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DEFINETTIAN CONSENSUS.

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DeFinettian Consensus

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Abstract

It is always possible to construct a real function ϕ, given random quantities X and Y with continuous distribution functions F and G, respectively, in such a way that $\phi(X)$ and $\phi(Y)$, also random quantities, have both the same distribution function, say H.

Based on this result, it is introduced an alternative way to somehow characterize the "opinion" of a group of experts about a continuous random quantity of interest, the construction of **Fields of coincidence of opinions**. A Field of coincidence of opinions is a finite union of intervals where the opinions of the experts coincide with respect to that quantity of interest.

1 Introduction

The main object of this paper is to review a result about transformations of continuous random quantities presented by De Finetti (1953). We will introduce the problem solved by De Finetti by first recalling a well-known theorem about transformations of random quantities.

Theorem 1 *Let X be a real random variable with continuous distribution function F and H any other distribution function. Then there is a real transformation f such that $Z = f(X)$ has distribution function H.*

Two corollaries are derived from Theorem 1:

Corollary 1 *(Probability Integral Transform)*
If X is a random variable with continuous distribution function F, then $F(X)$ is uniformly distributed on the interval $(0,1)$.

Corollary 2 *Let U be a random variable uniformly distributed on* $(0,1)$ *and H any other distribution function. Then $H^{-1}(U) = \sup\{x \in \mathbb{R} : H(x) \leq U\}$ is a random variable with distribution function H.*

The result of Theorem 1 applies to a single random variable in the sense that if X and Y are random variables with continuous, then $f(X)$ and $f(Y)$ will not have necessarily the same distribution function. In other words, given two continuous random variables X and Y, it is always possible to construct two new random variables $Z = f(X)$ and $W = g(Y)$, both having a certain distribution function H. However, in general, f and g may differ. In this context, a question arises: Is there a real function ϕ such that the random variables $\phi(X)$ and $\phi(Y)$ both have distribution function H?

This question is affirmatively answered by De Finetti in his 1953 paper. De Finetti shows a real function ϕ satisfying the conditions stated in the question above and outlines the construction of the random variables $\phi(X)$ and $\phi(Y)$ and the derivation of their distribution functions.

We will present, with full details, the theorem of the existence of such real functions ϕ and will discuss the interpretation of this result.

2 Main Result

In this section, we first state the theorem mentioned in the previous section as well as the basic argumentation of its proof. We next construct a real function φ which figures in the demonstration of the main result. Finally, we define the random variables $\varphi(X)$ and $\varphi(Y)$ and determine their distribution functions.

Initially, let us state the main theorem:

Theorem 2 *(Bruno de Finetti)*
Let X and Y be random variables with continuous distribution functions F and G, respectively, and H any other distribution function. Then there is a real function ϕ such that the random variables $\phi(X)$ and $\phi(Y)$ both have the same distribution function H.

Proof outline: It is sufficient to prove the existence of a real function φ such that the random variables $\varphi(X)$ and $\varphi(Y)$ have common uniform distribution on $(0,1)$, since if $\phi = H^{-1} \circ \varphi$, $\phi(X)$ and $\phi(Y)$ will both have distribution function H, by Corollary 2. Thus, we proceed to construct the uniformly distributed random variables $\varphi(X)$ and $\varphi(Y)$, without loss of generality.

2.1 Construction of the function φ

The construction of the aforementioned real function φ is based on some properties of continuous distribution functions described in the sequel.

Lemma 1 *Let X and Y be random variables with continuous distribution functions F and G, respectively. Let $C_X = \{(a,b) \in \mathbb{R}^2 : F(b) - F(a) = \frac{1}{2}\ ,\ -\infty < a < b < +\infty\}$. If F and G have no common median, then there are $(a_1, b_1], (a_2, b_2] \in C_X$ such that $G(b_1) - G(a_1) < \frac{1}{2}$ and $G(b_2) - G(a_2) > \frac{1}{2}$.*

Lemma 2 *Let X be a random variable with continuous distribution function F. The set $C_X = \{(a,b) \in \mathbb{R}^2 : F(b) - F(a) = \frac{1}{2}\ ,\ -\infty < a < b < +\infty\}$ is connected.*

Let us note that since to every point $(x,y) \in \mathbb{R}^2$, with $x < y$, corresponds a unique interval $(x,y]$ of real numbers, C_X may be seen as the set of all intervals of real numbers having X-probability $\frac{1}{2}$.

