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INSTABILITY OF EQUILIBRIUM POINTS OF SOME LAGRANGIAN SYSTEMS

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Instability of Equilibrium Points of Some Lagrangian Systems

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Abstract

In this work we show that, if $\mathcal{L}$ is a natural lagrangian system such that the k-jet of the potential energy ensures it does not have a minimum at the equilibrium and such that its hessian has rank at least $n-2$, then there is an asymptotic trajectory to the associated equilibrium point and so the equilibrium is unstable. This applies, in particular, to analytic potentials with a saddle point and a hessian with at most 2 null eigenvalues.

The result is proven for lagrangians in a specific form, and we show that the class of lagrangians we are interested can be taken into this specific form by a subtle change of spatial coordinates. We also consider the extension of this results to systems subjected to gyroscopic forces.

Keywords: Liapunov stability; Lagrange-Dirichlet theorem; Lagrangian systems

1 The problem

Consider the study of the Liapunov instability of equilibrium points of conservative Lagrangian systems in $\mathbb{R}^{2n}$, with lagrangians $\mathcal{L}(q,\dot{q}) = T(q,\dot{q}) - \pi(q)$, where π is the potential energy and T the kinetic energy.

Lagrange's equations for these systems are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \tag{1}$$

and its equilibrium points are of the form $(q_0, 0)$ where $\frac{\partial \pi}{\partial q}(q_0) = 0$.

In this context, Lagrange announced in 1788 a theorem, proved in 1846 by Dirichlet, which asserts that that if an equilibrium point is a local strict minimum for π, then this point is stable in the sense of Liapunov. This is the classic Lagrange-Dirichlet theorem. In 1904 Painlevé gave a counter-example to the reciprocal of this theorem in 1 degree of freedom.

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Since then the problem of studying the stability of an equilibrium point which is not a local strict minimum for the potential energy, usually giving sufficient conditions for its instability, has been known as the inversion of the Lagrange-Dirichlet theorem.

In [5], a striking example of the non inversibility of the Lagrange-Dirichlet theorem was given in two degrees of freedom. In the example there is a straight line passing through the origin in which the potential energy is strictly negative except at the origin where its value is 0 and, yet, the origin is stable. For that, it was considered the kinetic energy $T(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{\dot{q}_1^2+\dot{q}_2^2}{2}$ and the potential energy $\pi(q_1, q_2) = e^{-\frac{1}{q_1^2}} \cos\frac{1}{q_1} - e^{-\frac{1}{q_2^2}} \left(\cos\frac{1}{q_2} + q_2^2\right)$. This system is clearly seen to be separable and, each part is shown to be stable just like in the Painlevé example. Thus, it shows that even if there is a curve adherent to the origin at which the potential energy is negative, this does not imply the instability of the equilibrium.

Among the possible partial inversions to the Lagrange-Dirichlet theorem, there is a conjecture placed by Liapunov and, with the notations and language introduced in [1], restated by Barone Netto, which says that, being the origin an equilibrium point for the lagrangian system, if the k-jet of the potential energy shows that it does not have a minimum in the origin, then the origin is an unstable equilibrium point (see the next section for the definitions of k-jet and of it showing that the origin does not have a minimum). This conjecture, if true, would be the best result possible in the set of functions that have k-jet.

Several interesting results concerning this conjecture have been proved recently. It is worth noticing that in the works cited below, something stronger than the instability of the origin is proved, namely they show the existence of an asymptotic trajectory to the origin in the past.

In the context of 2 degrees of freedom, the conjecture was completely proved in [4]. This was done using a Četaev like function to show the existence of an asymptotic trajectory to the origin.

In the general case of n degrees of freedom, we have only partial results in the direction of this conjecture. For example, when the jet that shows that the origin is not a minimum for the potential energy is homogeneous, then in [7] and, independently, in [10] it is proved that with some additional hypothesis on the regularity of the lagrangian the origin is unstable.

Extending this result, in [6] the following theorem is proved:

Theorem (Maffei, Moauro and Negrini). *Consider the lagrangian system given by equation* (1) *in* $\mathbb{R}^{2(m+n)}$, *with* $q = (u, v)$, $\dot{q} = (\dot{u}, \dot{v})$ *and* $\mathcal{L} = T - \pi$. *Assume there is an integer* $k \geq 3$ *and reals* $\omega_1, \ldots, \omega_m$ *such that:*

1. $\pi(u, v) = \frac{1}{2}(u, l(u, v)u) + \pi_{[k]}(u, v) + R(u, v)$, *where* $l(u, v)$ *is an* $m \times m$ *matrix such that* $l(0, 0) = \operatorname{diag}(\omega_1^2, \ldots, \omega_m^2)$, $\pi_{[k]}$ *homogeneous of degree* k, $R(u, v) = O(\|(u, v)\|^{k+1})$ *and* $\min\left\{\pi_{[k]}(0, v) : \|v\| = 1\right\} = -1$;

2. $\mathcal{L}$ *is* $C^{k+3+\lfloor\frac{k-3}{2}\rfloor}$.

Then there is a trajectory $\phi(t)$ such that $(\phi(t), \dot{\phi}(t)) \to (0,0)$ as $t \to -\infty$.

This result, in particular, proves the conjecture under the assumption that there is a split of $\mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m$ where we have that the k-jet of π is $\pi_2(q_1, \ldots, q_n) + \pi_k(q_{n+1}, \ldots, q_m)$, such that π_2 is a positive definite quadratic form in $(q_1, \ldots, q_n)$ and π_k is an homogeneous polynomial of degree k that shows that the origin is not a minimum for the potential energy π; and that $\mathcal{L}$ is $C^{k+3+\lfloor \frac{k-3}{2} \rfloor}$.

In the context of analytic lagrangians, in [8] it is shown that if the lagrangian is analytic and the potential energy does not have a minimum in the origin, then the origin is unstable. In this paper only the instability is proved, and the question on whether an asymptotic trajectory to the origin exists remains open. Also, the conjecture as proposed by Barone Netto includes the analytical case, since in [2] it is proved that if π is analytic and does not have a minimum in the origin, then there is a positive integer k such that $j^k\pi$ shows this fact.

In the present work, we increase the class of jets that ensures the instability of the equilibrium point. For this, we prove the following result:

Theorem A. *Consider the lagrangian system given by equation* (1) *and assume that $\mathcal{L} = T - \pi$ is such that* 0 *is an equilibrium point, the k-jet of π shows that the origin is not a minimum for π, that $\mathcal{L}$ is $C^{k+\lfloor \frac{k-3}{2} \rfloor+3}$, that the rank of $j^2\pi(0,0)$ is at least $n-2$ and that T is a positive definite quadratic form in $\dot{q}$ for every q. Then there is a trajectory $\phi(t)$ such that $(\phi(t), \dot{\phi}(t)) \to (0,0)$ as $t \to -\infty$.*

Therefore, this result is an extension of [6] for the case of co-dimension 2 and it also covers the analytic case when the hessian of the potential energy has at most 2 null eigenvalues.

