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Resilience for loose Hamilton cycles☆

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Abstract

We study the emergence of loose Hamilton cycles in subgraphs of random hypergraphs. Our main result states that the minimum d-degree threshold for loose Hamiltonicity relative to the random k-uniform hypergraph $\mathrm{H}_k(n, p)$ coincides with its dense analogue whenever $p \geq n^{-(k-1)/2+o(1)}$. The value of p is approximately tight for $d > (k+1)/2$. This is particularly interesting because the dense threshold itself is not known beyond the cases when $d \geq k-2$.

Keywords: random hypergraph; Hamilton cycle; minimum degree; Dirac's theorem; resilience

1. Introduction

A widely studied topic in combinatorics is the existence of vertex spanning substructures in graphs and hypergraphs. Since the corresponding decision problems are in many cases computational intractable, a large branch of research has focused on obtaining sufficient and easily verifiable conditions assuring the existence of these structures. A classic example of such a result is Dirac's theorem, which states that every graph on $n \geq 3$ vertices and minimum degree at least $n/2$ contains a Hamilton cycle. Since its inception, Dirac's theorem has been generalised in numerous ways [1, 2]. While the situation is increasingly well-understood for graphs, many problems in the setting of hypergraphs remain largely open.

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In the past decades much effort has been dedicated to studying which phenomena of dense settings can be transferred to sparser but well-structured settings. The study of combinatorial theorems relative to a random set has seen much progress in the recent years, which has gone alongside the development of powerful new methods and tools. One of the most fascinating aspects of this field is that certain dense parameters can be transferred to the sparse setting, without knowing their precise value, a famous example being hypergraph Túran densities as shown independently by Conlon and Gowers [3] and Schacht [4]. For a more detailed exposition of these results, see the survey of Conlon [5].

Here we are interested in *transference principles* for spanning substructures. A commonly studied object in this line of research is the random graph introduced by Erdős and Rényi. We denote by $\mathrm{G}(n, p)$ the binomial random graph with n vertices that contains every possible edge independently with probability p. The study of Hamilton cycles in random graphs dates back to Pósa [6] and Korshunov [7], who, independently, showed that $\mathrm{G}(n, p)$ contains a Hamilton cycle with high probability[1] (w.h.p. for short) provided that p is somewhat greater than $(\log n)/n$. More precise and stronger results were obtained by Komlós and Szemerédi [8], Ajtai, Komlós and Szemerédi [9] and Bollobás [10].

Given this, we can ask which subgraphs of $\mathrm{G}(n, p)$ typically contain a Hamilton cycle. Sudakov and Vu [11] conjectured the following random analogue of Dirac's theorem, which was proved by Lee and Sudakov [12]: For any $\varepsilon > 0$, if p is somewhat greater than $(\log n)/n$, then $\mathrm{G}(n, p)$ typically has the property that every spanning subgraph with minimum degree at least $(1 + \varepsilon)np/2$ contains a Hamilton cycle. Since this work, there has been a lot of interest in such *resilience* theorems for other types of spanning or almost spanning subgraphs (see, e.g., [13, 14, 15, 16, 17]), introducing many new ideas and techniques.

We study this problem in the setting of hypergraphs. A *k-uniform hypergraph* (*k-graph* for brevity) G consists of a set of vertices $V(G)$ and a set of edges $E(G)$ where each edge is a set of k vertices.[2] The *degree* $\deg(S)$ of a set $S \subseteq V(G)$ is the number of edges that contain S. For $1 \leq d \leq k-1$, we denote the *minimum d-degree* by $\delta_d(G)$, which is defined as the maximum m such that every set of d vertices (*d-set* for short) has degree at least m in G.

An intriguing open question in extremal hypergraph theory is to determine asymptotically optimal minimum degrees conditions that force hypergraph analogues of Hamilton cycles. A *loose cycle* in a k-graph is a cyclic sequence of edges such that each two consecutive edges overlap in exactly one vertex, and no pair of non-consecutive edges have vertices in common. The *order* and the *length* of a loose cycle are, respectively, its number of vertices and its number of edges. A loose cycle is *Hamilton* if it spans all vertices.

