Evaluation of beam shape coefficients in T-matrix methods using a finite series technique: on blowing-ups using hypergeometric functions and generalized Bessel polynomials

GÉRARD GOUESBET[1,*] AND LUIZ F. M. VOTTO[2] AND LEONARDO A. AMBROSIO[2]

[1] *CORIA-UMR 6614- Normandie Université. CNRS-Université et INSA de Rouen. Campus Universitaire du Madrillet. 76800, Saint-Etienne-du Rouvray, France*

[2] *Department of Electrical and Computer Engineering, São Carlos School of Engineering, University of São Paulo, 400 Trabalhador são-carlense Ave., 13566-590, São Carlos, São Paulo, Brazil*

[*] *gouesbet@coria.fr*

Abstract: In T-matrix methods (Generalized Lorenz-Mie theories or Extended Boundary Condition method), beam shape coefficients encoding the shape of the illuminating structured beam have to be evaluated. This may be carried out by using the finite series technique which, however, generates blowing ups when the partial wave order of the beam shape coefficients increases. Using hypergeometric functions and generalized Bessel polynomials, we demonstrate in the case of on-axis Gaussian beams that these blowing-ups are genuine phenomena, not due to a lack of numerical precision, and we establish criteria to evaluate the critical partial wave order which implies blowing-ups.

1. Introduction

T-matrix methods relate the scattered fields and the illuminating fields when a beam illuminates a scatterer. Such a method is constituted by the analytical generalized Lorenz-Mie theory (GLMT), more generally by generalized Lorenz-Mie theories (using the plural) [1–3], dealing with scattering particles which possess a sufficient degree of symmetry to allow for the use of the method of separation of variables [4–6]. Another example is the semi-analytical Extended Boundary Condition Method (EBCM), which is very often erroneously taken as synonymous of T-matrix, and which may deal with irregular particles [7, 8]. EBCM is mostly used in the case of plane wave illumination but illumination by structured beams may be considered as well, e.g. [9–13], see as well [3], Section 8.1, for a more extended discussion.

In such light scattering approaches, the illuminating structured beam is decomposed over a set of Vector Spherical Wave Functions (VSWFs) and expansion coefficients are best expressed in terms of beam shape coefficients (BSCs)which, in spherical coordinates, are denoted as $g_{n,TM}^{m}$ and $g_{n,TE}^{m}$ (TM: Transverse Magnetic; TE: Transverse Electric). In the case of on-axis Gaussian beams considered in the present paper, the double set $g_{n,TM}^{m}$ and $g_{n,TE}^{m}$ of BSCs reduces to a single uni-index set of special BSCs denoted g_n (n from 1 to ∞), which does not distinguish any more between TM and TE waves.

There are several methods used to the evaluation of beam shape coefficients, including quadratures [14, 15], localized approximation [16–19], finite series [20, 21], or the use of Angular Spectrum Decomposition [22–28].

Due mainly to the success of the localized approximations, the finite series method until recently disappeared from the stage during about three decades, excepted for a very small number of cases, two in an acoustical context [29, 30], and one in an electromagnetic context [31]. However, the recent discovery of limitations of localized approximations when the illuminating beam exhibits an axicon angle [32–36] and/or a topological charge [37–40], induced a revival of the finite series technique [39, 41–44].

It has then been observed that BSCs blow-up when the partial wave order of the BSCs increases, as summarized in [6], pp.168-171. In this reference, numerical observations for on-axis Gaussian beams have lead to the conclusion that blowing-ups are not the result of computational inaccuracy but no firm mathematical demonstration of this conclusion were established. Also, whether blowing-ups were consistent with physics remained an open problem. Blowing-ups have been observed as well in [39] in the case of Laguerre-Gauss beams freely propagating and in [44] for Laguerre-Gauss beams focused by a lens. In this last paper, blowing-ups were attributed to numerical inaccuracies, but it is suspected that, although numerical inaccuracies may be a source of blowing-ups, they might exist a genuine mathematical reason (connected with a physical explanation) for blowing-ups. The aim of the present paper is then to demonstrate within the framework of a firm mathematical demonstration, using hypergeometric functions and generalized Bessel polynomials, that blowing-ups indeed are a real phenomenon. Beside criteria to evaluate critical values of the partial wave order are proposed.

The paper is organized as follows. Section 2 describes the on-axis Gaussian beams to be examined either from a Davis scheme of approximations or as a special case of Laguerre-Gauss beams and shows how these two different approaches may be unified. Section 3 demonstrates that blowing-ups must indeed occur and establishes criteria for the critical values of the partial wave order. Such theoretical criteria can be directly compared with the "empirical" criterion recently proposed by Ambrosio and Gouesbet when dealing with computational calculations of the beam shape coefficients in acoustic scattering from ultrasonic Bessel beams, using a finite series method [45]. Section 4 exhibits numerical examples of blowing-ups and, using these numerical examples, criteria for critical values are tested. Section 5 is a conclusion.