We establish the following propositions:

Proposition 1 *Let X and Y be random variables with continuous distribution functions F and G, respectively. There is an interval of real numbers $I_1 = (a,b]$, with $-\infty \le a < b < +\infty$, satisfying*

$$F(b) - F(a) = G(b) - G(a) = \frac{1}{2}$$

Proof: Two situations need to be considered:

(1) F and G have a common median:

In this case, the conclusion is immediate. It is sufficient to take a common median of F and G, say m_0, and consider the interval $(-\infty, m_0]$, as

$$F(m_0) - F(-\infty) = G(m_0) - G(-\infty) = \frac{1}{2} - 0 = \frac{1}{2}$$

(2) F and G have no common median:

In this case, we will prove the existence of a point $(a, b) \in C_X$ (as defined in Lemma 1) such that $G(b) - G(a) = \frac{1}{2}$. For this purpose, we define the function $D : C_X \to \mathbb{R}$ by:

$$D(x, y) = G(y) - G(x).$$

D is obviously continuous on its domain C_X. By Lemma 2, C_X is connected and, since D is continuous, it follows that the image of D is also connected and, in particular, is an interval of real numbers. But, by Lemma 1, there are points (a_1, b_1) and $(a_2, b_2) \in C_X$ such that $D(a_1, b_1) < \frac{1}{2}$ and $D(a_2, b_2) > \frac{1}{2}$. The image of the function D is therefore an interval containing a value smaller than $\frac{1}{2}$ and another greater than $\frac{1}{2}$. Thus, there is an interval $(a, b) \in C_X$ such that

$$D(a, b) = G(b) - G(a) = \frac{1}{2} = F(b) - F(a) \quad ,$$

since $(a, b) \in C_X$.

We should emphasize that in situations where there are more than one interval satisfying Proposition 1, we will denote by I_1 the interval having the lowest infimum among those satisfying this result, in order to avoid any ambiguity (here, we admit, in a misuse of notation, $-\infty$ as the infimum of an unbounded interval). This choice having been made and still existing more than one interval satisfying proposition 1, I_1 will represent the interval with lowest supremum. Furthermore, we will always consider I_1 closed at the right and opened at the left and will denote by I_0 the complementary set of I_1 relatively to $\mathbb{R}$.

We now state another property of continuous distribution functions.

Proposition 2 *Let us assume the conditions of proposition 1 and the sets I_0 and I_1 derived from it. Then:*

(i) there is a set $I_{01} \subset I_0$ such that $P(X \in I_{01}) = P(Y \in I_{01}) = \frac{1}{4}$ and
(ii) there is a set $I_{11} \subset I_1$ such that $P(X \in I_{11}) = P(Y \in I_{11}) = \frac{1}{4}$

Proof:

(i) Let $I_1 = (a, b]$. There are two situations to be considered:

(1) Let us suppose $-\infty < a < b < +\infty$. By Proposition 1, $F(b) - F(a) = G(b) - G(a) = \frac{1}{2}$. Let us define the following distribution function $\overline{F}$ derived from F:

$$\overline{F}(x) = \begin{cases} 2F(x) & \text{for } x < a \\ 2\{F(x+b-a) - \frac{1}{2}\} & \text{for } x \geq a \end{cases}$$

Analogously, let us define $\overline{G}$ by:

$$\overline{G}(x) = \begin{cases} 2G(x) & \text{for } x < a \\ 2\{G(x+b-a) - \frac{1}{2}\} & \text{for } x \geq a \end{cases}$$

By Proposition 1, $\exists\ (a_0, b_0] \subset \boldsymbol{R}$ such that $\overline{F}(b_0) - \overline{F}(a_0) = \overline{G}(b_0) - \overline{G}(a_0) = \frac{1}{2}$.

