RT-MAT 2004-30

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Novembro 2004

Esta é uma publicação preliminar ("preprint").

SMALL WORLD SEMIPLANES WITH GENERALISED SCHUBERT CELLS

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ABSTRACT. Regular generalised polygons in particular are: semiplanes; small world graphs, i.e. the diameter $d \leq c \log_{k-1}(v)$, where v is order and k is average degree; graphs that can be homomorphically mapped onto the ordinary polygons. The problem of the existence of regular graphs satisfying these conditions with the degree $\geq k$ and the diameter $\geq d$ for each pair $k \geq 3$ and $d \geq 3$ is addressed in the paper. This problem is positively solved via the explicit construction. Generalised Schubert cells are defined in the spirit of Gelfand-Macpherson theorem for the Grassmanian. Constructed graph, induced on the generalised largest Schubert cells, is isomorphic to the well-known Wenger's graph. The interpretation of Wenger graph in terms of affine Lie algebras allows to prove its edge-transitivity.

1. INTRODUCTION

It is well known that the diameter of a k-regular graph (or graph with the average degree k) of order v is at least $\log_{k-1}(v)$ and that the random k-regular graph has diameter close to this lower bound (see [2, X]). Only several explicit constructions of families of k-regular graphs with diameter close to $\log_{k-1}(v)$ are known [2, X, sec.1], [13]. Most of them have cycles C_3 or C_4.

The problem of constructing infinite families of given degree with small diameter (i.e. with diameter at most $c \log_{k-1}(v)$, $c \geq 1$ is a constant) with certain additional properties is far from trivial. This problem has many remarkable applications in economics, natural sciences, computer sciences and even in sociology. For instance, the "small world graph" of binary relation "two person shake their hands" on the set of people in the world, has small diameter.

The restriction of this problem on the class of bipartite graphs has additional motivations because such problem for random graphs has been studied by Klee, Larman and Wright, Harary and Robinson, Bollobas and others (see the survey in [2, c X, sec.5]).

One of the most important classes of small world bipartite graphs with additional geometric properties is a class of regular generalised

m-gon, i.e. regular tactical configurations of diameter m and girth $2m$. For each parameter m, a regular generalised m-gon has degree $q+1$ and order $2(1+q+\cdots+q^m)$. Up to parameters as above all known examples of regular generalised m-gons are geometries of finite Shevalley group $A_2(q)$, $B_2(q)$ and $G_2(q)$ for $m = 3$, 4 and 6, respectively (see [6]). According to the famous Feit-Higman theorem regular thick (i.e. degree ≥ 3) generalised m-gons exist for $m = 3, 4$ and 6 only. Thus generalised Pentagon does not exist, in particular. Each generalised m-gon has a homomorphism (retraction) onto the geometry of diheadral group D_m, which is ordinary m-gon.

We underline the following natural generalizations of regular generalised polygons.

(i) The class of graphs with logarithmic diameter $d \leq c_1 \log_{k-1}(v)$ and logarithmic girth $g \geq c_2 \log_{k-1}(v)$, where c_1, c_2 are some constants. Such graphs are important for communication networks. The problem of existence of an infinite family of such graphs with constant degree k has been solved explicitly by Margulis ([10], [11], [12]) and Lubotzky, Phillips and Sarnak [9]. These graphs are not bipartite, they are Cayley graphs of $PSL_2(p)$ (p is prime) introduced in [10] and investigated in [9]. In this construction the diameter is bounded by $2\log_{k-1}(v)+2$ and the girth $g \approx \frac{4}{3}\log_{k-1}(v)$. This construction supports the existence of graphs with unbounded logarithmic diameter and logarithmic girth $\geq g$ of degree $\geq k$ for each pair (k, g).

(ii) Other generalisation of generalised m-gon is a flag system with the Coxeter metric of dihedral group D_m (for the definition, see [4], [5]). This class of combinatorial objects is very close to the generalised m-gons. The examples of such systems different from generalised m-gons are unknown.

(iii) Let us consider the class of regular semiplanes, which are bipartite small world graphs and can be epimorphically mapped onto the ordinary polygons. These two conditions are not so restrictive as existence of flag systems with Coxeter metrics. The existence of a homomorphism onto the ordinary polygon allows to define naturally so called Schubert cells and small Schubert cells on the vertex-set of the graph.

The purpose of this paper is to prove the existence of graphs from this class with the diameter $\geq d$ and degree $\geq k$ for each pair (d, k) via explicit constructions. Our main result is the following statement.