2. On-axis Gaussian beams.

2.1. From the Davis scheme of approximations.

Gaussian beams may be described by using the Davis scheme of approximations [46,47] in which the fields take the form of series in terms of powers of a small parameter s, which may be named the beam confinement parameter, given by [6]:

$$s = \frac{w_0}{l} = \frac{1}{kw_0} \tag{1}$$

in which w_0 is the beam waist radius, l the diffraction length and k the wave-number. The successive approximations are called the first-, third-, fifth-order ... approximations and, although the scheme eventually diverges, in a way similar to the one observed in quantum electrodynamics [48], the first-order approximation is already a quite decent paraxial approximation to the description of Gaussian beams. Assuming a further simplification of the first-order scheme, we may obtain simple expressions for on-axis Gaussian beams finite series when the beam waist center of the beam coincides with the origin of coordinates used to describe the beam, according to Eqs.(39)-(40) in [20], and Eqs.(6.151)-(6.152) in [6]:

$$g_{2p+1} = \sum_{q=0}^{p} \begin{pmatrix} p \\ q \end{pmatrix} \frac{\Gamma(p+q+3/2)}{\Gamma(p+3/2)} (-4s^2)^q \tag{2}$$

$$g_{2p+2} = \sum_{q=0}^{p} \begin{pmatrix} p \\ q \end{pmatrix} \frac{\Gamma(p+q+5/2)}{\Gamma(p+5/2)} (-4s^2)^q \tag{3}$$

for n odd and even, respectively (p from 0 to ∞). Details will be provided in this paper for n odd only. The strong similarity between Eqs.2 and 3 should be enough to convince the reader that results obtained from n odd would become valid as well, *mutatis mutandis*, for n even as well.

2.2. From Laguerre-Gauss beam description.

In the case of paraxial Laguerre-Gauss beams freely propagating [39, 41], BSCs obtained from the finite series technique for n odd read as, e.g. Eqs.(68), (70) of [41]:

$$\begin{aligned}
[g_{n,TM}^{\nu+1}]_\mu &= \frac{i^n\sqrt{\pi}}{2^{\nu+2}}\frac{(-1)^{(n+\nu+1)/2}}{\Gamma(\frac{n+\nu}{2}+1)}(\frac{n-\nu-1}{2})! \\
&\times\sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_\mu, \\
\text{for } n &\geq \nu+1,\ (n-\nu) \text{ odd},\ \nu\geq 0
\end{aligned} \tag{4}$$

$$\begin{aligned}
[g_{n,TM}^{-1}]_\mu &= \frac{i^n\sqrt{\pi}}{4}\frac{(-1)^{(n+1)/2}}{\Gamma(\frac{n}{2}+1)}(\frac{n-1}{2})! \\
&\times\sum_{m=0}^{\leq n/2} 2^{\frac{1}{2}+n-2m}\frac{\Gamma(\frac{1}{2}+n-m)}{m!}[b_{n-2m}]_\mu, \\
\text{for } n &\geq 1,\ n \text{ odd}
\end{aligned} \tag{5}$$

in which $[b_{n-2m}]_\mu$ are coefficients which, for the time being, do not need to be specified, and ν, μ are supercript and subscript to specify the Laguerre-Gauss polynomial L_ν^μ used to describe the Laguerre-Gauss beams at hand. Gaussian beams may be deduced from Laguerre-Gauss beams by specifying $\mu=\nu=0$. The BSCs of Eqs.4 and 5 may then be unified in a single expression reading as:

$$\begin{aligned}
[g_{n,TM}^{\pm 1}]_0 &= \frac{i^n\sqrt{2\pi}}{4}\frac{(-1)^{(n+1)/2}}{\Gamma(\frac{n}{2}+1)}(\frac{n-1}{2})! \\
&\times\sum_{q=0}^{\leq n/2} 2^{n-2q}\frac{\Gamma(\frac{1}{2}+n-q)}{q!}[b_{n-2q}]_0
\end{aligned} \tag{6}$$

From Eq.(54) of [41], and using as well Eqs.(45) and (52), we obtain:

$$[b_{n-2q}]_0 = \frac{is}{\sqrt{2\pi}}\frac{(-s^2)^{(n-2q-1)/2}}{(\frac{n-2q-1}{2})!} \tag{7}$$

Inserting Eq.7 into Eq.6 and restricting ourselves to the case $n=2p+1$ odd, as in subsection 2.1, we obtain:

$$[g_{2p+1,TM}^{\pm 1}]_0 = P\sum_{q=0}^{p}\begin{pmatrix} p \\ q \end{pmatrix}\frac{\Gamma(2p-q+3/2)}{\Gamma(p+3/2)}(\frac{-1}{4s^2})^q \tag{8}$$

in which P is a pre-factor reading as:

$$P = (-1)^p s^{2p+1}2^{2p-1} \tag{9}$$

We then omit the pre-factor (which is equivalent to a choice of normalization), and rewrite Eq.8 as:

$$(g_{2p+1})_{LG} = S_p = \sum_{q=0}^{p}\begin{pmatrix} p \\ q \end{pmatrix}\frac{\Gamma(2p-q+3/2)}{\Gamma(p+3/2)}(\frac{-1}{4s^2})^q \tag{10}$$

2.3. Unified scheme

Since Eqs.2 and 10 both describe a paraxial Gaussian beam, we should expect a strong relationship between both of them. Indeed, using Eq.10, we establish:

$$(-4s^2)^p S_p = \sum_{q=0}^{p} \begin{pmatrix} p \\ q \end{pmatrix} \frac{\Gamma(2p-q+3/2)}{\Gamma(p+3/2)} (-4s^2)^{p-q} \tag{11}$$

Then, setting $(p-q) = u$ in Eq.11, and eventually changing u to q, we readily establish:

$$(-4s^2)^p S_p = \sum_{q=0}^{p} \begin{pmatrix} p \\ q \end{pmatrix} \frac{\Gamma(p+q+3/2)}{\Gamma(p+3/2)} (-4s^2)^{q} \tag{12}$$

which relates $S_p = (g_{2p+1})_{LG}$ and g_{2p+1} of Eq.2.

