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MODULE CATEGORIES WITH INFINITE RADICAL SQUARE ZERO ARE OF FINITE TYPE

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It is well known that an artin algebra A is of finite representation type if and only if $\mathrm{rad}^\infty(\mathrm{mod}A)=0$. In this note we deepen this result by showing that $(\mathrm{rad}^\infty(\mathrm{mod}A))^2=0$ implies that A is of finite representation type.

1 Introduction.

Let A be an artin algebra over a commutative artin ring R. By an A-module we mean a finitely generated, right A-module. We denote by mod$A$ the category of all A-modules, by ind$A$ the full subcategory of mod$A$ whose objects are the indecomposable A-modules, and then rad(mod$A$) is the Jacobson radical of mod$A$, that is, the ideal in mod$A$ generated by all non-invertible morphisms in ind$A$. The *infinite radical* $\mathrm{rad}^\infty(\mathrm{mod}A)$ of mod$A$ is the intersection of all powers $\mathrm{rad}^i(\mathrm{mod}\ A)$, $i \geq 1$, of rad(mod$A$). The algebra A is said to be of *finite representation type* if ind$A$ has only finitely many non-isomorphic A-modules.

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Recent investigations showed that $\text{rad}^\infty(\text{mod}A)$ contains important informations on the category $\text{mod}A$ (see the survey article [S3]). We are interested in describing artin algebras A with $\text{rad}^\infty(\text{mod}A)$ nilpotent. It is well known that an artin algebra A is of finite representation type if and only if $\text{rad}^\infty(\text{mod}A)= 0$ (see [KS] and [S3]). On the other hand, for every hereditary algebra of infinite representation type H, we have that $(\text{rad}^\infty(\text{mod}H))^2 \neq 0$. The aim of this paper is to show the following:

THEOREM. *Let* A *be an artin algebra such that* $(\text{rad}^\infty(\text{mod}A))^2 = 0$. *Then* A *is of finite representation type.*

In a forthcoming paper [CMMS] we shall study module categories with infinite radical cube zero. The present article was written when the last named author was visiting the University of São Paulo. He acknowledges support from FAPESP, Brazil and from Polish Scientific Grant KBN No. 1222/2/91. The first three authors acknowledge support from CNPq, Brazil.

2 Proof of the theorem.

We denote by D the *standard duality* $\text{Hom}_R(-, \overline{I})$ on $\text{mod}A$, where $\overline{I}$ is the injective envelope of $R/\text{rad } R$ in mod R. We use the notations Γ_A for the Auslander-Reiten quiver of A, and $\tau_A = DTr$ and $\tau_A^{-1} = TrD$ for the Auslander-Reiten translations in Γ_A. Also, as usual, we shall not distinguish vertices of Γ_A from the corresponding indecomposable modules. A path

$$X_o \to X_1 \to \cdots \to X_n$$

in Γ_A is said to be *sectional* if $\tau_A X_i \neq X_{i-2}$, for $2 \leq i \leq n$. Further, such a path is an *oriented cycle* if $X_o = X_n$. A connected quiver is said to be *non trivial* if it contains at least two vertices (and hence at least one arrow).

Let X be in $\text{ind}A$. Then X is called *periodic* if for some $n > 0$ we have that $\tau_A^n X = X$. Besides, such a module X is called *left stable* (respectively *right stable*) if $\tau_A^n X \neq 0$ for all positive (respectively negative) integers n and it is called *stable* if it is both left and right stable. The τ_A-orbit of X is the set of all possible modules of the form $\tau_A^i X$, $i \in \mathbb{Z}$.

For basic facts about Auslander-Reiten theory we refer to [**AR3**, **AR4**] and for tilting theory to [**As**], [**Ri1**] and [**Ri2**].

Proof of the theorem. Suppose that A is an artin algebra of infinite representation type such that $(\text{rad}^{\infty}(\text{mod}A))^2 = 0$. Since A is of infinite representation type, by a theorem of Auslander, [**A2**](3.1), there exists an infinite sequence of proper epimorphisms

$$\cdots \to M_r \xrightarrow{f_r} M_{r-1} \to \cdots \to M_1 \xrightarrow{f_1} M_o$$

with $M_i \in \text{ind}A$. Let $\mathcal{M} = \{M_i : i \in \mathbb{N}\}$. Fix $r \geq 0$ and consider a projective cover $h_r : \mathbf{P}(M_r) \to M_r$ of M_r. Then, for each $t \geq r+1$ there exists a morphism $g_t : \mathbf{P}(M_r) \to M_t$ such that $h_r = f_{r+1} \cdots f_t g_t$. Hence, $h_r \in \text{rad}^{\infty}(\mathbf{P}(M_r), M_r)$. Since $(\text{rad}^{\infty}(\text{mod}A))^2 = 0$ and h_r is an epimorphism, we deduce that $\text{rad}^{\infty}(M_r, -) = 0$.