- Suppose $b_0 < a$. In this case, $\overline{F}(b_0) - \overline{F}(a_0) = 2F(b_0) - 2F(a_0)$ and $\overline{G}(b_0) - \overline{G}(a_0) = 2G(b_0) - 2G(a_0)$. Then,

$$\overline{F}(b_0) - \overline{F}(a_0) = \overline{G}(b_0) - \overline{G}(a_0) = \frac{1}{2} \Rightarrow 2F(b_0) - 2F(a_0) = 2G(b_0) - 2G(a_0) = \frac{1}{2} \Rightarrow$$

$$\Rightarrow F(b_0) - F(a_0) = G(b_0) - G(a_0) = \frac{1}{4}$$

As the functions F and G are continuous, we can still write

$$P(X \in (a_0, b_0]) = P(Y \in (a_0, b_0]) = \frac{1}{4}$$

Defining $I_{01} = (a_0, b_0]$, the result is proved.

• Let us assume now $a_0 < a \le b_0$. In this situation, $\overline{F}(b_0) - \overline{F}(a_0) = 2\{F(b_0 + b - a) - \frac{1}{2}\} - 2F(a_0)$ and $\overline{G}(b_0) - \overline{G}(a_0) = 2\{G(b_0 + b - a) - \frac{1}{2}\} - 2G(a_0)$. Thus,

$$\overline{F}(b_0) - \overline{F}(a_0) = \overline{G}(b_0) - \overline{G}(a_0) = \frac{1}{2} \Rightarrow$$

$$\Rightarrow 2\{F(b_0 + b - a) - \frac{1}{2}\} - 2F(a_0) = 2\{G(b_0 + b - a) - \frac{1}{2}\} - 2G(a_0) = \frac{1}{2} \Rightarrow$$

$$\Rightarrow F(b_0 + b - a) - \underbrace{\frac{1}{2}}_{F(b)-F(a)} - F(a_0) = G(b_0 + b - a) - \underbrace{\frac{1}{2}}_{G(b)-G(a)} - G(a_0) = \frac{1}{4} \Rightarrow$$

$$\Rightarrow F(b_0+b-a) - \{F(b) - F(a)\} - F(a_0) = G(b_0+b-a) - \{G(b) - G(a)\} - G(a_0) = \frac{1}{4} \Rightarrow$$

$$\Rightarrow \{F(b_0+b-a) - F(b)\} + \{F(a) - F(a_0)\} = \{G(b_0+b-a) - G(b)\} + \{G(a) - G(a_0)\} = \frac{1}{4}$$

Since F and G are continuous, it follows that

$$P(X \in (a_0, a]) + P(X \in (b, b_0 + b - a]) = P(Y \in (a_0, a]) + P(Y \in (b, b_0 + b - a]) = \frac{1}{4}$$

and, therefore,

$$P(X \in (a_0, a] \bigcup (b, b_0 + b - a]) = P(Y \in (a_0, a] \bigcup (b, b_0 + b - a]) = \frac{1}{4} ,$$

so that we have $I_{01} = (a_0, a] \cup (b, b_0 + b - a)$

(ii) The proof of the existence of $I_{11} \subset I_1$ such that $P(X \in I_{11}) = P(Y \in I_{11}) = \frac{1}{4}$ is analogous to the demonstration of part (i) of this proposition. We have just to consider the distribution functions

$$\overline{F}(x) = \begin{cases} 0 & \text{if } x < a \\ 2(F(x) - F(a)) & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases} \quad \text{and} \quad \overline{G}(x) = \begin{cases} 0 & \text{if } x < a \\ 2(G(x) - G(a)) & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$$

and proceed as in part (i).

Similarly to what was established for the interval I_1, in the situation where we have more than one subset of I_1 satisfying proposition 2, part (ii), we will consider the interval with lowest extremes as I_{11}. We also will denote by I_{01} the subset of I_0 satisfying the part (i) of proposition 1.2 formed by the least number of intervals. Again, if there is more than one subset of I_0 in these conditions, we will denote by I_{01} that one with lowest infimum. Finally, the complementary set of I_{11} relatively to I_1 will be called I_{10} and the complementary set of I_{01} relatively to I_0 will be called I_{00}.