3. Blowing-ups of beam shape coefficients

3.1. Coefficients must blow-up

In this section, we shall demonstrate that BSCs must blow-up. We shall use hypergeometric functions whose relevant properties are summarized in the Appendix for the convenience of the reader. Let us then consider g_{2p+1} of Eq.2. It may be rewritten as a hypergeometric series (see Appendix for notations and properties):

$$g_{2p+1} = \sum_{q=0}^{p} c_q (4s^2)^q \tag{13}$$

in which c_q is a hypergeometric term:

$$c_q = (-1)^q \begin{pmatrix} p \\ q \end{pmatrix} \frac{\Gamma(p+q+3/2)}{\Gamma(p+3/2)} \tag{14}$$

from which we derive, after rearranging, using $\Gamma(p+1) = p\Gamma(p)$:

$$\frac{c_{q+1}}{c_q} = \frac{q-p}{q+1}(p+q+3/2) \tag{15}$$

which using Eq.50, leads to:

$$\begin{aligned} g_{2p+1} &= \sum_{q=0}^{p} c_q (4s^2)^q \\ &= {}_2F_0\left(\begin{matrix} -p, p+3/2 \\ - \end{matrix} ; 4s^2 \right) = {}_2F_0(-p, p+3/2; ; 4s^2) \end{aligned} \tag{16}$$

Another way to express g_{2p+1} is to use the identity [49]:

$${}_{N+1}F_M \left(\begin{matrix} -m, \mathbf{a} \\ \mathbf{b} \end{matrix} ; z \right) = \frac{(\mathbf{a})_m}{(\mathbf{b})_m} (-z)^m \cdot {}_{M+1}F_N \left(\begin{matrix} -m, 1-m-\mathbf{b} \\ 1-m-\mathbf{a} \end{matrix} ; \frac{(-1)^{N+M}}{z} \right) \tag{17}$$

In particular, we may check with a bit of effort that:

$$
\begin{aligned}
g_{2p+1} &= {}_2F_0\left(\begin{matrix} -p, p+3/2 \\ - \end{matrix} ; 4s^2\right) \qquad (18) \\
&= (p+3/2)_p(-4s^2)^p \cdot {}_1F_1\left(\begin{matrix} -p \\ -2p-1/2 \end{matrix} ; \frac{-1}{4s^2}\right)
\end{aligned}
$$

Instead of using hypergeometric functions, we may use generalized Bessel polynomials defined as [50,51]:

$$
y_p(x;a,b) = \sum_{q=0}^{p} \begin{pmatrix} p \\ q \end{pmatrix} (p+a-1)_q \left(\frac{x}{b}\right)^q \qquad (19)
$$

From now on, it will be sufficient to use $b = 2$ so that we may use the notation:

$$
y_p(x;a,b) = y_p(x;a,2) = y_p(x;a) \qquad (20)
$$

We then establish:

$$
g_{2p+1} = y_p(-4s^2; \frac{5}{2}, 1) = y_p(-8s^2; \frac{5}{2}, 2) = y_p(-8s^2; \frac{5}{2}) \qquad (21)
$$

However, there exists an asymptotic expression for the generalized Bessel polynomials which, for large p and $x \neq 0$, reads as ([51], Chapter 13, Theorem 3):

$$
y_p(x;a) = \left(\frac{2px}{e}\right)^p 2^{a-3/2} e^{1/x} \left[1 - \frac{1}{24p} - \frac{1}{4px^2} - (a-2)\frac{a-1+2/x}{4p} + O\left(\frac{1}{p^2}\right)\right] \qquad (22)
$$

When p is sufficiently large the behavior of $y_p(x;a)$ is then governed by $(px)^p$ which, therefore, must blow up.

3.2. Blowing-up criteria

Criteria for blowing-ups are defined by critical values of p denoted p_c and associated critical values of partial wave orders n denoted n_c which are equal to $2p_c + 1$. We shall propose and test three theoretical criteria. The first one relies on Eq.22. When p is sufficiently large, the blowing-up criterion is governed by the leading term $(2px/e)^p$ and the blowing-up criterion is then obtained when this term begins to increase indefinitely when p increases, so that we have a first blowing-up criterion given by:

$$
\frac{2p_c\,|x|}{e} \sim 1 \qquad (23)
$$

From Eq.21, we have $x = -8s^2$, so that:

$$
p_c \sim int\left(\frac{e}{16}\frac{1}{s^2}\right) \sim int(0.170/s^2) \qquad (24)
$$

leading to:

$$
n_c \sim int(0.340/s^2) + 1 \qquad (25)
$$

in which $int(x)$ is the integer closest to x.

