & x>0; \\ -\Big[\frac{(p+1)\omega}{2}\operatorname{sech}^2\Big(\frac{(p-1)\sqrt{\omega}}{2}(x-y_2)\Big)\Big]^{\frac{1}{p-1}}, & x<0, \end{cases} ,\omega > \frac{4}{\beta^2}\frac{p+1}{p-1},$$

where $y,\ y_1$ and y_2 are positive constants depending on β, p, ω (see [3, Theorem 5.3]). Moreover, in [3, 4] were established the following stability results. The standing wave $e^{i\omega t}\varphi^{odd}_{\omega,\beta}$ is stable in $H^1(\mathbb{R}\setminus\{0\})$ for $p > 1$, $\omega \in (\frac{4}{\beta^2}, \frac{4}{\beta^2}\frac{p+1}{p-1})$, and unstable in $H^1(\mathbb{R}\setminus\{0\})$ for $p > 1$, $\omega > \frac{4}{\beta^2}\frac{p+1}{p-1}$. The standing wave $e^{i\omega t}\varphi^{ass}_{\omega,\beta}$ is stable in $H^1(\mathbb{R}\setminus\{0\})$ for $1 < p \leq 5$, $\omega > \frac{4}{\beta^2}\frac{p+1}{p-1}$, and $p > 5$, $\omega \in (\frac{4}{\beta^2}\frac{p+1}{p-1}, \omega_1)$, meanwhile $e^{i\omega t}\varphi^{ass}_{\omega,\beta}$ is unstable in $H^1(\mathbb{R}\setminus\{0\})$ for $p > 5$, $\omega > \omega_2 > \omega_1$.

In what follows, we will use the notation $\varphi_\beta = \varphi^{odd}_{\omega,\beta}$. Due to Grillakis, Shatah and Strauss approach, we need to study the spectral properties of the following two self-adjoint operators

$$L_{1,\beta} = -\frac{d^2}{dx^2} + \omega - p|\varphi_\beta|^{p-1}, \quad L_{2,\beta} = -\frac{d^2}{dx^2} + \omega - |\varphi_\beta|^{p-1},$$
$$\operatorname{dom}(L_{j,\beta}) = D_{\beta,\delta'},\ \ j \in \{1,2\}.$$

The operators $L_{1,\beta}$ and $L_{2,\beta}$ are associated with the action functional

$$S_\beta(\psi) = \tfrac{1}{2}||\psi'||^2 + \tfrac{\omega}{2}||\psi||^2 - \tfrac{1}{p+1}||\psi||^{p+1}_{p+1} - \tfrac{1}{2\beta}|\psi(0+) - \psi(0-)|^2,$$

defined on $H^1(\mathbb{R}\setminus\{0\})$, in the following sense:

$(S_\beta)''(\varphi_\beta)(u,v) = (L_{1,\beta}u_1, v_1) + (L_{2,\beta}u_2, v_2)$, where $u = u_1 + iu_2$ and $v = v_1 + iv_2$. The well-posedness for (5.4) in $H^1(\mathbb{R}\setminus\{0\})$ was established in [3, Proposition 3.3]. Moreover, it was shown that $\ker(L_{2,\beta}) = \operatorname{span}\{\varphi_\beta\}$, and $\ker(L_{1,\beta}) = \{0\}$, and the sign of $\partial_\omega||\varphi_\beta||^2$ was computed.

The following result on the Morse index of $L_{1,\beta}$ was proved in [3] via variational approach. We propose an alternative proof in the framework of the extension theory.

Proposition 5.1. *Let $\beta > 0$ and $\omega > \frac{4}{\beta^2}$. Then*

(i) $n(L_{1,\beta}) = 1$ *for* $\omega \in (\frac{4}{\beta^2}, \frac{4}{\beta^2}\frac{p+1}{p-1}]$,

(ii) $n(L_{1,\beta}) = 2$ *for* $\omega \in (\frac{4}{\beta^2}\frac{p+1}{p-1}, \infty)$.