Definition 1.1. The minimum *d-degree threshold for loose Hamilton cycles* $\mu_d(k)$ is defined as the least $\mu \in [0, 1]$ such that for every $\gamma > 0$ and large enough n divisible by $k-1$, every n-vertex k-graph G with $\delta_d(G) \geq (\mu + \gamma)\binom{n-d}{k-d}$ contains a loose Hamilton cycle.

For $d = k-1$, the threshold $\mu_d(k)$ was determined by Kühn and Osthus [18] when $k = 3$ and in general independently by Keevash, Kühn, Osthus and Mycroft [19] and by Hàn and Schacht [20]. Other than this, $\mu_d(k)$ is only known for $d = 1$ and $k = 3$, a result due to Buß, Hàn and Schacht [21], which was later refined by Han and Zhao [22]. For later reference, we note that $\mu_1(k) \geq 2^{-(k-1)}$. This follows by considering the disjoint union of two cliques of the same order. Additional lower bounds can be found in the work of Han and Zhao [23]. We remark that other types of cycles have also been studied, and we refer the reader to a recent survey for a more detailed history of the problem [24].

Returning to the random setting, the binomial random k-graph $\mathrm{H}_k(n, p)$ on n vertices is defined analogously to $\mathrm{G}(n, p)$ and contains every possible edge independently with probability p. Loose Hamiltonicity in the random setting was investigated by Dudek, Frieze, Loh and Speiss [25], who showed that w.h.p. $\mathrm{H}_k(n, p)$ contains a loose Hamilton cycle provided that p is somewhat greater than $(\log n)/n^{k-1}$ after preliminary work of Frieze [26]. The value of p is asymptotically optimal, since below this threshold, w.h.p. $\mathrm{H}_k(n, p)$ contains isolated vertices. It should be noted that these results have been recovered recently in a much more general setting [27, 28].

Given the resilience results in the graph setting, it is natural to ask whether extensions to the hypergraph setting are possible. For loose cycles, this question was posed by Frieze [29, Problem 58]. Our main result gives an essentially optimal answer to this question for $d > (k+1)/2$, which is in particular interesting as the threshold $\mu_d(k)$ is not known beyond the cases in which we have $d \geq k-2$ [19, 20, 21, 30, 31].

[1] Meaning with probability going to 1 as n tends to infinity.

[2] For convenience, we write *subgraph* instead of *subhypergraph*.

Theorem 1.2 (Main result)**.** *For every $1 \leq d < k$ and $\gamma > 0$ there is a $C > 0$ such that the following holds. If $p \geq \max\{n^{-(k-1)/2+\gamma}, Cn^{-(k-d)} \log n\}$ and n is divisible by $k-1$, then w.h.p. $G \sim \mathrm{H}_k(n, p)$ has the property that every spanning subgraph $G' \subseteq G$ with $\delta_d(G') \geq (\mu_d(k) + \gamma)p\binom{n-d}{k-d}$ contains a loose Hamilton cycle.*

The value of p in this result is unlikely to be optimal for the whole range of d. On the other hand, $p = \Omega(n^{d-k} \log n)$ is a natural lower bound, since otherwise $\delta_d(G) = 0$ with high probability. Thus, our result is approximately tight whenever $d > (k + 1)/2$: for such d, we have $n^{-(k-1)/2+\gamma} \leq Cn^{d-k} \log n$ if γ is small enough, and hence our hypothesis on p in Theorem 1.2 matches the natural lower bound for p mentioned above. Very recently, Petrova and Trujić [32] gave a stronger bound for the case $(k, d) = (3, 1)$, as they proved that for $k = 3$ it suffices to have $p \geq C \max\left\{n^{-3/2}, n^{d-3}\right\} \log n$ for some $C > 0$. However, for all we know, the threshold for the property described in Theorem 1.2 could even be of order $n^{d-k} \log n$ for the whole range of d.

Resilience for Hamiltonicity has also been investigated for other types of hypercycles. In particular, Clemens, Ehrenmüller and Person [33] studied Hamilton Berge cycles (which are less restrictive than loose cycles) in 3-graphs and Allen, Parczyk and Pfenninger [34] studied tight Hamilton cycles (which are more restrictive than loose cycles) for $d = k - 1$. Moreover, Ferber and Kwan [35] proved an analogous result to Theorem 1.2 for perfect matchings, which gives approximately tight bounds for p whenever $d > k/2$.

