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EXPONENTIAL ESTIMATES FOR "NOT VERY LARGE DEVIATIONS" AND WARE FRONT PROPAGATION FOR A CLASS OF REACTION-DIFFUSION EQUATIONS

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EXPONENTIAL ESTIMATES FOR "NOT VERY LARGE DEVIATIONS" AND WAVE FRONT PROPAGATION FOR A CLASS OF REACTION-DIFFUSION EQUATIONS

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ABSTRACT. A Large Deviation Principle for a class of random processes depending on a small parameter $\varepsilon > 0$ is established. This class of processes arises from a random perturbation of a dynamical system. Then, exponential estimates for events of the type "not very large deviations" (deviations of order ε^{κ}) are obtained. Finally, the wave front propagation, as $\varepsilon \downarrow 0$, of the solution of some initial-boundary value problems is analyzed; these problems are formulated in terms of a reaction-diffusion equation whose diffusion coefficient is of order $\frac{1}{\varepsilon}$ and the non linear term is of order $\frac{1}{\varepsilon^{1-2\kappa}}$. The wave front is characterized in terms of the action functional corresponding to the Large Deviation Principle initially obtained

1. Introduction

This paper is concerned with a family of random processes $(X_t^\varepsilon : t \geq 0)$ depending on a small parameter $\varepsilon > 0$ and satisfying the system of differential equations

$$\dot{X}_t^\varepsilon = b(X_t^\varepsilon, Y_t^\varepsilon), \qquad X_0^\varepsilon = x \in \mathbf{R}^d \tag{1.1}$$

where $b(x,y) = (b^1(x,y), \cdots, b^d(x,y))$, $x \in \mathbf{R}^d$, $y \in \mathbf{R}^l$, is bounded as well as are its first and second derivatives. We define $Y_t^\varepsilon \equiv Y_{\frac{t}{\varepsilon}}$ where $(Y_t : t \geq 0)$ is a random process whose trajectories are continuous with probability one or have a finite number of descontinuities of first kind on any finite interval. These conditions are sufficient (see [10]) for system (1.1) having a unique solution with probability one.

We assume that there exists a vector field $\bar{b}(x)$ in $\mathbf{R}^d$ such that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T b(x, Y_s)\, ds = \bar{b}(x), \qquad \forall x \in \mathbf{R}^d, \tag{1.2}$$

uniformly in x, with probability one. Under (1.2) the trajectories of X_t^ε converge, as $\varepsilon \downarrow 0$, to the solution $(\bar{x}_t : t \geq 0)$ of

$$\dot{\bar{x}}_t = \bar{b}(\bar{x}_t), \qquad \bar{x}_0 = x \in \mathbf{R}^d. \tag{1.3}$$

Key words and phrases. Large deviation, action functional, "not very large deviations", wave front propagation.

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The convergence is in the space $(C_{[0,T]}(\mathbf{R}^d); \|\cdot\|)$ of the continuous functions on $[0,T]$ with values in $\mathbf{R}^d$, with the supremum norm $\|\cdot\|$.

As a consequence of the Averaging Principle,

$$\lim_{\varepsilon \downarrow 0} P\{\|X^\varepsilon_\cdot - \bar{x}_\cdot\| > \delta\} = 0, \quad \forall \delta > 0. \tag{1.4}$$

The event $[\|X^\varepsilon_\cdot - \bar{x}_\cdot\| > \delta]$ represents a "Large Deviation" (deviation of order 1) of $X^\varepsilon_\cdot$ from $\bar{x}_\cdot$. Problems related with deviations of order 1 were extensively studied by Freidlin (see [6]-[9]). Under some aditional conditions, he proved that $(X^\varepsilon_t : t \geq 0)$ has a normalized action functional in the space $(C_{[0,T]}(\mathbf{R}^d); \|\cdot\|)$ which is given by

$$S_{0T}(\varphi) = \begin{cases} \int_0^T L(\varphi_s; \dot{\varphi}_s)\, ds, & \varphi \text{ a.c.} \\ +\infty, & \text{in the rest of } C_{[0,T]}(\mathbf{R}^d), \end{cases} \tag{1.5}$$

the function $L(x,\beta)$ is the Legendre transform of $\lambda(x,\alpha)$ with

$$\lambda(x,\alpha) = \lim_{T\to+\infty} \frac{1}{T} \ln E \exp\left\{ \int_0^T < \alpha, b(x, Y_s) > ds \right\}, \quad x,\alpha,\beta \in \mathbf{R}^d, \tag{1.6}$$

where E is the expectation corresponding to the distribution of $(Y_t : t \geq 0)$. The normalized coefficient is $\frac{1}{\varepsilon}$.

According to the definition of action functional (see Freidlin [10]), $S_{0T}(\varphi)$ satisfies the following conditions:

(A.0) Compactness of the level sets: $\forall s > 0$, $\forall x \in \mathbf{R}^d$,

$$\Phi(s) = \{\varphi \in C_{[0,T]}(\mathbf{R}^d) : S_{0T}(\varphi) \leq s,\ \varphi_0 = x\}$$

are compact sets.

(A.I) Lower bound: $\forall \delta > 0$, $\forall \gamma > 0$, $\forall \varphi \in C_{[0,T]}(\mathbf{R}^d)$, $\exists \varepsilon_0 > 0$ such that

$$P\{\|X^\varepsilon_\cdot - \varphi\| < \delta\} \geq \exp\left\{-\frac{1}{\varepsilon}[S_{0T}(\varphi) + \gamma]\right\}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

(A.II) Upper bound: $\forall \delta > 0$, $\forall \gamma > 0$, $\forall s > 0$, $\exists \varepsilon_0 > 0$ such that

$$P\{\rho_{0T}(X^\varepsilon_\cdot, \Phi(s)) \geq \delta\} \leq \exp\left\{-\frac{1}{\varepsilon}(s - \gamma)\right\}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Under conditions **(A.0)**-**(A.II)**, one say that $S_{0T}(\cdot)$ is the normalized action functional for the family of random processes $(X^\varepsilon_t : t \geq 0)$ with normalizing coeficient $\frac{1}{\varepsilon}$. Moreover, it characterizes a Large Deviation Principle (LDP) for that family.

The functional (1.5) allows one to obtain exponential estimate for $P\{\|X^\varepsilon_\cdot - \bar{x}_\cdot\| > \delta\}$. Moreover, Freidlin, in many of his articles, considered a variety of applications of the Large Deviations Principle characterized by (1.5).

Taking into account the smoothness of $b(x,y)$ and assuming conditions of strong mixing for $(Y_t : t \geq 0)$, Khas'minskii [13] proved that

$$\zeta_t^\varepsilon = \frac{X_t^\varepsilon - \bar{x}_t}{\sqrt{\varepsilon}} \tag{1.7}$$

converges weakly, as $\varepsilon \downarrow 0$, to a Gaussian Markov process ζ_t^0 on $[0,T]$. The precise assumptions for the function $b(x,y)$ and the process $(Y_t : t \geq 0)$ may be found in [13] or in Theorem 7.3.1 in [10]. The limit process satisfies the system of linear differential equations

$$\dot{\zeta}_t^0 = \dot{W}_t^0 + \bar{B}(\bar{x}_t)\,\zeta_t^0, \quad \zeta_0^0 = 0; \tag{1.8}$$

the process W_t^0 is Gaussian with independent increments, $EW_t^0 = 0$, and correlation matrix $(R^{ki}(t))$ given by

$$R^{ki}(t) = EW_t^{0,k} W_t^{0,i} = \int_0^t A^{ki}(\bar{x}_s)\,ds,$$

with

$$A^{ki}(x) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_0^T A^{ki}(x,s,t)\,ds\,dt \tag{1.9}$$

and

$$A^{ki}(x,s,t) = E[b^k(x,Y_s) - Eb^k(x,Y_s)][b^i(x,Y_t) - Eb^i(x,Y_t)]. \tag{1.10}$$

The matrix $\bar{B}(x)$ is given by

$$\bar{B}_k^i(x) = \frac{\partial \bar{b}^i}{\partial x^k}(x) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T B_k^i(x,Y_s)\,ds \tag{1.11}$$

uniformly in x, with probability one, and $B_k^i(x,y) = \frac{\partial b^i}{\partial x^k}(x,y)$.

The study of the normalized difference (1.7) was carried out by considering the norma-lized linearized system obtained by linearization in the neighborhood of $\bar{x}_t$ and then verifying that the original system differs from the linearized one by an infinitely small quantity compared with $\sqrt{\varepsilon}$. The weak convergence of ζ_t^ε to ζ_t^0 characterizes the asymptotic behavior of deviations of order $\sqrt{\varepsilon}$:

$$\lim_{\varepsilon\downarrow 0} P\{\|X_\cdot^\varepsilon - \bar{x}_\cdot\| > \delta\sqrt{\varepsilon}\} = P\{\|\zeta_\cdot^0\| > \delta\}, \quad \forall \delta > 0. \tag{1.12}$$

In this paper we are mainly interested in the asymptotic behavior of $(Z_t^\varepsilon : t \geq 0)$, as $\varepsilon \downarrow 0$, where

$$Z_t^\varepsilon = \frac{X_t^\varepsilon - \bar{x}_t}{\varepsilon^\kappa}, \quad 0 < \kappa < \frac{1}{2}. \tag{1.13}$$

It turns out that , $\forall\delta > 0$,

(1.14)
$$\lim_{\varepsilon\downarrow 0} P\{\|X^\varepsilon_\cdot - \bar{x}_\cdot\| > \delta\varepsilon^\kappa\} = 0.$$

Deviations of order ε^κ of $X^\varepsilon_\cdot$ from $\bar{x}_\cdot$ are called "not very large deviations ". Baîer & Freidlin [3] and Freidlin [10] considered "not very large deviations" when the initial condition is an equilibrium point of the system (1.3). They studied the stability of the solution of (1.1) in a neighborhood of order ε^κ of the equilibrium point, as $\varepsilon\downarrow 0$. In this case, if 0 is the initial point, then $\bar{b}(0) = 0$ and the process Z^ε_t becomes

(1.15)
$$Z^\varepsilon_t = \frac{X^\varepsilon_t}{\varepsilon^\kappa}.$$

In [3] (or [10]) a LDP for the family of processes in (1.15) is enunciated and a suggestion for the proof of the lower bound **(A.I)** and upper bound **(A.II)** in the definition of action functional is given. Using this LDP, it was proved that

$$\lim_{\varepsilon\downarrow 0} P\{\|X^\varepsilon_\cdot\| > \delta\varepsilon^\kappa\} = 0.$$

By using the method suggested by Baîer & Freidlin we established a LDP for the fa-mily Z^ε_t in (1.13) when the initial point is not necessarily an equilibrium point. From the smoothness of $b(x,y)$ we write

(1.16)
$$\begin{aligned} X^\varepsilon_t - \bar{x}_t &= \int_0^t b(X^\varepsilon_s, Y^\varepsilon_s)\,ds - \int_0^t \bar{b}(\bar{x}_s)\,ds = \\ &= \int_0^t [b(\bar{x}_s, Y^\varepsilon_s) - \bar{b}(\bar{x}_s)]\,ds + \int_0^t B(\bar{x}_s, Y^\varepsilon_s)(X^\varepsilon_s - \bar{x}_s)\,ds + \\ &+ \int_0^t r^2(X^\varepsilon_s - \bar{x}_s)\,ds \end{aligned}$$

where $B(x,y)$ is given in (1.11) and $r^2(\cdot)$ is the rest of Lagrange in the Taylor's expansion of $b(x,y)$ in a neighborhood of $\bar{x}_t$.

Let us define

(1.17)
$$\eta^\varepsilon_t = \frac{1}{\sqrt{\varepsilon}}\int_0^t [b(\bar{x}_s, Y^\varepsilon_s) - \bar{b}(\bar{x}_s)]\,ds \equiv \frac{1}{\sqrt{\varepsilon}}\int_0^t \tilde{b}(\bar{x}_s, Y^\varepsilon_s)\,ds.$$

Then Z^ε_t in (1.13) satisfies

(1.18)
$$\dot{Z}^\varepsilon_t = \varepsilon^{\frac{1}{2}-\kappa}\dot{\eta}^\varepsilon_t + B(\bar{x}_t, Y^\varepsilon_t)\,Z^\varepsilon_t + \frac{r^2(X^\varepsilon_t - \bar{x}_t)}{\varepsilon^\kappa}, \quad Z^\varepsilon_0 = 0.$$

Our first result is a LDP for the family of random processes $\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_t$. Let us assume the following conditions:

Condition B-1. *There exists a matrix* $A(x) = (A^{ij}(x))_{i,j=1,\cdots,d}$ *nonnegative definite, symmetric, continuous in* x, *invertible, such that for any step functions* $\alpha, \psi : [0,T] \to \mathbf{R}^d$,

$$(1.19)\qquad \lim_{\varepsilon\downarrow 0} \varepsilon^{1-2\kappa} \ln E \exp\left\{ \frac{1}{\varepsilon^{1-\kappa}} \int_0^T < \alpha_s, \bar{b}(\psi_s, Y_s^\varepsilon) > ds \right\} = \\ = \frac{1}{2} \int_0^T < A(\psi_s)\alpha_s, \alpha_s > ds.$$

Condition B-2. $\exists t_0, 0 < t_0 \le 1$ *and a function* $\sigma(t) > 0$ *with* $\sigma(t) \to 0$ *as* $t \downarrow 0$ *such that*

$$(1.20)\qquad \overline{\lim_{\varepsilon\downarrow 0}} \sup_{\substack{\varepsilon\le t\le t_0 \\ 0\le h\le 1-t}} \varepsilon^{1-2\kappa} \left\| \ln E \exp\left\{ \frac{1}{\varepsilon^{1-\kappa}\sigma(t)} \int_h^{h+t} \bar{b}(\bar{x}_s, Y_s^\varepsilon)\, ds \right\} \right\| = \\ = l_{+\infty} < +\infty.$$

Condition B-3. $\forall \Delta > 1 - 2\kappa$, $\forall \delta > 0$,

$$(1.21)\qquad \lim_{\varepsilon\downarrow 0} \varepsilon^{\Delta} \ln P\left\{ \sup_{0\le t\le T} \left\| \varepsilon^{-\kappa} \int_0^t (B(\bar{x}_s, Y_s^\varepsilon) - \bar{B}(\bar{x}_s)) \right.\right. \\ \left.\left. \int_0^s e^{\int_u^s \bar{B}(\bar{x}_v)\,dv} . \bar{b}(\bar{x}_u, Y_u^\varepsilon)\, du\, ds \right\| > \delta \right\} = -\infty.$$

Remark 1.1. Condition **B-1** is equivalent to the existence of the limit in (1.19) for every continuous functions α and ψ.

