dynamics

MDPI

Article

A Combined Separation of Variables and Fractional Power Series Approach for Selected Boundary Value Problems

Gabriel Antonio Felipe, Carlos Alberto Valentim and Sergio Adriani David *

Department of Biosystems Engineering, University of São Paulo, Pirassununga, São Paulo 13635-900, Brazil; gabrielantoniofelipe0704@usp.br (G.A.F.); carlos.valentim@usp.br (C.A.V.)
* Correspondence: sergiodavid@usp.br

Abstract

Fractional modeling has emerged as an important resource for describing complex phenomena and systems exhibiting non-local behavior or memory effects, finding increasing application in several areas in physics and engineering. This study presents the analytical derivation of equations pertinent to the modeling of different systems, with a focus on heat conduction. Two specific boundary value problems are addressed: a Helmholtz equation modified with a fractional derivative term, and a fractional formulation of the Laplace equation applied to steady-state heat conduction in circular geometry. The methodology combines the separation of variables technique with fractional power series expansions, primarily utilizing the Caputo fractional derivative. An important aspect of this paper is its instructional emphasis, wherein the mathematical derivations are presented with detail and clarity. This didactic approach is intended to make the analytical methodology transparent and more understandable, thereby facilitating greater comprehension of the application of these established methods to non-integer-order systems. The final goal is not only to provide a different approach of solving these physical models analytically, but to provide a clear, guided pathway for those engaging in the treatment of fractional differential equations.

Keywords: fractional calculus; analytical methods; Helmholtz equation; Laplace equation; didactic approaches

check for updates

Academic Editor: José Manoel Balthazar

Received: 17 May 2025
Revised: 10 June 2025
Accepted: 14 June 2025
Published: 20 June 2025

Citation: Felipe, G.A.; Valentim, C.A.; David, S.A. A Combined Separation of Variables and Fractional Power Series Approach for Selected Boundary Value Problems. *Dynamics* **2025**, *5*, 24. https://doi.org/10.3390/dynamics5030024

1. Introduction

The mathematical modeling of several real-world problems in science and engineering often leads to formulations expressed through differential equations, which describe the evolution of quantities as functions of time and/or space. Such equations are fundamental tools for the advancement of many areas of scientific and technological knowledge, with solutions achievable by either analytical or numerical methods [1,2]. There is a myriad of phenomena that have been successfully modeled by means of ordinary differential equations (ODE) or partial differential equations (PDE).

A fundamental phenomenon in engineering and physics modeled by PDE is heat conduction, which plays a crucial role in numerous applications ranging from industrial processes to environmental systems. The accurate modeling of heat transfer is important for optimizing thermal management strategies, enhancing energy efficiency, and developing advanced materials with tailored thermal properties. Analytical and numerical methods are also used to study new problems related to this subject [3]. An efficient numerical model has been proposed to address steady-state heat conduction problems with local uncertainty [4]. In addition, analytical heat conduction models have been proposed to create temperature profiles that help predict high-power laser beam shapes helpful for specific materials processing [5].

Heat conduction problems have been well explored using classical integer-order differential equations based on Fourier's law. However, as the understanding of complex systems has evolved, it has become increasingly apparent that these conventional models may not always capture the intricacies of heat transfer in heterogeneous or anomalous diffusion media, particularly in porous media [6].

Fractional calculus (FC) has emerged as a powerful mathematical tool for modeling complex physical processes that exhibit memory effects or non-local behavior [7]. As an extension of classical calculus, FC introduces the concept of non-integer order derivatives and integrals, allowing for more flexible and accurate representations of various phenomena [7,8]. Due to these capabilities, fractional modeling has been applied to solve problems in several different fields including Mathematics, Physics, Engineering, Biology, Finance, and Economics, among others [9–11].

Podlubny's seminal work contributed to the comprehensive growth of FC, presenting several analytical solutions for fractional differential equations and specifically introducing the concept of short memory [12]. Applications in engineering and applied sciences can be directly related to FC concepts, particularly to special functions, such as the Mittag-Leffler function and the Wright function, as they serve as fractional approaches, or generalizations, of exponential and trigonometric functions, respectively [13].