Another line of research, going under the name of ‘random perturbation’, combines a deterministic graphs with a random model to study the emergence of large structures, including loose Hamilton cycles [36, 37]. Finally, a related set of questions has been studied in terms of random sparsifications of (hyper)graphs that robustly contain perfect matchings and tilings [38, 39, 40, 41], which has been coined ‘(random) robustness’.

Our proof is based on the absorption method in combination with embedding results for the (Weak) Hypergraph Regularity Lemma. We also benefit from a framework of Ferber and Kwan [35], which was introduced to tackle the resilience problems for matchings. The main difference to matchings is that Hamilton cycles come with a notion of connectivity. Hence, to prove Theorem 1.2 we need to extend this framework with further ideas in essentially every step.

An important constraint for our proof is that the value of $\mu_d(k)$ is unknown in almost all cases. In contrast to the aforementioned results for Hamiltonicity [34, 33, 32], we therefore cannot rely on any structural insights of past work regarding $\mu_d(k)$. Our strategy thus uses only the mere existence of loose Hamilton cycles to extract certain characteristic properties such as connectivity and covering most vertices and relate them to the random setting. Unfortunately, this is not sufficient in certain critical situations, where multiple Hamilton cycles have to be combined. We overcome the issues arising in this situation by showing that the threshold $\mu_d(k)$ actually allows us to find a Hamilton cycle with additional properties such as certain vertices being far apart from each other. Hence, in order to find a Hamilton cycle in the sparse setting, we develop in parallel a way to find an ‘enhanced’ Hamilton cycle in the dense setting.

The rest of the paper is organised as follows. In the next section, we present two main lemmas from which we derive Theorem 1.2. The rest of the paper, Sections 3–4, is dedicated to the proofs of the above mentioned two lemmas. We finish with some concluding remarks in Section 5.

2. Proof of Theorem 1.2 (Main result)

We apply the absorption method to find a Hamilton cycle in the proof of Theorem 1.2. Informally, this technique separates the argument into two parts. First we find a special (loose) path A that allows us to integrate any small set of vertices into a larger path. Then we cover all but few vertices with a loose cycle that contains A as a subpath. We conclude the proof by using the property of A. A diagram showing how to build the proof from the main lemmas is illustrated in Figure 1.

A *loose path* P in a k-graph is a sequence of edges such that each two consecutive edges overlap in exactly one vertex, and no pair of non-consecutive edges have vertices in common. If the context is clear, we simply speak of a *path*. The *order* of P is its number of vertices. For vertices u and v, we say that P is a *loose* (u, v)*-path* if u is in the first edge, v is in the last edge and no other edge contains u or v. For convenience, the constant hierarchies are expressed in standard $\ll$-notation in the remainder of the paper. To be precise, we write $x \ll y$ to mean that for any $y \in (0, 1]$ there exists an $x_0 \in (0, 1)$ such that for all $x \leq x_0$ the subsequent statements hold. Hierarchies with more constants

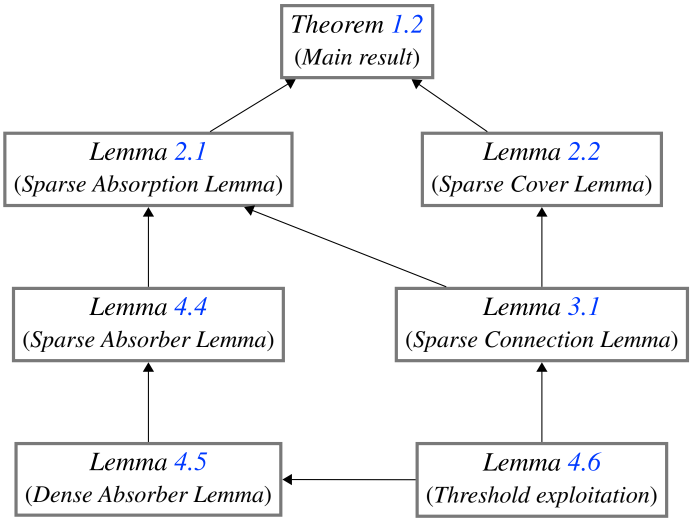

Figure 1: Proof diagram.

are defined in a similar way and should be read from the right to the left. The first constant in a hierarchy is always assumed to be positive. Finally, if we write $1/x$ in a hierarchy, we implicitly mean that $x \in \mathbb{N}$.