Theorem 1.1. *For each integer m, $m \geq 2$, and any prime power q, there exists a semiplane $SP_m(q)$ of diametrer d, $m \leq d \leq 2m-1$, of* *order* $2(1+q+\dots q^{m-1})$ *and degree* $q+1$, *which can be homomorphically mapped onto the geometry of the dihedral group* D_m.

Note that $SP_3(q)$ and $SP_4(q)$ are isomorphic to geometries of groups $A_2(q)$ and $B_2(q)$, respectively. Semiplane property insures that the girth of the graphs $SP_m(q)$ is ≥ 6. The Schubert geometry of $SP_m(q)$, i.e the totality of all points and lines at maximal distance from standard flag, turns out to be Wenger graph [14], which is useful for applications in Computer Science.

Other example of important problems is the construction of small world graphs of infinite degree and finite diameter which is greater or equal than given integer k with the fast algorithm of finding the pass between any two vertices.

Graphs $SP_m(q)$ are defined via equations over F_q written in terms of field addition and multiplication. If we change F_q onto general commutative field K we will get graphs $SP_m(K)$. If K is infinite then $SP_m(K)$ are infinite graphs of diameter $\geq m$ such that we can find a pass of length t, $t \leq 2m+1$ fast, i.e. with $O(m^2)$ arithmetic operations.

2. Graphs and incidence system

The missing definitions of graph-theoretical concepts which appears in this paper can be found in [1] or [2]. All graphs we consider are simple, i.e. undirected without loops and multiple edges. Let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. Then $|V(G)|$ is called the *order* of G, and $|E(G)|$ is called the *size* of G. A path in G is called *simple* if all its vertices are distinct. When it is convenient, we shall identify G with the corresponding anti-reflexive binary relation on $V(G)$, i.e. $E(G)$ is a subset of $V(G)\times V(G)$ and write vGu for the adjacent vertices u and v (or neighbors). The sequence of distinct vertices $v_0, v_1, \dots, v_t$, such that v_iGv_{i+1} for $i=1,\dots,t-1$ is the pass in the graph. The length of a pass is a number of its edges. The distance $\mathrm{dist}(u,v)$ between two vertices is the length of the shortest pass between them. The diameter of the graph is the maximal distance between two vertices u and v of the graph. Let C_m denote the cycle of length m i.e. the sequence of distinct vertices $v_0,\dots,v_m$ such that v_iGv_{i+1}, $i=1,\dots,m-1$ and v_mGv_1. The girth of a graph G, denoted by $g=g(G)$, is the length of the shortest cycle in G. The degree of vertex v is the number of its neighbors.

The incidence structure is the set V with partition sets P (points) and L (lines) and symmetric binary relation I such that the incidence of two elements implies that one of them is a point and another is a line. We shall identify I with the simple graph of this incidence relation (bipartite graph). If number of neighbours of each element is finite and depends only from its type (point or line), then the incidence structure is a tactical configuration in the sense of Moore (see [6]). An incidence structure is a semiplane if two distinct lines are intersecting not more than in one point and two distinct points are incident not more than one line. As it follows from the definition, graphs of the semiplane have no cycles C_3 and C_4.

The graph is k-regular if each of its vertex has degree k, where k is a constant.

Let us consider an incidence structure with point set P and line set L, which are two copies of n-dimensional vector space over F_q. It will be convenient for us to denote vectors from P as

$$\mathrm{x} = (x) = (x_{1,0}, x_{1,1}, x_{2,1}, x_{3,1}, \dots x_{i,1}, \dots)$$

and vectors from L as

$$\mathrm{y} = [y] = [y_{0,1}, y_{1,1}, y_{2,1}, y_{3,1}, \dots, y_{i,1}, \dots].$$

We say that point (x) is incident with the line $[y]$ and we write it xIy or $(x)\mathrm{I}[y]$ if and only if the following condition are satisfied:

$$y_{i,1} - x_{i,1} = y_{i-1,1}x_{1,0}$$

where $i = 1, 2, \dots$.

Let W(q) be the incidence graph of the structure $\Gamma(F_q) = (P, L, I)$. For each integer $k \geq 2$ let $\Gamma(l, F_q) = (P(k), L(k), I(k))$ be the incidence system, where $P(k)$ and $L(k)$ are the images of P and L under the projection of these spaces on the first k-coordinates and binary relation $I(k)$ is defined by the first k equations. Finally, let $\mathrm{W}_k(\mathrm{q})$ be the incidence graph for $\Gamma(k, F_q)$. This is exactly the graph which has been defined by Wenger. Graph W(q) is a projective limit of $\mathrm{W}_k(q)$ when k goes to infinity.