In this context, the works of Ortigueira and Machado and Valério et al. aim to elucidate and unify the numerous definitions and operators associated with fractional derivatives highlighted in the literature by using two parameters (order and asymmetry). Additionally, their contributions emphasize the need for a clear classification of fractional derivatives, underscoring and analyzing the validity of the definitions currently found in the literature [14,15]. In turn, Podlubny's research introduced innovative geometric interpretations related to fractional operators. His work includes the Riemann–Liouville and Stieltjes integrals and derivatives, promoting a deeper understanding of these operators and, consequently, expanding the theoretical scope of FC [16].

Tarasov emphasized the importance of power series within the scope of FC, exploring the concept of a fractional derivative as a non-integer power of the differential operator, using both the Taylor and the Fourier series [17]. Nonlinear fractional differential equations have been employed in the investigation of boundary value problems involving lower-order fractional derivatives [18]. Recent advances have also explored the use of alternative formulations of fractional derivatives, such as the modified Atangana–Baleanu derivative, to handle systems of fractional differential equations [19]. Furthermore, applications involving power fractional differential equations (PFDEs) have been studied, including the non-autonomous PFDEs in both the Riemann–Liouville and Caputo senses [20].

The use of fractional PID controllers in the areas of process control and automation has shown good results in mitigating vibrations and achieving specific desirable outputs [21]. Moreover, FC applied to artificial neural networks has gained attention for its potential in system stabilization and synchronization, as well as parameter training, in areas such as signal processing, robotics, medicine, and cryptography, among others [22]. The combination of FC with fuzzy logic and nonlocal conditions has been studied through fuzzy fractional differential equations and optimal control [23].

Regarding computational modeling, the applications of a fractional order have been emphasized in the unfolding dynamics of contagious diseases. Multiscale models concerning fractional-order tumor growth have been studied, including the study of chaotic behavior in a tumor growth model based on fractional-order differential equations, considering both commensurate and incommensurate cases [24]. A COVID-19 model was examined with a particular focus on nonlinear dynamic behavior using commensurate and incommensurate fractional-order derivatives via the Caputo operator [25]. Additionally, in finance and economics, the fractional dynamic behavior in an ethanol price series has been analyzed.

Several studies have focused on the resolution and application of the Laplace differential equation, as it describes various phenomena related to the fields of physics and mathematics, such as field theory, electrostatics, thermodynamics, and others, including steady-state heat conduction [26–28]. Furthermore, studies on boundary conditions, such as the study on the stability of the Laplace equation in the absence of Dirichlet conditions, are also noted [29]. It is also worth mentioning the use of FC in correlation with the Laplace equation in some investigations, as the determination of coefficients in the p-Laplace fractional equation from external measurements is currently being explored [30,31].

In this paper, we propose a didactic approach that combines the mathematical rigor of FC with the practical need for accurate modeling by exploring two cases of interest in engineering applications: 1D Helmholtz equation (Case 1) and steady-state heat conduction in a 2D circular plate (Case 2). Our framework integrates two methods to obtain analytical solutions: separation of variables and fractional power series. These formulations offer greater flexibility and deeper analysis.

We adopt the Caputo fractional derivative rather than other formulations described in the literature primarily because of its practicality and effectiveness in modeling physical phenomena, particularly in problems involving fractional differential equations.

While there are other strategies to solve PFDEs, including numerical algorithms such as Diethelm's variation of the Adams–Bashforth method [32], we have prioritized an analytical approach. This decision stems from the significant advantages of analytical methods: they yield exact solutions, provide precise insights, and offer a more direct and intuitive pathway to understanding the problem's structure.

We emphasize that this work offers a twofold contribution: First, it presents a novel combination of two methods—separation of variables and power series—both applied within the framework of FC. To the best of our knowledge, this combination has not yet been explored in the context of the problems addressed here. Second, we have made a deliberate effort to present the solutions in a clear and pedagogical manner, with the aim of providing readers with a comprehensive and accessible understanding.

Furthermore, by examining behavioral and accuracy differences among different fractional approaches, we seek deeper insights into the dynamics of systems under fractional-order structures. This didactic and analytical approach not only highlights the advantages of FC in modeling complex systems but emphasizes the physical implications of using FC as a modelling approach.

The remainder of this paper is structured as follows: Section 2 presents the preliminary concepts related to FC. In Section 3, the two cases addressed are introduced, and the respective mathematical developments are presented. Section 4 outlines some final remarks obtained through analytical developments and the validation of using power series within the fractional framework as an effective tool for obtaining interesting solutions.