Instead of using generalized Bessel polynomials, we may evaluate criteria using hypergeometric functions. Eq.12 shows that S_p and $(2p+1)$ generate identical critical values. As a variant to the use of g_{2p+1} to establish Eq.25, we shall now deal with Eq.10 to evaluate new critical values. We rewrite Eq.10 as:

$$S_p = \sum_{q=0}^{p} T_q \tag{26}$$

in which:

$$T_q = \begin{pmatrix} p \\ q \end{pmatrix} \frac{\Gamma(2p-q+3/2)}{\Gamma(p+3/2)} \left(\frac{-1}{4s^2}\right)^q \tag{27}$$

We observe that S_p of Eq.26 is an alternating series. Assuming that p is odd (the case p even would be treated similarly), we may rewrite S_p as:

$$\begin{aligned} S_p &= \sum_{q=0}^{p} T_q = (T_0+T_1)+(T_2+T_3)+\ldots+(T_{p-1}+T_p) \\ &= \sum_{q=0,even}^{p-1} (T_q+T_{q+1}) \end{aligned} \tag{28}$$

We then define, still with q even:

$$R_q = \frac{T_{q+2}+T_{q+3}}{T_q+T_{q+1}} \tag{29}$$

and set the criterion to a critical value of $p = p_c$ such as:

$$R_q(p_c) \sim 1 \tag{30}$$

From Eq.27, we establish that:

$$\begin{aligned} T_{q+2}+T_{q+3} &= \frac{\Gamma(2p-q-3/2)}{\Gamma(p+3/2)} \frac{p!}{(q+2)!(p-q-3)!} \left(\frac{-1}{4s^2}\right)^{q+2} \\ &\times\left(\frac{2p-q-3/2}{p-q-2} + \frac{1}{q+3}\frac{-1}{4s^2}\right) \end{aligned} \tag{31}$$

from which we immediately derive:

$$\begin{aligned} T_q+T_{q+1} &= \frac{\Gamma(2p-q+1/2)}{\Gamma(p+3/2)} \frac{p!}{(q)!(p-q-1)!} \left(\frac{-1}{4s^2}\right)^{q} \\ &\times\left(\frac{2p-q+1/2}{p-q} + \frac{1}{q+1}\frac{-1}{4s^2}\right) \end{aligned} \tag{32}$$

so that, after rearranging, we obtain:

$$\begin{aligned} R_q &= \frac{p-q-1}{(q+1)(q+2)} \left(\frac{-1}{4s^2}\right)^2 \\ &\times \frac{(2p-q-3/2)+(p-q-2)\frac{1}{q+3}\frac{-1}{4s^2}}{\frac{2p-q+1/2}{p-q}+\frac{1}{q+1}\frac{-1}{4s^2}} \end{aligned} \tag{33}$$

which can be rearranged to:

$$R_q = \frac{(p-q-1)(p-q)}{(q+2)(q+3)} \left(\frac{1}{4s^2}\right)^2 \times \frac{(2p-q-3/2)(q+3)4s^2-(p-q-2)}{(2p-q+1/2)(q+1)4s^2-(p-q)} \tag{34}$$

It is numerically observed that the successive terms of the series begin to decrease when p decreases (see next Section). Then, a blowing-up criterion is obtained when this trend begins to be false so that the most dangerous value of q is the one at the top level of the summation. Comparing Eq.28 in which the summation ends with T_p and Eq.29 in which the highest-order term has an order equal to $(q+3)$, we see that the most dangerous term in R_q is for $p = q+3$, i.e. $q = p-3$. Therefore to apply the criterion of Eq.30, we have to evaluate R_q for $q = p-3$ (note that q is even, which agrees with the fact that p is odd). We readily obtain:

$$R_{p-3} = \frac{6}{p(p-1)} \left(\frac{1}{4s^2}\right)^2 R_F \tag{35}$$

in which:

$$R_F = \frac{4p(p+3/2)s^2-1}{4(p-2)(p+7/2)s^2-3} \tag{36}$$

We shall now distinguish two regimes. In Regime 1, we assume that the critical p is so large (this will depend on the value of s) that we neglect the integers (-1) and (-3) in the numerators and denominators of Eq.36. Then, we have:

$$R_F \sim 1 \tag{37}$$

so that the critical p is given by:

$$\frac{6}{p_c^2} \left(\frac{1}{4s^2}\right)^2 \sim 1 \tag{38}$$

leading to:

$$p_c \sim int(\frac{\sqrt{6}}{4}\frac{1}{s^2}) \tag{39}$$

and to:

$$n_c \sim int(\frac{\sqrt{6}}{2}\frac{1}{s^2}) + 1 \sim int(1.224/s^2) + 1 \tag{40}$$

which departs significantly from the criterion of Eq.25.

In Regime 2, we assume the values of p and s work together so that we may neglect, in the numerators and denominators of Eq.12, the algebraic terms with respect to the integers. Then:

$$R_F \sim 1/3 \tag{41}$$

so that the critical p is given by:

$$p_c \sim int(\frac{\sqrt{2}}{4}\frac{1}{s^2}) \tag{42}$$

so that the critical value of n is:

$$n_c \sim int(\frac{\sqrt{2}}{2}\frac{1}{s^2}) + 1 \sim int(0.353/s^2) + 1 \tag{43}$$

which is close to the result given by Eq.25.

Furthermore, these last evaluations provide as well a complementary demonstration, using hypergeometric functions, that BSCs must blow up.

Another blow-up criterion has also been recently proposed in the framework of acoustic scattering when dealing with the Finite Series method for ultrasonic Bessel beams. Based on computational calculations, Ambrosio and Gouesbet suggested an empirical critical value for n of the form [45]:

$$n_c \sim int(\frac{3}{8s^2}) \sim int(0.375/s^2) \tag{44}$$

which shall be used in the next section in comparison with the three theoretical criteria discussed above.

4. Numerical results

In order to assess the accurateness of the critical criteria derived in the previous section and given by Eqs. (24), (39) and (42), an algorithm has been written using the commercial software *Mathematica* (Wolfram, 12.1 Student Edition) and is available upon request. All simulations were run on a personal laptop with the following configuration: Intel(R) Core(TM) i7-3630QM CPU @ 2.40GHz, 16.0 GB.