Let P_m be the incidence graph of the incidence structure of points (vertices) and lines (edges) of the ordinary m-gon.

For $k \geq 1$ and $m \geq 1$ define a family $F(k, m)$ of incidence structures satisfying the axioms (A1)-(A6) below.

(A1) $F(k, m)$ is a family of small world graphs;

(A2) Each $\gamma \in F(k, m)$ is a k-regular tactical configuration;

(A3) $\gamma \in F(k, m)$ is a semiplane;

(A4) For each $\gamma \in F(k, m)$ there is a homomorphism $\phi : \gamma \to P_m$ and monomorphism $\eta : P_m \to \gamma$ such that $\phi \circ \eta$ is the identity map and $\eta(P_m)$ is the set of fixed points of $\eta \circ \phi$;

(A5) there is a flag $\{p, l\} \in P_m$ such that $dist(u, \eta(p)) = dist(u, \eta(p))$ and $dist(u, \eta(l)) = dist(u, \eta(l))$ if and only if $\phi(u) = \phi(v)$;

The axioms (A4) and (A5) alow us to define the *generalised Schubert cells* in the following way: vertices u and v are in the same cell if and only if $\phi(u) = \phi(v)$ (or distances from u and v to the elements of standard flag $\{p, l\}$ are the same). We can also consider *generalised small Schubert cells*: u and v are in the same cell if $dist(u, x) = dist(v, x)$ for each $x \in \eta(P_m)$. Last equivalence relation is defined in the spirit of Gelfand-Mac Pherson theorem for the Grassmanian [7].

In the next section we construct explicitly a family of graphs satisfying the axioms $A(1) - A(5)$.

3. Main construction

Let us consider the dihedral group D_m and its geometry. The Coxeter group D_m is defined as group with generators a and b and generic relations $(ab)^m = e$, $a^2 = e$ and $b^2 = e$. The order of D_m is $2m$. The point set and the line set for the geometry D_m is the totality of cosets $D_m : (a)$ and $D_m : (b)$, respectively. Two classes α and β are incident $\alpha I \beta$ if and only if $|\alpha \cap \beta| = 0$ It is easy to see that the geometry is just the incidence structure P_m of vertices (points) and edges (lines) of theordinary m-gon.

The totality of mirror symmetries (reflections) of ordinary m-gon is the set of elements with odd length with respect to the irreducible decomposition into letters of the alphabet $\{a, b\}$. It contains the words $a, b, aba, bab, \ldots$, and the longest element is $(ab)^r a = (ba)^r b$, $2r + 1 = 2[m/2]$.

Let $l(\alpha)$, $\alpha \in (D_m : (a)) \cup (D_m : (b))$ be the length of the coset α, i. e. the minimal length of the irreducible representation for representatives of α. Let Δ be the totality of all reflections of the Coxeter group D_m. To each element $\alpha \in \Gamma(D_m)$ we construct the set $\Delta(\alpha) = \{w \in \Delta | l(w\alpha) \leq l(\alpha)\}$. and the vector space $V(\alpha) = (F_q)^{\Delta(\alpha)} = \{f : \Delta(\alpha) \to F_q\}$. We can consider such a vector space as a subspace of ${F_q}^{\Delta}$ consisting of elements satisfying condition $f(x) = 0$ for $x \in \Delta - \Delta(\alpha)$. The natural basis of ${F_q}^{\Delta}$ is the totality of e_r, where $e_r(r) = 1$ and $e_r(r') = 0$, $r \neq r'$. Let us use "double index notation" for the basis elements: $e_a = e_{1,0}$, $e_b = e_{0,1}$, $e_{aba} = e_{2,1}$, $e_{bab} = e_{3,1}, \ldots, e_{ab^{[m/2]}a} = e_{m-2,1}$.

We can turn ${F_q}^{\Delta}$ into an alternating linear algebra with the multiplication $*$, such that $e_{1,0} * e_{0,1} = e_{1,1}$, $e_{1,0} * e_{i,1} = e_{i+1,1}$, $i = 1, \ldots, m-3$ and product of other basis elements is zero. Note that this operation is not associative. In fact it is a Lie bracket (see the last section of the paper).