In general, proceeding successively in this way, we can obtain, $\forall n \in I\!N$, 2^n disjoint sets $I_{i_1 \dots i_n}$, $(i_1, \dots, i_n) \in \{0,1\}^n$, such that

$$P(X \in I_{i_1 \dots i_n}) = P(Y \in I_{i_1 \dots i_n}) = \left(\frac{1}{2}\right)^n \quad , \quad \forall (i_1, \dots, i_n) \in \{0,1\}^n \quad \text{and}$$

$$I_{i_1 \dots i_n} = I_{i_1 \dots i_n, 0} \bigcup I_{i_1 \dots i_n, 1} \ , \ \forall n \in I\!N \ .$$

Another characteristic of the sets $I_{i_1 \dots i_n}$ is presented in Lemma 3 below.

Lemma 3 *Any set of the form $I_{i_1 \dots i_n}$, $(i_1, \dots, i_n) \in \{0,1\}^n$, constructed by the procedure described above is formed, at most, by $n+1$ intervals of real numbers.*

Stated propositions 1 and 2 and their extensions yield the real function φ which makes the random variables $\varphi(X)$ and $\varphi(Y)$ uniformly distributed over $(0,1)$.

We define $\varphi : \mathbb{R} \to [0,1]$ by:

$$\varphi(x) = \sum_{n=1}^{\infty} i_n \left(\frac{1}{2}\right)^n \ ,$$

where $i_1, i_2, \dots$ are such that $x \in I_{i_1 \dots i_n}$, $\forall n \ge 1$ (here, the sets $I_{i_1 \dots i_n}$ correspond to the sets constructed via the distribution functions F and G).

It should be noted that φ is also a function of the distribution functions F and G, but we will omit these arguments when referring to the function φ in order to keep the notation easy.

Analysing the expression of φ, we see that the function associates to each real number x the element of the interval $[0,1]$ having a dyadic representation (expansion) given by $0, i_1 i_2 \ldots$, with $x \in I_{i_1 \ldots i_n}$, $\forall n \geq 1$. It can be proved in a straightforward manner that φ is well-defined.

2.2 Determining the distribution of φ

Let us now prove that $\varphi(X)$ and $\varphi(Y)$ both have uniform distribution on the interval $(0,1)$. We will determine the distribution function of $\varphi(X)$,

$$F_{\varphi(X)}(t) = P(\varphi(X) < t) = P(X \in \varphi^{-1}((-\infty, t)) = P_X(\{x \in \mathbb{R} : \varphi(x) < t\}) ,$$

where P_X is the probability measure on $(\mathbb{R}, \mathcal{B})$ induced by the random variable X and $\varphi^{-1}(A)$ is the inverse image of the set $A \in \mathcal{B}$ by the function φ. We then have:

(1) $t \leq 0 \Rightarrow \{x \in \mathbb{R} : \varphi(x) < t\} = \emptyset$ and $P_X(\{x \in \mathbb{R} : \varphi(x) < t\}) = P_X(\emptyset) = 0$.

(2) $t > 1 \Rightarrow \{x \in \mathbb{R} : \varphi(x) < t\} = \mathbb{R}$. So, $P_X(\{x \in \mathbb{R} : \varphi(x) < t\}) = P_X(\mathbb{R}) = 1$.

(3) If $0 < t \leq 1$, it follows that:

$$\{x \in \mathbb{R} : \varphi(x) < t\} = \bigcup_{n=1}^{\infty} A_n \text{ , so that}$$

$$F_{\varphi(X)}(t) = P_X(\bigcup_{n=1}^{\infty} A_n) ,$$

where $\{A_n : n \geq 1\}$ is the sequence of sets defined by:

$$A_1 = \begin{cases} \emptyset & \text{if } d_1(t) = 0 \\ I_0 & \text{if } d_1(t) = 1 \end{cases} \quad \text{and} \quad A_n = \begin{cases} \emptyset & \text{if } d_n(t) = 0 \\ I_{d_1(t) \ldots d_{n-1}(t), 1-d_n(t)} & \text{if } d_n(t) = 1 \end{cases} , n > 1$$

