MARSHALL'S CONJECTURE FOR PYTHAGOREAN FIELDS

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Let F be a formally real, Pythagorean field. We describe here the structure of our affirmative answer to the following well-known problem in the theory of quadratic forms, posed by M. Marshall in 1974 (see [L]) :

Problem I : *Let φ be a quadratic form over F. Suppose that for all orders P in F, the signature of φ at P is $\equiv 0 \bmod 2^n$. Is it true that φ belongs to the n^{th} -power of the fundamental ideal of the Witt ring of F ?*

The detailed presentation of the solution of this problem, together with the proof of certain important local-global principles in cohomology and K-theory, will appear as [DM2]. Some of the constructions in [DM1] proved useful in showing that Problem I has an affirmative answer. We write

$$F \models [\mathrm{MC}],$$

to indicate that, for F, the answer to Problem I is affirmative.

There is a more general formulation of this problem, also due to M. Marshall, originally stated in terms of abstract order spaces. By the Duality Theorem between abstract order spaces and special groups (see [DM1]), we can state this generalization, still an open problem, as

Problem II : *Let G be a reduced special group and φ a form over G. Suppose that for all SG-character σ of G, the signature of φ at σ is $\equiv 0 \bmod 2^n$. Is it true that $\varphi \in I^n(G)$?*

We now set down some of the notation that will be in use in all that follows. For F a

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formally real, Pythagorean field :

- $\dot{F} = \{x \in F : x \neq 0\}$; - $\dot{F}^2 = \{x^2 : x \in \dot{F}\}$.

- $G(F) = \dot{F}/\dot{F}^2 = \{\overline{a} : a \in \dot{F}\}$, is the group of classes of elements of F, modulo squares.

- The classes of 1 and -1 in $G(F)$, will be written 1 and -1, respectively.

- $\langle a_1, \ldots, a_n \rangle$ ($\in \dot{F}^n$) denotes a quadratic form of dimension n over F;

- Write $\langle a_1, \ldots, a_n \rangle \equiv_F \langle b_1, \ldots, b_n \rangle$ for the **isometry** relation of quadratic forms over F;

- $X(F)$ is the space of orders of F; it is a boolean space (compact, Hausdorff, totally disconnected). For $a \in \dot{F}$

$$\varepsilon(\overline{a}) =_{def} \{P \in X(F) : a < 0 \text{ in the order } P\} =_{def} [\overline{a} = -1],$$

is a sub-basic clopen in $X(F)$. Let $B(F)$ be the Boolean algebra of clopens in $X(F)$.

Let $\triangle$ be the operation of **symmetric difference** in $B(F)$. Define the relation $\equiv_{B(F)}$ in $B(F)$ by

$$\langle a, b \rangle \equiv_{B(F)} \langle c, d \rangle \quad \text{iff} \quad \begin{cases} a \triangle b = c \triangle d \\ \quad \textit{and} \\ a \cap b = c \cap d \end{cases}$$

We extend $\equiv_{B(F)}$ to $B(F)^n$ by the rule : $\langle x_1, \ldots, x_n \rangle \equiv_{B(F)} \langle y_1, \ldots, y_n \rangle$ iff

$$\exists\, a,\, b,\, z_3,\, \ldots,\, z_n \in B(F) \text{ such that} \quad \begin{cases} \langle x_1, a \rangle \equiv_{B(F)} \langle y_1, b \rangle & \text{and} \\ \langle x_2, \ldots, x_n \rangle \equiv_{B(F)} \langle a, z_3, \ldots, z_n \rangle & \text{and} \\ \langle y_2, \ldots, y_n \rangle \equiv_{B(F)} \langle b, z_3, \ldots, z_n \rangle. \end{cases}$$

If B is a Boolean algebra, let $\wedge$ (meet), $\vee$ (join) and $(\cdot)^c$ (complement) be the basic operations in B. Write $\perp$ (bottom) and $\top$ (top) for the smallest and largest element of B, respectively. We may define *symmetric difference* in B by

$$a \bigtriangleup b = (a \wedge b^c) \vee (a^c \wedge b).$$

Thus, we may define in any Boolean algebra B, a relation $\equiv_B$, by exactly the same formulas as above, with $\wedge$ in place of $\cap$. We use this observation in the statement of the next result, showing that $B(F)$ satisfies a certain universal property that characterizes it among all Boolean algebras.