2. Preliminaries

This section recalls fundamental concepts, definitions, and propositions, while introducing some notations that are relevant to the subsequent mathematical procedures developed in the paper.

Definition 1. *The Caputo derivative [7] is defined as follows:*

$$D_*^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \quad m-1 < \alpha < m, \tag{1}$$

where m is considered the smallest integer greater than α, which allows the Caputo's fractional derivative of order α > 0 to be defined.

Proposition 1. *Let $\mathcal{D} \subset \mathbb{R}$ be a bounded domain that defines the functional space below [33]:*

$$C(\mathcal{D}) = \{u: \mathcal{D} \to \mathbb{R} : u \text{ is continuous}\},$$
$$C(\overline{\mathcal{D}}) = \{u: \overline{\mathcal{D}} \to \mathbb{R} : u \text{ is uniformly continuous}\},$$
$$C^m(\mathcal{D}) = \{u: \mathcal{D} \to \mathbb{R} : u \text{ is } m - \text{times continuously differentiable}\},$$
$$C^m(\overline{\mathcal{D}}) = \{u \in C^m(\mathcal{D}) : \mathcal{D}^{\Gamma} u \text{ is uniformly continuous for all } |\Gamma| \le m\},$$

with $\partial\mathcal{D} \in C^{1,\beta}$ and $C^{m,\beta}(\mathcal{D})\left[C^{m,\beta}(\overline{\mathcal{D}})\right]$ being a subspace of $C^m(\mathcal{D})\left[C^m(\overline{\mathcal{D}})\right]$, consisting of functions whose partial derivatives of order m are uniformly continuous.

One assumes that $\alpha \in (0,1)$ and $\mathcal{D}_\infty = (0,\infty) \times D$. Consequently, the following holds:

$$\mathcal{H}_\Delta(\mathcal{D}_\infty) = \Big\{u: \mathcal{D}_\infty \to \mathbb{R} : \tfrac{\partial u}{\partial t}, \tfrac{\partial^\alpha u}{\partial t^\alpha}, \Delta u \in C(\mathcal{D}_\infty),$$
$$\left|\tfrac{\partial u(x,y)}{\partial t}\right| \le g(x,t)t^{\alpha-1}, g \in C(\mathcal{D}), t > 0\Big\}.$$

The solution to the previous equation and the corresponding boundary conditions are as follows:

$$\begin{aligned} D_t^\alpha u(x,t) = \Delta u(x,t), \quad & x \in \mathcal{D};\ t \ge 0, \\ u(x,t) = 0, \quad & x \in \partial\mathcal{D};\ t \ge 0, \\ u(x,0) = f(x), \quad & x \in \mathcal{D}; \end{aligned} \tag{2}$$

is equivalent to the following equation:

$$u(x,t) = \sum_{n=1}^{\infty} \overline{f}(n) E_\alpha\left(-A_n^2 t^\alpha\right) \phi_n(x),$$

whose detailed derivation for the function u(x,t) is available in Appendix A.

Firstly, assuming that the function u(x,t) is a solution to Equation (2), and subsequently applying Green's second identity [34], it can be shown that $\phi_n(x)$is an eigenfunction corresponding to the eigenvalue A^2. Since u(x,t) is uniformly continuous, which implies that it is uniformly bounded on the domain under consideration, the dominated convergence theorem allows us to conclude the following:

$$\lim_{t\to 0} \int_D u(x,t)\phi_n(x)dx = \overline{f}(n).$$

Proposition 2. *A power series expanded around x = 0 is considered. For such, one supposes a solution in the form as follows:*

$$u(x) = \sum_{n=0}^{\infty} C_n x^{n\alpha}. \tag{3}$$

The next step consists in determining $D_x^\alpha u(x)$, where D_x^α is an operator representing a derivative of order α regarding the independent variable.