To attain this, we initially prove a theorem that ensures the instability of the origin when the potential energy has a particular form. Then, we show that the class of systems above mentioned can be taken to this form by means of a subtle change of spatial coordinates. The method we use to prove the theorem is a modification of the method used in [4], that keeps all the good properties of that construction, and is the key step in this work.

Also related to the problem at hand, we consider Lagrangian systems under the action of gyroscopic forces, known as the Routh problem. Lagrange's equations for a system in this conditions are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q\,, \tag{2}$$

where $Q = Q(q)\dot{q}$ is linear in $\dot{q}$ and such that $\langle Q(q)\dot{q}, \dot{q}\rangle = 0$. The equilibria are again of the form $(q_0, 0)$ where $\frac{\partial \pi}{\partial q}(q_0) = 0$, exactly as in the conservative case.

Gyroscopic stabilization is a known fact that shows the importance of additional hypothesis on the gyroscopic force and, also, that some techniques used in the study of the inversion of the Lagrange-Dirichlet theorem may fail to work in this context. See, for example, reference [9].

In [7], it is shown that the method used to treat the case when the k-jet of π that shows the origin is not a minimum for π is homogeneous works, under the additional hypothesis that there is an integer s such that $s \geq \frac{k+2}{2}$ and $j^s Q$ is the first non null jet of Q. This result is an extension of the analytic case in [7].

We show in the same way, under slightly weaker conditions on the gyroscopic force, that our results are valid in this context as well.

This text is organized as follows: in the following section, we introduce the definitions and notations that we use along the text. In section 3, we prove some lemmas and estimates that are important to prove the key theorem that we use to prove theorem A above. In section 4 we prove this key theorem, and apply it in section 5, proving theorem A and extending the results of [4]. Finally, in section 6 we give a natural extension of these results for the case of lagrangians systems with gyroscopic forces.

2 Hypothesis and notations

The definitions and notations introduced in the present section are used along the text.

We need two basic definitions from k-decidability introduced by Barone Netto and presented, respectively, in [1] and [4].

Definition. *Let $\Omega \subset \mathbb{R}^n$ be an open set which contains the origin. A function $f : \Omega \to \mathbb{R}$ is said to have punctual jet of order k in the origin, with k a positive integer, if there is a polynomial $P : \mathbb{R}^n \to \mathbb{R}$ with degree less or equal to k such that $\lim_{x \to 0} \frac{f(x)-P(x)}{\|x\|^k} = 0$. In this case, we denote the polynomial P by $j^k f$.*

In this text, we call the punctual jet of order k simply by k-jet, always at the origin.

A simple consequence of this definition is that if there is the k-jet of f, then it is unique. Therefore, polynomial of order k, it is its k-jet.

Definition. *We say that $j^k f$ shows that f does not have minimum at the origin (or, equivalently, $j^k f$ shows that the origin is not a minimum for f) if f is a function which has punctual jet of order k at the origin and for every function g such that $j^k f = j^k g$ we have that g does not have a minimum (not even strictly weak) at the origin.*

In order to correctly specify the system which we work in the beginning, we need the following definition.

Definition. *Given real numbers $\alpha > 0$ and $\beta > 1$, and a C^1 function $P : \mathbb{R}^m \to \mathbb{R}$, we say that P satisfies the (α, β)-property if there are real numbers $\delta \in (0, \frac{\alpha}{4})$ and $\sigma > 0$ such that there is a connected component C of*

$$\Lambda = \left\{(q_1, \ldots, q_m) : \, 0 < q_m < \sigma, \; P(q_1, \ldots, q_m) \leq \delta q_m^\beta \right\}$$

which contains $\{(0, \ldots, 0, q_m), \, 0 < q_m < \sigma\}$ where the following inequality holds

$$q_m \frac{\partial P}{\partial q_m}(q_1, \ldots, q_m) \leq (k-1)\delta q_m^\beta \, ,$$

with $k = \lceil \beta \rceil$.

With this, we can finally define the class of lagrangians for whom we prove our initial result.

Definition. *We say that the lagrangian* $\mathcal{L}(q,\dot{q}) = T(q,\dot{q}) - \pi(q)$ *satisfies the hypothesis* $\mathcal{H}_0$ *if the following conditions hold*

1. $\mathcal{L}$ *is defined for* q *in* $\Omega \cap \{q : q_n > 0\} \cup \{0\}$, *for some open neighborhood* Ω *of* 0 *in* $\mathbb{R}^n$, *and* $\dot{q}$ *in* $\mathbb{R}^n$;

2. T *is positive definite quadratic form in* $\dot{q}$ *for every* q *and letting* $T(q,\dot{q}) = \frac{1}{2}\sum_{l,s=1}^{n} g_{ls}(q)\dot{q}_l\dot{q}_s$, *we have that* $[g_{ls}(q)]$ *is symmetric for every* q *and there are reals* $\mu_1 > 0$ *and* $\mu_2 > -1$ *such that* $[g_{ls}(q)] = I + h(q)$, *where* $\|h(q)\| = o(\|q\|^{\mu_1})$ *and* $\|h'(q)\| = o(\|q\|^{\mu_2})$;

3. π *is of class* C^2 *for* $q_n > 0$ *and is continuous in* $\Omega \cap \{q : q_n > 0\} \cup \{0\}$, $\pi(0) = 0$ *and there are a positive integer* N *and reals* $\alpha > 0$ *and* $\beta > 1$ *such that*
$$\pi(q) = U(q_1,\ldots,q_N) + \pi_2(q_{N+1},\ldots,q_n),$$
where:

   (a) $U(q_1,\ldots,q_N) \geq 0$ *for all* $(q_1,\ldots,q_N)$;

   (b) $\pi_2(q_{N+1},\ldots,q_n) = -\alpha q_n^{\beta} + P(q_{N+1},\ldots,q_n) + R(q_{N+1},\ldots,q_n)$, *with*

      i. $P(q_{N+1},\ldots,q_n)$ *satisfies the* (α,β)*-property in* $\mathbb{R}^{n-N}$;

      ii. $P(0,\ldots,0,q_n) = 0$ *and* $P \geq 0$ *for all* $(q_{N+1},\ldots,q_n) \in C$, *where* C *is given by the* (α,β)*-property of* P;

      iii. $R(q_{N+1},\ldots,q_n) = o(\|(q_{N+1},\ldots,q_n)\|^{\beta})$;

      iv. $\frac{\partial R}{\partial q_j} = o(\|(q_{N+1},\ldots,q_n)\|^{\beta-1})$ *for* $j = N+1,\ldots,n$.

Yet, for the lagrangians in the section where we extend the results of [4], the following definition is important to keep the hypothesis together.