Let us consider now the following new incidence structure on the set $\tilde{\Gamma}(D_n)$ of elements (α, x), $\alpha \in \Gamma(D_n)$ (element of ordinary n-gon), $\mathrm{x} \in F_q$, We shall assume that (α, x) is a point if and only if α is a point of ordinary n-gon. Two pairs (α, x) and (β, y) are incident (relation I') if and only if the following two conditions hold

(i) $\alpha I \beta$ within geometry of ordinary m-gon

(ii) $\mathrm{x} - \mathrm{y}|_{\Delta(\alpha)\cap\Delta(\beta)} = \mathrm{x} * \mathrm{y}$.

The graph of the incidence relation I' will be denoted as $SP_m(q)$.

We can identify elements of kind $(\alpha, 0)$, where $0(x) = 0$ for each $x \in \Delta$ with the elements of $\tilde{\Gamma}$. Thus we have a natural embedding η of Γ into $\tilde{\Gamma}$. Let us use the term *standard flag* for $(a), (b)$.

Proposition 3.1. *The degree of each element of* $\tilde{\Gamma}$ *is* $q+1$. *The diameter of* $\tilde{\Gamma}$ *is bounded by* $2m-1$. *The map* $\phi : \tilde{\Gamma} \to \Gamma$, $\phi(\alpha, \bar{x}) = \alpha$, *is the homomorphism onto the geometry of ordinary m-gon, the map* $\eta : \Gamma \to \tilde{\Gamma}$ *is monomorphism,* $\phi \circ \eta$ *is an identity map and* $\eta(\Gamma)$ *is the set of fixed elements of* $\eta \circ \phi$.

Proof. The definition of the incidence relation for $\tilde{\Gamma}$ implies that ϕ is an epimorphism. Let (α, f) be the vertex of $\tilde{\Gamma}$. The element α has two neighbors α_1 and α_2 in the polygon. Without loss of generality we may assume that $l(\alpha_1) < l(\alpha_2)$. It is clear that $\Delta(\alpha_1) \subset \Delta(\alpha_2)$ and, as it follows from the definition of the incidence, there is a unique neighbour u of (α, f) such that $\phi(u) = \alpha$. In fact, it is $(\alpha_1, f|_{\Delta(\alpha_1)})$. For the neighbour of (α, f) of kind α_2 we have two different cases: if $l(\alpha_2) > l(\alpha_1)$, then $\Delta(\alpha_2)$ includes $\Delta(\alpha_1)$ and $|\Delta(\alpha_2) \setminus \Delta(\alpha_1)| = 1$ and we have q-neighbours of kind (α, g), such that $g|_{\alpha_1} = f$. Let $l(\alpha_2) = l(\alpha)$, i.e. the cosets α_2 and α have maximal length. Then $|\Delta(\alpha_2)| = |\Delta(\alpha)| = m-1$ and $|\Delta(\alpha) \cap \Delta(\alpha)| = m-2$. As it follows from the definition of the incidence relation, the neighbour of kind (α_2, g) is uniquely determined by $g(w)$, where $\{w\} = \Delta(\alpha_2) \setminus (\Delta(\alpha_2) \cap \Delta(\alpha))$. Thus we have exactly q options there. It means that the degree of each vertex of $\tilde{\Gamma}$ has degree $q+1$.

Let v and u be the vertices of $\tilde{\Gamma}$, their minimal distance to some element of the standard flag is restricted by $m-1$. If v and u are elements of the same type then the shortest walks from them to elements of the standard flag have same last element. Thus $\mathrm{dist}(u, v) = 2m-2$. If these elements are of different type then we can combine the shortest walk from the first element, edge of the standard flag and reverse for the shortest walk from the second element to the standard flag. It means that the $\mathrm{dist}(u, v) \leq 2m-1$.

□

Proposition 3.2. *The graph $SP_m(q)$ is a semiplane.*

Proof. We have to prove that the common neighbourhood for two distinct vertices u and v of the same type (both points or both lines) contains at most one element. Let us consider the case $\phi(u) \neq \phi(v)$. Without loss of generality we may assume that $\Delta(\phi(u))$ contains $\Delta(\phi(v))$ and write u as (α, f). There is a unique common neighbour β of $\phi(u)$ and $\phi(v)$ and $\Delta(\beta)$ is a subset of $\Delta(\alpha)$. It means that the only possible option for the common neighbour is $(\beta, f|_{\Delta(\beta)})$. In fact, the condition of the existence of the unique common neighbour is $v = (\phi(v), f_{\Delta(\phi(v))})$.