Equation (2) for g_{2p+1} has been implemented and investigated as a function of p for different values of s. The three criteria in Eqs. (24), (39) and (42) are called "C1", "C2" and "C3", respectively, while the empirical criterion $n_c \sim int(3/8s^2)$ ($n_c = 2p_c + 1$) [45], is here denoted as "Emp".

Figure 1 shows the natural logarithm of g_{2p+1} as a function of p for $s = 1/2\pi$, $1/3\pi$, $1/6\pi$, $1/10\pi$, $1/15\pi$, respectively. Inset figures have been included in Figs. **??**(a)-(c) for better visualization of the curves at regions close to the actual critical points.

For $s = 1/2\pi$, it is seen that the critical point above which g_{2p+1} starts to increase occurs at $p = p_c = 7$. In addition, for this particular value of s, Eqs. (24), (39) and (42) gives $p_{c,C1} = 7$, $p_{c,C2} = 24$ and $p_{c,C3} = 14$, while $p_{c,Emp} = 7$. Therefore, C2 and C3 predict critical points two and more than three times the actual one. On the other hand, C1 and Emp exactly agrees with the turning point seen in Fig. 1(a).

As the value of s increases, the critical p values predicted by criteria C1, C2 and C3 deviate more and more from those actually observed through the figures. For example, $p_{c,C1} = p_{c,Emp} = p_c = 7$ for $s = 1/2\pi$, but it is seen from Fig. 1(e) for $s = 1/15\pi$ that $p_c = 418$, but $p_{c,C1} = 377$, $p_{c,C2} = 1360$ and $p_{c,C3} = 785$, while $p_{c,Emp} = 416$.

The discrepancy between Emp and C_i ($i = 1, 2, 3$) is expected, since the blowing-up criteria established in Sec. 3.2 relies upon approximations and observations related to asymptotic expressions, while the empirical criterion was previously postulated in Ref. [45] based upon computational simulations of similar finite series and hypergeometric functions associated with the beam shape coefficients of acoustic beams. Figure 2 shows the behavior of Eqs. (24), (39) and (42) as a function of s, for $1/65 \leq s \leq 16/100$. The empirical criterion $n_c \sim int(3/8s^2)$ is actually shown in terms of p_c. Notice that, as s increases, all criteria tend to converge to small values of p_c, since they all carry a s^{-2}-dependence. The largest bound of $s = 0.16$ implies on a Gaussian beam whose beam waist radius w_0 is approximate equal to the wavelength [52].

It becomes clear from the above results that C2 is the worst to be used as a blow-up criterion, followed by C3 and C1. On the other hand, Emp is accurate for essentially the entire range of s.

Regardless of the criterion, however, it remains the fact that, counter-intuitive as it may seen at first, the beam shape coefficients derived from the finite series must eventually blow-up.