Let be $g(z) = \sum_{n=0}^{\infty} C_n z^n$, *for* $0 \le z < R^\alpha$, *where* R^α *is the convergence radius of the series and the derivative follows the Caputo definition [35], one finds the following:*

$$\begin{aligned} D^\alpha g(z) &= \tfrac{1}{\Gamma(m-\alpha)} \int_0^z (z-\tau)^{m-\alpha-1} g^{(m)}(\tau) d\tau \\ &= \tfrac{1}{\Gamma(m-\alpha)} \int_0^z (z-\tau)^{m-\alpha-1} \left(\tfrac{d^m}{d\tau^m} \sum_{n=0}^{\infty} C_n \tau^{n\alpha} \right) d\tau \\ &= \tfrac{1}{\Gamma(m-\alpha)} \int_0^z (z-\tau)^{m-\alpha-1} \left(\sum_{n=0}^{\infty} C_n \tfrac{d^m}{d\tau^m} \tau^{n\alpha} \right) d\tau \\ &= \sum_{n=0}^{\infty} C_n \left[\tfrac{1}{\Gamma(m-\alpha)} \int_0^z (z-\tau)^{m-\alpha-1} \left(\tfrac{d^m}{d\tau^m} \tau^{n\alpha} \right) d\tau \right] \end{aligned} \tag{4}$$

$$= \textstyle\sum_{n=0}^{\infty} (D^\alpha z^n).$$

Considering then the definition with $0 \le \tau < z$ *and* $0 \le z < R^\alpha$ *as follows:*

$$D^\alpha z^n = \frac{1}{\Gamma(m-\alpha)} \int_0^z (z-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} \tau^n d\tau, \tag{5}$$

plugging $z = x^\alpha,\ x \ge 0$ *into Equation (5) one obtains the following equation:*

$$\begin{aligned} D^\alpha u(x) &= D^\alpha g(x^\alpha) = D^\alpha \sum_{n=0}^{\infty} C_n (x^\alpha)^n \\ &= D^\alpha \sum_{n=0}^{\infty} C_n x^{n\alpha} = \sum_{n=0}^{\infty} C_n D^\alpha (x^\alpha)^n,\ \ 0 < x^\alpha < R^\alpha. \end{aligned}$$

Following Equation (5), if one admits that $y = x^m$, *then*

$$\frac{d^\alpha y}{dx^\alpha} = D_x^\alpha y = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}.$$

Analogously, if $y = x^{n\alpha}$, *then*

$$\frac{d^\alpha y}{dx^{n\alpha}} = D_x^\alpha y = \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha-\alpha+1)} x^{n\alpha-\alpha} = \frac{\Gamma(n\alpha+1)}{\Gamma[(n-1)\alpha+1]} x^{(n-1)\alpha}.$$

Therefore,

$$\begin{aligned} D^\alpha u(x) &= \sum_{n=1}^{\infty} C_n \tfrac{\Gamma(n\alpha+1)}{\Gamma[(n-1)\alpha+1]} x^{(n-1)\alpha}, \\ D^\alpha u(x) &= \sum_{n=0}^{\infty} C_{n+1} \tfrac{\Gamma[(n+1)\alpha+1]}{\Gamma[n\alpha+1]} x^{n\alpha}. \end{aligned} \tag{6}$$

By considering $n-1 = m$, *one can adjust the series index as follows:*

$$\begin{aligned} D^\alpha u(x) &= \sum_{n=1}^{\infty} C_n \tfrac{\Gamma(n\alpha+1)}{\Gamma[(n-1)\alpha+1]} x^{(n-1)\alpha} \\ &= \sum_{m=0}^{\infty} C_{m+1} \tfrac{\Gamma[(m+1)\alpha+1]}{\Gamma[m\alpha+1]} x^{m\alpha} \\ &= \sum_{n=0}^{\infty} C_{n+1} \tfrac{\Gamma[(n+1)\alpha+1]}{\Gamma[n\alpha+1]} x^{n\alpha}. \end{aligned}$$

Similarly, the operator $D^{2\alpha}$ can be written as follows:

$$\begin{aligned} D^{2\alpha}u(x) &= \sum_{n=2}^{\infty} C_n \frac{\Gamma(n\alpha+1)}{\Gamma[(n-2)\alpha+1]} x^{(n-2)\alpha} \\ &= \sum_{n=0}^{\infty} C_{n+2} \frac{\Gamma[(n+2)\alpha+1]}{\Gamma[n\alpha+1]} x^{n\alpha}. \end{aligned} \tag{7}$$

One can now apply Equations (3), (6), and (7) to solve generalized differential equations.