Proof. It is easily seen that $L_{1,\beta}$ is the self-adjoint extension of the symmetric operator $L_{\min}$ defined by

$$L_{\min} = -\frac{d^2}{dx^2} + \omega - p|\varphi_\beta|^{p-1}, \quad \mathrm{dom}(L_{\min}) = \{v \in H^2(\mathbb{R}) : v(0) = v'(0) = 0\}. \tag{5.7}$$

Since $\varphi_\beta \in L^\infty(\mathbb{R})$, we obtain $\mathrm{dom}(L^*_{\min}) = H^2(\mathbb{R} \setminus \{0\})$. Moreover, the operator $L_{\min}$ is non-negative for $\beta > 0$. Indeed, it is easy to verify that for $\beta > 0$ and $v \in H^2(\mathbb{R} \setminus \{0\})$ the following identity holds

$$-v'' + \omega v - p|\varphi_\beta|^{p-1}v = \frac{-1}{\varphi'_\beta}\frac{d}{dx}\Big[(\varphi'_\beta)^2\frac{d}{dx}\Big(\frac{v}{\varphi'_\beta}\Big)\Big], \quad x \neq 0. \tag{5.8}$$

Using (5.8) and integrating by parts, we get

$$(L_{\min}v, v) = \Big(\int_{-\infty}^{0-} + \int_{0+}^{\infty}\Big)(\varphi'_\beta)^2\Big[\frac{d}{dx}\Big(\frac{v}{\varphi'_\beta}\Big)\Big]^2 dx + \Big[v'v - v^2\frac{\varphi''_\beta}{\varphi'_\beta}\Big]_{0-}^{0+}. \tag{5.9}$$

The integral terms in (5.9) are non-negative. Due to the conditions $v(0) = v'(0) = 0$, non-integral term vanishes, and we get $L_{\min} \geq 0$. Note that

$$\mathrm{dom}(L^*_{\min}) = H^2(\mathbb{R} \setminus \{0\}) = \mathrm{dom}(L_{\min}) \oplus \mathrm{span}\{v^1_i, v^2_i\} \oplus \mathrm{span}\{v^1_{-i}, v^2_{-i}\},$$

where

$$v^1_{\pm i} = \begin{cases} e^{i\sqrt{\pm i}x} & x > 0; \\ 0 & x < 0. \end{cases}, \quad v^2_{\pm i} = \begin{cases} 0 & x > 0; \\ e^{-i\sqrt{\pm i}x} & x < 0. \end{cases}, \quad \Im(\sqrt{\pm i}) > 0.$$

Indeed, due to the fact that $\varphi_\beta \in L^\infty(\mathbb{R})$, we get $\mathrm{dom}(L^*_{\min}) = \mathrm{dom}(L^*)$, where

$$L = -\frac{d^2}{dx^2}, \quad \mathrm{dom}(L) = \mathrm{dom}(L_{\min}).$$

Moreover, $n_\pm(L_{\min}) = n_\pm(L) = 2$. Since $L_{1,\beta}$ is the self-adjoint extension of the non-negative symmetric operator $L_{\min}$ and $n_\pm(L_{\min}) = 2$, by Proposition 3.9, $n(L_{1,\beta}) \leq 2$. Otherwise, we obtain from (5.5) that $(L_{1,\beta}\varphi_\beta, \varphi_\beta) < 0$, and therefore $n(L_{1,\beta}) \geq 1$. Thus, we get $1 \leq n(L_{1,\beta}) \leq 2$.

(i) Note that $L_{1,\beta}$ is the self-adjoint extension of the following symmetric operator

$$L_0' = -\frac{d^2}{dx^2} + \omega - p|\varphi_\beta|^{p-1}, \quad \mathrm{dom}(L_0') = \left\{v \in H^2(\mathbb{R}) : v'(0) = 0\right\}.$$

Let us show that $L_0' \geq 0$. Using (5.8) and integrating by parts,

$$(L_0'v, v) = \Big(\int_{-\infty}^{0-} + \int_{0+}^{\infty}\Big)(\varphi_\beta')^2\Big[\frac{d}{dx}\Big(\frac{v}{\varphi_\beta'}\Big)\Big]^2 dx + \Big[v'v - v^2\frac{\varphi_\beta''}{\varphi_\beta'}\Big]_{0-}^{0+}. \quad (5.10)$$