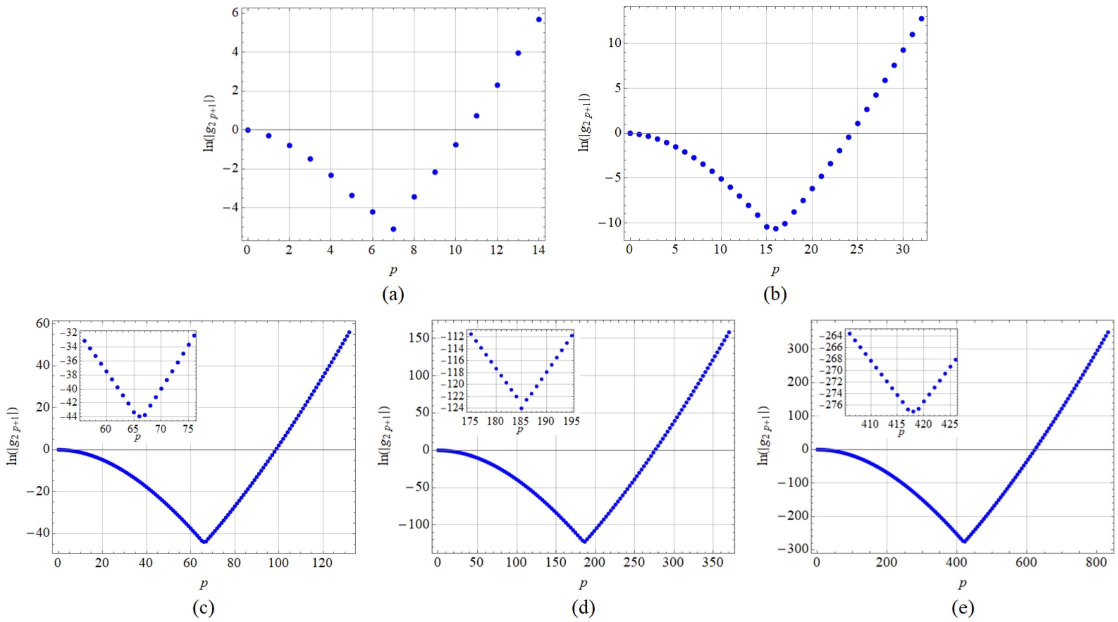

Fig. 1. Natural logarithmic of g_{2p+1} as a function of p for (a) $s = 1/2\pi$, (b) $s = 1/3\pi$, (c) $s = 1/6\pi$, (d) $s = 1/10\pi$ and (e) $1/15\pi$. Observed value of p_c and critical values associated with C1, C2, C3 and Emp are, respectively: (a) $p_c = 7$, $p_{c,C1} = 7$, $p_{c,C2} = 24$, $p_{c,C3} = 14$ and $p_{c,Emp} = 7$. (b) $p_c = 16$, $p_{c,C1} = 15$, $p_{c,C2} = 54$, $p_{c,C3} = 31$ and $p_{c,Emp} = 16$. (c) $p_c = 66$, $p_{c,C1} = 60$, $p_{c,C2} = 618$, $p_{c,C3} = 126$ and $p_{c,Emp} = 66$. (d) $p_c = 185$, $p_{c,C1} = 168$, $p_{c,C2} = 604$, $p_{c,C3} = 349$ and $p_{c,Emp} = 185$. (e) $p_c = 148$, $p_{c,C1} = 377$, $p_{c,C2} = 1360$, $p_{c,C3} = 785$ and $p_{c,Emp} = 416$.

5. Conclusion

In this paper we have therefore demonstrated, relying on the case of on-axis Gaussian beams, that blowing-ups observed when using the finite series technique to the evaluation of BSCs are not artefacts, but genuine mathematical consequences of the beam description. Using different methods, it may furthermore be demonstrated that these blowing-ups are not physical artefacts as well. They actually are physical ingredients required to describe evanescent waves [53].

Appendix.

The generalized hypergeometric function ${}_NF_M$ represents a series according to:

$$ {}_NF_M\left(\begin{matrix} a_1, ..., a_N \\ b_1, ..., b_M \end{matrix}; z\right) = \sum_{q=0}^{\infty} \frac{(a_1)_q ... (a_N)_q}{(b_1)_q ... (b_M)_q} \frac{z^q}{q!} \tag{45} $$

in which $(x)_n$ is the Pochhammer symbol reading as:

$$ (x)_0 = 1 \tag{46} $$

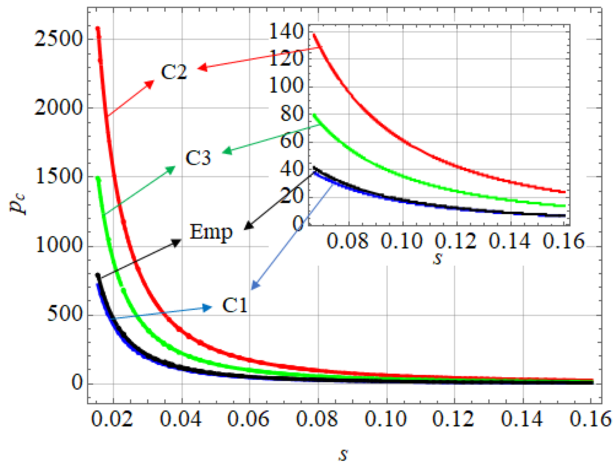

Fig. 2. p_c as a function of s for all four criteria (C1, C2, C3 and Emp). As s increases, the curves tend to converge to small values of p_c due to their s^{-2}-dependence.

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\ldots(x+n-1),\ n \neq 0 \tag{47}$$

Eq.45 may be written more concisely as:

$${}_N F_M\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix};z\right) =_N F_M(\mathbf{a};\mathbf{b};z) = \sum_{q=0}^{\infty} \frac{(\mathbf{a})_q}{(\mathbf{b})_q}\frac{z^q}{q!} \tag{48}$$

Next, a series $\sum_{k=0}^{\infty} c_k$ is called a hypergeometric series, and c_k a hypergeometric term, if $c_0 = 1$ and if the ratio of two successive terms my be written as [54]:

$$\frac{c_{k=1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{1}{k+1}\frac{(a_1+k)\ldots(a_N+k)}{(b_1+k)\ldots(b_M+k)} \tag{49}$$

We may then easily demonstrate that:

$$\sum_{k=0}^{\infty} c_k z^k =_N F_M\left(\begin{matrix}a_1,\ldots,a_N\\ b_1,\ldots,b_M\end{matrix};z\right) \tag{50}$$

Incidentally, one of the parameter row may be empty (if $N = 0$ or $M = 0$), then the row is noted by using a strike. As an example, the reader may check that the geometric series is equivalent to ${}_1F_0$ according to:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} =_1 F_0\left(\begin{matrix}1\\ -\end{matrix};x\right) =_1 F_0(1;;x) \tag{51}$$

References

1. G. Gouesbet, "T-matrix formulation and generalized Lorenz-Mie theories in spherical coordinates," Opt. Commun. **283, 4**, 517–521 (2010).
2. G. Gouesbet and J. Lock, "On the electromagnetic scattering of arbitrary shaped beams by arbitrary shaped particles: A review," J. Quant. Spectrosc. Radiat. Transf. **162**, 31–49 (2015).
3. G. Gouesbet, "T-matrix methods for electromagnetic structured beams: A commented reference database for the period 2014-2018," J. Quant. Spectrosc. Radiat. Transf. **230**, 247–281 (2019).