Definition. *We say that the lagrangian* $\mathcal{L}(q,\dot{q}) = T(q,\dot{q}) - \pi(q)$ *satisfies the hypothesis* $\mathcal{H}_1$ *if the following conditions hold*

1. $\mathcal{L} = T - \pi$ *is of class* C^2 *and is defined for* q *in some open neighborhood of* 0 *in* $\mathbb{R}^n$ *and* $\dot{q} \in \mathbb{R}^n$;

2. $T(0,\dot{q}) = \frac{1}{2}\sum_{i=1}^{n}\dot{q}_i^2$ *and* T *is a positive definite quadratic form in* $\dot{q}$ *for all* q;

3. *there are positive integers* N *and* k *such that*
$$\pi(q_1,\ldots,q_n) = U(q_1,\ldots,q_N) + \pi_2(q_{N+1},\ldots,q_n),$$
where:

(a) $U(q_1, \ldots, q_N) \geq 0$ *for all* $(q_1, \ldots, q_N)$;

(b) $\pi(0) = \|\frac{\partial \pi}{\partial q}(0)\| = 0$;

(c) $j^k \pi_2$ *shows that* π_2 *does not have a minimum in the origin;*

(d) *there is* $j^{k-1}\nabla \pi_2$ *in the origin.*

We should note that the last condition is to be understood as every coordinate of $\nabla \pi_2$ has a $(k-1)$-jet. This hypothesis imply that $\pi_2 = j^k \pi_2 + R$, with $R(q_{N+1}, \ldots, q_n) = o(\|(q_{N+1}, \ldots, q_n)\|^k)$. Also, since there is $j^{k-1}\nabla \pi_2$ at the origin, it is shown in the appendix of [4] that $\frac{\partial R}{\partial q_j} = o(\|(q_{N+1}, \ldots, q_n)\|^{k-1})$ for $j = N+1, \ldots, n$.

3 Some preliminary lemmas

We keep the notation introduced in the last section, and assume that $\mathcal{L}$ satisfies $\mathcal{H}_0$. In particular, δ, σ and C are the ones given by the (α, β)-property of P. Also, denote by $E(q, \dot{q}) = T(q, \dot{q}) + \pi(q)$ the total energy of the system.

In this conditions, we can write Lagrange's equations (1) in coordinates and, after a brief development, get into

$$\frac{d}{dt}\left(\sum_{s=1}^{n} g_{rs}\dot{q}_s\right) = -\frac{\partial \pi}{\partial q_r} + \frac{1}{2}\sum_{l,s=1}^{n} \frac{\partial g_{ls}}{\partial q_r}\dot{q}_l\dot{q}_s \qquad (3)$$

for $r = 1, \ldots, n$. Although these equations are not in the normal form, this will be convenient later.

With these notations, we make a subtle modification in the auxiliary function used in [4] that, as shown in the following lemmas, retains the same estimates obtained in [4], and enables us to follow the same route in proving the instability theorem in the next section.

Let $p_n(q, \dot{q}) = \sum_{s=1}^{n} g_{ns}\dot{q}_s$ and consider the function

$$V(q, \dot{q}) = \alpha q_n^\beta - \frac{1}{2g_{nn}}p_n^2 - R(q_{N+1}, \ldots, q_n)$$

and $\widetilde{V}(q, \dot{q}) = \frac{V(q,\dot{q})}{q_n^\beta}$. Then, for $\sigma_1 \in (0, \sigma)$ and $\lambda > 0$, we define

$$C_{\sigma_1} = \left\{ (q, \dot{q}) : \begin{array}{ll} E(q, \dot{q}) = 0, & (q_{N+1}, \ldots, q_n) \in C, \\ \widetilde{V}(q, \dot{q}) < \delta, & \text{and} \quad q_n \in (0, \sigma_1) \end{array} \right\}$$

and

$$C_{\sigma_1, \lambda} = C_{\sigma_1} \cap \{(q, \dot{q}) : \|(q_1, \ldots, q_{n-1})\|_\infty < \lambda q_n \}.$$

Notice that C_{σ_1} is not empty since all the points such that $\frac{g_{nn}}{2}\dot{q}_n^2 = \alpha q_n^\beta - R$, with all the other coordinates equal to 0 and q_n sufficiently small are in the set.

Since we are searching for asymptotic trajectories to the origin, and by the definition of C_{σ_1}, it makes sense to work in the relative topology of the energy level $E = 0$, and we do so.

It is worth noticing that, since $E = T + U - \alpha q_n^\beta + P + R$, in C_{σ_1} we have that $V = T - \frac{1}{2g_{nn}} p_n^2 + U + P$. Since $T(0, \dot{q}) = \frac{1}{2}\sum_{i=1}^{n} \dot{q}_i^2$ and, from the definition of p_n, $T(q, \dot{q}) - \frac{1}{2g_{nn}} p_n^2(q, \dot{q})$ does not have any term in $\dot{q}_n$, given $\varepsilon > 0$, for small enough $\|q\|$, $\frac{(1-\varepsilon)}{2}\sum_{i=1}^{n-1} \dot{q}_i^2 \leq T(q, \dot{q}) - \frac{1}{2g_{nn}} p_n^2(q, \dot{q}) \leq \frac{(1+\varepsilon)}{2}\sum_{i=1}^{n-1} \dot{q}_i^2$.

Thus, recalling that U and P are positive, we have that

$$0 \leq \frac{(1-\varepsilon)}{2}\sum_{i=1}^{n-1} \dot{q}_i^2 \leq \frac{(1-\varepsilon)}{2}\sum_{i=1}^{n-1} \dot{q}_i^2 + U + P \leq V < \delta q_n^\beta . \tag{4}$$

It follows that, in $C_{\sigma_1,\lambda}$,

$$\dot{q}_j^2 = O(q_n^\beta) \tag{5}$$

for $j = 1, \ldots, n-1$, which will be used in the following.

An important bound for $\dot{q}_n$, which will be used in the next lemma, is given by:

Lemma 1. *Given $\lambda > 0$ and $\varepsilon \in (0, 1)$, there is $\sigma_0 > 0$ such that if $(q, p) \in \overline{C}_{\sigma_0,\lambda}$, then*

$$\frac{\alpha}{1+\varepsilon} q_n^\beta \leq \dot{q}_n^2 \leq \frac{4\alpha}{1-\varepsilon} q_n^\beta .$$

Proof. It follows the hypothesis on g_{ls} and equation 5 that there is $\sigma_1 > 0$ such that, in $C_{\sigma_1,\lambda}$,

$$\frac{1-\varepsilon}{2}\dot{q}_n^2 + o(q_n^\beta) \leq \frac{1}{2g_{nn}}\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)^2 \leq \frac{1+\varepsilon}{2}\dot{q}_n^2 + o(q_n^\beta) . \tag{6}$$

Let $\sigma_2 \in (0, \sigma_1]$ be such that, as remarked after the definition of V, $V \geq 0$ in $C_{\sigma_2,\lambda}$. Recall that $R = o(q_n^\beta)$ in $C_{\sigma_2,\lambda}$, and using the definition of V we get

$$\alpha q_n^\beta - \frac{1}{2g_{nn}} p_n^2 + o(q_n^\beta) \geq 0$$

and then $\frac{1}{2g_{nn}} p_n^2 \leq \alpha q_n^\beta + o(q_n^\beta)$. Using (6), we get $\frac{1-\varepsilon}{2}\dot{q}_n^2 \leq \alpha q_n^\beta + o(q_n^\beta)$. From here, it follows that $\dot{q}_n^2 \leq \frac{2\alpha}{1-\varepsilon} q_n^\beta + o(q_n^\beta)$ and, for $\sigma_3 \in (0, \sigma_2]$, $\dot{q}_n^2 \leq \frac{4\alpha}{1-\varepsilon} q_n^\beta$ in $C_{\sigma_3,\lambda}$.