In §3 we proof the following theorem:

Theorem 1. *Under conditions* **B-1** *and* **B-2**, *the action functional for the family of random processes* $\varepsilon^{\frac{1}{2}-\kappa}\eta_t^\varepsilon$ *is given by* $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}^1(\varphi)$, *where*

$$(1.22)\qquad S_{0T}^1(\varphi) = \begin{cases} \frac{1}{2}\int_0^T < A^{-1}(\bar{x}_s)\dot{\varphi}_s, \dot{\varphi}_s > ds, & \varphi \text{ a.c.} \\ +\infty, & \text{in the rest of } C_{[0,T]}(\mathbf{R}^d) \end{cases}$$

where $A^{-1}(x)$ *is the inverse of* $A(x)$.

Theorem 1 is an extension of a result obtained by Gärtner [11]. He considered a family of random processes converging weakly, as $\varepsilon \downarrow 0$, to a Wiener process in $\mathbf{R}$. He established sufficient conditions for this family of random processes, conveniently rescaled, having the same action functional of the limit process in the new scale. In Theorem 1 we extend Gärtner's result in two ways: the space variable has dimension $d \ge 1$ and the family of random processes η_t^ε converges weakly, as $\varepsilon \downarrow 0$, to a Gaussian process W_t^0 introduced in (1.8) if we assume the hypothesis for Khas'minskii's result being valid. It is worth to observe that the weak convergence above cited is not an hypothesis of Theorem 1. But if the matrix $A(x)$ in (1.19) satisfies (1.9) then the action functional for $\varepsilon^{\frac{1}{2}-\kappa}W_t^0$ is $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}^1(\varphi)$ (for action functionals for families of Gaussian processes see Freidlin & Wentzell [10]).

The main result in this paper is a LDP for the family $(Z_t^\varepsilon : t \ge 0)$ in (1.13). In §4 we prove the following theorem:

Theorem 2. *If conditions* **B-1, B-2,** *and* **B-3** *are satisfied then the action functional for* $(Z_t^\varepsilon : t \geq 0)$ *is given by* $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}(\varphi)$ *with*

$$(1.23) \qquad S_{0T}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T < A^{-1}(\bar{x}_s)(\dot{\varphi}_s - \bar{B}(\bar{x}_s)\varphi_s), (\dot{\varphi}_s - \bar{B}(\bar{x}_s)\varphi_s) > ds, \\ \qquad \text{if } \varphi \text{ is a.c.} \\ +\infty, \quad \text{in the rest of } C_{[0,T]}(\mathbf{R}^d) \end{cases}$$

where $\bar{B}(x)$ *satisfies (1.11).*

Notice that (1.23) is also the normalized action functional for a Gaussian process ζ_t^ε satisfying the linear system

$$(1.24) \qquad \dot{\zeta}_t^\varepsilon = \bar{B}(\bar{x}_t)\,\zeta_t^\varepsilon + \varepsilon^{\frac{1}{2}-\kappa}\sigma(\bar{x}_t)\,\dot{W}_t, \quad \zeta_0^\varepsilon = 0,$$

where $\sigma(x)\sigma^*(x) = A(x)$ and W_t is a Wiener process in $\mathbf{R}^d$ starting at zero. This result may be found in [10], Chapter 3.

Using Theorem 2 we prove (1.14) and we also obtain exponential estimates for probabi-lities of "not very large deviations". Theorem 2 may be interpreted as follows: probabilities of deviations of order ε^κ of X_t^ε from $\bar{x}_t$ have the same asymptotics as deviations of order 1 caused by the Gaussian process in (1.24). The asymptotics of probabilities of "not very large deviations" are essentially different of the corresponding to "large deviations". As in the case of "normal deviations", the study of deviations of order ε^κ is reduced to the study of deviations of the same order of the linearized system obtained from (1.16).

Now we sketch the proof of Theorem 2. Firstly we consider the linearized system

$$(1.25) \qquad \dot{\tilde{\Upsilon}}_t^\varepsilon = \tilde{b}(\bar{x}_t, Y_t^\varepsilon) + B(x_t, Y_t^\varepsilon)\,\tilde{\Upsilon}_t^\varepsilon, \quad \tilde{\Upsilon}_0^\varepsilon = 0.$$

We prove that, if $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}(\varphi)$ is the action functional for $\frac{\tilde{\Upsilon}_t^\varepsilon}{\varepsilon^\kappa}$ then it is the action functional for Z_t^ε. Then we take a simplified linearized system

$$(1.26) \qquad \dot{\hat{\Upsilon}}_t^\varepsilon = \tilde{b}(x_t, Y_t^\varepsilon) + B(x_t)\,\hat{\Upsilon}_t^\varepsilon, \quad \hat{\Upsilon}_0^\varepsilon = 0.$$

It turns out that, under Condition **B-3**, $\frac{\tilde{\Upsilon}^\varepsilon}{\varepsilon^\kappa}$ and $\frac{\hat{\Upsilon}_t^\varepsilon}{\varepsilon^\kappa}$ have the same action functional. Finally, using Theorem 1, we prove that $\frac{\hat{\Upsilon}_t^\varepsilon}{\varepsilon^\kappa}$ has $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}(\varphi)$ as its action functional.

In §5 we study the asymptotics of the solution for a class of reaction-diffusion equations depending on a small parameter $\varepsilon > 0$, as $\varepsilon \downarrow 0$. Using Theorem 1 and Theorem 2 we prove that the solution converges to a function of the wave-type.

Solutions of wave-type for reaction-diffusion equations have been studied since 1930's by Kolmogorov, Petrovskii, Piskounov [14] (such equation is called KPP equation), Aronson & Weinberger [2], by using classical methods and later, after 1970, by Freidlin, Gärtner, McKean, and others, via stochastic approach. Freidlin [7] introduced a small parameter $\varepsilon > 0$ in the generalized KPP equation whose diffusion coefficient became small, of order ε. He described the wave front for the solution of certain class of problems, as $\varepsilon \downarrow 0$, by using the Feynman-Kac formula and Large Deviations for some families of random processes.

Carmona [4] generalized Freidlin's work in one direction by introducing a "fast variable" y of order $\frac{1}{\varepsilon}$ in some initial-boundary value problems for the equation

$$\frac{\partial u^\varepsilon(t,x,y)}{\partial t} = \frac{1}{2\varepsilon}\frac{\partial^2 u^\varepsilon(t,x,y)}{\partial y^2} + \frac{\varepsilon}{2}a(x,y)\frac{\partial^2 u^\varepsilon(t,x,y)}{\partial x^2} + \\ + \frac{1}{\varepsilon}f(x,y,u^\varepsilon), \quad x \in \mathbf{R}, \quad |y| < a, \quad t > 0.$$

In §5 of this paper, we consider problems of the type

$$(1.27) \qquad \begin{cases} \frac{\partial u^\varepsilon(t,x,y)}{\partial t} = \frac{1}{2\varepsilon}\frac{\partial^2 u^\varepsilon(t,x,y)}{\partial y^2} + \frac{1}{\varepsilon^{1-2\kappa}}f(\varepsilon^\kappa x, y, u^\varepsilon(t,x,y)) + \\ \frac{1}{\varepsilon^\kappa}b(\varepsilon^\kappa x, y)\frac{\partial u^\varepsilon(t,x,y)}{\partial x}, \quad x \in \mathbf{R}^d, \quad y \in (-a,a), \\ u^\varepsilon(0,x,y) = g(x) \\ \frac{\partial u^\varepsilon(t,x,y)}{\partial y}/_{y=\pm a} = 0 \end{cases}$$

where $0 < \kappa < \frac{1}{2}$, $b(x,y)$ satisfies the conditions specified in the introduction of this paper, the initial function is nonnegative, and its support $G_0 \neq \mathbf{R}^d$, $[(G_0)] = [G_0]$ where $[A]$ is the closure of A and (A) its interior. For each x, y, the nonlinear term $f(x,y,u)$ belongs to the class $\mathcal{F}_1$ (see Freidlin [6]), i.e., $f(x,y,\cdot) \in C^1$, $c(x,y) = f'(x,y,0) = \sup_{0\le u\le 1} \frac{f(x,y,u)}{u} > 0$, and $c(x,y,u) = \frac{f(x,y,u)}{u}$.

To analyze the solution $u^\varepsilon(t,x,y)$ of this type of problem we shall use the Feynman-Kac formula and "not very large deviations" for families of random processes as in (1.1) or, equivalently, large deviations for families of random processes as in (1.13) and (1.15). This is done, roughly speaking, in the following way: To the differential operator

$$(1.28) \qquad L^\varepsilon = \frac{1}{2\varepsilon}\frac{\partial^2}{\partial y^2} + \frac{1}{\varepsilon^\kappa}b(\varepsilon^\kappa x, y)\frac{\partial}{\partial x}$$

it is associated a random process $(X_t^\varepsilon, Y_t^\varepsilon; P_{xy}^\varepsilon)$ where

$$(1.29) \qquad X_t^\varepsilon = x + \frac{1}{\varepsilon^\kappa}\int_0^t b(\varepsilon^\kappa X_s^\varepsilon, Y_s^\varepsilon)\,ds, \qquad x \in \mathbf{R}^d,$$

$(Y_t; \tilde{P}_y)$ is a Brownian motion on $[-a,a]$ starting at $y \in (-a,a)$, with instantaneous reflection at $\pm a$, and $Y_t^\varepsilon \equiv Y_{\frac{t}{\varepsilon}}$. Notice that the diffusion coefficient of the variable y is of order $\frac{1}{\varepsilon}$; so it is called "fast variable".

The Feynman-Kac formula allow us to express the solution of (1.27) as

$$(1.30) \quad u^\varepsilon(t,x,y) = E_{xy}^\varepsilon g(X_t^\varepsilon)\exp\left\{\frac{1}{\varepsilon^{1-2\kappa}}\int_0^t c(\varepsilon^\kappa X_s^\varepsilon, Y_s^\varepsilon, u^\varepsilon(t-s, X_s^\varepsilon, Y_s^\varepsilon))\,ds\right\}.$$

Using the action functional for certain families of random processes as in (1.13) and (1.15) one can verify that $u^\varepsilon(t,x,y)$ converges, as $\varepsilon \downarrow 0$, to a step function $u^0(t,x,y)$ given by

$$u^0(t,x,y) = \begin{cases} 1, & V(t,x) > 0, \quad |y| \le a \\ 0, & V(t,x) < 0, \quad |y| \le a, \end{cases}$$

for some function $V(t,x)$ which will be specified in §5.

2. Auxiliary Results

Proposition 2.1. *If condition* **B-1** *holds then* $\forall x,\ \alpha \in \mathbf{R}^d$,

$$\lim_{T\to+\infty} T^{2\kappa-1} \ln E \exp\left\{T^{-\kappa}\int_0^T < \alpha, \bar{b}(x,Y_s) > ds\right\} = \tag{2.1}$$
$$= \frac{1}{2} < A(x)\alpha, \alpha > .$$

Proof: Condition **B-1** implies that, $\forall x,\ \alpha \in \mathbf{R}^d$

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{1-2\kappa} \ln E \exp\left\{\frac{1}{\varepsilon^{1-\kappa}}\int_0^T < \alpha, \bar{b}(x,Y_{\frac{s}{\varepsilon}}) > ds\right\} =$$
$$= \frac{1}{2} T < A(x)\alpha, \alpha > .$$

Then

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{1-2\kappa} \frac{1}{T} \ln \bar{E} \exp\left\{\frac{1}{\varepsilon^{1-\kappa}}\int_0^T < \frac{\alpha}{T^\kappa}, \bar{b}(x,Y_{\frac{t}{\varepsilon}}) > dt\right\} =$$
$$= \frac{1}{2}\frac{1}{T^{2\kappa}} < A(x)\alpha, \alpha >$$

The result follows by changing variables in the last equality.

■

The proof of the following proposition is similar to the one of Lemma 7.4.3 in [10] and we omit it.

Proposition 2.2. *Suppose that* $(Y_t; \bar{P}_y)$ *is a homogeneous Markov process with values in a compact set* $D \subset \mathbf{R}^l$ *and (2.1) holds uniformly in the initial point* $y \in (D)$, *where* (D) *is the interior of* D. *Then Condition* **B-1** *is satisfied.*

Now we shall characterize a class of random processes $(Y_t : t \ge 0)$ which satisfies conditions **B-1** and **B-2**.