Let us fix $t \in (0,1]$. This point t may be written as $t = \sum_{n=1}^{\infty} d_n(t)(\frac{1}{2})^n$, where $d_1(t), d_2(t)$... are such that $0, d_1(t)d_2(t)$... is a dyadic expansion of t. In order to avoid any ambiguity in the definition of the dyadic expansion of a real number $t \in (0,1]$, we will here consider for t the infinite dyadic representation, that is, the representation having an infinite number of $1's$ (for instance, for $t = \frac{1}{2}$, we will consider the expansion $0,01111...$ instead of the expansion $0,10000...$). Then, taking $x_0 \in \{x \in I\!R : \varphi(x) < t\}$ and considering $\varphi(x_0) = \sum_{n=1}^{\infty} i_n(\frac{1}{2})^n$, we obtain:

$$x_0 \in \{x \in I\!R : \varphi(x) < t\} \Leftrightarrow \varphi(x_0) < t \Leftrightarrow \sum_{n=1}^{\infty} i_n(\frac{1}{2})^n < \sum_{n=1}^{\infty} d_n(t)(\frac{1}{2})^n$$

Since we are considering an infinite dyadic expansion for t, it follows that the last inequality above is true if, and only if,

$$\exists\, n_0 \in I\!N \text{ such that } n_0 = \inf\{n \in I\!N : i_n \neq d_n(t) \text{ and } i_n = 1 - d_n(t) = 0\} \Leftrightarrow$$

$$\Leftrightarrow \exists\, n_0 \in I\!N \text{ such that } x_0 \in I_{d_1(t)\dots d_{n_0-1}(t),0} \text{ and } d_{n_0}(t) = 1 \Leftrightarrow$$

$$\Leftrightarrow \exists\, n_0 \in I\!N \text{ such that } x_0 \in I_{d_1(t)\dots d_{n_0-1}(t),1-d_{n_0}(t)} \text{ and } d_{n_0}(t) = 1$$

$$\Leftrightarrow \exists\, n_0 \in I\!N \text{ such that } x_0 \in A_{n_0} \Leftrightarrow x_0 \in \bigcup_{n=1}^{\infty} A_n \text{ , so that}$$

$$x_0 \in \{x \in I\!R : \varphi(x) < t\} \Leftrightarrow x_0 \in \bigcup_{n=1}^{\infty} A_n \text{ . Therefore,}$$

$$\{x \in I\!R : \varphi(x) < t\} = \bigcup_{n=1}^{\infty} A_n.$$

However, for every $n \in I\!N$, each set of the form $I_{i_1 \dots i_n}$ is formed by a finite union of intervals (at most $n+1$ intervals), as stated in Lemma 3, and since $A_n = \emptyset$ or A_n is of the form $I_{i_1 \dots i_n}$, it follows that $A_n \in \mathcal{B}$, $\forall n \geq 1$. Since $A_n \in \mathcal{B}$, $\forall n \geq 1$, we have that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$ and, consequently, $\{x \in I\!R : \varphi(x) < t\} \in \mathcal{B}$, $\forall t \in (0,1]$.

Finally, let us determine the distribution function of $\varphi(X)$.

$$F_{\varphi(X)}(t) = P_X(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P_X(A_n) ,$$

since $\{A_n : n \geq 1\}$ is a sequence of pairwise disjoint sets.

However, if $d_n(t) = 0$, then $A_n = \emptyset$ and $P_X(A_n) = 0$. If $d_n(t) = 1$, then $P_X(A_n) = P_X(I_{d_1(t)\ldots d_{n-1}(t),1-d_n(t)}) = (\frac{1}{2})^n$, as, by construction, the set $I_{d_1(t)\ldots d_{n-1}(t),1-d_n(t)}$ contains $(\frac{1}{2})^n$ of the distribution of X. In this way:

$$P_X(A_n) = \begin{cases} 0 & \text{if } d_n(t) = 0 \\ (\frac{1}{2})^n & \text{if } d_n(t) = 1 \end{cases} \text{, or, } \quad P_X(A_n) = (\frac{1}{2})^n d_n(t).$$