3. Case Studies

To apply the methodology presented in the previous section, we now explore two cases of interest in engineering, the first one involving an ODE and the second one, significantly more complex, modeled by means of a PDE. In both cases, the didactic purpose of presenting the conduction of the solution process by means of FC is highlighted.

3.1. Case 1—1D Helmholtz Equation with Constant Term k

The first case addresses the generalized 1D Helmholtz equation with an additional derivative term, represented by the constant k. The Helmholtz equation is widely used in modeling physical phenomena, particularly in contexts where wave-like or oscillatory behavior is present (e.g., acoustics, electromagnetic theory, heat conduction, and diffusion) [36]. The analyzed case aims to deepen and broaden the understanding of heat conduction in an engine by using a simple 1D fin, since the modeling presented in this work is intended to assist in determining fin efficiency and optimizing maximum heat dissipation. It is known that heat conduction in internal combustion engine fins is strongly dependent on several factors, among which geometry (considered one-dimensional in this work) plays as important role. In addition, the use of numerical techniques can also contribute to achieve the optimal fin design [37–39].

Figure 1 schematically illustrates a 1D domain $[0, L]$, such as a long metallic rod, where the function $u(x)$ is discretized at distinct locations along the spatial axis by means of five arbitrary points. This configuration serves solely as a representative physical application, as the primary objective of this study is the analytical derivation of the proposed fractional problem.

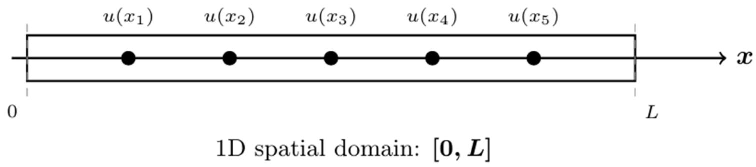

Figure 1. Conceptual illustration of the fractional Helmholtz model in a 1D domain, with multiple evaluation points $u(x_i)$.

The fractional differential equation is written as follows:

$$D_x^{2\alpha}u(x) + D_x^{\alpha}u(x) + k^2u(x) = 0, \qquad x \geq 0. \tag{8}$$

We consider the interval $0.5 < \alpha \leq 1$, with the boundary conditions given by Equation (9):

$$\begin{aligned} \mathrm{u}(0) = C_0 = \mu_0, \\ D_x^{\alpha}u(0) = p_0. \end{aligned} \tag{9}$$

Following the procedure established in the methodology, after plugging the equivalent series for the fractional operators, one obtains the following equation:

$$\sum_{n=0}^{\infty} C_{n+2} \frac{\Gamma[(n+2)\alpha+1]}{\Gamma[n\alpha+1]} x^{n\alpha} + \sum_{n=0}^{\infty} C_{n+1} \frac{\Gamma[(n+1)\alpha+1]}{\Gamma[n\alpha+1]} x^{n\alpha} + \sum_{n=0}^{\infty} k^2 C_n x^{n\alpha} = 0,$$

which, after grouping and simplifying yields, the recurrence equation for $n = 0, 1, 2, 3, \ldots$ is as follows:

$$C_{n+2} = -k^2 C_n \frac{\Gamma(n\alpha+1)}{\Gamma[(n+2)\alpha+1]} - C_{n+1} \frac{\Gamma[(n+1)\alpha+1]}{\Gamma[(n+2)\alpha+1]}. \tag{10}$$

Using Equation (10), one can find the following coefficients for the different values of n :

$$\begin{aligned}
(n=0) \rightarrow C_2 &= -k^2 C_0 \tfrac{\Gamma(1)}{\Gamma(2\alpha+1)} - C_1 \tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \\
&= \tfrac{1}{\Gamma(2\alpha+1)} [-k^2 C_0 - C_1 \Gamma(\alpha+1)], \\
(n=1) \rightarrow C_3 &= -k^2 C_1 \tfrac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)} - C_2 \tfrac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} \\
&= \tfrac{1}{\Gamma(3\alpha+1)} [C_1 \Gamma(\alpha+1)(1-k^2) + k^2 C_0], \\
(n=2) \rightarrow C_4 &= -k^2 C_2 \tfrac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} - C_3 \tfrac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)} \\
&= \tfrac{1}{\Gamma(4\alpha+1)} [k^4 C_0 - C_1 \Gamma(\alpha+1) - k^2 C_0].
\end{aligned}$$