Teorema 0.1 : *(see [DM1]) The map* $\quad \varepsilon : G(F) \longrightarrow B(F), \ \ \overline{a} \mapsto [\overline{a} = -1]$

is an injective group homomorphism, such that $\varepsilon(1) = \emptyset$ *and* $\varepsilon(-1) = X(F)$. *Moreover,*

a) For $a_1, \ldots, a_n, b_1, \ldots, b_n \in \dot{F}$, *the following conditions are equivalent :*

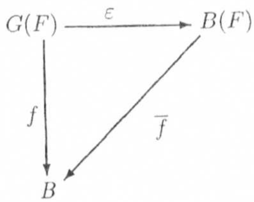
1. $\langle a_1, \ldots, a_n \rangle \equiv_F \langle b_1, \ldots, b_n \rangle$; $\quad$ 2. $\langle \varepsilon(\overline{a_1}), \ldots, \varepsilon(\overline{a_n}) \rangle \equiv_{B(F)} \langle \varepsilon(\overline{b_1}), \ldots, \varepsilon(\overline{b_n}) \rangle$.

b) For all Boolean algebras B *and group homomorphisms* $f : G(F) \longrightarrow B$, *satisfying*

(i) $f(1) = \bot$ *and* $f(-1) = \top$;

(ii) For all $a, b \in \dot{F}$, $\langle a, b \rangle \equiv_F \langle c, d \rangle$ *implies* $\langle f(\overline{a}), f(\overline{b}) \rangle \equiv_B \langle f(\overline{c}), f(\overline{d}) \rangle$,

there is a unique Boolean algebra homomorphism $\overline{f} : B(F) \longrightarrow B$, *such that the diagram*



is commutative.

Because of Theorem 0.1, $B(F)$ is called the **boolean hull** of $G(F)$.

There are two basic operations that are defined for quadratic forms over F : sum and product. For forms $\varphi = \langle a_1, \dots, a_n \rangle$, $\psi = \langle b_1, \dots, b_m \rangle$ over F, set

$$\varphi \oplus \psi = \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle \qquad \text{(sum)}$$

$$\varphi \otimes \psi = \langle a_1 b_1, \dots, a_k b_j, \dots, a_n b_m \rangle \qquad \text{(product)}$$

When $\varphi = \langle a \rangle$, we write $a\psi$ for $\langle a \rangle \otimes \psi$; $a\psi$ is called a *scalar multiple* of ψ.

Write $\mathcal{W}(F)$ for the Witt ring of F, that is, the set of equivalence classses of quadratic forms under isometry, with the operations induced by the sum aand product above. The form $\langle 1 \rangle$ is the identity in $\mathcal{W}(F)$, while its zero is the class of hyperbolic forms, $\oplus_{i=1}^{k} \langle 1, -1 \rangle$. The **fundamental ideal** $I(F)$ of $W(F)$, is defined by

$$I(F) = \{\text{classes of forms of even dimension}\}$$

For $n \geq 1$, $I^n(F)$ is the n^{th} -power of $I(F)$. As an additive group, I^n is generated by scalar multiples of the **Pfister forms of degree n**, that is, sums of forms of the type

$$a \otimes_{i=1}^{n} \langle 1, a_i \rangle,$$

where $a, a_1, \dots, a_n \in \dot{F}$. For each $n \geq 1$, set

$$\overline{I^n} = I^n(F)/I^{n+1}(F).$$

The **graded Witt ring of F** is the system

$$\mathcal{W}_{gr}(F) = (\overline{I^1}, \dots, \overline{I^n}, \dots)$$

For $n \geq 1$, we have a homomorphism

$$t_n : \overline{I^n} \longrightarrow \overline{I^{n+1}}, \text{ given by } \quad \varphi \mapsto \langle 1, 1 \rangle \otimes \varphi =_{def} 2\varphi.$$

Now consider the condition

$$\text{[WMC] : For all } n \geq 1, \quad t_n \text{ is injective.}$$

We then have

Teorema 0.2 : $F \models$ [MC] *sse* $F \models$ [WMC].

The proof of Theorem 0.2 is a consequence of the following results and constructions : Consider the inductive systems

$$IW(F) = (\overline{I^1} \xrightarrow{t_1} \overline{I^2} \ldots \overline{I^n} \xrightarrow{t_n} \overline{I^{n+1}} \ldots)$$

$$\mathcal{K}(F) = (k_1 F \xrightarrow{l(-1)} k_2 F \ldots k_n F \xrightarrow{l(-1)} k_{n+1} F \ldots)$$