On the other hand, $V < \delta q_n^\beta$ in $C_{\sigma_3,\lambda}$ and from the definition of V and the order of R we have

$$\alpha q_n^\beta - \frac{1}{2g_{nn}} p_n^2 + o(q_n^\beta) < \delta q_n^\beta$$

so $\frac{1}{2g_{nn}} p_n^2 > (\alpha - \delta) q_n^\beta + o(q_n^\beta)$. Again, by (6), and recalling that from the (α, β)-property $\delta < \frac{1}{4}\alpha$, we get $\frac{1+\varepsilon}{2}\dot{q}_n^2 \geq (\alpha - \delta) q_n^\beta + o(q_n^\beta) \geq \frac{3}{4}\alpha q_n^\beta + o(q_n^\beta)$, and it follows that $(1+\varepsilon)\dot{q}_n^2 \geq \frac{3}{2}\alpha q_n^\beta + o(q_n^\beta)$. Finally, we can take $\sigma_0 \in (0, \sigma_3]$ such that in $C_{\sigma_0,\lambda}$ we have $\dot{q}_n^2 \geq \frac{\alpha}{(1+\varepsilon)} q_n^\beta$. □

Now we determine the sign of $\dot{\tilde{V}}$, which is a main step in proving the instability theorem. Recall that the border of the set in the next lemma is taken in the relative topology of the energy level $E = 0$.

Lemma 2. *For every $\lambda > 0$, there is $\sigma_0 \in (0, \sigma)$ such that in*

$$(\partial C_{\sigma_0} \setminus \{(0,0)\}) \cap \{(q, \dot{q}) : \|(q_1, \ldots, q_{n-1})\|_\infty \leq \lambda q_n\}$$

the function $\dot{\tilde{V}}$ is not zero. More precisely, $\dot{\tilde{V}}$ has the opposite sign of $\dot{q}_n$ in each connected component of this set.

Proof. Since $\tilde{V} = \frac{V}{q_n^\beta}$, we have that $\dot{\tilde{V}} = \frac{\dot{V}}{q_n^\beta} - \frac{\beta V}{q_n^{\beta+1}} \dot{q}_n$, and thus

$$q_n^{\beta+1} \dot{\tilde{V}} = q_n \dot{V} - \beta V \dot{q}_n \,. \tag{7}$$

Since $q_n > 0$ in $C_{\sigma,\lambda}$, it is enough to show that $q_n^{\beta+1} \dot{\tilde{V}}$ does not vanish. Let us calculate $\dot{V}$, using the definition of V

$$\dot{V} \;=\; \alpha\beta q_n^{\beta-1} \dot{q}_n - \frac{1}{g_{nn}} p_n \dot{p}_n + \frac{1}{2 g_{nn}^2} \dot{g}_{nn} p_n^2 - \sum_{i=N+1}^{n} \frac{\partial R}{\partial q_i} \dot{q}_i \,.$$

From here, using the definition of p_n we have

$$\begin{aligned} \dot{V} \;=\;& \alpha\beta q_n^{\beta-1} \dot{q}_n - \frac{1}{g_{nn}} \left(\sum_{s=1}^{n} g_{ns} \dot{q}_s \right) \frac{d}{dt} \left(\sum_{s=1}^{n} g_{ns} \dot{q}_s \right) \\ & + \frac{1}{2 g_{nn}^2} \dot{g}_{nn} \left(\sum_{s=1}^{n} g_{ns} \dot{q}_s \right)^2 - \sum_{i=N+1}^{n} \frac{\partial R}{\partial q_i} \dot{q}_i \end{aligned}$$

and by equations (3), we get

$$\begin{aligned} \dot{V} \;=\;& \alpha\beta q_n^{\beta-1} \dot{q}_n - \frac{\left(\sum_{s=1}^{n} g_{ns} \dot{q}_s\right)}{g_{nn}} \left(-\frac{\partial \pi}{\partial q_n} + \frac{1}{2} \sum_{l,s=1}^{n} \frac{\partial g_{ls}}{\partial q_n} \dot{q}_l \dot{q}_s \right) \\ & + \frac{1}{2 g_{nn}^2} \dot{g}_{nn} \left(\sum_{s=1}^{n} g_{ns} \dot{q}_s \right)^2 - \sum_{i=N+1}^{n} \frac{\partial R}{\partial q_i} \dot{q}_i \,. \end{aligned}$$

. Using the hypotheses on the form of π, we have

$$\begin{aligned} \dot{V} \;=\;& \alpha\beta q_n^{\beta-1} \dot{q}_n - \frac{\left(\sum_{s=1}^{n} g_{ns} \dot{q}_s\right)}{g_{nn}} \left(\alpha\beta q_n^{\beta-1} - \frac{\partial P}{\partial q_n} - \frac{\partial R}{\partial q_n} \right) \\ & - \frac{\left(\sum_{s=1}^{n} g_{ns} \dot{q}_s\right)}{2 g_{nn}} \left(\sum_{l,s=1}^{n} \frac{\partial g_{ls}}{\partial q_n} \dot{q}_l \dot{q}_s \right) + \frac{1}{2 g_{nn}^2} \dot{g}_{nn} \left(\sum_{s=1}^{n} g_{ns} \dot{q}_s \right)^2 \\ & - \sum_{i=N+1}^{n} \frac{\partial R}{\partial q_i} \dot{q}_i \,, \end{aligned}$$

$$
\begin{aligned}
\dot{V} &= -\frac{\left(\sum_{s=1}^{n-1} g_{ns}\dot{q}_s\right)}{g_{nn}}\left(\alpha\beta q_n^{\beta-1}\right) + \frac{\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)}{g_{nn}}\frac{\partial P}{\partial q_n} \\
&+ \frac{\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)}{g_{nn}}\frac{\partial R}{\partial q_n} - \frac{\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)}{2g_{nn}}\left(\sum_{l,s=1}^{n}\frac{\partial g_{ls}}{\partial q_n}\dot{q}_l\dot{q}_s\right) \\
&+ \frac{1}{2g_{nn}^2}\dot{g}_{nn}\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)^2 - \sum_{i=N+1}^{n}\frac{\partial R}{\partial q_i}\dot{q}_i ,
\end{aligned}
$$

$$
\begin{aligned}
\dot{V} &= -\frac{\left(\sum_{s=1}^{n-1} g_{ns}\dot{q}_s\right)}{g_{nn}}\left(\alpha\beta q_n^{\beta-1} - \frac{\partial P}{\partial q_n}\right) + \dot{q}_n\frac{\partial P}{\partial q_n} \\
&+ \frac{\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)}{g_{nn}}\frac{\partial R}{\partial q_n} - \frac{\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)}{2g_{nn}}\left(\sum_{l,s=1}^{n}\frac{\partial g_{ls}}{\partial q_n}\dot{q}_l\dot{q}_s\right) \\
&+ \frac{1}{2g_{nn}^2}\dot{g}_{nn}\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)^2 - \sum_{i=N+1}^{n}\frac{\partial R}{\partial q_i}\dot{q}_i .
\end{aligned}
$$