Lemma 2.1 (Sparse Absorption Lemma). *Let $\eta \ll \alpha \ll 1/k,\ 1/d,\ \gamma$ and $1/C \ll 1/k, \gamma$ with $k \geq 3$ and $p \geq \max\{n^{-(k-1)/2+\gamma}, Cn^{-(k-d)} \log n\}$. Then w.h.p. $G \sim \mathrm{H}_k(n, p)$ has the following property.*

For any spanning subgraph $G' \subseteq G$ with $\delta_d(G') \geq (\mu_d(k)+\gamma)p\binom{n-d}{k-d}$, there is a set $A \subseteq V(G')$ with $|A| \leq \alpha n$ and two vertices $u, v \in A$ such that for any subset $W \subseteq V(G') \setminus A$ with $|W| \leq \eta n$ divisible by $k-1$, the induced graph $G'[A \cup W]$ has a loose (u, v)-path covering $A \cup W$.

The next lemma allows us to cover most vertices with a single loose path.

Lemma 2.2 (Sparse Cover Lemma). *Let $1 \leq d \leq k-1$ with $k \geq 3$, $\eta > 0$, $\alpha \ll 1/k,\ 1/d,\ \gamma$ and $1/C \ll 1/k, \gamma$ and $p \geq \max\{Cn^{-(k-d)} \log n,\ Cn^{-(k-2)} \log n\}$, then w.h.p. $G \sim \mathrm{H}_k(n, p)$ has the following property.*

For any spanning subgraph $G' \subseteq G$ with $\delta_d(G') \geq (\mu_d(k)+\gamma)p\binom{n-d}{k-d}$, $Q \subseteq V(G')$ with $|Q| \leq \alpha n$ and $u, v \in V(G') \setminus Q$, there is a loose (u, v)-path P in $G' - Q$ that covers all but at most ηn vertices of $G' - Q$.

Lemmas 2.1 and 2.2 combine easily to a proof of our main result.

Proof of Theorem 1.2. The case when $k = 2$ is covered by the work of Lee and Sudakov [12]. So in the following, we assume that $k \geq 3$. Consider α, η with $\eta \ll \alpha \ll 1/k,\ \gamma$. Then w.h.p. $G \sim \mathrm{H}_k(n, p)$ satisfies the outcomes of Lemmas 2.1 and 2.2. Now let $G' \subseteq G$ be a spanning subgraph with $\delta_d(G') \geq (\mu_d(k) + \gamma)p\binom{n-d}{k-d}$. By assumption on G, we may pick a set $A \subseteq V(G')$ and two vertices $u, v \in A$ with the property described in Lemma 2.1. Let $Q = A \setminus \{u, v\}$. By assumption on G, there is a loose (u, v)-path P in $G' - Q$ that covers all but at most ηn vertices W of $G' - Q$. To finish the proof, we use the property of A to find a loose (u, v)-path P' covering $A \cup W$. It follows that $P \cup P'$ is a loose Hamilton cycle. □

The remaining sections are dedicated to the proofs of Lemmas 2.1 and 2.2.

3. Connecting and covering vertices

In the proofs of Lemmas 2.1 and 2.2, we need to connect vertices with a short path of uniform order while avoiding a few other vertices. This is the purpose of the following lemma.