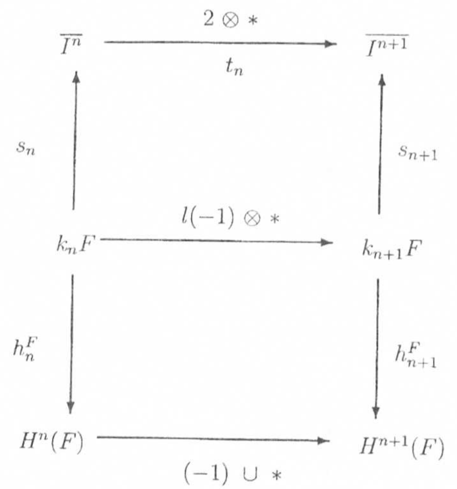
where the system $\mathcal{K}(F)$ is constructed from Milnor's mod 2 K-theory of F (see [Mi]). Let $W(F)$ and $k(F)$ be the inductive limit of the systems $IW(F)$ and $\mathcal{K}(F)$, respectively. Then, Theorem 0.1 is an important ingredient in the proof of

Teorema 0.3 : *$W(F)$ e $k(F)$ are Boolean rings, naturally isomorphic to $B(F)$.*

It follows from Theorem 0.2 that to verify that $F \models$ [MC], it suffices to show that $F \models$ [WMC]. We now describe the main points involved in the proof of

Teorema 0.4 : *Every formally real Pythagorean field satisfies [WMC].*

Fundamental, for the proof of Theorem 0.4, is the following commutative diagram :



(D)

Here, $H^*(F)$ is the (full) mod 2 cohomology ring of the field F, while k_*F is the mod 2 K-theory of F. The top row consists of the graded Witt ring of F. The upper vertical arrows (s_n) are due to John Milnor, while lower ones (h_n^F), are due to J. Tate and H. Bass (see [Mi]). We first show

Teorema 0.5 : *Cup product by (-1) is an injective homomorphism from $H^n(F)$ to $H^{n+1}(F)$.*

The recent and beautiful result of V. Voevodsky (see [V]), namely

$$\forall\, n \geq 0, \quad h_n^F \text{ is an isomorphism,}$$

and Theorem 0.5 yield that multiplication by $l(-1)$ is an injective homomorphism in K-theory. Now, a well-known argument, using the K-theoretic Stiefel-Whitney invariant of order 2^{n-1} (see [Mi]), yields Milnor's graded Witt ring conjecture for F :

$$\forall\, n \geq 0,\ s_n \text{ is an isomorphism.}$$

Consequently, the horizontal homomorphisms in the first row of **(D)** are also injective, that is, F satisfies [WMC]. Thus, Theorem 0.4 follows from Theorem 0.2.

For the proof of Theorem 0.5 we need the following two results, the second of which is due to J. Kr. Arason (see [A]).

Teorema 0.6 : *Let $E = F(d^{1/2})$ be a quadratic extension of F and let $n \geq 2$ be an integer. Then, for each $\eta \in k_nE$, there is a finite set of indices I, together with $a_1^i, \ldots, a_{n-1}^i \in \dot{F}$ and $b^i \in \dot{E}$, $i \in I$, such that*

$$\eta = \Sigma_{i\in I}\ l(a_1^i)\ldots l(a_{n-1}^i)l(b^i).$$

Theorem (see [A]) : *Let $E = F(d^{1/2})$ be a quadratic extension of F. Then, there is a long exact sequence in Galois cohomology, which for each $n \geq 0$ consists of*

$$\cdots\ H^n(F) \xrightarrow{Res} H^n(E) \xrightarrow{Cor} H^n(F) \xrightarrow{\mu} H^{n+1}(F)\ \cdots,$$

wher μ is cup product with (d), while Res e Cor are the restriction and corestriction homomorphisms in cohomology, respectively.

References

[A] J. Kr. Arason, *Cohomologische Invarianten quadratischer Formen*, J. Algebra, **36**, 448-491 (1975).

[DM1] M. Dickmann, F. Miraglia, *Special Groups : Applications of Boolean Algebras to the Algebraic Theory of Quadratic Forms*, October 1996, 210 pgs., submitted

[DM2] M. Dickmann, F. Miraglia, *On Quadratric Forms whose Total Signature is $\equiv 0$ mod 2^n. Solution to a Problem of M. Marshall.*, 30 pgs., to appear in *Inventiones Mathematicae*, 1997.

[L] T. Y. Lam, *Ten Lectures on Quadratic Forms over Fields*, in Conf. on Quadratic Forms (G. Orzech, ed.), Queen's Papers on Pure and Applied Math. **46**, Queen's University, Ontario, Canada, 1 - 102, 1977.

[Ma] M. Marshall, *A reduced theory of quadratic forms*, in Conf. on Quadratic Forms (G. Orzech, ed.), Queen's Papers on Pure and Applied Math. **46**, Queen's University, Ontario, Canada, 569 - 579, 1977.

[Mi] J.Milnor, *Algebraic K-theory and Quadratic Forms*, Inventiones Math. **9** (1970), 318-344.

[V] V. Voevodsky, *Milnor's Conjecture*, preprint, 1996.

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