Therefore, we can write

$$
\begin{aligned}
q_n\dot{V} &= -\frac{\left(\sum_{s=1}^{n-1} g_{ns}\dot{q}_s\right)}{g_{nn}}\left(\alpha\beta q_n^{\beta} - q_n\frac{\partial P}{\partial q_n}\right) + \dot{q}_n q_n\frac{\partial P}{\partial q_n} \\
&+ q_n\frac{\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)}{g_{nn}}\frac{\partial R}{\partial q_n} - q_n\frac{\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)}{2g_{nn}}\left(\sum_{l,s=1}^{n}\frac{\partial g_{ls}}{\partial q_n}\dot{q}_l\dot{q}_s\right) \\
&+ \frac{q_n}{2g_{nn}^2}\dot{g}_{nn}\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)^2 - q_n\left(\sum_{i=N+1}^{n}\frac{\partial R}{\partial q_i}\dot{q}_i\right) .
\end{aligned}
$$

And, with exception of the term $\dot{q}_n q_n \frac{\partial P}{\partial q_n}$, for $\sigma_0 > 0$ small enough, the other terms are $o(q_n^{\frac{3\beta}{2}})$. In fact, noting that by hypothesis $g_{ij} \to 0$ if $i \neq j$ and $g_{ii} \to 1$, let's analyze each remaining term:

- $\frac{\left(\sum_{s=1}^{n-1} g_{ns}\dot{q}_s\right)}{g_{nn}}\alpha\beta q_n^{\beta}$: recall that $\dot{q}_j = O(q_n^{\frac{\beta}{2}})$ from (5), so we get that each parcel is $o(q_n^{\frac{3\beta}{2}})$;
- $\frac{\left(\sum_{s=1}^{n-1} g_{ns}\dot{q}_s\right)}{g_{nn}} q_n \frac{\partial P}{\partial q_n}$: from (5) and the (α,β)-the property of P, we conclude that each parcel is $o(q_n^{\frac{3\beta}{2}})$;

- $q_n \frac{\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)}{g_{nn}} \frac{\partial R}{\partial q_n}$ e $q_n \left(\sum_{i=N+1}^{n} \frac{\partial R}{\partial q_i}\dot{q}_i\right)$: from the hypothesis on R, we have that $\frac{\partial R}{\partial q_n} = o(q_n^{\beta-1})$, from (5) and lemma 1 we conclude that each parcel is again $o(q_n^{\frac{3\beta}{2}})$;
- $q_n \frac{\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)}{2g_{nn}} \left(\sum_{l,s=1}^{n} \frac{\partial g_{ls}}{\partial q_n}\dot{q}_l\dot{q}_s\right)$: from (5), lemma 1 and the fact that $\frac{\partial g_{ls}}{\partial q_n}$ is by hypothesis of order $\mu_2 > -1$, and then $q_n \frac{\partial g_{ls}}{\partial q_n}$ has strictly positive order, we get that each parcel is $o(q_n^{\frac{3\beta}{2}})$;
- $\frac{q_n}{2g_{nn}^2}\dot{g}_{nn}\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)^2$: notice that $\dot{g}_{nn} = \sum_{i=1}^{n} \frac{\partial g_{nn}}{\partial q_i}\dot{q}_i$ and recalling that by hypothesis $\frac{\partial g_{ls}}{\partial q_n}$ is of order $\mu_2 > -1$, from what $q_n \frac{\partial g_{ls}}{\partial q_n}$ has strictly positive order, it follows from (5) and lemma 1 that each parcel is $o(q_n^{\frac{3\beta}{2}})$.

Therefore, we get that

$$q_n \dot{V} = \dot{q}_n q_n \frac{\partial P}{\partial q_n} + o(q_n^{\frac{3\beta}{2}}).$$

Notice that in $\partial C_{\sigma_0,\lambda}$ we have $V = \delta q_n^{\beta}$, and thus $\beta V \dot{q}_n = \beta\delta q_n^{\beta}\dot{q}_n$. So, substituting the results in equation (7), we get

$$\begin{aligned} q_n^{\beta+1}\dot{\tilde{V}} &= \dot{q}_n q_n \frac{\partial P}{\partial q_n} - \beta\delta q_n^{\beta}\dot{q}_n + o(q_n^{\frac{3\beta}{2}}) \\ &= -\left(\beta\delta q_n^{\beta} - q_n \frac{\partial P}{\partial q_n}\right)\dot{q}_n + o(q_n^{\frac{3\beta}{2}}). \end{aligned} \quad (8)$$

Since in $\partial C_{\sigma_0,\lambda}$ we have $P \leq \delta q_n^{\beta}$, by the (α,β)-property of P we have that $q_n \frac{\partial P}{\partial q_n} \leq \delta(k-1)q_n^{\beta}$. Then

$$\beta\delta q_n^{\beta} - q_n \frac{\partial P}{\partial q_n} \geq (\beta - (k-1))\,\delta q_n^{\beta}.$$

Recalling that $(\beta - (k-1)) > 0$ and that for σ_0 given by lemma 1 sufficiently small, we have that $\dot{q}_n \neq 0$ and by the inequalities of the same lemma 1, we have

$$\begin{aligned} |q_n^{\beta+1}\dot{\tilde{V}}| &\geq (\beta - (k-1))\,\delta q_n^{\beta}|\dot{q}_n| - |o(q_n^{\frac{3\beta}{2}})| \\ &\geq (\beta - (k-1))\,\delta\sqrt{\alpha}\, q_n^{\frac{3\beta}{2}} - |o(q_n^{\frac{3\beta}{2}})|. \end{aligned}$$

Finally, getting σ_0 smaller if necessary, the result follows. In particular, equation (8) shows that the sign of $\dot{\tilde{V}}$ in

$$(\partial C_{\sigma_0} \setminus \{(0,0)\}) \cap \{(q,\dot{q}) : \|(q_1,\dots,q_{n-1})\|_\infty \leq \lambda q_n\}$$

is given by the opposite sign of $\dot{q}_n$, as we wanted. □

4 Instability theorem

We prove now the key result of the text, which is the the instability theorem showing the existence of an asymptotic orbit to the origin as $t \to -\infty$, what assures the instability of the origin as an equilibrium point. The proof, now that we have obtained the estimates of lemmas 1 and 2, is a direct extension of the main theorem in [4].