4. G. Gouesbet, B. Maheu, and G. Gréhan, “Light scattering from a sphere arbitrarily located in a Gaussian beam, using a Bromwich formulation,” J. Opt. Soc. Am. A **5,9**, 1427–1443 (1988).
5. G. Gouesbet, G. Gréhan, and B. Maheu, *Combustion measurements, edited by N. Chigier* (Hemisphere Publishing Corporation, New-York, USA, 1991), chap. : Generalized Lorenz-Mie theory and applications to optical sizing, pp. 339–384.
6. G. Gouesbet and G. Gréhan, *Generalized Lorenz-Mie theories, second edition* (Springer International Publishing AG, 2017).
7. P. Waterman, “Symmetry, unitarity, and geometry in electromagnetic scattering,” Phys. Rev. D **3, 4**, 825–839 (1971).
8. M. Mishchenko, L. Travis, and A. Lacis, *Scattering, absorption, and emission of light by small particles* (Cambridge University Press, Cambridge, 2002).
9. D. Mackowski and M. Mishchenko, “Direct simulation of multiple scattering by discrete random media illuminated by Gaussian beams,” Phys. Rev. A **83**, Article number 013804 (2011).
10. Z. Chen, H. Zhang, Z. Huang, and X. Wu, “Scattering of on-axis Gaussian beam by a uniaxial anisotropic object,” J. Opt. Soc. Am. A **31, 11**, 2545–2550 (2014).
11. M. Zheng, H. Zhang, Y. Sun, and Z. Wang, “Scattering of on-axis Gaussian beam by an arbitrarily shaped chiral object,” J. Quant. Spectrosc. Radiat. Transf. **151**, 192–197 (2015).
12. P. Briard, J. Wang, and Y. Han, “Shaped beam scattering by an aggregate of particles using generalized Lorenz-Mie theory,” Opt. Commun. **365**, 186–193 (2016).
13. M. Mishchenko and J. Dlugach, “Scattering of Gaussian beams by disordered particulate media,” J. Quant. Spectrosc. Radiat. Transf. **183**, 85–89 (2016).
14. G. Gouesbet, C. Letellier, K. Ren, and G. Gréhan, “Discussion of two quadrature methods of evaluating beam shape coefficients in generalized Lorenz-Mie theory,” Appl. Opt. **35,9**, 1537–1542 (1996).
15. G. Gouesbet, L. Ambrosio, and J. Lock, “On an infinite number of quadratures to evaluate beam shape coefficients in generalized Lorenz-Mie theory and extended boundary condition method for structured EM fields,” J. Quant. Spectrosc. Radiat. Transf. **242**, Article 196779 (2020).
16. G. Gouesbet, J. Lock, and G. Gréhan, “Generalized Lorenz-Mie theories and description of electromagnetic arbitrary shaped beams: localized approximations and localized beam models, a review,” J. Quant. Spectrosc. Radiat. Transf. **112**, 1–27 (2011).
17. G. Gouesbet and J. Lock, “Comments on localized and integral localized approximations in spherical coordinates.” J. Quant. Spectrosc. Radiat. Transf. **179**, 132–136 (2016).
18. J. Wang and G. Gouesbet, “Note on the use of localized beam models for light scattering theories in spherical coordinates,” Appl. Opt. **51, 17**, 3832–3836 (2012).
19. G. Gouesbet, “Second modified localized approximation for use in generalized Lorenz-Mie theories and other theories revisited,” J. Opt. Soc. Am. A **30, 4**, 560–564 (2013).
20. G. Gouesbet, G. Gréhan, and B. Maheu, “Computations of the g_n coefficients in the generalized Lorenz-Mie theory using three different methods,” Appl. Opt. **27,23**, 4874–4883 (1988).
21. G. Gouesbet, G. Gréhan, and B. Maheu, “Expressions to compute the coefficients g_n^m in the generalized Lorenz-Mie theory, using finite series,” J. Opt. (Paris) **19,1**, 35–48 (1988).
22. G. Sherman, “Introduction to the angular-spectrum representation of optical fields,” In: Appl. mathematics modern optics, SPIE **358**, 31–38 (1982).
23. R. Borghi, “On the angular-spectrum representation of multipole wave fields,” J. Opt. Soc. Am. A **21**, 1805–1810 (2004).
24. G. Gouesbet, J. Lock, Y. Han, and J. Wang, “Efficient computation of arbitrary beam scattering on a sphere: Comments and rebuttal, with a review on the angular spectrum decomposition,” J. Quant. Spectrosc. Radiat. Transf. **276**, Article 107913 (2021).
25. J. Shen, Z. Liu, and J. Liu, “Expanding non-axisymmetric beams in spherical coordinates with cylindrical wave spectrum decomposition,” Results Phys. **https://doi.org/10.1016/j.rinp.2022.106068** (2022).
26. J. Shen, J. Liu, Z. Liu, and H. Yu, “Angular spectrum decomposition method and the quadrature method in the generalized Lorenz-Mie theory for evaluating the beam shape coefficients of $TEM01^*$ doughnut beam,” Opt. Commun. **515**, Article 128224 (2022).
27. J. Shen, J. Liu, Y. Wang, Z. Liu, and H. Yu, “Cylindrical wave spectrum decomposition method for evaluating the expansion coefficients of the shaped beam in spherical coordinates,” J. Quant. Spectrosc. Radiat. Transf. **283**, Article 108138 (2022).
28. J. Shen, Y. Wang, H. Yu, L. Ambrosio, and G. Gouesbet, “Angular spectrum representation of the Bessel-Gauss beam and its approximation: A comparison with the localized approximation,” J. Quant. Spectrosc. Radiat. Transf. **284**, Article 108167 (2022).
29. X. Zhang, Z. Song, D. Chen, G. Zhang, and H. Cao, “Finite series expansion of Gaussian beam for the acoustic radiation force calculation of cylindrical particles in water,” J. Acoust. Soc. Am. **137, 4**, 1826–1833 (2015).
30. C. Jiang, X. Liu, J. Liu, Y. Mao, and P. Marston, “Acoustic radiation force on a sphere in a progressive and standing zero-order quasi-Bessel-Gauss beam,” Ultrasonics **76**, 1–9 (2017).
31. Y.-M. Bi, L.-F. Wang, Y.-X. Li, and J.-G. Zhao, “The generalized Lorenz-Mie scattering theory and algorithm of Gaussian beam,” Proc. SPIE 8 196, Int. symposium on photoelectronic detection imaging 2011: Space exploration technologies applications, 81861R **doi: 10 1117/12.900965** (2011).