Lemma 2.1. *Let* $(Y_t; \bar{P}_y)$ *be a homogeneous Markov process on the phase space* $(D, \mathcal{B}(D))$, $D \subset \mathbf{R}^l$ *compact, and* $\mathcal{B}(D)$ *the σ-field of the Borel subsets of* D

in the topology inherited from the Euclidean norm in $\mathbf{R}^l$. *Assume conditions* (L.1)-(L.5) *in Theorem 2.2, [5]. Then Condition* B-1 *is satisfied.*

Proof: Let us suppose that $\bar{b}(0) = 0$. For each $\alpha \in \mathbf{R}^d$ we introduce the semigroup of operators

$$T_t^\alpha f(y) = \bar{E}_y f(Y_t) \exp\left\{\int_0^t < \alpha, b(0, Y_s) > ds\right\},$$

where f is a continuous numerical function on D.

From Theorem 2.2 in [5] we know that

$$\lim_{T\to+\infty} \frac{1}{T} \ln \bar{E}_y \exp\left\{\int_0^T < \alpha, b(0, Y_s) > ds\right\} = \lambda(\alpha), \tag{2.2}$$

where $\lambda(\alpha)$ is the maximal eigenvalue of $\mathcal{A}^\alpha$, the infinitesimal generator of T_t^α. It is real and simple, the corresponding eigenvector ϕ is positive, and $\|\phi\| = 1$. Moreover, $T_t^\alpha \phi(y) = e^{\lambda(\alpha)t}\phi(y)$.

From Theorem 7.1.8 in [10] we can write

$$\lambda(\alpha) = \lambda'(0) + \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 \lambda}{\partial\alpha_i\partial\alpha_j}(0)\alpha_i\alpha_j + o(\alpha^2).$$

Taking into account that 1 is the maximal eigenvalue of T_t^α for $\alpha = 0$, we have $\lambda(0) = 0$. On the other hand, from (2.2)

$$\lambda(\alpha) \geq \lim_{T\to+\infty} \frac{1}{T}\bar{E}_y \int_0^T < \alpha, b(0, Y_s) > ds =$$
$$=< \alpha, \lim_{T\to+\infty} \frac{1}{T}\int_0^T \bar{E}_y b(0, Y_s)\, ds >=< \alpha, \bar{b}(0) > .$$

Since $\bar{b}(0) = 0$ we have $\lambda(\alpha) \geq 0$, for all $\alpha \in \mathbf{R}^d$. Therefore $\lambda'(0) = 0$ and

$$\lambda(\alpha) = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 \lambda}{\partial\alpha_i\partial\alpha_j}(0)\alpha_i\alpha_j + o(\alpha^2). \tag{2.3}$$

Now, the compactness of D implies that $\exists K > 0$ such that $0 < K \leq \phi(y) \leq 1$, $\forall y \in D$. Then,

$$t^{2\kappa-1}\ln K + t^{2\kappa-1}\ln(T_t^{t^{-\kappa}\alpha}1)(y) \leq t^{2\kappa-1}\ln(T_t^{t^{-\kappa}\alpha}\phi)(y) =$$
$$= t^{2\kappa}\lambda(t^{-\kappa}\alpha) + t^{2\kappa-1}\ln\phi(y) \leq t^{2\kappa-1}\ln(T_t^{t^{-\kappa}\alpha}1)(y).$$

Hence, using (2.3) we get

$$\lim_{t\to+\infty} t^{2\kappa-1}\ln\left(T_t^{t^{-\kappa}\alpha}1\right)(y) = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2\lambda}{\partial\alpha_i\partial\alpha_j}(0)\alpha_i\alpha_j \equiv$$
$$\equiv \frac{1}{2} < A\alpha, \alpha >$$

which is Condition **B-1** in the case $\bar{b}(0)=0$.

When the initial point in (1.3) is not an equilibrium point, the arguments are the same as above if one recall that

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \bar{b}(x,Y_s)\,ds=0, \qquad \forall x\in \mathbf{R}^d,\ \text{w.p. } 1$$

and then $\lambda(x,\alpha)\geq 0$, $\forall x,\alpha$. The matrix $A(x)$ in Condition **B-1** is given by

$$A(x)=\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2\lambda}{\partial\alpha_i\partial\alpha_j}(x,0)\alpha_i\alpha_j. \tag{2.4}$$

■

Let $\psi\in C_{[0,T]}(\mathbf{R}^d)$ be a step function, constant in the intervals $[\frac{j}{r}T,\frac{j+1}{r}T]$, $j=0,1,2,\cdots,r-1$. For each $\alpha=(\alpha^1,\cdots,\alpha^r)\in(\mathbf{R}^d)^r$, define

$$H^\psi(\alpha)=\frac{1}{2r}\sum_{j=0}^{r-1}< A(\psi_{\frac{jT}{r}})\alpha^{j+1},\alpha^{j+1}>. \tag{2.5}$$

The function in (2.5) is convex, $H^\psi(0)=0$, it is lower semicontinuous, $H^\psi(\alpha)<+\infty$, $\forall\alpha$. Let $L^\psi(\beta)$ be its Legendre transform:

$$\begin{aligned} L^\psi(\beta)&=\sup_\alpha\{<\alpha,\beta>-H^\psi(\alpha)\}=\\ &=\frac{r}{2}\sum_{j=0}^{r-1}< A^{-1}(\psi_{\frac{jT}{r}})\beta^{j+1},\beta^{j+1}>, \qquad \beta\in(\mathbf{R}^d)^r. \end{aligned} \tag{2.6}$$

The function in (2.6) is convex, lower semicontinuous, assuming values in $(-\infty,+\infty]$, and it is not identically equal to $+\infty$.

Define for each $s>0$,

$$\begin{aligned} \Phi^r(s)&=\{\beta\in(\mathbf{R}^d)^r: L^\psi(\beta)\leq s\}=\\ &=\left\{\beta\in(\mathbf{R}^d)^r:\frac{r}{2}\sum_{j=0}^{r-1}< A^{-1}(\psi_{\frac{jT}{r}})\beta^{j+1},\beta^{j+1}>\leq s\right\}. \end{aligned} \tag{2.7}$$

The following proposition is similar to Theorem 5.1.1 in [10] and we omit its proof.

Proposition 2.3. $\forall\delta>0$, $\forall s>0$, $\exists\alpha_1,\cdots,\alpha_N\in(\mathbf{R}^d)^r$ *such that*

$$\Phi^r(s)\subset\bigcap_{i=1}^{N}\{\beta:<\alpha_i,\beta>-H^\psi(\alpha_i)\leq s\}\subset\Phi^r_{+\delta}(s),$$

where $\Phi^r_{+\delta}(s) = \{\beta : dist(\beta, \Phi^r(s)) < \delta\}$.

Let us define, for each $x, \alpha \in \mathbf{R}^d$

$$H(x,\alpha) = \frac{1}{2} < A(x)\alpha, \alpha > = \frac{1}{2}\sum_{i,j=1}^{d} A_{ij}(x)\alpha_i\alpha_j. \tag{2.8}$$

This function is convex in the second argument and jointly continuous (by hypothesis $A(x)$ is continuous). Let $L(x,\beta)$ be its Legendre transform

$$L(x,\beta) = \sup_\alpha\{< \alpha, \beta > - H(x,\alpha)\} = \frac{1}{2} < A^{-1}(x)\beta, \beta >, \quad \beta \in \mathbf{R}^d. \tag{2.9}$$

This function is convex in β and jointly lower semicontinuous in all variables (see the proof of Lemma 4.1.7 in [10]).

Let us define

$$\begin{aligned} G_\varepsilon(\alpha) \equiv \ln \bar{E}_y \exp\left\{\int_0^T \alpha_t\, d\eta_t^\varepsilon\right\} = \\ = \ln \bar{E}_y \exp\left\{\frac{1}{\sqrt{\varepsilon}}\int_0^T < \alpha_t, \tilde{b}(\bar{x}_t, Y_t^\varepsilon) > dt\right\}, \quad \alpha : [0,T] \to \mathbf{R}^d. \end{aligned} \tag{2.10}$$

The process η_t^ε was introduced in (1.17). Condition **B-1** may be written, for $\psi_t = \bar{x}_t$, as

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{1-2\kappa} G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha\right) = \frac{1}{2}\int_0^T < A(\bar{x}_t)\alpha_t, \alpha_t > dt. \tag{2.11}$$

3. Proof of Theorem 1

The following theorem is an extension of results obtained by Gärtner [11]. We shall use the same approach for proving it.

It is well known (see [10]) that the level sets of the functional $S^1_{0T}(\cdot)$ in (1.22) are compact sets. So condition **(A.0)** is verified. The following theorem gives us condition **(A.I)** (the lower bound).

Theorem 3.1. *If conditions* **B-1** *and* **B-2** *are satisfied then* $\forall\gamma > 0$, $\forall\delta > 0$, $\forall\varphi \in C_{[0,T]}(\mathbf{R}^d)$, $\varphi_0 = 0$, $\exists\varepsilon_0 > 0$ *such that*

$$P\{\|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_\cdot - \varphi\| < \delta\} \geq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[S^1_{0T}(\varphi) + \gamma\right]\right\}, \quad 0 < \varepsilon \leq \varepsilon_0, \tag{3.1}$$

where η_t^ε *is the process introduced in (1.17).*

Proof: For simplifying notation we assume $T = 1$. Let $r > 0$ be an integer and $\bar{\eta}_t^\varepsilon$ be the random polygonal line with vertices at the points $\frac{i}{r}$ and $\bar{\eta}^\varepsilon_{\frac{i}{r}} = \eta^\varepsilon_{\frac{i}{r}}$,

$j = 1, \cdots, r$. Let $n \equiv n(\varepsilon) = r[\frac{1}{\varepsilon r}]$ and $\bar{\eta}_t^\varepsilon$ be the random polygonal line with vertices at $\frac{i}{n}$ with $\bar{\eta}_{\frac{i}{n}}^\varepsilon = \eta_{\frac{i}{n}}^\varepsilon$, $j = 1, \cdots, n$. Notice that $\bar{\eta}_{\frac{j}{r}}^\varepsilon = \bar{\eta}_{\frac{j}{r}}^\varepsilon$, $j = 1, \cdots, r$.

Let $(Q_m)_{m=1,2,\ldots}$ be a sequency of sets in $C_{[0,1]}(\mathbf{R}^d)$ which will be defined later. Then for any $\varphi \in C_{[0,1]}(\mathbf{R}^d)$ and $\delta > 0$,

$$
\begin{aligned}
&P\{\|\varepsilon^{\frac{1}{2}-\kappa}\eta_\cdot^\varepsilon - \varphi\| < \delta\} \geq \\
&\qquad \geq P\left\{\|\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}_\cdot^\varepsilon - \varphi\| < \frac{\delta}{2}\right\} - P\left\{\|\varepsilon^{\frac{1}{2}-\kappa}\eta_\cdot^\varepsilon - \varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}_\cdot^\varepsilon\| \geq \frac{\delta}{2}\right\} \geq \\
&\qquad \geq P\{\|\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}_\cdot^\varepsilon - \varphi\| < \frac{\delta}{2}\} - P\{\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}_\cdot^\varepsilon \notin Q_{n(\varepsilon)}\} - \\
&\qquad - P\{\|\varepsilon^{\frac{1}{2}-\kappa}\eta_\cdot^\varepsilon - \varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}_\cdot^\varepsilon\| \geq \frac{\delta}{2}, \varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}_\cdot^\varepsilon \in Q_{n(\varepsilon)}\} \equiv \\
&\qquad \equiv I_1 - I_2 - I_3.
\end{aligned}
\tag{3.2}
$$

Let $\varphi \in C_{[0,1]}(\mathbf{R}^d)$ with $S_{01}^1(\varphi) < +\infty$ and $\bar{\varphi}_t$ be the polygonal line with step $\frac{1}{r}$ such that $\bar{\varphi}_{\frac{k}{r}} = \varphi_{\frac{k}{r}}$, $k = 0, 1, \cdots, r$. Then, $\dot{\bar{\varphi}}_t$ is a step function. Let us define

$$
\alpha(t, x) = \frac{\partial L}{\partial \beta}(x, \dot{\bar{\varphi}}_t) \tag{3.3}
$$

where $L(x, \beta)$ was introduced in (2.9). Then

$$
\frac{\partial H}{\partial \alpha}(x, \alpha(t, x)) = \dot{\bar{\varphi}}_t \tag{3.4}
$$

where $H(x, \alpha)$ is given in (2.8). Since $\dot{\bar{\varphi}}_t$ is a step function then $\alpha(\cdot, x)$ is also a step function. Besides, $\alpha(t, x)$ is bounded because the matrix $A(x)$ and $\dot{\bar{\varphi}}_t$ are bounded.