Theorem 1. *Consider the lagrangian systems given by equations* (1). *Assume that $\mathcal{L}$ satisfies $\mathcal{H}_0$. Then, there is $\rho > 0$ and a trajectory $\phi(t)$ such that $(\phi(t), \dot{\phi}(t)) \to (0,0)$ as $t \to -\infty$, and that for $|t|$ great enough, $(\phi(t), \dot{\phi}(t)) \in C_{\rho,\sqrt{2\alpha}}$.*

Proof. Given $\gamma > 0$, let's denote by Ω_γ the set

$$\Omega_\gamma = C_{\gamma,\sqrt{2\alpha}} \cap \{(q,\dot{q}) : \dot{q}_n > 0\} .$$

By lemma 1, there is $\rho_1 > 0$ such that in Ω_{ρ_1}, $\dot{q}_n^2 > \frac{1}{2}\alpha q_n^\beta$. Since $\dot{q}_n > 0$, we have that $\dot{q}_n > \sqrt{\frac{1}{2}\alpha q_n^{\frac{\beta}{2}}}$.

Also, from equation (4), we have that there is $\rho_2 \in (0, \rho_1]$ such that in Ω_{ρ_2}

$$\dot{q}_j^2 < 4\delta q_n^\beta < 2\alpha \dot{q}_n^2$$

for $j = 1, \ldots, n-1$.

Thus, for any solution $\varphi(t) = (q_1(t), \ldots, q_n(t))$ of (3), if there is a t_1 such that $(\varphi(t_1), \dot{\varphi}(t_1)) \in \overline{\Omega}_{\rho_2}$ and $|q_j(t_1)| = q_n(t_1)$, then there is $\epsilon_1 > 0$ such that for all $t \in (t_1, t_1 + \epsilon_1)$ we have $|q_j(t)| < \sqrt{2\alpha} q_n(t)$ and for all $t \in (t_1 - \epsilon, t_1)$, we have $(\varphi(t), \dot{\varphi}(t)) \notin \overline{\Omega}_{\rho_2}$.

On the other side, lemma 2 implies that there is $\rho \in (0, \rho_2]$ such that if $(\varphi(t_2), \dot{\varphi}(t_2)) \in \overline{\Omega}_\rho$ and, for some instant t_2, $V(\varphi(t_2), \dot{\varphi}(t_2)) = \delta q_n(t_2)^\beta$ then there is $\epsilon_2 > 0$ such that for all $t \in (t_2, t_2 + \epsilon_2)$ we have $V(\varphi(t), \dot{\varphi}(t)) < \delta q_n(t)^\beta$ and for all $t \in (t_2 - \epsilon_2, t_2)$ we have $V(\varphi(t), \dot{\varphi}(t)) > \delta q_n(t)^\beta$.

Therefore, each solution φ, with $0 < q_n(t) < \rho$, that in some instant $\tilde{t}$ is in the (relative) border of Ω_ρ was outside Ω_ρ for some time interval before $\tilde{t}$. And, besides, this solution will be in the relative interior of Ω_ρ for a time interval after $\tilde{t}$.

Let's take a sequence $z_j = (q_{1,j}, \ldots, q_{n,j}, \dot{q}_{1,j}, \ldots, \dot{q}_{n,j}) \in \partial\Omega_\rho$, with $0 < q_{n,j} < \rho$ and such that $\lim_{j\to\infty} z_j = (0,0)$ and consider the solutions ϕ_j of (3) such that $(\phi_j(0), \dot{\phi}_j(0)) = z_j$.

In some positive instant, these solutions are going to be in Ω_ρ and they cannot leave Ω_ρ in a point with $q_n < \rho$. Also, since in Ω_ρ we have $\dot{q}_n > \sqrt{\frac{1}{2}\alpha q_n^{\frac{\beta}{2}}}$, there are sequences $t_j > 0$ and $w_j = (\tilde{q}_{1,j}, \ldots, \rho, \dot{\tilde{q}}_{1,j}, \ldots, \dot{\tilde{q}}_{n,j}) \in \partial\Omega_\rho$, such that $(\phi_j(t_j), \dot{\phi}_j(t_j)) = w_j$ and $(\phi_j(t), \dot{\phi}_j(t)) \in \Omega_\rho$ for all $t \in (0, t_j)$.

It is clear that there is a subsequence w_{j_i} converging to a point $\bar{w}$. Without loss of generality, let's suppose that $w_{j_i} = w_j$.

We assert that the solution $\phi(t)$ of (3) with initial conditions $\bar{w}$ is asymptotic to $(0,0)$ as $t \to -\infty$.

Suppose by contradiction that it does not occur. Then, there would be a time $\bar{t}$ such that $(\phi(\bar{t}), \dot{\phi}(\bar{t})) \notin \overline{\Omega}_\rho$.

Let

$$d = \left\{ \text{dist}((\phi(\bar{t}), \dot{\phi}(\bar{t})), \partial\Omega_\rho), \min_{t\in[\bar{t},0]} \|(\phi(t), \dot{\phi}(t))\| \right\},$$

with $\text{dist}((\phi(\bar{t}), \dot{\phi}(\bar{t}), \partial\Omega_\rho)$ being the distance from $(\phi(\bar{t}), \dot{\phi}(\bar{t}))$ to $\partial\Omega_\rho$.

Since w_j converges to $\bar{w}$, continuous dependence guarantees that there is j_0 such that for $j > j_0$ we have

$$\min_{t\in[\bar{t},0]} \|(\phi(t), \dot{\phi}(t)) - (\phi_j(t_j + t), \dot{\phi}_j(t_j + t))\| < \frac{d}{2},$$

what implies that $t_j + \bar{t} < 0$ for $j > j_0$, because

$$\|(\phi(\bar{t}), \dot{\phi}(\bar{t})) - (\phi_j(t_j + \bar{t}), \dot{\phi}_j(t_j + \bar{t}))\| < \frac{d}{2} \implies ((\phi_j(t_j + \bar{t}), \dot{\phi}_j(t_j + \bar{t})) \notin \overline{\Omega}_\rho.$$

But then we have that $z_j = (\phi_j(0), \dot{\phi}_j(0)) = (\phi_j(t_j + (-t_j)), \dot{\phi}_j(t_j + (-t_j)))$ does not converge to $(0,0)$ as $j \to \infty$, what is a contradiction with the choice of z_j and completes the proof. □

5 The splitting of the potential energy

In this section we study a particular case of systems that, after a subtle change of spatial coordinates, satisfies the hypothesis $\mathcal{H}_0$. Therefore, as an application of theorem 1, we obtain an instability theorem for these systems. The interest in this is that the conditions required are more natural to state and verify than the ones in hypothesis $\mathcal{H}_0$.