As the coefficients are calculated, the solution is written as follows:

$$\begin{aligned}
u(x) = C_0 &\left[1 - \tfrac{k^2}{\Gamma(2\alpha+1)} x^{2\alpha} + \tfrac{k^2}{\Gamma(3\alpha+1)} x^{3\alpha} + \tfrac{(k^4-k^2)}{\Gamma(4\alpha+1)} x^{4\alpha} + \cdots\right] + \\
C_1 &\left[x^{\alpha} - \tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} x^{2\alpha} + \tfrac{(1-k^2)\Gamma(\alpha+1)}{\Gamma(3\alpha+1)} x^{3\alpha} + \tfrac{(2k^2-1)\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} x^{4\alpha} + \cdots\right],
\end{aligned}$$

and after once again considering the boundary conditions and substituting coefficients C_0 and C_1, one finds the generalized solution for Equation (8) as follows:

$$\begin{aligned}
u(x) = \mu_0 &\left[1 - \tfrac{k^2}{\Gamma(2\alpha+1)} x^{2\alpha} + \tfrac{k^2}{\Gamma(3\alpha+1)} x^{3\alpha} + \tfrac{(k^4-k^2)}{\Gamma(4\alpha+1)} x^{4\alpha} + \cdots\right] + \\
p_o &\left[\tfrac{1}{\Gamma(\alpha+1)} x^{\alpha} - \tfrac{1}{\Gamma(2\alpha+1)} x^{2\alpha} + \tfrac{(1-k^2)}{\Gamma(3\alpha+1)} x^{3\alpha} - \tfrac{(2k^2-1)}{\Gamma(4\alpha+1)} x^{4\alpha} + \cdots\right].
\end{aligned} \tag{11}$$

3.2. Case 2—Steady-State Heat Conduction in a 2D Circular Plate Using Laplace's Equation

The second case explores a heat conduction problem involving a circular plate with isolated faces. In this context, there are several studies in the literature that deal with the thermal modeling of vehicle brake disc systems [40–42]. Nonetheless, none of them has employed the combination of methods used in this paper under the FC approach. Once again, we intend to deepen and broaden the understanding of the thermal behavior dynamics of the braking system, represented as a simplified circular plate in this Case 2.

In this scenario, the upper and lower boundaries are kept at temperatures u_1 and u_2, respectively. Figure 2 provides a schematic representation of the problem.

To obtain an analytical solution of arbitrary order of the Laplace equation, one applies the fractional derivative previously defined, thus obtaining Equation (12):

$$D_{\rho}^{2\alpha} u(\rho,\varphi) + \frac{1}{\rho} D_{\rho}^{\alpha} u(\rho,\varphi) + \frac{1}{\rho^2} D_{\varphi}^{2\alpha} u(\rho,\varphi) = 0. \tag{12}$$

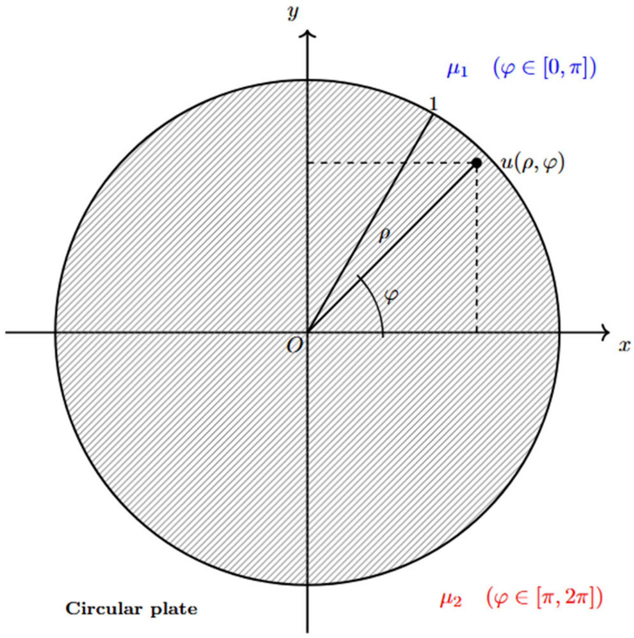

Figure 2. Schematic of the circular plate domain in polar coordinates.