Recalling the expression of the distribution function of $\varphi(X)$, we have

$$F_{\varphi(X)}(t) = \sum_{n=1}^{\infty} d_n(t)(\frac{1}{2})^n = t ,$$

since $0, d_1(t)d_2(t)\ldots$ is a dyadic expansion of t. We then obtain, from (1), (2) and (3),

$$F_{\varphi(X)}(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t & \text{if } 0 < t \leq 1 \\ 1 & \text{if } t > 1 \end{cases}$$

which is the distribution function of a random variable uniformly distributed on $(0, 1)$. The proof that $\varphi(Y) \sim \mathcal{U}(0, 1)$ is analogous.

It is interesting to emphasize that in the conditions of Theorem 2 nothing is mentioned about the probability spaces where X and Y are defined; references are made only on the structure of the distribution functions F e G, so that the result of theorem 2 is valid also for random variables defined in distinct probability spaces. We also note that X and Y need neither to be absolutely continuous random variables nor to possess moments.

In the next section, we present an interpretation of the result just proved.

3 Interpretation

The most interesting focus of Theorem 2, according to De Finetti, corresponds to the situation in which the random variables X and Y are related to a unique random quantity of interest, instead of two distinct quantities of interest, and F and G are the the opinions of two individuals about this unique random quantity. This formulation is, of course, natural from the subjectivistic standpoint. probability, natural at all.

In this context, when two persons express their opinions about a given random quantity of interest, we have the following fact from the construction of the real function φ (and of the random variables $\varphi(X)$ and $\varphi(Y)$): it is always possible to construct a finite union of intervals, which De Finetti named Fields of Coincidence of Opinions, where the opinions of both individuals about the quantity of interest (represented by F and G) coincide. In other words, it is always possible to construct a finite union of intervals that contains, for any level $\alpha \in (0,1)$, at least $1-\alpha$ of the distributions of X and Y simultaneously. Formally, we state this fact in the following proposition:

Proposition 3 *Let X and Y be random variables with continuous distribution functions F and G, respectively. Then, $\forall \alpha \in (0,1)$, there is a finite union of intervals $B = B(\alpha)$ such that*

$$P_X(B) = P_Y(B) \geq 1 - \alpha.$$

Proof: Let us fix $\alpha \in (0,1)$. We know that there is a natural number $n_0 = n_0(\alpha) \in I\!N$ such that $1 - (\frac{1}{2})^{n_0} \geq 1 - \alpha$. We then need only to define a sequence of sets $\{B_n : n \geq 1\}$ by

$$\begin{aligned} B_1 &= I_1 \\ B_2 &= I_{01} \\ B_3 &= I_{001} \\ &\vdots \\ B_n &= I_{\underbrace{0\ldots01}_{n-1 \text{ zero's}}} \\ &\vdots \end{aligned}$$

and take $B = \bigcup_{n=1}^{n_0} B_n$. We will have

$$P_X(B) = P_X(\bigcup_{n=1}^{n_0} B_n) = \sum_{n=1}^{n_0} P_X(B_n)$$

where the last equality follows from the fact that $\{B_n : n \geq 1\}$ is a sequence of pairwise disjoint sets. And since, by construction, $P_X(B_n) = (\frac{1}{2})^n$, $\forall n \geq 1$, it follows that:

$$P_X(B) = \sum_{n=1}^{n_0} (\frac{1}{2})^n = 1 - (\frac{1}{2})^{n_0} \geq 1 - \alpha$$

In an analogous way, we verify that $P_Y(B) = 1 - (\frac{1}{2})^{n_0} \geq 1 - \alpha$, concluding the demonstration of Proposition 3.

We emphasize that the result of Proposition 3 can be extended to any finite number of continuous distribution functions. Thus, it is possible to establish fields of coincidence of opinions for a finite group of individuals. This fact, according to De Finetti, hints a possibility of characterizing the "conjoint opinion" of a group of experts, as discussed below.