Lemma 3.1 (Sparse Connection Lemma)**.** *Let $k \geq 3$ and suppose $1 \leq d < k$, $\alpha \ll 1/k$, γ and $1/K \ll 1/k, \gamma$. Suppose further that $0 < \nu \leq 1 - \alpha$ and $\varrho \ll 1/k$, ν. If $p \geq \max\{Kn^{-(k-d)} \log n,\ \omega(n^{-(k-2)})\}$, then w.h.p. $G \sim \mathrm{H}_k(n, p)$ has the following property.*

Let $G' \subseteq G$ be a spanning subgraph with $\delta_d(G') \geq (\mu_d(k) + \gamma)p\binom{n-d}{k-d}$ and $Q \subseteq V(G')$ with $|Q| \leq \alpha n$. Let $C \subseteq V(G') \setminus Q$ be a νn-set taken uniformly at random. Then with probability at least $2/3$ the following holds. For any $R \subseteq C$ with $|R| \leq \varrho n$ and distinct $u, v \in V(G') \setminus (Q \cup R)$, there is a loose (u, v)-path P in G' of order $8(k-1) + 1$ with $V(P) \setminus \{u, v\} \subseteq C \setminus R$.

The basic idea for the proof of Lemma 3.1 is to apply a sparse version of the (Weak) Hypergraph Regularity Lemma. This reduces the problem to performing the required connections in a dense setting, where we can benefit from the assumptions on the minimum degrees. We remark that with the help of Lemma 3.1 we already have all the requirements for proving Lemma 2.2 (Sparse Cover Lemma). Both proofs will appear in the full version of the paper.

4. Absorbing vertices

This section is dedicated to the proof of Lemma 2.1. The basic idea is to combine many small absorbing structures to a larger one. We define the former as follows.

Definition 4.1 (Absorber)**.** Let $X = \{x_1, \ldots, x_{k-1}\}$ be a set of vertices in a k-graph G. A collection $A = \{P_1^j, P_2^j\}_{j \in [q]}$ is an *absorber rooted in* X if the following hold:

- $P_i^1, \ldots, P_i^q$ are pairwise vertex-disjoint loose paths for $i \in [2]$;
- $V(\bigcup_{j \in [q]} P_1^j) = V(\bigcup_{j \in [q]} P_2^j) \cup X$ and $X \cap V(\bigcup_{j \in [q]} P_2^j) = \emptyset$;
- P_1^j has the same starting and terminal vertices as P_2^j for each $j \in [q]$.

The *vertices of* A are the vertices in $\bigcup_{j \in [q]} P_2^j$ and we let $V(A) = \bigcup_{j \in [q]} V(P_2^j)$. Note that $V(A) \cap X = \emptyset$. The *order* of A is $|V(A)|$.

Next we define templates, which encode the relative position of the absorbers in our construction.

Definition 4.2 (Template)**.** An r-graph T is an (r, z)-*template* if there is a z-set $Z \subseteq V(T)$ such that $T - W$ has a perfect matching for any set $W \subseteq Z$ of size less than $z/2$ with $v(T - W)$ divisible by r. We call Z the *flexible* set of T.

Templates were introduced by Montgomery in [42]. We remark that $r = k - 1$ in our proof, because we intend to absorb $k - 1$ vertices at a time. The next lemma was derived by Ferber and Kwan [35] from the work of Montgomery and states that there exist sparse templates.

Lemma 4.3 (Lemma 7.3 in [35])**.** *For $1/z$, $1/L \ll 1/r$, there is an (r, z)-template T with $v(T)$, $e(T) \leq Lz$ and $v(T) \equiv 0 \bmod r$.*

To use templates we require the following key lemma, which provides us with a single absorber avoiding any specified small set of vertices.

Lemma 4.4 (Sparse Absorber Lemma)**.** *Let $k \geq 3$ and suppose $1/M \ll \alpha \ll 1/k$, $1/d$, γ and $1/C \ll 1/k, \gamma$. If $p \geq \max\{n^{-(k-1)/2+\gamma}, Cn^{-(k-d)} \log n\}$, then w.h.p. $G \sim \mathrm{H}_k(n, p)$ has the following property.*

For any spanning subgraph $G' \subseteq G$ with $\delta_d(G') \geq (\mu_d(k) + \gamma)p\binom{n-d}{k-d}$, any set Q of at most αn vertices and any $(k-1)$-set X in $V(G) \setminus Q$, there is an absorber A in G' rooted in X that avoids Q and has order at most M.