The boundary conditions for the proposed problem are defined as follows:

$$\begin{aligned} u(1,\varphi) &= u_1, \quad 0 < \varphi < \pi, \\ u(1,\varphi) &= u_2, \quad \pi < \varphi < 2\pi. \end{aligned} \tag{13}$$

The method of separation of variables is employed to advance with the solution, that is, $u(\rho,\varphi) = R(\rho)\Phi(\varphi)$.

Substituting the relationship into Equation (12), one obtains Equation (14):

$$\Phi(\varphi)D_\rho^{2\alpha}R(\rho) + \frac{\Phi(\varphi)}{\rho}D_\rho^{\alpha}R(\rho) + \frac{R(\rho)}{\rho^2}D_\varphi^{2\alpha}\Phi(\varphi) = 0, \tag{14}$$

and, after some further algebraic manipulations, one obtains the following equations:

$$\Phi(\varphi)D_\rho^{2\alpha}R(\rho) + \frac{\Phi(\varphi)}{\rho}D_\rho^{\alpha}R(\rho) = -\frac{R(\rho)}{\rho^2}D_\varphi^{2\alpha}\Phi(\varphi), \tag{15}$$

$$\frac{1}{R(\rho)}D_\rho^{2\alpha}R(\rho) + \frac{1}{\rho R(\rho)}D_\rho^{\alpha}R(\rho) = -\frac{1}{\rho^2\Phi(\varphi)}D_\varphi^{2\alpha}\Phi(\varphi), \tag{16}$$

$$\frac{\rho^2}{R(\rho)}D_\rho^{2\alpha}R(\rho) + \frac{\rho}{R(\rho)}D_\rho^{\alpha}R(\rho) = -\frac{1}{\Phi(\varphi)}D_\varphi^{2\alpha}\Phi(\varphi), \tag{17}$$

$$\frac{\rho^2}{R(\rho)}D_\rho^{2\alpha}R(\rho) + \frac{\rho}{R(\rho)}D_\rho^{\alpha}R(\rho) = -\frac{1}{\Phi(\varphi)}D_\varphi^{2\alpha}\Phi(\varphi) = -A^2. \tag{18}$$

From Equation (18), it is possible to highlight two resulting ODE to subsequently obtain the final solution within the fractional framework as follows:

$$\rho^2 D_\rho^{2\alpha}R(\rho) + \rho D_\rho^{\alpha}R(\rho) + A^2R(\rho) = 0, \tag{19}$$

$$D_\varphi^{2\alpha}\Phi(\varphi) - A^2\Phi(\varphi) = 0. \tag{20}$$

To obtain the final solution, one solves each ODE obtained in the previous step separately and then develops each equation with the boundary conditions. It is important to emphasize that power series were the chosen method to approach this problem, as represented in Equation (21), and by applying the fractional derivative, using the function t^x as an example [43]:

$$R(\rho) = \sum_{n=0}^{\infty} C_n \rho^{n\alpha}. \tag{21}$$

According to the definition of the derivative for function t^x, Equation (22), the values for $D_\rho^{2\alpha} R(\rho)$ and $D_\rho^{\alpha} R(\rho)$ are consequently obtained for Equations (23) and (24), respectively, as follows:

$$D^n x^m = \frac{\Gamma(m+1) t^{m-n}}{\Gamma(m-n+1)} = \frac{m! t^{m-n}}{(m-n)!}. \tag{22}$$

$$D_\rho^{2\alpha} R(\rho) = C_n \sum_{n=2}^{\infty} \frac{\Gamma(n\alpha+1)}{\Gamma[(n-2)\alpha+1]} \rho^{(n-2)\alpha}. \tag{23}$$

$$D_\rho^{\alpha} R(\rho) = C_n \sum_{n=1}^{\infty} \frac{\Gamma(n\alpha+1)}{\Gamma[(n-1)\alpha+1]} \rho^{(n-1)\alpha}. \tag{24}$$