Now we apply Cramér's method by introducing a new probability measure $\tilde{P}^\varepsilon$ defined by

$$
\tilde{P}^\varepsilon(A) = E\mathcal{X}_A \exp\left\{\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}} \int_0^1 \alpha(t, \bar{x}_t)\, d\eta_t^\varepsilon - G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha(\cdot, \bar{x}_\cdot)\right)\right\}.
$$

Since φ is continuous, then for r sufficiently large and $0 < \delta' < \frac{\delta}{2}$ we have

$$
I_1 \equiv P\left\{\|\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}_\cdot^\varepsilon - \varphi\| < \frac{\delta}{2}\right\} \geq P\left\{\max_{k=1,\cdots,r} \|\varepsilon^{\frac{1}{2}-\kappa}\eta_{\frac{k}{r}}^\varepsilon - \varphi_{\frac{k}{r}}\| < \delta'\right\},
$$

where $G_\varepsilon(\alpha)$ is given in (2.10). But
(3.5)

$$P\left\{\max_{k=1,\cdots,r}\|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_{\frac{k}{r}}-\varphi_{\frac{k}{r}}\|<\delta'\right\}=$$

$$=\tilde{E}^\varepsilon \mathcal{X}_{[\max_{k=1,\cdots,r}\|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_{\frac{k}{r}}-\varphi_{\frac{k}{r}}\|<\delta']}\exp\left\{-\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\int_0^1\alpha(t,\bar{x}_t)\,d\eta^\varepsilon_t+\right.$$

$$\left.+G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha(\cdot,\bar{x}_\cdot)\right)\right\}=$$

$$=\tilde{E}^\varepsilon \mathcal{X}_{[\max_{k=1,\cdots,r}\|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_{\frac{k}{r}}-\varphi_{\frac{k}{r}}\|<\delta']}\exp\left\{-\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\left[\int_0^1\alpha(t,\bar{x}_t)\,d(\eta^\varepsilon_t-\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\bar{\varphi}_t)\right]\right\}\times$$

$$\times\exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[\int_0^1\alpha(t,\bar{x}_t)\,\dot{\bar{\varphi}}_t\,dt-\varepsilon^{1-2\kappa}G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha(\cdot,\bar{x}_\cdot)\right)\right]\right\}.$$

¿From Condition **B-1** we have

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{1-2\kappa}G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha(\cdot,\bar{x}_\cdot)\right)=\frac{1}{2}\int_0^1<A(\bar{x}_t)\alpha(t,\bar{x}_t),\alpha(t,\bar{x}_t)>dt=$$

$$=\int_0^1 H(\bar{x}_t,\alpha(t,\bar{x}_t))\,dt.$$

Then $\forall\gamma>0$, $\exists\varepsilon_0>0$ such that

$$\frac{1}{\varepsilon^{1-2\kappa}}\left[\int_0^1 H(\bar{x}_t,\alpha(t,\bar{x}_t)\,dt-\frac{\gamma}{3}\right]<G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha(\cdot,\bar{x}_\cdot)\right)<$$

$$<\frac{1}{\varepsilon^{1-2\kappa}}\left[\int_0^1 H(\bar{x}_t,\alpha(t,\bar{x}_t))\,dt+\frac{\gamma}{3}\right],\qquad 0<\varepsilon\le\varepsilon_0.$$

Since $\alpha(t,x)\dot{\bar{\varphi}}_t-H(x,\alpha(t,x))=L(x,\dot{\bar{\varphi}}_t)$ and taking into account that

$$\int_0^1 L(\bar{x}_t,\dot{\bar{\varphi}}_t)\,dt\le\int_0^1 L(\bar{x}_t,\dot{\varphi}_t)\,dt,$$

the second exponential in (3.5) is greater than

$$\exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[\int_0^1 L(\bar{x}_t,\dot{\varphi}_t)\,dt+\frac{\gamma}{3}\right]\right\},\qquad 0<\varepsilon\le\varepsilon_0.$$

On the other hand, if $\max_{k=1,\cdots,r}|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_{\frac{k}{r}}-\varphi_{\frac{k}{r}}|<\delta''$, $0<\delta''<\delta'$ and δ'' sufficiently small, then we have

$$\int_0^1\alpha(t,\bar{x}_t)\,d(\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_t-\bar{\varphi}_t)<\frac{\gamma}{3}$$

because $\alpha(t,\bar{x}_t)$ is bounded. Hence, returning to (3.5), we obtain

$$I_1 \geq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\frac{\gamma}{3}\right\}.\exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[\int_0^1 L(\bar{x}_t,\dot{\varphi}_t)\,dt+\frac{\gamma}{3}\right]\right\}\times \tag{3.6}$$
$$\times \tilde{P}^\varepsilon\left\{\max_{k=1,\cdots,r}\|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_{\frac{k}{r}}-\varphi_{\frac{k}{r}}\|<\delta''\right\}, \qquad 0<\varepsilon\leq\varepsilon_0.$$

Now we shall prove that

$$\lim_{\varepsilon\downarrow 0}\tilde{P}^\varepsilon\left\{\max_{k=1,\cdots,r}\|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_{\frac{k}{r}}-\varphi_{\frac{k}{r}}\|\geq\delta''\right\}=0.$$

For this, it is sufficient to show that for $t\in\{\frac{1}{r},\frac{2}{r},\cdots,\frac{r}{r}\}$, the following relations are valid:

$$\lim_{\varepsilon\downarrow 0}\tilde{P}^\varepsilon\{\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon,i}_t-\bar{\varphi}^i_t-\delta''\geq 0\}=0, \quad \text{and}$$
$$\lim_{\varepsilon\downarrow 0}\tilde{P}^\varepsilon\{-\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon,i}_t+\bar{\varphi}^i_t-\delta''\geq 0\}=0, \qquad \text{for } i=1,\cdots,d. \tag{3.7}$$

From the Chebyshev's exponential inequality we can write, for all $\gamma^*>0$,

$$\tilde{P}^\varepsilon\left\{\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon,i}_t-\bar{\varphi}^i_t-\delta''\geq 0\right\}\leq\tilde{E}^\varepsilon\exp\left\{\frac{\gamma^*}{\varepsilon^{1-2\kappa}}[\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon,i}_t-\bar{\varphi}^i_t-\delta'']\right\}=$$
$$=E^\varepsilon\exp\left\{\frac{\gamma^*}{\varepsilon^{1-2\kappa}}[\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon,i}_t-\bar{\varphi}^i_t-\delta'']\right\}.$$
$$\exp\left\{\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\int_0^1\alpha(s,\bar{x}_s)\,d\eta^\varepsilon_s-G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha(\cdot,\bar{x}_\cdot)\right)\right\}=$$
$$=E^\varepsilon\exp\left\{\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\int_0^1[\alpha(s,\bar{x}_s)+\gamma^*\mathcal{X}_{[0,t]}e^{(i)}]\,d\eta^\varepsilon_s-\frac{\gamma^*}{\varepsilon^{1-2\kappa}}(\bar{\varphi}^i_t+\delta'')-\right.$$
$$\left.-G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha(\cdot,\bar{x}_\cdot)\right)\right\}$$

where $e^{(i)}$ is the component of order i of the canonical basis of $\mathbf{R}^d$. Hence,

$$\tilde{P}^\varepsilon\{\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon,i}_t-\bar{\varphi}^i_t-\delta''\geq 0\}\leq\exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[\varepsilon^{1-2\kappa}G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha(\cdot,\bar{x}_\cdot)\right)-\right.\right. \tag{3.8}$$
$$\left.\left.-\varepsilon^{1-2\kappa}G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\left[\alpha(\cdot,\bar{x}_\cdot)+\gamma^*\mathcal{X}_{[0,t]}e^{(i)}\right]\right)+\gamma^*(\bar{\varphi}^i_t+\delta'')\right]\right\}$$

From Condition **B-1**,

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{1-2\kappa}G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha(\cdot,\bar{x}_\cdot)\right)=\frac{1}{2}\int_0^1 <A(\bar{x}_s)\alpha(s,\bar{x}_s),\alpha(s,\bar{x}_s)>\,ds.$$

But

$$\alpha(s,\bar{x}_s)+\gamma^*\mathcal{X}_{[0,t]}e^{(i)}=\begin{cases}\alpha(s,\bar{x}_s)+\gamma^*e^{(i)}, & \text{if } s\leq t\\ \alpha(s,\bar{x}_s), & \text{if } s>t.\end{cases}$$

Then

$$\int_0^1 < A(\bar{x}_s)(\alpha(s,\bar{x}_s) + \gamma^* \mathcal{X}_{[0,t]} e^{(i)}), (\alpha(s,\bar{x}_s) + \gamma^* \mathcal{X}_{[0,t]} e^{(i)}) > ds =$$

$$= \int_0^1 < A(\bar{x}_s)\alpha(s,\bar{x}_s), \alpha(s,\bar{x}_s) > ds + 2\int_0^t \sum_{j=1}^d A_{ij}(\bar{x}_s)\alpha^j(s,\bar{x}_s)\gamma^* \, ds +$$

$$+ \gamma^{*2} \int_0^t A_{ii}(\bar{x}_s)\, ds.$$

Hence, Condition **B-1** implies that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{1-2\kappa} G_\varepsilon \left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}} [\alpha(\cdot, \bar{x}_\cdot) + \gamma^* \mathcal{X}_{[0,t]} e^{(i)}] \right) =$$

$$= \frac{1}{2} \int_0^1 < A(\bar{x}_s)\alpha(s,\bar{x}_s), \alpha(s,\bar{x}_s) > ds +$$

$$+ \gamma^* \int_0^t \sum_{j=1}^d A_{ij}(\bar{x}_s)\alpha^j(s,\bar{x}_s)\, ds + \frac{1}{2}\gamma^{*2} \int_0^t A_{ii}(\bar{x}_s)\, ds.$$

Therefore, the expression in brackets in (3.8) converges, as $\varepsilon \downarrow 0$, to

$$(3.9) \qquad \begin{aligned} &- \gamma^* \int_0^t \sum_{j=1}^d A_{ji}(\bar{x}_s)\alpha^j(s,\bar{x}_s)\, ds - \frac{1}{2}\gamma^{*2} \int_0^t A_{ii}(\bar{x}_s)\, ds + \gamma^*(\bar{\varphi}_t^i + \delta'') = \\ &= \gamma^* \left[\delta'' + \bar{\varphi}_t^i - \int_0^t \sum_{j=1}^d A_{ji}(\bar{x}_s)\alpha^j(s,\bar{x}_s)\, ds - \frac{\gamma^*}{2} \int_0^t A_{ii}(\bar{x}_s)\, ds \right]. \end{aligned}$$

We should find $\gamma^* > 0$ such that the above expression be strictly positive. For such γ^* we get (3.7) from (3.8).

Notice that

$$\frac{\partial H}{\partial \alpha^k}(x,\alpha) = \sum_{i=1}^d A_{ik}(x)\, \alpha^i$$

and

$$\frac{\partial^2 H}{\partial \alpha^k \partial \alpha^l}(x,\alpha) = \frac{\partial}{\partial \alpha^l} \left[\sum_{i=1}^d A_{ik}(x)\, \alpha^i \right] = A_{lk}(x).$$

¿From (3.4) we have

$$\frac{\partial H}{\partial \alpha^k}(x, \alpha(t,x)) = \sum_{i=1}^d A_{ik}(x)\alpha^i(t,x) = \dot{\bar{\varphi}}_t^k.$$

Therefore the limit in (3.9) reduces to

$$\gamma^* \left[\delta'' - \frac{\gamma^*}{2} \int_0^t A_{ii}(\bar{x}_s)\, ds \right].$$

¿From the hypothesis on $A(x)$ we have $\int_0^t A_{ii}(\bar{x}_s)\,ds > 0$. We choose $\gamma^* > 0$ such that $\delta" - \frac{\gamma^*}{2}\int_0^t A_{ii}(\bar{x}_s)\,ds > 0$.

Now we can say that $\forall\bar{\gamma} > 0$, $\exists\varepsilon_0 > 0$ such that

$$\bar{P}^\varepsilon\{\varepsilon^{\frac{1}{2}-\kappa}\eta_t^{\varepsilon,i} - \bar{\varphi}_t^i - \delta" \geq 0\} \leq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[\gamma^*\delta" - \frac{1}{2}\gamma^{*2}\int_0^t A_{ii}(\bar{x}_s)\,ds - \bar{\gamma}\right]\right\},$$

for all $0 < \varepsilon \leq \varepsilon_0$. Choose $0 < \bar{\gamma} < \gamma^*\delta" - \frac{1}{2}\gamma^{*2}\int_0^t A_{ii}(\bar{x}_s)\,ds$ and we get

$$\bar{P}^\varepsilon\{\varepsilon^{\frac{1}{2}-\kappa}\eta_t^{\varepsilon,i} - \bar{\varphi}_t^i - \delta" \geq 0\} \leq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}C\right\}, \quad 0 < \varepsilon \leq \varepsilon_0,$$

for some $C > 0$. Then, $\forall\delta > 0$, $\exists\varepsilon_0 > 0$ such that

$$\bar{P}^\varepsilon\left\{\max_{k=1,\cdots,r}\|\varepsilon^{\frac{1}{2}-\kappa}\eta_{\frac{k}{r}}^\varepsilon - \varphi_{\frac{k}{r}}\| < \delta"\right\} > 1 - \delta, \quad 0 < \varepsilon \leq \varepsilon_0.$$

But, for ε sufficiently small, $1 - \delta > \exp\{-\frac{\frac{2}{3}}{\varepsilon^{1-2\kappa}}\}$. Therefore, returning to (3.6), we conclude that $\forall\gamma > 0$, $\exists\varepsilon_0 > 0$ such that

$$I_1 > \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[S_{01}^1(\varphi) + \gamma\right]\right\}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

For estimating

$$I_2 \equiv P\left\{\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}_\cdot^\varepsilon \notin Q_{n(\varepsilon)}\right\}$$

we shall use arguments analogous to the ones in Gärtner [11]. In what follows, we outline them.