Once Lemma 4.4 has been established, we may combine the above results to obtain a proof of Lemma 2.1, whose outline is given in the following. A formal proof will appear in the full version of the paper.

Sketch of the proof of Lemma 2.1 (Sparse Absorption Lemma). Consider ν and M with $\eta \ll \nu$, $1/M \ll \alpha$. We begin by reserving a set $Z \subseteq V(G')$ and vertices $u' \in Z$ and $v \notin Z$ such that, for any set $W \subseteq V(G') \setminus Z$ with $|W| \leq \eta n$, we may cover $W' = W \cup \{v\}$ with a loose (u', v)-path $P_{W'}$ of order at most $\sqrt{\eta}n$ in $G'[Z \cup W']$. This is possible by applying Lemma 3.1 with Z playing the role of C and an auxiliary result proved by Ferber and Kwan [35, Lemma 7.5].

Next, we use Lemma 4.3 to find a $(k-1, \nu n)$-template T. We assume that $V(T) \subseteq V(G')$ and that Z considered above is the flexible set of T. In the next step, we find absorbers rooted in the edges of T. More precisely, by Lemma 4.4 there is an absorber A_e in G' rooted in e of order at most M for each $e \in E(T)$. Denote by $\mathcal{P}_e$ the collection of loose paths of A_e that do not cover e, that is, the paths corresponding to the passive state of A_e. We can assume that the collections $\mathcal{P}_e$ are pairwise vertex-disjoint. This can be guaranteed by choosing the above absorbers one after another, avoiding the already involved vertices with the help of the set Q in Lemma 4.4. Finally, fix $u \in V(G') \setminus (V(T) \cup \{v\})$ arbitrarily and integrate the absorbers A_e ($e \in E(T)$) into a single loose (u, u')-path P_1 with $V(P_1) \cap (V(T) \cup \{v\}) = \{u'\}$ using Lemma 3.1 several times.

Set $A = V(P_1) \cup V(T) \cup \{v\}$. We claim that A has the properties detailed in Lemma 2.1. We may pick the involved constants so that $|A| \leq \alpha n$. For the absorption property, consider an arbitrary subset $W \subseteq V(G') \setminus A$ with $|W| \leq \eta n$ and $|W|$ divisible by $k-1$. We have to show that the induced graph $G'[A \cup W]$ has a loose (u, v)-path covering $A \cup W$.

To that end, let $W' = W \cup \{v\}$. By the choice of Z, we can find a loose (u', v)-path $P_{W'}$ of order at most $\sqrt{\eta}n$ in $G'[Z \cup W']$ that covers W'. We now concatenate P_1 and $P_{W'}$ to obtain a loose (u, v)-path P that covers W and uses at most $|Z|/2$ vertices of Z. Moreover, one can check that $|V(T) \setminus V(P)| \equiv 0 \bmod (k-1)$. Recall that T is a $(k-1, \nu n)$-template with flexible set Z. It follows that $T - V(P)$ admits a perfect matching $\mathcal{M}$. We then 'activate' each absorber A_e with $e \in \mathcal{M}$ and leave all other absorbers in their passive state. For each $e \in \mathcal{M}$, let $\mathcal{P}'_e$ be the collection of paths of A_e that cover e. Let P' be obtained from P by replacing $\mathcal{P}_e$ with $\mathcal{P}'_e$ for each $e \in \mathcal{M}$. It follows that P' is a loose (u, v)-path in $G'[A \cup W]$ covering $A \cup W$. □

4.1. Absorbers in the sparse setting

Let us now discuss the proof of Lemma 4.4 (Sparse Absorber Lemma). The idea of the proof is to find an absorbing structure A in the dense setting and embed it in the random graph via an embedding lemma for hypergraph regularity. One issue arising in this approach is that the probability p depends on the 'densest spots' of A. So to minimise p, we have to find an absorbing structure A whose edges are nowhere too cluttered. This can be formalised by the following definition, which we adopt from Ferber and Kwan [35].