The integral terms in (5.10) are non-negative. Let us focus on the non-integral term. Due to the conditions $v'(0) = 0, v(0+) = v(0-)$, and formula (5.6), we deduce

$$\begin{aligned}\Big[v'v - v^2\frac{\varphi_\beta''}{\varphi_\beta'}\Big]_{0-}^{0+} &= -\Big[v^2\frac{\varphi_\beta''}{\varphi_\beta'}\Big]_{0-}^{0+} = v^2(0)\frac{\varphi_\beta''(0-) - \varphi_\beta''(0+)}{\varphi_\beta'(0-)} \\ &= -v^2(0)\frac{\beta\omega}{2}\Big(p - 1 - (p+1)\frac{4}{\beta^2\omega}\Big) \geq 0.\end{aligned}$$

The last inequality follows from $\omega \leq \frac{4}{\beta^2}\frac{p+1}{p-1}$.

Using arguments numerously repeated above, we get

$\mathrm{dom}((L_0')^*) = \left\{v \in H^2(\mathbb{R}\setminus\{0\}) : v'(0+) = v'(0-)\right\}$, and $\mathrm{dom}((L_0')^*) = \mathrm{dom}(L_0') \oplus \mathrm{span}\{v_i\} \oplus \mathrm{span}\{v_{-i}\}$, where

$$v_{\pm i} = \begin{cases} e^{i\sqrt{\pm i}x} & x > 0, \\ -e^{-i\sqrt{\pm i}x} & x < 0, \end{cases}, \quad \Im(\sqrt{\pm i}) > 0.$$

Then $n_\pm(L_0') = 1$, and by Proposition 3.9, we obtain $n(L_{1,\beta}) \leq 1$, and finally $n(L_{1,\beta}) = 1$.

(ii) The quadratic form of the operator $L_{1,\beta}$ is defined on $H^1(\mathbb{R}\setminus\{0\})$ by $F_{1,\beta}(u) = ||u'||^2 + \omega||u||^2 - p(|\varphi_\beta|^{p-1}u, u) - \frac{1}{\beta}|u(0+) - u(0-)|^2$. Noting that $\varphi_\beta'(0+) = \varphi_\beta'(0-)$ and integrating by parts, we get

$$\begin{aligned} F_{1,\beta}(\varphi_\beta') &= \Big(\int_{-\infty}^{0-} + \int_{0+}^{+\infty}\Big)\varphi_\beta'\Big(-\varphi_\beta''' + \omega\varphi_\beta' - p|\varphi_\beta|^{p-1}\varphi_\beta'\Big)dx \\ &+ \varphi_\beta'(0+)(\varphi_\beta''(0-) - \varphi_\beta''(0+)) = \varphi_\beta'(0+)(\varphi_\beta''(0-) - \varphi_\beta''(0+)) \\ &= -\tfrac{2}{\beta}\omega\left[\left(\tfrac{(p+1)\omega}{2}\right)\left(1 - \tfrac{4}{\beta^2\omega}\right)\right]^{\frac{2}{p-1}}\left(p - 1 - (p+1)\tfrac{4}{\beta^2\omega}\right).\end{aligned}$$

The last one expression is negative due to $\omega > \frac{4}{\beta^2}\frac{p+1}{p-1}$. Since $F_{1,\beta}(\varphi_\beta) = (L_{1,\beta}\varphi_\beta, \varphi_\beta) < 0$, and the functions $\varphi_\beta, \varphi'_\beta$ have different parity, we obtain $F_{1,\beta}(s\varphi_\beta + r\varphi'_\beta) = |s|^2 F^\beta_{1,\omega}(\varphi_\beta) + |r|^2 F^\beta_{1,\omega}(\varphi'_\beta) < 0$. Therefore, we have that $F_{1,\beta}$ is negative on two-dimensional subspace $\mathcal{M} = \mathrm{span}\{\varphi_\beta, \varphi'_\beta\} \subset H^1(\mathbb{R}\setminus\{0\})$. Thus, minimax principle induces $n(L_{1,\beta}) \geq 2$, and consequently $n(L_{1,\beta}) = 2$. □

In [3, Proposition 6.5] it was shown that $\partial_\omega ||\varphi_\beta||^2$ is positive for any $p > 1$ and $\omega \in (\frac{4}{\beta^2}, \frac{4}{\beta^2}\frac{p+1}{p-1})$. Thus, due to Proposition 5.1, we conclude that $e^{i\omega t}\varphi_\beta$ is orbitally stable in this case.