For each number $I > 0$ there exists a monotonic function $\Gamma(t)$ with $\Gamma(t) \downarrow 0$ as $t \downarrow 0$ such that

$$\inf_{0<t\leq t_0}\frac{2\Gamma(t)}{\sigma(t)} - l_\infty > 2I,$$

where t_0, l_∞, and $\sigma(t)$ come from Condition **B-2** in (1.20). We choose $r > 0$ sufficiently large such that $\Gamma(\frac{1}{r}) < \frac{\delta}{12}$ and $\frac{1}{r} < t_0$.

Define

$$(3.10)\qquad Q_n = \bigcap_{j=1}^{[nt_0]}\bigcap_{k=0}^{n-j}\left\{f \in C_{[0,1]}(\mathbf{R}^d) : \left\|f(\frac{k+j}{n}) - f(\frac{k}{n})\right\| < \Gamma(\frac{j}{n})\right\}.$$

Let $n(\varepsilon) = r[\frac{1}{\varepsilon r}]$. Then

$$I_2 \leq \sum_{j=1}^{[nt_0]}\sum_{k=0}^{n-j} P\left\{\|\varepsilon^{\frac{1}{2}-\kappa}\eta_{\frac{k+j}{n}}^\varepsilon - \varepsilon^{\frac{1}{2}-\kappa}\eta_{\frac{k}{n}}^\varepsilon\| \geq \Gamma(\frac{j}{n})\right\} \leq$$

$$\leq \sum_{j=1}^{[nt_0]}\sum_{k=0}^{n-j}\sum_{i=1}^{d} P\left\{|\varepsilon^{\frac{1}{2}-\kappa}\eta_{\frac{k+j}{n}}^{\varepsilon,i} - \varepsilon^{\frac{1}{2}-\kappa}\eta_{\frac{k}{n}}^{\varepsilon,i}| > \frac{\Gamma(\frac{j}{n})}{\sqrt{d}}\right\}.$$

¿From the Chebyshev's inequality we obtain

$$I_2 \le \sqrt{d}\, n^2(\varepsilon)\, t_0\, 2\exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[\inf_{0\le t\le t_0}\frac{\Gamma(t)}{\sigma(t)} - \right.\right.$$

$$\left.\left. - \sup_{\substack{\varepsilon<t\le t_0\\ 0<h\le 1-t}} \varepsilon^{1-2\kappa}\left\| \ln E\exp\left\{\pm\frac{1}{\varepsilon^{\frac12-\kappa}\sigma(t)}\int_h^{h+t} d\eta_t^{\varepsilon,i}\right\}\right\|\right]\right\}.$$

But $n^2(\varepsilon) = (r[\frac{1}{\varepsilon r}])^2 \le \frac{1}{\varepsilon^2}$, $\frac{1}{\varepsilon^2} = \exp\{-\frac{2\varepsilon^{1-2\kappa}\ln\varepsilon}{\varepsilon^{1-2\kappa}}\}$ and $\varepsilon^{1-2\kappa}\ln\varepsilon \to 0$ as $\varepsilon \downarrow 0$. Then, as in Gärtner [11]. we may conclude that

$$I_2 \le 2\, d\, t_0 \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}(2\varepsilon^{1-2\kappa}\ln\varepsilon)\right\}\exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[\inf_{0<t\le t_0}\frac{\Gamma(t)}{\sigma(t)} - l_\infty - \frac{I}{2}\right]\right\} \le$$

$$\le \exp\left\{-\frac{I}{\varepsilon^{1-2\kappa}}\right\},$$

$0 < \varepsilon \le \varepsilon_0$.

For estimating

$$I_3 \equiv P\left\{\|\varepsilon^{\frac12-\kappa}\eta^\varepsilon_\cdot - \varepsilon^{\frac12-\kappa}\bar\eta^\varepsilon_\cdot\| > \frac{\delta}{2},\, \varepsilon^{\frac12-\kappa}\bar\eta^\varepsilon_\cdot \in Q_{n(\varepsilon)}\right\},$$

as in Gärtner [11], one can prove that $\forall I > 0$, $\exists\varepsilon_0 > 0$ such that

$$I_3 < \exp\left\{-\frac{I}{\varepsilon^{1-2\kappa}}\right\}, \qquad 0 < \varepsilon \le \varepsilon_0.$$

Returning to (3.2) the result follows.

■

Now we shall prove the upper bound **(A.II)**.

Theorem 3.2. $\forall\delta > 0$, $\forall\gamma > 0$, $\forall s > 0$, $\exists\varepsilon_0 > 0$ *such that*

(3.11) $$P\left\{\rho_{0T}\left(\varepsilon^{\frac12-\kappa}\eta^\varepsilon_\cdot, \Phi(s)\right) > \delta\right\} < \exp\left\{-\frac{s-\gamma}{\varepsilon^{1-2\kappa}}\right\}, \qquad 0 < \varepsilon \le \varepsilon_0,$$

where

$$\Phi(s) = \{\varphi \in C_{[0,T]}(\mathbb{R}^d) : S^1_{0T}(\varphi) \le s,\ \varphi_0 = 0\}$$

and $S^1_{0T}(\varphi)$ *is defined in (1.22).*

Proof: Again we take $T = 1$.

$$P\left\{\rho_{01}\left(\varepsilon^{\frac12-\kappa}\eta^\varepsilon_\cdot, \Phi(s)\right) > \delta\right\} \le$$

$$\le P\left\{\rho_{01}\left(\varepsilon^{\frac12-\kappa}\bar\eta^\varepsilon_\cdot, \Phi(s)\right) > \frac{\delta}{2}\right\} + P\left\{\|\varepsilon^{\frac12-\kappa}\eta^\varepsilon_\cdot - \varepsilon^{\frac12-\kappa}\bar\eta^\varepsilon_\cdot\| \le \frac{\delta}{2}\right\} \le$$

(3.12) $$\le P\left\{\rho_{01}\left(\varepsilon^{\frac12-\kappa}\bar\eta^\varepsilon_\cdot, \Phi(s)\right) > \frac{\delta}{2}\right\} + P\left\{\varepsilon^{\frac12-\kappa}\bar\eta^\varepsilon_\cdot \notin Q_{n(\varepsilon)}\right\} +$$

$$+ P\left\{\|\varepsilon^{\frac12-\kappa}\eta^\varepsilon_\cdot - \varepsilon^{\frac12-\kappa}\bar\eta^\varepsilon_\cdot\| > \frac{\delta}{2}, \varepsilon^{\frac12-\kappa}\bar\eta^\varepsilon_\cdot \in Q_{n(\varepsilon)}\right\} \equiv$$

$$\equiv I_4 + I_2 + I_3,$$

where $Q_{n(\varepsilon)}$ is defined in (3.10).

¿From the proof of Theorem 3.1 we know that forall $I > 0$, there exists a sequence $\{Q_m\}_{m=1,2,\cdots}$, a number r, and $\varepsilon_0 > 0$ such that

$$I_2 + I_3 < \exp\left\{-\frac{I}{\varepsilon^{1-2\kappa}}\right\}, \quad 0 < \varepsilon \le \varepsilon_0. \tag{3.13}$$

Now we shall estimate

$$I_4 \equiv P\left\{\rho_{01}\left(\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^{\varepsilon}_{\cdot}, \Phi(s)\right) > \frac{\delta}{2}\right\}.$$

It is known that $L(x,\beta)$ is jointly semicontinuous in all variables. Then, the functional

$$\int_0^1 L(\psi_s, \dot{\varphi}_s)\,ds = \frac{1}{2}\int_0^1 < A^{-1}(\psi_s)\dot{\varphi}_s, \dot{\varphi}_s > ds$$

is lower semicontinuous in ψ and φ. Let $\psi^n \to \psi$ as $n \to +\infty$. Then, for φ fixed and using Fatou's Lemma,

$$\underline{\lim}_{n\to+\infty} \int_0^1 L(\psi^n_s; \dot{\varphi}_s)\,ds \ge \int_0^1 \underline{\lim}_{n\to+\infty} L(\psi^n_s; \dot{\varphi}_s)\,ds \ge \int_0^1 L(\psi_s; \dot{\varphi}_s)\,ds.$$

Then, $\forall \Delta > 0$, $\exists \delta > 0$ such that if $\|\bar{x}_{\cdot} - \psi\| < \delta$,

$$\frac{1}{2}\int_0^1 < A^{-1}(\psi_s)\dot{\varphi}_s, \dot{\varphi}_s > ds > \frac{1}{2}\int_0^1 < A^{-1}(\bar{x}_t)\dot{\varphi}_s, \dot{\varphi}_s > ds - \Delta. \tag{3.14}$$

We choose ψ as a step function with $\psi_{\frac{j}{r}} = \bar{x}_{\frac{j}{r}}$, $j = 0, \cdots, r-1$ satisfying $\|\bar{x}_{\cdot} - \psi\| < \delta$ and we define

$$\Phi^{\psi}(s) = \left\{\varphi \in C_{[0,1]}(\mathbf{R}^d) : \frac{1}{2}\int_0^1 < A^{-1}(\psi_t)\dot{\varphi}_t, \dot{\varphi}_t > dt \le s\right\}.$$

Since $\Phi^{\psi}(s-\Delta) \subset \Phi(s)$ we have

$$P\left\{\rho_{01}\left(\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^{\varepsilon}_{\cdot}, \Phi(s)\right) > \frac{\delta}{2}\right\} \le P\left\{\rho_{01}\left(\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^{\varepsilon}_{\cdot}, \Phi^{\psi}(s-\Delta)\right) > \frac{\delta}{2}\right\} \equiv \mathcal{P}.$$

Define

$$\lambda^{\varepsilon} = (\lambda^{\varepsilon}_1, \cdots, \lambda^{\varepsilon}_r) = \varepsilon^{\frac{1}{2}-\kappa}(\bar{\eta}^{\varepsilon}_{\frac{1}{r}}, \bar{\eta}^{\varepsilon}_{\frac{2}{r}} - \bar{\eta}^{\varepsilon}_{\frac{1}{r}}, \bar{\eta}^{\varepsilon}_{\frac{3}{r}} - \bar{\eta}^{\varepsilon}_{\frac{2}{r}}, \cdots, \bar{\eta}^{\varepsilon}_{\frac{r}{r}} - \bar{\eta}^{\varepsilon}_{\frac{r-1}{r}}).$$

Notice that $\lambda^{\varepsilon} \in (\mathbf{R}^d)^r$. Fix $\alpha = (\alpha^1, \cdots, \alpha^r) \in (\mathbf{R}^d)^r$. Then,

$$\begin{aligned} &\lim_{\varepsilon\to0} \varepsilon^{1-2\kappa} \ln E \exp\left\{\frac{1}{\varepsilon^{1-2\kappa}} < \alpha, \lambda^{\varepsilon} >\right\} = \\ &= \lim_{\varepsilon\to0} \varepsilon^{1-2\kappa} \ln E \exp\left\{\varepsilon^{\kappa-1} \sum_{j=1}^{r} < \alpha^j, \int_{\frac{j-1}{r}}^{\frac{j}{r}} \bar{b}(\bar{x}_t, Y^{\varepsilon}_t)\,dt >\right\} = \\ &= \lim_{\varepsilon\downarrow0} \varepsilon^{1-2\kappa} \ln E \exp\left\{\varepsilon^{\kappa-1} \int_0^1 < \alpha_t, \bar{b}(\bar{x}_t, Y^{\varepsilon}_t) > dt\right\} = \\ &= \frac{1}{2}\int_0^1 < A(\bar{x}_t)\alpha_t, \alpha_t > dt, \end{aligned} \tag{3.15}$$

where the last equality follows from Condition **B-1** and $\alpha_t = \alpha^j$, $\frac{i-1}{r} \le t < \frac{i}{r}$, $j = 1, \cdots, r$. Now,

$$\mathcal{P} \le P\left\{\frac{1}{2}\int_0^1 < A^{-1}(\psi_t)\varepsilon^{\frac{1}{2}-\kappa}\dot{\tilde{\eta}}_t^{\varepsilon}, \varepsilon^{\frac{1}{2}-\kappa}\dot{\tilde{\eta}}_t^{\varepsilon} > dt > s - \Delta\right\}.$$

¿From (2.6) and (2.7) we get

$$\mathcal{P} \le P\left\{L^{\psi}(\lambda^{\varepsilon}) > s - \Delta\right\} \le P\left\{\lambda^{\varepsilon} \notin \Phi^r(s - \Delta)\right\}.$$