We claim that, for each $r \geq 0$, there exists a natural number s_r such that $\text{rad}^{s_r}(M_r, -) = 0$. Indeed, since $\text{rad}^{\infty}(M_r, -) = 0$, we infer that $\text{rad}^{\infty}(M_r, DA) = 0$. Observe that there exists an s_r such that $\text{rad}^{s_r}(M_r, DA) = \text{rad}^{\infty}(M_r, DA)$ because $\text{Hom}(M_r, DA)$ is an artinian R-module. Since for each A-module X there is a monomorphism of the form $X \to (DA)^m$, for some $m \geq 1$, we get our claim.

We have then that none of the morphisms f_r can belong to $\text{rad}^{\infty}(\text{mod}A)$, and so, that all modules in $\mathcal{M}$ are in the same connected component of Γ_A. Furthermore, for each $r \geq 0$, any sectional path starting at M_r has length bounded by s_r. Indeed, if there is such a path

$$M_r = Z_o \to Z_1 \to \cdots \to Z_l$$

with $l > s_r$, choosing irreducible morphisms f_i for each arrow $Z_{i-1} \to Z_i$, it follows from [**B**] and [**IT**] that the composition $f_l \cdots f_1$ is not zero and clearly belongs to $\text{rad}^l(M_r, Z_l)$, a contradiction.

Let now $\mathcal{C}$ be the connected component of Γ_A which contains all modules of $\mathcal{M}$ and let us denote by $\mathcal{C}_l$ (respectively, by $\mathcal{C}_r$) the left (respectively, the right) stable part of $\mathcal{C}$. It is obtained by deleting from $\mathcal{C}$ the τ_A-orbits of the indecomposable projective (respectively, injective) modules.

Since $\mathcal{C}$ is infinite, either $\mathcal{C}_l$ or $\mathcal{C}_r$ has a connected component which is non-trivial. In fact, if there is, say, an infinite family of left stable trivial components $\{\tau_A^i X\}, i \geq 0$, with X non-periodic, then, for N big enough and $i \geq N$, we will have that $\tau_A^i X$ is not a neighbor of a projective module, which leads to a contradiction. Further, if that is not the case, there are infinitely many periodic orbits of trivial components that would be neighbors of orbits of projective or injective indecomposable modules, which is again a contradiction.