Let β be one of two common neighbours for α in the pentagon. We can write u and v as (α, f_1) and (α, f_2), respectively. Then a possible common neighbour of u and v can be written as (β, g). Consider the following cases:

(i) If $l(\beta) > l(\alpha)$ then $\Delta(\beta)$ contains $\Delta(\alpha)$ and $f_1 = f_2 = g|_{\Delta(\alpha)}$. Thus $u = v$ and we get a contradiction in this case.

(ii) Let $l(\beta) < l\alpha$, then possible neighbours have form $(\beta, f_1|_{\Delta(\alpha)})$. The condition of the existence of common neighbour for u and v is $f_1(x) = f_2(x)$ for $x \in \Delta$. Then the unique neighbour of u and v exist in the case $f_1(x) = f_2(x)$ for $x \in \Delta(\beta)$. Notice that $f_1(r) \neq f_2(r)$ for the single root r in $\Delta(\alpha) \setminus \Delta(\beta)$.

(iii) Let $l(\beta) = l(\alpha)$ and $g(r') = x$ for $r' \neq \Delta(\alpha)$. The values f_1 and f_2 are the following tuples $(a_r, a_{1,1}, \ldots a_{m-2,1})$ and $(b_r, b_{11}, \ldots, b_{m-2,1})$, where r is a simple root different from r'. Let $e(r) = 1$ for $r = (1,0)$ and $e(r) = 0$ for $r = (0,1)$. If $a_r \neq b_r$ then possible x is uniquely defined from the system of two equations

$$a_{1,1} - x_{1,1} = e(r)a_r x, \quad b_{1,1} - x_{1,1} = e(r)b_r x.$$

Notice that in this case $a_{1,1} \neq b_{1,1}$ and there is no neighbour w with $l(\phi(w)) = m-1$. Let $a_r = b_r$ then from the incidence equations we are getting $f_1 = f_2$ which contradicts to $u \neq v$.

Thus u and v have at most one common neighbour.

□

Proposition 3.3. *The Schubert substructure of $\tilde{\Gamma} = SP_m(q)$ is well defined. It is isomorphic to the Wenger graph* $W_{m-1}(q)$.

Proof. Let us consider point p and lines l of $\tilde{\Gamma}$ with the property $l(\phi(p)) = m-1$, $l(\phi(l)) = m-1$ and pIl. Then distances from p and l to the nearest vertex from the standard flag equal $m-1$. Thus the generalised largest Schubert cells are well-defined. Le t $p = (\alpha, f)$, $l = (\beta, g)$, f and g are defined by tuples $(a_{1,0}, a_{1,1}, \ldots, a_{m-2,1})$ and

$(b_{0,1}, b_{1,1}, \ldots, b_{m-2,1})$. Then incidence condition of $\tilde{\Gamma}$ implies

$$a_{i+1,1} - b_{i+1,1} = a_{1,0}b_{i,1}, i = 0, 1, \ldots, m-3.$$

These are the equations that define the Wenger graph. □

Propositions 3.1-3.3 imply immediately Theorem 1.1 and show that the family of graphs $SP_m(q)$ satisfies to the axioms $A(1) - A(5)$.

4. Schubert transitivity

Let us consider the affine Kac-Moody Lie algebra $L = \tilde{A}_1$ over the field K defined via 2×2 symmetric extended Cartan matrix (a_{ij}) with $a_{11} = a_{22} = 2$ and $a_{12} = -2$ see [8]. It has a Cartan decomposition $L^- \oplus H \oplus L^+$, where H and $H \oplus L^+$ are the Cartan and the Borel algebras respectively. The algebra L^+ is a direct sum of one dimensional root subalgebras corresponding to positive roots. The set of positive roots in the standard basis of simple roots α_1 and α_2 can be written as tuples $(i+1, i)$, (i, i), $(i, i+1)$, $i = 0, 1, \ldots$. Let $<$ be the lexicographical order on the set of positive roots. Let e_α be the basic element from the root subalgebra L_α. We choose a basis of L such that $[e_\alpha, e_\beta] = e_{\alpha+\beta}$ if $\alpha < \beta$ and $\alpha + \beta$ is a root, and identify the elements of L with the tuples in this basis.