Below, we briefly discuss how to demonstrate the orbital instability of $e^{i\omega t}\varphi_\beta$ for $p > 1$ and $\omega > \frac{4}{\beta^2}\frac{p+1}{p-1}$ proved in [3, Theorem 6.11]. To do that, we need the following key result.

Proposition 5.2. *Let* $\omega > \frac{4}{\beta^2}$, $\beta > 0$, *and operator* $\widetilde{L}_{1,\beta}$ *be defined as*

$$\widetilde{L}_{1,\beta} = -\frac{d^2}{dx^2} + \omega - p|\varphi_\beta|^{p-1}, \quad \mathrm{dom}(\widetilde{L}_{1,\beta}) = D_{\beta,\delta'} \cap X_{\mathrm{odd}},$$

where X_{odd} *is the set of odd functions in* $L^2(\mathbb{R})$. *Then* $n(\widetilde{L}_{1,\beta}) = 1$.

Proof. It is obvious that $n(\widetilde{L}_{1,\beta}) \leq 1$ in X_{odd}. Indeed, $n_\pm(L_{\min}) = 1$ in X_{odd} for $L_{\min}$ defined by (5.7). Since $\varphi_\beta \in \mathrm{dom}(\widetilde{L}_{1,\beta})$ and $(\widetilde{L}_{1,\beta}\varphi_\beta, \varphi_\beta) < 0$, then we get $n(\widetilde{L}_{1,\beta}) = 1$. □

Well-posedness of the Cauchy problem in $H^1(\mathbb{R}\setminus\{0\}) \cap X_{\mathrm{odd}}$ associated with equation (5.4) was shown in [3, Theorem 6.11]. Thus, we induce orbital instability of $e^{i\omega t}\varphi_\beta$ for $p > 1$ and $\omega > \frac{4}{\beta^2}\frac{p+1}{p-1}$. Indeed, when $\partial_\omega||\varphi_\beta||^2 > 0$, instability follows from Proposition 5.1-(*ii*) and from the results by Ohta in [45]. In the case $\partial_\omega||\varphi_\beta||^2 < 0$, we can conclude by Proposition 5.2 orbital instability of $e^{i\omega t}\varphi_\beta$ in $H^1(\mathbb{R}\setminus\{0\}) \cap X_{\mathrm{odd}}$ which naturally induces orbital instability in $H^1(\mathbb{R}\setminus\{0\})$.

5.3. **The NLS-δ equation on the line.** The existence of standing wave solutions $u(t,x) = e^{i\omega t}\varphi$ to equation (5.3) requires that the profile $\varphi \in D_{\gamma,\delta}$ satisfies the semi-linear elliptic equation

$$H^\delta_\gamma \varphi + \omega\varphi - |\varphi|^{p-1}\varphi = 0. \tag{5.11}$$

The authors in [30] (see also [32]) showed that (5.11) for $\omega > \frac{\gamma^2}{4}$ has a unique positive even solution modulo rotation

$$\varphi_\gamma(x) = \Big[\frac{(p+1)\omega}{2}\operatorname{sech}^2\Big(\frac{(p-1)\sqrt{\omega}}{2}|x| + \tanh^{-1}\Big(\frac{\gamma}{2\sqrt{\omega}}\Big)\Big)\Big]^{\frac{1}{p-1}}, \quad (5.12)$$

$x \in \mathbb{R}$. For the sake of completeness, we recall the main results on the stability of soliton solutions to (5.3). For $\gamma = 0$, the orbital stability has been extensively studied in [18, 25, 26, 47]. Namely, $e^{i\omega t}\varphi_0$ is stable in $H^1(\mathbb{R})$ for any $\omega > 0$ and $1 < p < 5$ (see [25]), and unstable in $H^1(\mathbb{R})$ for any $\omega > 0$ and $p \geq 5$ (see [18] for $p > 5$ and [47] for $p = 5$).