As discussed in the introduction, the initial problem that motivates this section is obtaining a generalization of the main result of [4] in a splitting of $\mathbb{R}^n$, analogous – although with very distinct techniques – to what is done in [6] to extend the results of [7].

In this spirit, we study the system of equations (3) for a lagrangian $\mathcal{L} = T - \pi$ satisfying $\mathcal{H}_1$ and, we further suppose that $n - N = 2$, that is, $\pi_2 = \pi_2(q_{n-1}, q_n)$ is a function of the plane, whose k-jet shows that π_2 does not have a minimum in the origin.

Then, according to section 3 of [4], there is a change of coordinates in which π_2 may be rewritten such that it satisfies the respective hypothesis in $\mathcal{H}_0$. In particular, lemma 1 of [4] proves the (α, β)-property of P.

To complete the demonstration, it is enough to verify that the kinetic energy, in the new coordinates, satisfies the hypothesis on its order. For that, let's recall briefly the construction of the change of coordinates made in [4].

Suppose, initially, that $j^k\pi_2$ is not homogeneous or it is homogeneous and there is (q_{n-1}, q_n) such that $j^k\pi_2(q_{n-1}, q_n) > 0$, that is, it is an homogeneous saddle. Under this conditions, it is shown in the appendix of [4] that there are reals σ, λ_0, $\alpha > 0$, $(k-1) < \beta \leq k$ and an algebraic curve $\gamma : I \to \mathbb{R}$, where $I = [0, \sigma)$, with $\gamma(0) = 0$, such that, after an eventual rotation, the following relation holds

$$\min_{-\lambda_0 q_n < q_{n-1} < \lambda_0 q_n} j^k \pi_2(q_{n-1}, q_n) = j^k \pi_2(\gamma(q_n), q_n) = -\alpha q_n^\beta + o(q_n^\beta)$$

for all $q_n \in [0, \sigma)$.

Besides, we can write $\gamma(q_n) = \sum_{i=1}^{\infty} b_i q_n^{\beta_i}$, with $b_i \in \mathbb{R}$ and (β_i) a strictly increasing sequence of rationals, with $\beta_1 > 1$. From here, it follows that, with an eventually smaller σ, there are positive constants c_1, c_2 and c_3 such that, for $q_n \in [0, \sigma)$, we have

$$\begin{array}{rcl} |\gamma(q_n)| & < & c_1 q_n^{\beta_1} \\ |\gamma'(q_n)| & < & c_2 q_n^{\beta_1 - 1} \\ |\gamma''(q_n)| & < & c_3 q_n^{\beta_1 - 2}. \end{array} \tag{9}$$

The change of coordinates we are looking for is made in

$$\mathcal{F} = \{(q_{n-1}, q_n) : 0 < q_n < \sigma\},$$

where it is C^∞ and admits an extension to $\mathcal{F} \cup \{0\}$, which is an homeomorphism, preserving the asymptotic trajectories to the origin tangent to the semi-axis $\{q_n > 0\}$. This change of coordinates takes the curve γ into this semi-axis.

Thus, consider $\Phi(q_{n-1}, q_n) = (\tilde{q}_{n-1}, q_n)$, where $\tilde{q}_{n-1} = q_{n-1} - \gamma(q_n)$, with $(q_{n-1}, q_n) \in \mathcal{F}$; and $\Phi(0,0) = (0,0)$. In these coordinates, we have that π_2 satisfies the relative hypothesis in $\mathcal{H}_0$, a result shown in details in the section 3 of [4]. We notice that $P \geq 0$ for $|\tilde{q}_{n-1}| \leq \lambda_0 q_n$, but the proof of lemma 1 in [4] shows that $P \geq 0$ in C, as desired.

Let's denote by Ψ the change of coordinates in $\mathbb{R}^n$ such that

$$\Psi(q_1, \ldots, q_{n-1}, q_n) = (q_1, \ldots, q_{n-2}, \Phi(q_{n-1}, q_n)).$$

Now, we can calculate the kinetic energy in the new coordinates and verify that it satisfies the hypothesis on its order.

Let $q \in \mathbb{R}^n$ and let's denote the new variables by $\tilde{q} = (q_1, \ldots, q_{n-1} - \gamma(q_n), q_n)$ and $\dot{\tilde{q}} = (\dot{q}_1, \ldots, \dot{q}_{n-1} - \gamma'(q_n)\dot{q}_n, \dot{q}_n)$.

Denoting $T(q, \dot{q}) = \frac{1}{2}\langle G(q)\dot{q}, \dot{q}\rangle$, we have that in the new coordinates, the kinetic energy $\tilde{T}$ is given by

$$\tilde{T}(\tilde{q}, \dot{\tilde{q}}) = \frac{1}{2}\langle G(\Psi^{-1}(\tilde{q}))A(\tilde{q})\dot{\tilde{q}}, A(\tilde{q})\dot{\tilde{q}}\rangle,$$

where $A(\tilde{q})$ is the inverse matrix of the transformation induced in $\dot{q}$ by Ψ, that is , $A^{-1} = \Psi'$.

Thus, introducing $\tilde{G}(\tilde{q}) = G(\Psi^{-1}(\tilde{q}))$, it is clear that the matrix of $\tilde{T}$ is given by $A^T \tilde{G} A$. Then, it is enough to calculate A. For this, notice that

$$\Psi' = \begin{pmatrix} I_{N \times N} & 0 \\ 0 & \Phi' \end{pmatrix},$$

where, recalling the definition of Φ, we have

$$\Phi' = \begin{pmatrix} 1 & -\gamma' \\ 0 & 1 \end{pmatrix},$$

whose inverse is

$$(\Phi')^{-1} = \begin{pmatrix} 1 & \gamma' \\ 0 & 1 \end{pmatrix}$$

and, finally, it is clear that

$$A = (\Psi')^{-1} = \begin{pmatrix} I_{N\times N} & 0 \\ 0 & (\Phi')^{-1} \end{pmatrix},$$

from where we conclude, using relations (9), that $A^{\mathrm{T}}\tilde{G}A = I + h$, with $\|h(\tilde{q})\| = o(\|\tilde{q}\|^{\mu_1})$ and $\|h'(\tilde{q})\| = o(\|\tilde{q}\|^{\mu_2})$, with $\mu_1 = \min\{1, \beta_1 - 1\} > 0$ and $\mu_2 = \min\{0, \beta_1 - 2\} = \mu_1 - 1 > -1$, as we desired.

With this, using theorem 1 we have proved the following:

Theorem 2. *Consider the lagrangian system given by equations* (1). *Assume that* $\mathcal{L}$ *satisfies* $\mathcal{H}_1$, *that* $n - N = 2$ *and that* $j^k\pi_2$ *is not homogeneous or is an homogeneous saddle. Then there is a trajectory* $\phi(t)$ *such that* $(\phi(t), \dot{\phi}(t)) \to (0,0)$ *as* $t \to -\infty$.