Since $\Phi^r(s - 2\Delta) \subset \Phi^r(s - \Delta)$, $\partial\Phi^r(s - \Delta) = \{\beta \in (\mathbf{R}^d)^r : L^{\psi}(\beta) = s - \Delta\}$ is a compact set, and $\Phi^r(s - 2\Delta) \cap \partial\Phi^r(s - \Delta) = \emptyset$, we have $d \equiv dist(\phi^r(s - 2\Delta), \partial\Phi^r(s - \Delta)) > 0$. Then, from Proposition 2.3 and Chebyshev's inequality, there exist $\alpha_1, \cdots, \alpha_N \in (\mathbf{R}^d)^r$ such that

$$\begin{aligned}
&\mathcal{P} \le P\left\{\lambda^{\varepsilon} \notin \Phi^r(s - \Delta)\right\} \le P\left\{\rho(\lambda^{\varepsilon}, \Phi^r(s - 2\Delta)) \ge d\right\} \le \\
&\le P\left\{\lambda^{\varepsilon} \in \cup_{i=1}^N \{\beta : < \alpha_i, \beta > - H^{\psi}(\alpha_i) > s - 2\Delta\}\right\} \le \\
&\le \sum_{i=1}^N P\left\{\exp\left\{\frac{1}{\varepsilon^{1-2\kappa}}[< \alpha_i, \lambda^{\varepsilon} > - H^{\psi}(\alpha_i)]\right\} > \exp\left\{\frac{s - 2\Delta}{\varepsilon^{1-2\kappa}}\right\}\right\} \le \\
(3.16) \qquad &\le \sum_{i=1}^N \exp\left\{\frac{s - 2\Delta}{\varepsilon^{1-2\kappa}}\right\} E\exp\left\{\frac{1}{\varepsilon^{1-2\kappa}}\left[< \alpha_i, \lambda^{\varepsilon} > - H^{\psi}(\alpha_i)\right]\right\} = \\
&= \exp\left\{-\frac{s - 2\Delta}{\varepsilon^{1-2\kappa}}\right\} \sum_{i=1}^N \exp\left\{-\frac{H^{\psi}(\alpha_i)}{\varepsilon^{1-2\kappa}}\right\}. \\
&. \exp\left\{\frac{1}{\varepsilon^{1-2\kappa}}\left[\varepsilon^{1-2\kappa} \ln E\exp\left\{\frac{1}{\varepsilon^{1-2\kappa}} < \alpha_i, \lambda^{\varepsilon} >\right\}\right]\right\}.
\end{aligned}$$

From (3.15) we know that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{1-2\kappa} \ln E\exp\left\{\frac{1}{\varepsilon^{1-2\kappa}} < \alpha_i, \lambda^{\varepsilon} >\right\} = \frac{1}{2}\int_0^1 < A(\bar{x}_t)\alpha_{t,i}, \alpha_{t,i} > dt,$$

where $\alpha_{t,i} = \alpha_i^j$, $\frac{i-1}{r} \le t \le \frac{i}{r}$, $j = 1, \cdots, r$. Since $\frac{1}{2}\int_0^1 < A(\psi_t)\alpha_t, \alpha_t > dt$ is lower semicontinuous in ψ, take the same step function ψ_t (with $\psi_{\frac{j}{r}} = \bar{x}_{\frac{j}{r}}$, $j = 0, \cdots, r - 1$), and write for every $\gamma > 0$,

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \varepsilon^{1-2\kappa} \ln E\exp\left\{\frac{1}{\varepsilon^{1-2\kappa}} < \alpha_i, \lambda^{\varepsilon} >\right\} < \\
&\qquad < \frac{1}{2}\int_0^1 < A(\psi_t)\alpha_{t,i}, \alpha_{t,i} > dt + \Delta = \\
&\qquad = H^{\psi}(\alpha_i) + \Delta < H^{\psi}(\alpha_i) + \frac{\gamma}{4},
\end{aligned}$$

for $\Delta > 0$ sufficiently small. Returning to (3.16) we get

$$\mathcal{P} \leq \exp\left\{-\frac{s-2\Delta}{\varepsilon^{1-2\kappa}}\right\}\sum_{i=1}^{N}\exp\left\{-\frac{H^{\psi}(\alpha_i)}{\varepsilon^{1-2\kappa}}\right\}.\exp\left\{\frac{1}{\varepsilon^{1-2\kappa}}\left[H^{\psi}(\alpha_i)+\gamma\right]\right\} =$$

$$= \exp\left\{-\frac{s}{\varepsilon^{1-2\kappa}}\right\}\exp\left\{\frac{2\Delta}{\varepsilon^{1-2\kappa}}\right\}N\exp\left\{\frac{\frac{\gamma}{4}}{\varepsilon^{1-2\kappa}}\right\} \leq$$

$$\leq \exp\left\{-\frac{s-\frac{\gamma}{2}}{\varepsilon^{1-2\kappa}}\right\}, \quad 0<\varepsilon\leq\varepsilon_0.$$

and we can say that $\forall\gamma > 0$, $\exists\varepsilon_0 > 0$ such that

$$I_4 \leq \exp\left\{-\frac{s-\frac{\gamma}{2}}{\varepsilon^{1-2\kappa}}\right\}, \quad 0<\varepsilon\leq\varepsilon_0. \tag{3.17}$$

Therefore, by taking $I = s - \frac{\gamma}{2}$ in (3.13), there exists $\varepsilon_0 > 0$ such that

$$P\left\{\rho_{01}\left(\varepsilon^{1-2\kappa}\eta^{\varepsilon}_{\cdot},\Phi(s)\right) > \delta\right\} \leq I_4 + I_2 + I_3 \leq$$
$$= \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}(s-\gamma)\right\}, \quad 0<\varepsilon\leq\varepsilon_0.$$

■

Remark 3.1. $A(x)$ being Lipschitz continuous, there exists a matrix $\sigma(x)$ such that $A(x) = \sigma(x)\sigma^*(x)$, $x \in \mathbf{R}^d$. Define $W^0_t = \int_0^t \sigma(\bar{x}_s)\,dW_s$ where W_t is a d-dimensional Wiener process starting at zero and $\bar{x}_t$ is the function introduced in (1.3), W^0_t is a Gaussian process with independent increments, $EW^0_t = 0$, and correlation matrix $(R^{ij}(t))_{i,j=1,\cdots,d}$ given by

$$R^{ij}(t) = EW^{0,i}_tW^{0,j}_t = \int_0^t A^{ij}(\bar{x}_s)\,ds.$$

It is known (see [10]) that the action functional for $\varepsilon^{\frac{1}{2}-\kappa}W^0_t$ is given by $\frac{1}{\varepsilon^{1-2\kappa}}S^1_{0T}(\varphi)$ where $S^1_{0T}(\varphi)$ is given by (1.22). Then, from theorems 3.1 and 3.2 we conclude that $\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon}_t$ and $\varepsilon^{\frac{1}{2}-\kappa}W^0_t$ have the same action functional. Recall that under Khas'minskii's conditions, η^{ε}_t converges weakly to W^0_t where $A(x)$ satisfies (1.9).

Remark 3.2. When $\bar{b}(0) = 0$, Theorem 1 gives the normalized action functional for $\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon}_t = \int_0^t \frac{f(Y^{\varepsilon}_s)ds}{\varepsilon^{\kappa}}\,ds$ which is

$$S^1_{0T}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T < A^{-1}\dot{\varphi}_s, \dot{\varphi}_s > ds, & \varphi \text{ a.c.} \\ +\infty, & \text{in the rest of } C_{[0,T]}(\mathbf{R}^d), \end{cases}$$

with normalizing coefficient $\frac{1}{\varepsilon^{1-2\kappa}}$ where A is the matrix in Condition **B-1**. It is easy to verify that the action functional for the family of processes

$$V^{\varepsilon}_t = x + \frac{\int_0^t f(Y^{\varepsilon}_s)\,ds}{\varepsilon^{\kappa}}, \quad x \in \mathbf{R}^d$$

is also given by $\frac{1}{\varepsilon^{1-2\kappa}} S^1_{0T}(\varphi)$. However, in this case, the level sets are $\Phi(s) = \{\varphi \in C_{[0,T]}(\mathbf{R}^d) : S^1_{0T}(\varphi) < s,\ \varphi_0 = x\}$.

4. Proof of Theorem 2

Now we shall prove Theorem 2 in the most general situation, when the initial point is not necessarily an equilibrium point for the system (1.3).

We consider X^ε_t satisfying (1.1) with $X^\varepsilon_0 = x \in \mathbf{R}^d$ and $\bar{x}_t$ the solution of (1.3). Then

$$X^\varepsilon_t = x + \int_0^t b(X^\varepsilon_s, Y^\varepsilon_s)\, ds. \tag{4.1}$$

Define

$$\Upsilon^\varepsilon_t = \int_0^t [b(X^\varepsilon_s, Y^\varepsilon_s) - \bar{b}(\bar{x}_s)]\, ds. \tag{4.2}$$

The process Z^ε_t in (1.13) can be written as

$$Z^\varepsilon_t = \frac{\Upsilon^\varepsilon_t}{\varepsilon^\kappa}, \quad 0 < \kappa < \frac{1}{2}.$$

We define

$$\tilde{Z}^\varepsilon_t = \frac{\tilde{\Upsilon}^\varepsilon_t}{\varepsilon^\kappa} \text{ and } \hat{Z}^\varepsilon_t = \frac{\hat{\Upsilon}^\varepsilon_t}{\varepsilon^\kappa}$$

where $\tilde{\Upsilon}^\varepsilon_t$ and $\hat{\Upsilon}^\varepsilon_t$ were introduced in (1.25) and (1.26).

First we recall that the action functional for $\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_t$ with η^ε_t given in (1.17) is $\frac{1}{\varepsilon^{1-2\kappa}} S^1_{0T}(\varphi)$ in (1.22). The contraction principle implies that the normalized action functional for $\hat{Z}^\varepsilon_t$ is $S_{0T}(\varphi)$ in (1.23) with normalizing coefficient $\frac{1}{\varepsilon^{1-2\kappa}}$. Now we shall prove that $\hat{Z}^\varepsilon_t$ and $\tilde{Z}^\varepsilon_t$ have the same action functional.

Proposition 4.1. *If Condition* **B-3** *in (1.21) holds then $\hat{Z}^\varepsilon_t$ and $\tilde{Z}^\varepsilon_t$ have the same action functional.*

Proof: Given $\gamma > 0$, $\delta > 0$, and $\varphi \in C_{[0,T]}(\mathbf{R}^d)$, $\varphi_0 = x$,

$$\begin{aligned} P\{\|\tilde{Z}^\varepsilon_\cdot - \varphi\| < \delta\} &\geq P\{\|\tilde{Z}^\varepsilon_\cdot - \hat{Z}^\varepsilon_\cdot\| < \frac{\delta}{2}, \|\hat{Z}^\varepsilon_\cdot - \varphi\| < \frac{\delta}{2}\} \geq \\ &\geq P\{\|\hat{Z}^\varepsilon_\cdot - \varphi\| < \frac{\delta}{2}\} - P\{\|\tilde{Z}^\varepsilon_\cdot - \hat{Z}^\varepsilon_\cdot\| \geq \frac{\delta}{2}\} \equiv \\ &\equiv I_1 - I_2. \end{aligned} \tag{4.3}$$

Since $S_{0T}(\varphi)$ in (1.23) is the action functional for $\hat{Z}^\varepsilon_t$, $\exists \varepsilon_0 > 0$ such that

$$I_1 \geq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}[S_{0T}(\varphi) + \gamma]\right\}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

On the other hand,

$$I_2 = P\{\|\tilde{Z}^\epsilon_\cdot - \hat{Z}^\epsilon_\cdot\| \geq \frac{\delta}{2}\} = P\left\{\sup_{0\leq t\leq T}\left|\int_0^t B(Y^\epsilon_s)\,\tilde{Z}^\epsilon_s\,ds - \int_0^t \bar{B}\hat{Z}^\epsilon_s\,ds\right| \geq \frac{\delta}{2}\right\}.$$

Since the processes $\tilde{\Upsilon}^\epsilon_t$ and $\hat{\Upsilon}^\epsilon_t$ satisfy the linear differential equations

$$\dot{\tilde{\Upsilon}}^\epsilon_t - B(\bar{x}_t, Y^\epsilon_t)\,\tilde{\Upsilon}^\epsilon_t = \tilde{b}(\bar{x}_t, Y^\epsilon_t), \quad \text{and}$$
$$\dot{\hat{\Upsilon}}^\epsilon_t - \bar{B}(\bar{x}_t)\,\hat{\Upsilon}^\epsilon_t = \tilde{b}(\bar{x}_t, Y^\epsilon_t),$$

we may write

$$\tilde{\Upsilon}^\epsilon_t = \exp\left\{\int_0^t B(\bar{x}_s, Y^\epsilon_s)\,ds\right\}\int_0^t \exp\left\{-\int_0^s B(\bar{x}_u, Y^\epsilon_u)\,du\right\}\tilde{b}(\bar{x}_s, Y^\epsilon_s)\,ds$$

and

$$\hat{\Upsilon}^\epsilon_t = \exp\left\{\int_0^t \bar{B}(\bar{x}_s)\,ds\right\}\int_0^t \exp\left\{-\int_0^s \bar{B}(\bar{x}_u)\,du\right\}\tilde{b}(\bar{x}_s, Y^\epsilon_s)\,ds.$$