We claim now that every non-trivial connected component of $\mathcal{C}_l$ (respectively, of $\mathcal{C}_r$) contains an oriented cycle. Indeed, let $\mathcal{C}'$ be a non-trivial connected component of $\mathcal{C}_l$ that does not contain an oriented cycle. Then, by **[L]**(3.6), there exists a valued quiver Δ, containing no oriented cycle, such that $\mathcal{C}'$ is isomorphic to a full translation subquiver of $\mathbb{Z}\Delta$ which is closed under predecessors. Let us fix a copy of Δ in $\mathcal{C}'$ such that no module in Δ is a successor of a projective module in $\mathcal{C}$. Let $\mathcal{D}$ be the full translation subquiver of $\mathcal{C}'$ whose vertices are all predecessors of Δ in $\mathcal{C}'$. Note that $\mathcal{D}$ is also closed under predecessors in $\mathcal{C}$. Let I be the annihilator of $\mathcal{D}$ in A, $B = A/I$ and M the direct sum of the modules in $\tau_A\Delta$. We claim that $\mathrm{Hom}_A(M, \tau_A M) = 0$. Indeed, if this were not so, there would exist direct summands Y and Z of M and a non-zero morphism $f : Y \to \tau_A Z$. Observe that such a morphism would belong to $\mathrm{rad}^\infty(\mathrm{mod}A)$, because $\mathcal{D}$ is closed under predecessors and has no oriented cycles. Now, if $\pi : \mathbf{P}(Y) \to Y$ is a projective cover, our choice of Δ implies that $\pi \in \mathrm{rad}^\infty(\mathrm{mod}A)$, and hence $f\pi$ is a non-zero morphism in $(\mathrm{rad}^\infty(\mathrm{mod}A))^2$, a contradiction. Consequently, $\mathrm{Hom}_A(M, \tau_A M) = 0$ and, by **[S1]**, Lemma 2, Δ is finite. Then, $I = \mathrm{ann}M$ (see **[S2]**, Lemma 3) and hence M is a faithful B-module. Observe also that $\mathcal{D}$ consists of B-modules, so that $\tau_B X = \tau_A X$ for any X in $\mathcal{D}$. Therefore, $\mathrm{Hom}_B(M, \tau_B M) = 0$ and, similarly, $\mathrm{Hom}_B(\tau_B^{-1} M, M) = 0$. Moreover, if $\mathrm{Hom}_B(M, X) \neq 0$ for an $X \in \mathrm{ind}B$ which is not a direct summand of M, then $\mathrm{Hom}_B(\tau_B^{-1} M, X) \neq 0$. Therefore, by **[RSS]** (1.5) and (1.6) (see also **[S3]**(3.2)), M is a tilting B-module. Further, by **[S3]**(3.4), $H = \mathrm{End}_B(M)$ is a hereditary algebra. This means that B is a tilted algebra and that $\mathcal{D}$ is a full translation subquiver of the connecting component Σ of Γ_B which is closed under predecessors and Δ is a slice in Σ. Since Σ has no projective modules, we infer that B is given by a tilting module without preinjective direct summands (see **[Ri2]**, p. 42). Then, by a result of Strauss **[St]** (7.5), there exists a factor algebra C of B which is concealed. Observe then that, since Γ_C has regular components, $(\mathrm{rad}^\infty(\mathrm{mod}\ C))^2 \neq 0$. Indeed, let Z be a vertex of a regular connected component of Γ_C and let us consider a projective cover $\mathbf{P}(Z) \to Z$ and an injective envelope $Z \to \mathbf{I}(Z)$ of Z. Then their composite is clearly a non-zero morphism in $(\mathrm{rad}^\infty(\mathrm{mod}\ C))^2$ and hence in $(\mathrm{rad}^\infty(\mathrm{mod} A))^2$, which contradicts our assumption. We show, in a similar way, that also every non-trivial connected component of $\mathcal{C}_r$ contains oriented cycles.

We shall show now that there are at most finitely many τ-orbits in $\mathcal{C}$ which are not τ_A-periodic. Let $\mathcal{E}$ be a non-trivial connected component of $\mathcal{C}_l$. If $\mathcal{E}$ contains a periodic module, then either $\mathcal{E}$ is a stable tube or it is of the form $\mathbb{Z}Q/G$, where Q is a Dynkin quiver and G is a group of automorphisms of $\mathbb{Z}Q$ [**HPR**]. In this case, all τ_A-orbits of $\mathcal{E}$ are periodic. On the other hand, if $\mathcal{E}$ has no periodic module, then, by [**L**](2.3), $\mathcal{E}$ has only finitely many τ_A-orbits. Therefore, we infer that there is at most a finite number of non-periodic τ_A-orbits in $\mathcal{C}$ (see [**BC**](4.2)), and our claim is proved.

Let us observe now that a stable tube $\mathcal{T}$ in Γ_A has no module from $\mathcal{M}$, because all modules in $\mathcal{T}$ are starting vertices of infinite sectional paths. Hence, there exists a τ-orbit $\mathcal{O}$ in $\mathcal{C}$ containing infinitely many modules from $\mathcal{M}$. Without loss of generality, we can assume that, for some $Y \in \mathcal{O}$, $\Omega = \{\tau_A^i Y, i \geq 0\}$ contains infinitely many modules of $\mathcal{M}$. Obviously then there is a connected component $\mathcal{F}$ of $\mathcal{C}_l$ that contains all but finitely many modules of Ω. Since $\mathcal{F}$ has no periodic modules but contains oriented cycles, it follows from [**L**](2.3) that there exists an infinite sectional path

$$\cdots \to \tau_A^{2t} X_1 \to \tau_A^t X_s \to \cdots \to \tau_A^t X_1 \to X_s \to \cdots \to X_1,$$

in $\mathcal{F}$, where $t > s$, at least one of the modules X_j is not stable, and $\{X_1, ..., X_s\}$ is a complete set of representatives of τ_A-orbits in $\mathcal{F}$.

It follows that $\mathcal{M}$ contains a module of the form $\tau_A^i X_j$, for some $i \geq t$ and $1 \leq j \leq s$. Observe that there exists an infinite sectional path starting in $\tau_A^i X_j$, which is a contradiction to the fact that this module belongs to $\mathcal{M}$.

This completes the proof of the theorem.

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