The case $\gamma > 0$ was studied in [31]. In particular, the authors showed that the standing wave $e^{i\omega t}\varphi_\gamma$ is stable in $H^1(\mathbb{R})$ for any $\omega > \frac{\gamma^2}{4}$ and $1 < p \leq 5$, and if $p > 5$, there exists a critical ω^* such that $e^{i\omega t}\varphi_\gamma$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (\frac{\gamma^2}{4}, \omega^*)$ and unstable in $H^1(\mathbb{R})$ for any $\omega > \omega^*$. In the case $\gamma < 0$, the standing wave $e^{i\omega t}\varphi_\gamma$ is unstable "almost for sure" in $H^1(\mathbb{R})$ for any $p > 1$ (see [28, 30, 45]).

Linearization of the NLS-δ equation on the line gives the following two self-adjoint linear operators

$$L_{1,\gamma} = -\frac{d^2}{dx^2} + \omega - p\varphi_\gamma^{p-1}, \quad L_{2,\gamma} = -\frac{d^2}{dx^2} + \omega - \varphi_\gamma^{p-1},$$

with $\operatorname{dom}(L_{j,\gamma}) = D_{\gamma,\delta}$, $j \in \{1, 2\}$. The operators $L_{1,\gamma}$ and $L_{2,\gamma}$ are associated with the key action functional

$$S_\gamma(\psi) = \tfrac{1}{2}||\psi'||^2 + \tfrac{\omega}{2}||\psi||^2 - \tfrac{1}{p+1}||\psi||_{p+1}^{p+1} - \tfrac{\gamma}{2}|\psi(0)|^2, \ \ \psi \in H^1(\mathbb{R}),$$

by $(S_\gamma)''(\varphi_\gamma)(u, v) = (L_{1,\gamma}u_1, v_1) + (L_{2,\gamma}u_2, v_2)$, where $u = u_1 + iu_2$ and $v = v_1 + iv_2$.

The initial value problem associated to the NLS-δ equation is locally well-posed in $H^1(\mathbb{R})$ (see [26, Theorem 4.6.1]) for any $p > 1$. Making use of the explicit form (5.12) for φ_γ, the sign of $\partial_\omega||\varphi_\gamma||^2$ was computed in [30, 31]. By variational methods, it was shown in [30] that $n(L_{1,\gamma}) = 1$ in $H^1_{\text{rad}}(\mathbb{R})$, for arbitrary γ. Moreover, by using analytic perturbation theory and continuation argument, it was shown in [28] that $n(L_{1,\gamma}) = 1$ in $H^1(\mathbb{R})$ for any $\gamma > 0$, as well as $n(L_{1,\gamma}) = 2$ for $\gamma < 0$.

Below, we establish two novel proofs of the equality $n(L_{1,\gamma}) = 1$ in $H^1(\mathbb{R})$ for any $\gamma > 0$. The first one is based on a generalization of the classical Sturm oscillation theorem to the case of the δ-interaction (see [7, 19] and Lemma 5.3 below). The second one uses the extension theory. Note also that the equality $\ker(L_{2,\gamma}) = \operatorname{span}\{\varphi_\gamma\}$ and Lemma 5.3 imply $n(L_{2,\gamma}) = 0$.

Lemma 5.3. *Let $V(x)$ be real-valued continuous function on $\mathbb{R}$ such that $\lim\limits_{|x|\to\infty} V(x) = c$. Let also $\varphi_1, \varphi_2 \in L^2(\mathbb{R})$ be eigenfunctions of the operator*

$$L_V = -\frac{d^2}{dx^2} + V(x), \quad \mathrm{dom}(L_V) = D_{\gamma,\delta},$$

corresponding to the eigenvalues $\lambda_1 < \lambda_2 < c$ respectively. Suppose that n_1 and n_2 are the number of zeroes of φ_1, φ_2 respectively. Then $n_2 > n_1$.