Let us consider the situation where a group of experts has to make a decision jointly and, for this purpose, they have to tell their opinions about a certain random quantity of interest via the elicitation of their respective distribution functions (that we will suppose continuous). The construction of the fields of coincidence of opinions sketches an alternative method of expressing what the "opinion" of these experts would look like, as some properties of the fields of coincidence of opinions are requirements to characterize the "opinion" of a group of experts.

At first, we emphasize that the fields of coincidence can be seen as a genuine attribution of probability from the group, differently of the usual procedures of mixing probabilities, which produce probability distributions having no meaning in the subjectivistic paradigm. In other words, the probability distributions resulting from the processes of mixture do not correspond to the opinion of anybody, opposing De Finetti's viewpoint.

Another positive point of this method is that the individual opinions are preserved at the construction of the fields of coincidence of opinions. This property gives to this method an objective character in the sense that individual opinions are preserved and there is no reason for the members of the group to reject the fields of coincidence of opinions as a consensus expression (in fact, no member of the group needs to give up his beliefs about the random quantity in the construction of these fields of coincidence).

The method based on fields of coincidence also presents some limitations. Initially, we observe that a field of coincidence of opinions does not determine a proper probability distribution as it consists only of a finite union of intervals and their respective uncertainty rates according to all members of the group. Thus, the normative Bayesian theory for decision-making (based on expected utility maximization) does not apply to any procedure of decision-making based on the fields of coincidence as a description of the "opinion" of the group.

Another deficiency that we can point out in this method is the absence of an axiomatic support, based on coherence, which would justify the adoption of the fields of coincidence as a representation of the "opinion" of a group of experts. This disadvantage arises because there is no concept of joint coherence (or rationality). Notwithstanding Arrow's [1] impossibility result, much of current research in group decision theory has been devoted to establish such a concept and, consequently, a numerical transcription of the uncertainty of a group of experts.

In this context, where there is no normative theory for decision-making, but many attempts to characterize joint coherence, we think that the existence of the fields of coincidence of opinions may contribute to the discussion on this question. This discussion turns out to be even more open as there is " a point which is becoming increasingly better understood in group decision theory, namely that a group of Bayesians cannot always be fully Bayesian even when its members would want it to be" (Genest and Zidek [4]).

We now show some examples of fields of coincidence of opinions.

Example 1 *Let X and Y be random variables uniformly distributed on* $(1,2)$ *and* $(2,3)$, *respectively.*

A field of coincidence of opinions with $\alpha = 0{,}125$ would be the union of intervals $(1, \frac{11}{8}] \cup (\frac{3}{2}, \frac{23}{8}]$ (corresponding to $I_1 \cup I_{01} \cup I_{001}$, as the construction in section 2.1), or the interval $(\frac{9}{8}, \frac{23}{8}]$. As the supports of the distributions of X and Y are disjoint, no field of coincidence will be contained in the intersection of these supports.

Example 2 *Let X and Y be random variables normally distributed with common mean 0 and variances 1 and 4, respectively.*

Here, a field of coincidence of opinions with $\alpha = 0{,}25$ is the set $(-\infty, 0] \cup (0,635\,;\, 2,35]$. In this case, it seems that no field of coincidence is formed by a unique interval, differently from the previous example.

These two examples show that there are a number of points to be better understood in the characterization of the fields of coincidence: When is a field of coincidence for a given value of α unique? Under what conditions over the supports of the distribution functions F and G is it possible to obtain a unique interval of real numbers as a field of coincidence of opinions? How does the number of intervals vary in function of α? (As to the last question, a rough upper bound for the number of intervals is $\frac{n_0(n_0+1)}{2}$, where $n_0 = n_0(\alpha) = \min\{n \in I\!N : 1 - (\frac{1}{2})^n \geq 1 - \alpha\}$).

Apart from the mathematical questions just mentioned, there are also some philosophical inquiries: Does it make sense to construct fields of coincidence of opinions when the supports of the distributions of X and Y are mutually exclusive? Do fields of coincidence provide a more precise interpretation of the uncertainty of a group of experts when F and G have the same support than in situations when F and G are more generic? How better is to have a group's uncompromising field instead of a group's (mathematical) probability as a measure of its "opinion"? As we can see, there is much to be studied and understood on De Finetti's fields of coincidence of opinions.

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