Then,

$$\begin{aligned}
|\tilde{Z}^\epsilon_t - \hat{Z}^\epsilon_t| &= \left|\int_0^t B(\bar{x}_s, Y^\epsilon_s)\,\tilde{Z}^\epsilon_s\,ds - \int_0^t \bar{B}(\bar{x}_s)\hat{Z}^\epsilon_s\,ds\right| \leq \\
&\leq \left|\int_0^t B(\bar{x}_s, Y^\epsilon_s)\tilde{Z}^\epsilon_s\,ds - \int_0^t B(\bar{x}_s, Y^\epsilon_s)\hat{Z}^\epsilon_s\,ds\right| + \\
&+ \left|\int_0^t B(\bar{x}_s, Y^\epsilon_s)\hat{Z}^\epsilon_s\,ds - \int_0^t \bar{B}(\bar{x}_s)\hat{Z}^\epsilon_s\,ds\right| \leq \\
&\leq K\int_0^t |\tilde{Z}^\epsilon_s - \hat{Z}^\epsilon_s|\,ds + \frac{1}{\epsilon^\kappa}\left|\int_0^t (B(\bar{x}_s, Y^\epsilon_s) - \bar{B}(\bar{x}_s))\;\hat{Z}^\epsilon_s\,ds\right| = \\
&= K\int_0^t |\tilde{Z}^\epsilon_s - \hat{Z}^\epsilon_s|\,ds + \\
&+ \frac{1}{\epsilon^\kappa}\left|\int_0^t (B(\bar{x}_s, Y^\epsilon_s) - \bar{B}(\bar{x}_s))\int_0^s e^{\int_u^s \bar{B}(\bar{x}_v)\,dv}\tilde{b}(\bar{x}_u, Y^\epsilon_u)\,du\,ds\right|,
\end{aligned}$$

for some $K > 0$. Using Lemma 2.1.1 in [10] we obtain

$$\begin{aligned}
&|\tilde{Z}^\epsilon_t - \hat{Z}^\epsilon_t| \leq \\
&\leq e^{Kt}\frac{1}{\epsilon^\kappa}\left|\int_0^t (B(\bar{x}_s, Y^\epsilon_s) - \bar{B}(\bar{x}_s))\int_0^s e^{\int_u^s \bar{B}(\bar{x}_v)\,dv}\,\tilde{b}(\bar{x}_u, Y^\epsilon_u)\,du\,ds\right|.
\end{aligned}$$

Hence, $\forall\delta > 0$

$$\begin{aligned}
&P\left\{\|\tilde{Z}^\epsilon_\cdot - \hat{Z}_\cdot\| \geq \frac{\delta}{2}\right\} \leq \\
&e^{KT}P\left\{\sup_{0\leq t\leq T}\frac{1}{\epsilon^\kappa}|(B(\bar{x}_s, Y^\epsilon_s) - \bar{B}(\bar{x}_s))\;\int_0^s E^{\int_u^s \bar{B}(\bar{x}_v)\,dv}\,\tilde{b}(\bar{x}_u, Y^\epsilon_u)\,du\,ds\right| \geq \frac{\delta}{2}\right\}.
\end{aligned}$$

From Condition **B-3** we have that $\forall M > 0$, $\exists \varepsilon_0 > 0$ such that

$$P\left\{\|\bar{Z}^\varepsilon_\cdot - \hat{Z}^\varepsilon_\cdot\| \geq \frac{\delta}{2}\right\} \leq \exp\left\{-\frac{M}{\varepsilon^{1-2\kappa}}\right\}, \quad 0 < \varepsilon \leq \varepsilon_0. \tag{4.4}$$

Returning to (4.3) we get

$$P\left\{\|\bar{Z}^\varepsilon_\cdot - \varphi\| < \delta\right\} \geq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left(S_{0T}(\varphi) + \gamma\right)\right\}, \quad 0 < \varepsilon \leq \varepsilon_0,$$

which is the lower bound in **(A.I)**.

The upper bound is easily obtained. Let

$$\Phi(s) = \left\{\varphi \in C_{0T}(\mathbf{R}^d) : S_{0T}(\varphi) \leq s,\ \varphi_0 = x \in \mathbf{R}^d\right\}.$$

Then

$$\begin{aligned}
&P\left\{\rho_{0T}\left(\bar{Z}^\varepsilon_\cdot, \Phi(s)\right) \geq \delta\right\} \leq \\
&P\left\{\rho_{0T}\left(\bar{Z}^\varepsilon_\cdot, \Phi(s)\right) \geq \delta, \|\bar{Z}^\varepsilon_\cdot - \hat{Z}_\cdot\| < \frac{\delta}{4}\right\} + P\left\{\|\bar{Z}^\varepsilon_\cdot - \hat{Z}^\varepsilon_\cdot\| \geq \frac{\delta}{4}\right\} \leq \\
&\leq P\left\{\rho_{0T}\left(\hat{Z}^\varepsilon_\cdot, \Phi(s)\right) \geq \frac{\delta}{2}\right\} + P\left\{\|\bar{Z}^\varepsilon_\cdot - \hat{Z}^\varepsilon_\cdot\| \geq \frac{\delta}{4}\right\} \leq \\
&\leq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left(s - \frac{\gamma}{2}\right)\right\} + \exp\left\{-\frac{M}{\varepsilon^{1-2\kappa}}\right\},
\end{aligned}$$

where the last inequality follows from (4.4) and the fact that $S_{0T}(\varphi)$ is the normalized action functional for $\hat{Z}^\varepsilon_t$. By taking $M = s$,

$$P\left\{\rho_{0T}\left(\bar{Z}^\varepsilon_\cdot, \Phi(s)\right) \geq \delta\right\} \leq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}[s - \gamma]\right\}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

■

Proposition 4.2. *If $S_{0T}(\varphi)$ is the normalized action functional for $\bar{Z}^\varepsilon_t$ with normalizing coefficient $\frac{1}{\varepsilon^{1-2\kappa}}$ then it is the action functional for Z^ε_t with the same coefficient.*

Proof: Given $\delta > 0$ and $\gamma > 0$,

$$\begin{aligned}
P\{\|Z^\varepsilon_\cdot - \varphi\| < \delta\} &\geq P\left\{\|Z^\varepsilon_\cdot - \bar{Z}^\varepsilon_\cdot\| < \frac{\delta}{2}, \|\bar{Z}^\varepsilon_\cdot - \varphi\| < \frac{\delta}{2}\right\} \geq \\
&\geq P\left\{\|\bar{Z}^\varepsilon_\cdot - \varphi\| < \frac{\delta}{2}\right\} - P\left\{\|Z^\varepsilon_\cdot - \bar{Z}^\varepsilon_\cdot\| \geq \frac{\delta}{2}\right\} \equiv \\
&\equiv I_1 - I_2.
\end{aligned} \tag{4.5}$$

A lower bound for I_1 is obtained from the hypothesis that $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}(\varphi)$ is the action functional for $\bar{Z}^\varepsilon_t$.

From (4.2) we have

$$\Upsilon_t^\varepsilon + \int_0^t \bar{b}(\bar{x}_s)\, ds = \int_0^t b(X_s^\varepsilon, Y_s^\varepsilon)\, ds.$$

We introduce a new process

$$V_t^\varepsilon = \int_0^t b(\tilde{\Upsilon}_s^\varepsilon + \bar{x}_s, Y_s^\varepsilon)\, ds - \int_0^t \bar{b}(\bar{x}_s)\, ds.$$

Then,

$$V_t^\varepsilon + \int_0^t \bar{b}(\bar{x}_s)\, ds = \int_0^t b(\tilde{\Upsilon}_s^\varepsilon + \bar{x}_s, Y_s^\varepsilon)\, ds.$$

From the smoothness of $b(x,y)$ we get

$$b(\tilde{\Upsilon}_s^\varepsilon + \bar{x}_s, Y_s^\varepsilon) = b(\bar{x}_s, Y_s^\varepsilon) + B(\bar{x}_s, Y_s^\varepsilon)\, \tilde{\Upsilon}_s^\varepsilon + r^{(2)}(\tilde{\Upsilon}_s^\varepsilon),$$

where $r^2(\cdot)$ is the rest of Lagrange of order 2. Then,

$$V_t^\varepsilon = \int_0^t \tilde{b}(\bar{x}_s, Y_s^\varepsilon)\, ds + \int_0^t B(\bar{x}_s, Y_s^\varepsilon)\, \tilde{\Upsilon}_s^\varepsilon\, ds + \\ + \int_0^t r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)\, ds$$

which implies that

$$\tilde{\Upsilon}_t^\varepsilon = V_t^\varepsilon - \int_0^t r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)\, ds.$$

Therefore,

$$\begin{aligned}
|Z_t^\varepsilon - \tilde{Z}_t^\varepsilon| &= \frac{1}{\varepsilon^\kappa} |\Upsilon_t^\varepsilon - \tilde{\Upsilon}_t^\varepsilon| = \\
&= \frac{1}{\varepsilon^\kappa} \left| \int_0^t b(X_s^\varepsilon, Y_s^\varepsilon)\, ds - \int_0^t b(\tilde{\Upsilon}_s^\varepsilon + \bar{x}_s, Y_s^\varepsilon)\, ds + \int_0^t r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)\, ds \right| \le \\
&\le \frac{1}{\varepsilon^\kappa} \left[K \int_0^t |X_s^\varepsilon - (\tilde{\Upsilon}_s^\varepsilon + \bar{x}_s)|\, ds + \int_0^t |r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)|\, ds \right] = \\
&= \frac{1}{\varepsilon^\kappa} \left[K \int_0^t |\Upsilon_s^\varepsilon - \tilde{\Upsilon}_s^\varepsilon|\, ds + \int_0^t |r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)|\, ds \right] = \\
&= K \int_0^t |Z_s^\varepsilon - \tilde{Z}_s^\varepsilon|\, ds + \frac{1}{\varepsilon^\kappa} \int_0^t |r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)|\, ds.
\end{aligned}$$

From Lemma 2.1.1 in [10] we get

$$|Z_t^\varepsilon - \tilde{Z}_t^\varepsilon| \le e^{Kt} \frac{1}{\varepsilon^\kappa} \int_0^t |r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)|\, ds$$

for some $K > 0$. Since the second order derivatives of $b(x, y)$ are bounded, we get

$$|r^{(2)}(\tilde{\Upsilon}^\varepsilon_s)| \leq \frac{M}{2} |\tilde{\Upsilon}^\varepsilon_s|^2.$$

for some $M > 0$. Therefore,

$$\sup_{0 \leq t \leq T} |Z^\varepsilon_t - \tilde{Z}^\varepsilon_t| \leq \frac{1}{\varepsilon^\kappa} e^{KT} \int_0^T |r^{(2)}(\tilde{\Upsilon}^\varepsilon_s)|\, ds \leq$$
$$\leq \frac{1}{\varepsilon^\kappa} e^{KT} \frac{M}{2} \int_0^T |\tilde{\Upsilon}^\varepsilon_s|^2\, ds \leq \frac{1}{2} e^{KT} M \varepsilon^\kappa T \|\tilde{Z}^\varepsilon_\cdot\|^2$$

which implies that

$$I_2 = P\left\{ \|Z^\varepsilon_\cdot - \tilde{Z}^\varepsilon_\cdot\| \geq \frac{\delta}{2} \right\} \leq P\left\{ \|\tilde{Z}^\varepsilon_\cdot\|^2 \geq \frac{\delta}{2} \frac{1}{\varepsilon^\kappa} \frac{2e^{-KT}}{TM} \right\} =$$
$$= P\left\{ \|\tilde{Z}^\varepsilon_\cdot\| \geq \frac{1}{\varepsilon^{\frac{\kappa}{2}}} \sqrt{\delta \frac{e^{-KT}}{TM}} \right\}.$$

On the other hand, for $s > 0$ $\Phi(s) = \{\varphi : S_{0T}(\varphi) \leq s, \varphi_0 = 0\}$ is compact. Moreover, $\varphi \equiv 0 \in \Phi(s)$, $\forall s > 0$ because $S_{0T}(0) = 0$. Let $\rho = dist(0; \partial\Phi(s))$. Notice that $\rho > 0$ since $\partial\Phi(s)$ is compact. Besides, for $\varepsilon > 0$ sufficiently small, $\sqrt{\frac{\delta e^{-KT}}{TM}} \frac{1}{\varepsilon^{\frac{\kappa}{2}}} > 2\rho$. Then, $\exists \varepsilon_0 > 0$ such that

$$\begin{aligned} P\left\{ \|Z^\varepsilon_\cdot - \tilde{Z}^\varepsilon_\cdot\| \geq \frac{\delta}{2} \right\} &\leq P\left\{ \|\tilde{Z}^\varepsilon_\cdot\| \geq 2\rho \right\} \leq \\ &\leq P\left\{ \rho_{0T}\left(\tilde{Z}^\varepsilon_\cdot, \Phi(s)\right) \geq \rho \right\} \leq \\ &\leq \exp\left\{ -\frac{1}{\varepsilon^{1-2\kappa}} \left(s - \frac{\gamma}{2}\right) \right\}, \quad 0 < \varepsilon \leq \varepsilon_0. \end{aligned} \tag{4.6}$$

The last inequality follows from the properties of the action functional. By choosing $s = S_{0T}(\varphi) + \gamma$, we have

$$P\left\{ \|Z^\varepsilon_\cdot - \tilde{Z}^\varepsilon_\cdot\| \leq \frac{\delta}{2} \right\} \geq \frac{1}{2} \exp\left\{ -\frac{1}{\varepsilon^{1-2\kappa}} \left[S_{0T}(\varphi) + \frac{\gamma}{2} \right] \right\}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

¿From (4.5) we get

$$P\{\|Z^\varepsilon_\cdot - \varphi\| < \delta\} \geq \exp\left\{ -\frac{1}{\varepsilon^{1-2\kappa}} \left[S_{0T}(\varphi) + \frac{\gamma}{2} \right] \right\} -$$
$$- \frac{1}{2} \exp\left\{ -\frac{1}{\varepsilon^{1-2\kappa}} \left[S_{0T}(\varphi) + \frac{\gamma}{2} \right] \right\} \geq$$
$$\geq \exp\left\{ -\frac{1}{\varepsilon^{1-2\kappa}} \left[S_{0T}(\varphi) + \gamma \right] \right\}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

The upper bound follows easily from (4.6):

$$P\{\rho_{0T}(Z^\varepsilon_\cdot,\Phi(s))\geq\delta\}\leq$$
$$\leq P\left\{\rho_{0T}(Z^\varepsilon_\cdot,\Phi(s))\geq\delta,\|Z^\varepsilon_\cdot-\tilde{Z}^\varepsilon_\cdot\|<\frac{\delta}{4}\right\}+P\left\{\|Z^\varepsilon_\cdot-\tilde{Z}^\varepsilon_\cdot\|\geq\frac{\delta}{4}\right\}\leq$$
$$\leq P\left\{\rho_{0T}\left(\tilde{Z}^\varepsilon_\cdot,\Phi(s)\right)\geq\frac{\delta}{2}\right\}+P\left\{\|Z^\varepsilon_\cdot-\tilde{Z}^\varepsilon_\cdot\|\geq\frac{\delta}{4}\right\}\leq$$
$$\leq\exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}(s-\gamma)\right\},\qquad 0<\varepsilon\leq\varepsilon_0.$$

■

Remark 4.1. Assume $\bar{b}(0)=0$. Let

$$Z^\varepsilon_t=\varepsilon^\kappa x+\int_0^t b(Z^\varepsilon_s,Y^\varepsilon_s)\,ds,\quad x\in\mathbf{R}^d.$$

Then,

$$\dot{Z}^\varepsilon_t=b(Z^\varepsilon_t,Y^\varepsilon_t),\qquad Z^\varepsilon_0=\varepsilon^\kappa x.$$

It is not difficult to show that the action functional for $\frac{Z^\varepsilon_t}{\varepsilon^\kappa}$ does not change if we start with $Z^\varepsilon_0=0$.