Proposition 5.4. *Let $\gamma > 0$ and $\omega > \frac{\gamma^2}{4}$. Then $n(L_{1,\gamma}) = 1$.*

The first proof of Proposition 5.4. Initially, we obtain from (5.11) that $(L_{1,\gamma}\varphi_\gamma, \varphi_\gamma) < 0$, and therefore $n(L_{1,\gamma}) \geq 1$. To evaluate $n(L_{1,\gamma})$ precisely consider the following self-adjoint operator

$$\widetilde{L}_{1,\gamma} = -\frac{d^2}{dx^2} + \omega - p\varphi_0^{p-1}, \qquad \mathrm{dom}(\widetilde{L}_{1,\gamma}) = D_{\gamma,\delta},$$

where $\varphi_0 = [\frac{(p+1)\omega}{2}\,\mathrm{sech}^2(\frac{(p-1)\sqrt{\omega}}{2}x)]^{\frac{1}{p-1}}$ is the classical soliton solution for the NLS equation. It is easily seen that $\varphi_0' \in \ker(\widetilde{L}_{1,\gamma})$. From Lemma 5.3 and the fact that $x = 0$ is the only zero of φ_0', we have $n(\widetilde{L}_{1,\gamma}) \leq 1$. Since $\varphi_0(x) > \varphi_\gamma(x)$ for all $x \in \mathbb{R}$ and $\gamma > 0$, we get the following inequality

$$(L_{1,\gamma}v, v) \geq (\widetilde{L}_{1,\gamma}v, v), \quad \text{for all } \; v \in D_{\gamma,\delta}.$$

Therefore, we get $1 \leq n(L_{1,\gamma}) \leq n(\widetilde{L}_{1,\gamma}) \leq 1$. Thereby, in the case $\gamma > 0$, we get $n(L_{1,\gamma}) = 1$. $\square$

The second proof of Proposition 5.4. Recall that $L_{1,\gamma}$ is the self-adjoint extension of the following symmetric operator

$$L_0 = -\frac{d^2}{dx^2} + \omega - p\varphi_\gamma^{p-1}, \quad \mathrm{dom}(L_0) = \left\{v \in H^2(\mathbb{R}) : \, v(0) = 0\right\}.$$

Moreover, it is known (see [5, Chapter I.3]) that

$$\mathrm{dom}(L_0^*) = H^1(\mathbb{R}) \cap H^2(\mathbb{R}\setminus\{0\}) = \mathrm{dom}(L_0) \oplus \mathrm{span}\{e^{i\sqrt{i}|x|}\} \oplus \mathrm{span}\{e^{i\sqrt{-i}|x|}\},$$

with $\Im(\sqrt{\pm i}) > 0$. Indeed, since $\varphi_\gamma \in L^\infty(\mathbb{R})$, we have $\mathrm{dom}(L_0^*) = \mathrm{dom}(L^*)$, where $L = -\frac{d^2}{dx^2}$, $\mathrm{dom}(L) = \mathrm{dom}(L_0)$. In particular, $n_\pm(L_0) = n_\pm(L) = 1$. Next, it is easy to verify that for $\gamma > 0$ and $v \in H^2(\mathbb{R}\setminus\{0\})$ the following identity holds

$$-v'' + \omega v - p\varphi_\gamma^{p-1}v = \frac{-1}{\varphi_\gamma'}\frac{d}{dx}\Big[(\varphi_\gamma')^2\frac{d}{dx}\Big(\frac{v}{\varphi_\gamma'}\Big)\Big], \qquad x \neq 0. \tag{5.13}$$

Then, using (5.13) and integrating by parts, we get

$$(L_0 v, v) = \Big(\int_{-\infty}^{0-} + \int_{0+}^{\infty}\Big)(\varphi_\gamma')^2\Big[\frac{d}{dx}\Big(\frac{v}{\varphi_\gamma'}\Big)\Big]^2 dx + \Big[v'v - v^2\frac{\varphi_\gamma''}{\varphi_\gamma'}\Big]_{0-}^{0+}.$$

Due to the condition $v(0) = 0$, non-integral term vanishes, and we get $L_0 \geq 0$ on $\mathrm{dom}(L_0)$. Then, using Proposition 3.9, we get $n(L_{1,\omega}^{\gamma}) \leq 1$. This finishes the proof due to the inequality $(L_{1,\gamma}\varphi_\gamma, \varphi_\gamma) < 0$. □

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