32. G. Gouesbet, “On the validity of localized approximations for Bessel beams: all N-Bessel beams are identically equal to zero,” J. Quant. Spectrosc. Radiat. Transf. **176**, 82–86 (2016).
33. G. Gouesbet, J. Lock, L. Ambrosio, and J. Wang, “On the validity of localized approximation for an on-axis zeroth-order Bessel beam,” J. Quant. Spectrosc. Radiat. Transf. **195**, 18–25 (2017).
34. L. Ambrosio, J. Wang, and G. Gouesbet, “On the validity of the integral localized approximation for Bessel beams and associated radiation pressure forces,” Appl. Opt. **56, 19**, 5377–5387 (2017).
35. A. Chafiq, L. Ambrosio, G. Gouesbet, and A. Belafhal, “On the validity of the integral localized approximation for on-axis zeroth-order Mathieu beams,” J. Quant. Spectrosc. Radiat. Transf. **204**, 27–34 (2018).
36. L. Ambrosio, L. Votto, G. Gouesbet, and J. Wang, “Assessing the validity of the localized approximation for discrete superposition of Bessel beams,” J. Opt. Soc. Am. B **35, 11**, 2690–2698 (2018).
37. G. Gouesbet and L. Ambrosio, “On the validity of the use of a localized approximation for helical beams. I. Formal aspects,” J. Quant. Spectrosc. Radiat. Transf. **208**, 12–18 (2018).
38. L. Ambrosio and G. Gouesbet, “On the validity of the use of a localized approximation for helical beams. II. Numerical aspects,” J. Quant. Spectrosc. Radiat. Transf. **215**, 41–50 (2018).
39. L. Votto, L. Ambrosio, and G. Gouesbet, “Evaluation of beam shape coefficients of paraxial Laguerre-Gauss beam freely propagating by using three remodeling methods,” J. Quant. Spectrosc. Radiat. Transf. **239**, Article 106618 (2019).
40. L. Ambrosio and G. Gouesbet, “On localized approximations for Laguerre-Gauss beams focused by a lens,” J. Quant. Spectrosc. Radiat. Transf. **218**, 100–114 (2018).
41. G. Gouesbet, L. Votto, and L. Ambrosio, “Finite series expressions to evaluate the beam shape coefficients of a Laguerre-Gauss beam freely propagating,” J. Quant. Spectrosc. Radiat. Transf. **227**, 12–19 (2019).
42. G. Gouesbet, L. Ambrosio, and L. Votto, “Finite series expressions to evaluate the beam shape coefficients of a Laguerre-Gauss beam focused by a lens in an on-axis configuration,” J. Quant. Spectrosc. Radiat. Transf. **242**, Article 106759 (2019).
43. N. Valdivia, L. Votto, G. Gouesbet, J. Wang, and L. Ambrosio, “Bessel-Gauss beams in the generalized Lorenz-Mie theory using three remodeling techniques,” J. Quant. Spectrosc. Radiat. Transf. **256**, Article 107292 (2020).
44. L. Votto, L. Ambrosio, G. Gouesbet, and J. Wang, “Finite series algorithm design for lens-focused Laguerre-Gauss beams in the generalized Lorenz-Mie theory,” J. Quant. Spectrosc. Radiat. Transfer, virtual special issue LIP2020 **261**, Article 107 488 (2021).
45. L. A. Ambrosio and G. Gouesbet, “Finite series approach for the calculation of beam shape coefficients in ultrasonic and other acoustic scattering,” Submitt. to Wave Motion (2023).
46. L. Davis, “Theory of electromagnetic beams,” Phys. Rev. **19, 3**, 1177–1179 (1979).
47. G. Gouesbet, J. Lock, and G. Gréhan, “Partial wave representations of laser beams for use in light scattering calculations,” Appl. Opt. **34,12**, 2133–2143 (1995).
48. G. Gouesbet, J. Shen, and L. Ambrosio, “Diverging and converging schemes of approximations to describe fundamental EM Gaussian beams beyond the paraxial approximation,” J. Quant. Spectrosc. Radiat. Transf. **291**, Article 108344 (2022).
49. “"DLMF: NIST digital library of mathematic functions",” Tech. rep.
50. H. Krall and O. Frink, “A new class of orthogonal polynomials: the Bessel polynomials,” Trans. Amer. Math. Soc. **65**, 100–115 (1949).
51. E. Grosswald, *Bessel polynomials, Lecture notes in mathematics* (Springer-Verlag, 1978).
52. G. Gouesbet and G.Gréhan, *Generalized Lorenz-Mie theories, 3rd edition* (Springer, 2023).
53. J. Shen, S. Tang, L. A. Ambrosio, and G. Gouesbet, “On blowing-ups of the finite series technique in spherical wave expansion of shaped beams,” In preparation (2023).
54. W. Koepf, *Hypergeometric summation: an algorithm approach to summation and special function identities* (Springer, 2nd edition, 2014).