For each positive root α and $l \in K$ we consider the automorphism $t_\alpha(l) = \exp(\mathrm{ad}(le_\alpha))$ of the infinite dimensional Lie algebra L^+. This automorphism can change infinitely many components of the vector from L^+, but the close formulae for the i-th component of $t_\alpha(l)(x)$, $x \in L^+$, is the polynomial expression in variables $x_1, \ldots x_i$.

Let us consider the direct sum $L(\alpha)$ of L_β such that $\beta \leq \alpha$. Then $t_r(l)$ acts naturally on $L(\alpha)$. Let U and $U(\alpha)$ be the groups generated by $t_r(l)$ where $e_r \in L^+$ and $e_r \in L(\alpha)$, respectively. Then U an $U(\alpha)$ act regularly, i.e. transitively with a trivial point stabilizer, on the vector spaces L and $L(\alpha)$, respectively.

Consider the subalgebra P of L generated by elements e_{α_1} and e_β, where $\beta = \alpha_1 + \alpha_2$. Then P is a direct sum of L_r, where $r = (i+1, i)$ and (i, i). Let $P(\alpha) = P \cap L(\alpha)$, where $e_\alpha \in P$. Groups $U(P) =< t_r(l) | e_r \in P >$ and $UP(\alpha) = U(P) \cap U(\alpha)$ act regularly on P and P_α, respectively. We will write any root $\alpha = l\beta + \alpha_1$ corresponding to a root subspace from P as $(l, 1)$. We will also restrict the order $<$ on this set of roots: $(l, 1) < (l', 1)$ if and only if $l < l'$.

The following statement is immediate corollary from the definitions.

Proposition 4.1. *The Lie algebra $(F_q^\Delta, *)$, which defines the graphs $SP_m(q)$, is isomorphic to $L(\alpha)$ for $\alpha = (m-2, 1)$, considered as a Lie algebra over the ground field F_q.*

Next statement is equivalent to the flag transitivity of the Schubert substructure (*Schubert transitivity*) for the semiplane $SP_m(q)$.

Theorem 4.2. *Wenger graph $W_m(q)$ is edge transitive.*

Proof. Consider first the case of char$F_q \geq m$. Let α^* be the dual root for $\alpha = (1,0)$. Then α^* is a basis element of the Cartan subalgebra H. The multiplication rule in $H \oplus L^+$ for α is $[\alpha^*, e_r] = 2e_r$, where $r \neq (0,1)$ and $[\alpha^*, e_{0,1}] = 0$.

Let us consider the external derivation β^* which is "dual" to $\beta = (0,1)$: $[\beta^*, e_r] = \beta^*(r)e_r$, where $\beta^*(i,1) = i$ and consider the the subalgebra $\tilde{L} =< \alpha^*, \beta^*, L^+ >$. We shall identify points $(x_{1,0}, x_{1,1}, \ldots, x_{m-1,1})$ and lines $[y_{0,1}, y_{1,1}, \ldots, y_{m-1,1}]$ with the elements

$$\tilde{x} = \alpha^* + \sum_{i=1}^{m-1} \frac{1}{i} x_{i,1} e_{i,1} + x_{1,0} e_{1,0}$$

and

$$\tilde{y} = \beta^* + \frac{1}{2} \sum_{i=0}^{m-1} y_{i,1} e_{i,1},$$

respecively.

We can rewrite the incidence condition of Wenger graph in the form $[\tilde{x}, \tilde{y}] = 0$. Elements $u = t_r(l)$ preserve the Lie bracket and the group $UP(\alpha)$, $\alpha = (m-1, 1)$ acts regularly on the set of pairs $(\tilde{x}, \tilde{y})$ such that $[\tilde{x}, \tilde{y}] = 0$ according to the rule: $\tilde{x} \to \tilde{x}^u|_{(L^+ - L_{0,1})}$, $\tilde{y} \to \tilde{y}^u|_{(L^+ - L_{1,0})}$. Thus Wenger graph is an edge transitive for $p = \mathrm{char}(F_q) \geq m$.

We can write close formula for each transformation $t_\alpha(l)$ acting on $P \cup L$ in the form $x_r \to x_r + f_r(x_{1,0}, \ldots, x_{r'})$, $y_r \to y_r = g_r(y_{0,1}, \ldots, y_{r'})$, $r' < r$, which preserve the incedence relation for the case of small characteristic as well.

These transformation generate the group which acts regularly on the vertices of Wenger graph.

□

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