5. Wave front Propagation

In this part we shall describe the wave front for the solution of the initial-boundary value problem introduced in (1.27). The main results may be proved by using the same approach as in Freidlin [6], Chapter VI, or in [7].

The conditions under $b(x,y)$, the initial function $g(x)$, and the non-linear term $f(x,y,u)$ were specified in the introduction. We assume an aditional condition: $\bar{b}(0)=0$ where $\bar{b}(x)$ satisfies (1.2).

Using the Feynman-Kac formula, the solution $u^\varepsilon(t,x,y)$ of (1.27) satisfies the equality (1.30). The properties of $f(x,y,u)$ imply that

$$u^\varepsilon(t,x,y)\leq E^\varepsilon_{xy}g(X^\varepsilon_t)\exp\left\{\frac{1}{\varepsilon^{1-2\kappa}}\int_0^t c(\varepsilon^\kappa X^\varepsilon_s,Y^\varepsilon_s)\,ds\right\}\tag{5.1}$$

where $(X^\varepsilon_t)_{t\geq0}$ is the process in (1.29), $c(x,y,u)=\frac{f(x,y,u)}{u}$, and $c(x,y)=\sup_{0\leq u\leq1}c(x,y,u)$.

Define

$$\Upsilon^\varepsilon_t\equiv\frac{1}{\varepsilon^\kappa}\int_0^t c(\varepsilon^\kappa X^\varepsilon_s,Y^\varepsilon_s)\,ds=\frac{1}{\varepsilon^\kappa}\int_0^t c(Z^\varepsilon_s,Y^\varepsilon_s)\,ds$$

where

$$Z^\varepsilon_t=\varepsilon^\kappa X^\varepsilon_t=\varepsilon^\kappa x+\int_0^t b(Z^\varepsilon_s,Y^\varepsilon_s)\,ds,\quad x\in\mathbf{R}^d.$$

Notice that

$$\dot{Z}^\varepsilon_t=b(Z^\varepsilon_t,Y^\varepsilon_t),\qquad Z^\varepsilon_0=\varepsilon^\kappa x.$$

Moreover, $\frac{1}{t}\int_0^t b(z,Y_s)\,ds \underset{t\to+\infty}{\to} \bar{b}(z)$ with probability 1. The Averaging Principle implies that $Z_t^\varepsilon \underset{\varepsilon\downarrow 0}{\to} \bar{z}_t \equiv 0$ where $\bar{z}_t$ satisfies (1.3) with $\bar{z}_0 = 0$. On the other hand, it is known that there exists a function $\bar{c}(z)$ such that $\frac{1}{t}\int_0^t c(z,Y_s)\,ds \underset{t\to+\infty}{\to} \bar{c}(z)$ with probability one, for all $z \in \mathbf{R}^d$. Since $\bar{b}(0) = 0$ we conclude that

$$\left(\int_0^t b(Z_s^\varepsilon, Y_s^\varepsilon)\,ds, \int_0^t c(Z_s^\varepsilon, Y_s^\varepsilon)\,ds\right) \underset{\varepsilon\downarrow 0}{\to} (0, \bar{c}(0)t)$$

with probability one.

Define

$$\eta_t^\varepsilon = \left(\varepsilon^\kappa x + \int_0^t b(0,Y_s^\varepsilon)\,ds, \int_0^t c(0,Y_s^\varepsilon)\,ds - \bar{c}(0)t\right).$$

According to Remark 4.1 and Theorem 1, the normalized action functional for $\frac{\eta_t^\varepsilon}{\varepsilon^\kappa}$ is

$$(5.2)\qquad S_{0T}^1(\varphi,\eta) = \begin{cases} \frac{1}{2}\int_0^T < A^{-1}(0,\bar{c}(0)t)(\dot\varphi_t,\dot\eta_t),(\dot\varphi_t,\dot\eta_t) > dt, \\ \varphi,\eta \text{ a.c. }, \\ +\infty, \quad \text{in the rest of } C_{[0,T]}(\mathbf{R}^d\times\mathbf{R}^d) \end{cases}$$

with normalizing coefficient $\frac{1}{\varepsilon^{1-2\kappa}}$, where $A(0,z)$ is the matrix satisfying

$$< A(0,z)(\alpha,\beta),(\alpha,\beta) >=$$
$$= \lim_{T\to+\infty}\frac{1}{T^{1-2\kappa}}\ln \bar{E}_y \exp\left\{T^{-\kappa}\int_0^T < (\alpha,\beta),(b(0,Y_t),c(z,Y_t)-\bar{c}(0)t > ds\right\}.$$

Using Theorem 2, we obtain the action functional for $\left(X_t^\varepsilon, \Upsilon_t^\varepsilon - \frac{\bar{c}(0)t}{\varepsilon^\kappa}\right)$ with initial point $(x,0)$ which is given by $\frac{1}{\varepsilon^{1-2\kappa}}S_{0T}(\varphi,\eta)$ where

$$(5.3)\qquad S_{0T}(\varphi,\eta) = \begin{cases} \frac{1}{2}\int_0^T < A^{-1}(0,\bar{c}(0)t)\left((\dot\varphi_t,\dot\eta_t) - \bar{B}(0,\bar{c}(0)t)(\varphi_t,\eta_t)\right), \\ \left((\dot\varphi_t,\dot\eta_t) - \bar{B}(0,\bar{c}(0)t)(\varphi_t,\eta_t)\right) > dt, \\ \varphi,\eta \text{ a.c.} \\ +\infty, \quad \text{in the rest of } C_{[0,T]}(\mathbf{R}^d\times\mathbf{R}^d). \end{cases}$$

Let us define, for each $t > 0$ and $x \in \mathbf{R}^d$,

$$V(t,x) = \sup\left\{\bar{c}(0)t - S_{0t}(\varphi,\eta) : \varphi,\eta \in C_{[0,+\infty)}(\mathbf{R}^d),\ \varphi_0 = x,\ \varphi_t \in G_0,\ \eta_0 = 0\right\}.$$

By using the properties of the action functional, one can prove, similarly to Lemma 6.2.1 in [6], that

$$(5.4)\qquad \lim_{\varepsilon\downarrow 0}\varepsilon^{1-2\kappa}\ln E_{xy}^\varepsilon g(X_t^\varepsilon)\exp\left\{\frac{1}{\varepsilon^{1-2\kappa}}\int_0^t c(Z_s^\varepsilon, Y_s^\varepsilon)\,ds\right\} = V(t,x).$$

Using (5.1) and (5.4) we obtain

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 0 \quad \text{if } V(t,x) < 0 \text{ and } |y| \leq a.$$

For proving that $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 1$ in the region $V(t,x) > 0$ and $|y| \leq a$, we shall assume that Condition (N) (see [6]) holds:

Condition (N): $\forall (t,x)$ such that $V(t,x) = 0$,

$$V(t,x) = \sup\{\bar{c}(0)t - S_{0t}(\varphi,\eta) : (\varphi,\eta) \in C_{[0,+\infty)}(\mathbf{R}^d \times \mathbf{R}^d),\ \varphi_0 = x,$$
$$\varphi_t \in G_0,\ V(t-s,\varphi_s) < 0 \text{ for } s \in (0,t),\ \eta_0 = 0\}.$$

Similarly to the proof of Theorem 6.2.1 in [6] one can prove that, under Condition (N), $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 1$ uniformly in any compact subset of $\{(t,x,y) : V(t,x) = 0,\ |y| \leq a\}$.

The following examples show the form of the wave front in some particular cases of problem (1.27).

Example 5.1 Assume $b(x,y) \equiv b(y)$ and $f(x,y,u) \equiv f(u)$. The differential operator in (1.28) becomes

$$L^\varepsilon = \frac{1}{2\varepsilon}\frac{\partial^2}{\partial y^2} + \frac{1}{\varepsilon^\kappa} b(y)\frac{\partial}{\partial x}$$

and $(X_t^\varepsilon : t \geq 0)$ in (1.29) is given by

$$X_t^\varepsilon = x + \frac{1}{\varepsilon^\kappa}\int_0^t b(Y_s^\varepsilon)\, ds.$$

In this case, $\bar{b}(x) \equiv \bar{b} = 0$. Let $c(u) = uf(u)$, $\bar{c} \equiv f'(0) = \sup_{0\leq u\leq 1} \frac{f(u)}{u}$, $\bar{c} > 0$.

Now,

$$V(t,x) = \sup\left\{\bar{c}t - \frac{1}{2}\int_0^t < A^{-1}\dot{\varphi}_s, \dot{\varphi}_s > ds : \varphi \in C_{[0,+\infty)},\ \varphi_0 = x,\ \varphi_t \in G_0\right\}.$$

For simplifying this function we assume $G_0 = \{x \in \mathbf{R}^d : \|x\| < 0\}$. ¿From the Euler-Lagrange equation (see [1]) we obtain

$$V(t,x) = \bar{c}t - \frac{1}{2t} < A^{-1}x, x >, \quad x \in \mathbf{R}^d.$$

The set $\{(t,x,y) : 2t^2\bar{c} =< A^{-1}x, x >\}$ describes the position of the wave front, as $\varepsilon \downarrow 0$. If $x \in \mathbf{R}$, the velocity of propagation is $\alpha^* = \sqrt{2A\bar{c}}$. It is not difficult to show that Condition (N) is satisfied.

Example 5.2 Assume $f \equiv f(y,u)$, $b \equiv b(u)$. The operator L^ε remains the same as in Example 5.1 as well as the process $(X_t^\varepsilon : t \geq 0)$. Define $c(y,u) = \frac{f(y,u)}{u}$,

$c(y) = \sup_{0\le u\le 1} c(y,u)$. We assume that $\exists k_1, k_2$ such that $0 \le k_1 \le c(y) \le k_2$, $\forall y$ and $\lim_{t\to+\infty} \frac{1}{t}\int_0^t c(Y_s)\,ds = \bar{c} > 0$. According to Remark 3.1, the family of processes $\left(X_t^\epsilon, \frac{1}{\epsilon^\kappa}\left[\int_0^t c(Y_s^\epsilon)\,ds - \bar{c}t\right]\right\}$ has the same action functional as $\left(\frac{1}{\epsilon^\kappa}\int_0^t b(Y_s^\epsilon)\,ds, \frac{1}{\epsilon^\kappa}[\int_0^t c(Y_s^\epsilon)\,ds - \bar{c}t]\right)$. The function $V(t,x)$ becomes

$$V(t,x) = \sup\Big\{\bar{c}t - \frac{1}{2}\int_0^t < A^{-1}(\dot\varphi_s, \dot\eta_s), (\dot\varphi_s, \dot\eta_s) > ds :$$
$$(\varphi,\eta) \in C_{[0,+\infty)}(\mathbf{R}^d \times \mathbf{R}^d),\ \varphi_0 = x,\ \eta_0 = 0, \varphi_t \in G_0\Big\} =$$
$$= \bar{c}t - \inf_{\gamma\in\mathbf{R}^d} \frac{1}{2t} < A^{-1}(x,\gamma),(x,\gamma) > .$$

Let γ^* be the point of minimum. Then

$$V(t,x) = \bar{c}t - \frac{1}{2t} < A^{-1}(x,\gamma^*),(x,\gamma^*) > .$$

Condition (N) is satisfied and the position of the wave front is

$$\{(t,x,y) : 2\bar{c}t =< A^{-1}(x,\gamma^*),(x,\gamma^*) >\} .$$

Example 5.3: In (1.27) assume $f \equiv f(u)$. The differential operator is L^ϵ in (1.28) and the process satisfies (1.29). The function $V(t,x)$ is given by

$$V(t,x) = \sup\Big\{\bar{c}t - \frac{1}{2}\int_0^t < A^{-1}(\dot\varphi_s - \bar{B}\varphi_s), (\dot\varphi_s - \bar{B}\varphi_s) > ds :$$
$$\varphi \in C_{[0,+\infty)}(\mathbf{R}^d),\ \varphi_0 = x,\ \varphi_t \in G_0\Big\},$$

where A is the matrix in Condition **B-1** and $\bar{B}$ is given in Remark 4.1.

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