A *Berge cycle* in a (possibly non-uniform) hypergraph is a sequence of distinct edges $e_1, \ldots, e_\ell$ such that there exist distinct vertices $v_1, \ldots, v_\ell$ with $v_i \in e_i \cap e_{i+1}$ for all i (where $e_{\ell+1} = e_1$). The *length* of such a cycle is its number of edges ℓ. The *girth* of a hypergraph is the length of the shortest Berge cycle it contains (if the hypergraph contains no Berge cycle we say it has infinite girth, or is *Berge acyclic*). We say that an absorber rooted on $\{x_1, \ldots, x_{k-1}\}$ is K*-sparse* if it has girth at least K, even after adding the extra edge $\{x_1, \ldots, x_{k-1}\}$. The following lemma allows us to find a sparse absorber in the dense setting.

Lemma 4.5 (Dense Absorber Lemma)**.** *Let $1/n \ll \eta \ll 1/M \ll \gamma, 1/k, 1/d, 1/K$. Let G be an n-vertex k-graph in which all but ηn^d d-sets have degree at least $(\mu_d(k) + \gamma)\binom{n-d}{k-d}$. Let X be a $(k-1)$-set of vertices each of degree at least $(\mu_d(k) + \gamma)\binom{n-1}{k-1}$. Then G contains a K-sparse absorber A rooted in X of order at most M.*

To prove Lemma 4.4 we apply a sparse version of the (Weak) Hypergraph Regularity Lemma. This reduces the problem to a dense setting, where we may use Lemma 4.5 to find the desired absorbers. The proof will appear in the full version of the paper.

4.2. Absorbers in the dense setting

Finally, we discuss the proof of Lemma 4.5. We require the following strengthening of the minimum degree threshold for loose Hamilton cycles, which ensures the existence of a loose Hamilton cycle under slightly more general conditions and with some additional properties. We say that a set $X \subseteq V(C)$ in a loose cycle C is K*-spread* if the distance (in terms of vertices) between any pair of (distinct) vertices of X is at least K with respect to the ordering of C.

For $1 \leq d \leq k-1$, we define $\mu^*_d(k)$ as the least $\mu \in [0, 1]$ such that for all $\gamma > 0$, positive integers t, K and n divisible by $k-1$ and sufficiently large the following holds: if G is an n-vertex k-graph, $y \in V(G)$ and $X \subseteq V(G)$ is a t-set such that $\delta_d(G - X) \geq (\mu + \gamma)\binom{n-d}{k-d}$ and $\deg(x) \geq (\mu + \gamma)\binom{n-1}{k-1}$ for every $x \in X$, then G contains a loose Hamilton cycle C in which X is K-spread and y has degree 2.

Note that we trivially have $\mu^*_d(k) \geq \mu_d(k)$. The following lemma, whose proof will appear in the full version of the paper, shows that these two thresholds actually coincide.

Lemma 4.6 (Threshold exploitation)**.** *We have* $\mu_d(k) = \mu^*_d(k)$ *for all* $1 \leq d \leq k - 1$.

Given this, the proof idea for Lemma 4.5 is to find a set of M vertices $S \subseteq V(G)$ that contains X and appropriately reflects the minimum degree structure of the host graph G. One can then apply Lemma 4.6 to the induced graph $G[S]$ in order to obtain the desired absorber. The details of this argument will appear in the full version of the paper.

5. Conclusion

In this paper, we investigated for which probabilities $p = p(n) \in [0, 1]$ a subgraph $G' \subseteq G$ of a k-uniform binomial random graph $G \sim \mathrm{H}_k(n, p)$ with minimum d-degree $\delta_d(G') \geq (\mu_d(k) + o(1))p\binom{n-d}{k-d}$ typically contains a loose Hamilton cycle. Our main result determines the optimal value for p when $d > (k + 1)/2$. While we do provide bounds for p when $d \leq (k + 1)/2$, it is unlikely that our results are optimal in this range. Hence a natural question is whether one can improve on this. It would furthermore be very interesting to understand whether one can obtain similar results for 'tighter' cycles. A first step in this direction was undertaken by Allen, Parczyk and Pfenninger [34] for $d = k - 1$. Note that in this situation, the problem of finding a tight Hamilton cycle in a dense graph is quite well-understood. Going beyond this, one potentially challenging problem would be to prove such a result for values of d and k for which we do not yet know the precise value of the (dense) minimum degree threshold.

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