In particular, this theorem together with the result of [6] for the homogeneous case, gives the extension to the results of [4] that we are looking for. In order to substitute the coupling condition presented in [6], we need the lagrangian to be of a higher differentiability class. The interested reader can find a weaker coupling condition in the same reference.

Theorem A. *Consider the lagrangian system given by equation* (1) *and assume that* $\mathcal{L} = T - \pi$ *is such that* 0 *is an equilibrium point, the k-jet of* π *shows that the origin is not a minimum for* π, *that* $\mathcal{L}$ *is of class* $C^{k+\lfloor\frac{k-3}{2}\rfloor+3}$, *that* $j^2\pi(0,0)$ *is a positive semi-definite quadratic form of rank at least* $n-2$ *and that* T *is a positive definite quadratic form in* $\dot{q}$ *for every* q. *Then there is a trajectory* $\phi(t)$ *such that* $(\phi(t), \dot{\phi}(t)) \to (0,0)$ *as* $t \to -\infty$.

Proof. Without loss of generality, we may suppose that $T(0,\dot{q}) = \frac{\|\dot{q}\|}{2}$, so the hypothesis on the kinetic energy are valid.

We note that since $\mathcal{L}$ is at least C^{k+3}, it is clear that there is $j^{k-1}\nabla\pi$, as required.

Then, a simple application of the splitting lemma – see [1] – leads to a system which directly satisfies either the hypotheses of the theorem in [6] (stated in the introduction) or the hypotheses from theorem 2 above, and the results follows. □

6 Systems with gyroscopic forces

We consider now a lagrangian system with the presence of gyroscopic forces given by equation (2), $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q$, where $Q = Q(q)\dot{q}$ is linear in $\dot{q}$ and such that $\langle Q(q)\dot{q}, \dot{q}\rangle = 0$.

We show that theorem 1 can be easily extended to this context, much in the same way that [7] does to the results of [3].

We can rewrite equation (2) in coordinates in a convenient manner, as the equations (3). Denoting the j-th component of $Q(q)\dot{q}$ by $\sum_{i=1}^{n} Q_{ji}(q)\dot{q}_i$, we get

$$\frac{d}{dt}\left(\sum_{s=1}^{n} g_{rs}\dot{q}_s\right) = -\frac{\partial \pi}{\partial q_r} + \frac{1}{2}\sum_{l,s=1}^{n} \frac{\partial g_{ls}}{\partial q_r}\dot{q}_l\dot{q}_s + \sum_{i=1}^{n} Q_{ri}(q)\dot{q}_i \qquad (10)$$

for $r = 1, \ldots, n$.

It is a well known fact that the total energy for these systems, $E(q,\dot{q}) = T(q,\dot{q}) + \pi(q)$, is still conserved. Thus, admitting that $\mathcal{L}$ satisfies $\mathcal{H}_0$ and some additional hypothesis regarding Q, the constructions made to prove theorem 1 are still valid with small changes. In fact, it is easily seen that verifying the validity of lemma 2 will suffice.

For this purpose, keeping the same notations and definitions of the conservative case, one can notice that in the calculations of $q_n\dot{V}$, due to equations (10), only the following additional terms will appear

$$-q_n \frac{\left(\sum_{s=1}^{n} g_{ns}\dot{q}_s\right)}{g_{nn}} \sum_{i=1}^{n} Q_{ni}(q)\dot{q}_i$$

and we would like them to be $o(q_n^{\frac{3\beta}{2}})$. Then, considering that $\dot{q}_j = O(q_n^{\frac{\beta}{2}})$, it is enough that $Q_{ni}(q) = o(\|q\|^{\frac{\beta}{2}-1})$ as $\|q\| \to 0$, for $i = 1, \ldots, n$ and we get the result desired with the same proof as in the conservative case.

Thus, we have proved the following

Theorem 3. *Consider the lagrangian systems with gyroscopic force Q given by equations (2). Assume that $\mathcal{L}$ satisfies $\mathcal{H}_0$ and that Q is of class C^1 and $Q_{ni}(q) = o(\|q\|^{\frac{\beta}{2}-1})$ as $\|q\| \to 0$, for $i = 1, \ldots, n$. Then, there is $\rho > 0$ and a trajectory $\phi(t)$ such that $(\phi(t), \dot{\phi}(t)) \to (0,0)$ as $t \to -\infty$, and that for $|t|$ great enough, $(\phi(t), \dot{\phi}(t)) \in C_{\rho,\sqrt{2\alpha}}$.*

For the case that $n - N = 2$ and $\mathcal{L}$ satisfies $\mathcal{H}_1$, as in the previous section, due to the change of coordinates that is made, the last two coordinates mixes, so we ask that $Q_{ni}(q) = o(\|q\|^{\frac{k}{2}-1})$ and $Q_{(n-1)i}(q) = o(\|q\|^{\frac{k}{2}-1})$ as $\|q\| \to 0$, for $i = 1, \ldots, n$. In this way, the extension of the results of that section is immediate, and we write them down as the following corollaries.

Corollary 1. *Consider the lagrangian system with gyroscopic force Q given by equations (2). Assume that $\mathcal{L}$ satisfies $\mathcal{H}_1$, that $n - N = 2$, that $j^k\pi_2$ in the origin is not homogeneous or is an homogeneous saddle and that $Q_{ni}(q) =$*

$o(\|q\|^{\frac{k}{2}-1})$ *and* $Q_{(n-1)i}(q) = o(\|q\|^{\frac{k}{2}-1})$ *as* $\|q\| \to 0$, *for* $i = 1, \dots, n$. *Then there is a trajectory* $\phi(t)$ *such that* $(\phi(t), \dot{\phi}(t)) \to (0,0)$ *as* $t \to -\infty$.

Corollary 2. *Consider the lagrangian system with gyroscopic force* Q *given by equations* (2). *Assume that* $\mathcal{L} = T - \pi$ *is such that the the* 0 *is an equilibrium point, the* k*-jet of* π *shows that the origin is not a minimum for* π, $\mathcal{L}$ *is of class* $\mathcal{C}^{k+\lfloor\frac{k-3}{2}\rfloor+3}$, *that* $j^2\pi(0,0)$ *is a positive semi-definite quadratic form of rank at least* $n-2$, *that* T *is a positive definite quadratic form in* $\dot{q}$ *for every* q *and that* $Q_{ni}(q) = o(\|q\|^{\frac{k}{2}-1})$ *and* $Q_{(n-1)i}(q) = o(\|q\|^{\frac{k}{2}-1})$ *as* $\|q\| \to 0$, *for* $i = 1, \dots, n$. *Then there is a trajectory* $\phi(t)$ *such that* $(\phi(t), \dot{\phi}(t)) \to (0,0)$ *as* $t \to -\infty$.

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