By substituting Equations (23) and (24) into the ODE given by Equation (19), the following expression is obtained:

$$\rho^2 \sum_{n=2}^{\infty} C_n \frac{\Gamma(n\alpha+1)}{\Gamma[(n-2)\alpha+1]} \rho^{(n-2)\alpha} + \rho \sum_{n=1}^{\infty} C_n \frac{\Gamma(n\alpha+1)}{\Gamma[(n-1)\alpha+1]} \rho^{(n-1)\alpha} + A^2 \sum_{n=0}^{\infty} C_n \rho^{n\alpha} = 0. \tag{25}$$

The previous equation can be rewritten as follows:

$$\sum_{n=2}^{\infty} C_n \frac{\Gamma(n\alpha+1)}{\Gamma[(n-2)\alpha+1]} \rho^{(n-2)\alpha+2} + \sum_{n=1}^{\infty} C_n \frac{\Gamma(n\alpha+1)}{\Gamma[(n-1)\alpha+1]} \rho^{(n-1)\alpha+1} + A^2 \sum_{n=0}^{\infty} C_n \rho^{n\alpha} = 0. \tag{26}$$

The next step consists of two variable changes, that is, $n-2=m$ with $m \to 0$ for $n \to 2$ and $m \to \infty$ for $n \to \infty$, as well as $n-1=t$, with $t \to 0$ for $n \to \infty$ and $t \to \infty$ for $n \to \infty$.

$$\sum_{n=2}^{\infty} C_n \frac{\Gamma(n\alpha+1)}{\Gamma[(n-2)\alpha+1]} \rho^{(n-2)\alpha+2} \equiv \sum_{m=0}^{\infty} C_{m+2} \frac{\Gamma[(m+2)\alpha+1]}{\Gamma(m\alpha+1)} \rho^{m\alpha+2}. \tag{27}$$

$$\sum_{n=1}^{\infty} C_n \frac{\Gamma(n\alpha+1)}{\Gamma[(n-1)\alpha+1]} \rho^{(n-1)\alpha+1} \equiv \sum_{t=0}^{\infty} C_{t+1} \frac{\Gamma[(t+1)\alpha+1]}{\Gamma(t\alpha+1)} \rho^{t\alpha+1}. \tag{28}$$

For a better representation, the substitution of t and m by n is allowed. By rewriting Equation (26) with the necessary changes, we obtain Equation (29):

$$\sum_{n=0}^{\infty} C_{n+2} \frac{\Gamma[(n+2)\alpha+1]}{\Gamma(n\alpha+1)} \rho^{n\alpha+2} + \sum_{n=0}^{\infty} C_{n+1} \frac{\Gamma[(n+1)\alpha+1]}{\Gamma(n\alpha+1)} \rho^{n\alpha+1} + A^2 \sum_{n=0}^{\infty} C_n \rho^{n\alpha} = 0. \tag{29}$$

The next step is also a variable change, as the exponents of the base ρ, $n\alpha+2$ and $n\alpha+1$, must be converted to $n\alpha$. We have $m = n + \frac{2}{\alpha}$, with $m \to \frac{2}{\alpha}$ for $n \to 0$ and $m \to \infty$ for $n \to \infty$.

$$\sum_{n=0}^{\infty} C_{n+2} \frac{\Gamma[(n+2)\alpha+1]}{\Gamma(n\alpha+1)} \rho^{n\alpha+2} \equiv \sum_{m=\frac{2}{\alpha}}^{\infty} C_{m-\frac{2}{\alpha}+2} \frac{\Gamma[(m+2)\alpha-1]}{\Gamma(m\alpha-1)} \rho^{m\alpha}. \tag{30}$$

We have $t = s + \frac{1}{\alpha}$, with $t \to \frac{1}{\alpha}$ for $s \to 0$ and $t \to \infty$ for $s \to \infty$.

$$\sum_{n=0}^{\infty} C_{n+1} \frac{\Gamma[(n+1)\alpha+1]}{\Gamma(n\alpha+1)} \rho^{n\alpha+1} \equiv \sum_{t=\frac{1}{\alpha}}^{\infty} C_{t-\frac{1}{\alpha}+1} \frac{\Gamma[(t+1)\alpha]}{\Gamma(t\alpha)} \rho^{t\alpha}. \tag{31}$$

The substitution of t and m by n in Equation (29), leads to Equation (32):

$$\sum_{n=\frac{2}{\alpha}}^{\infty} C_{n-\frac{2}{\alpha}+2}\frac{\Gamma[(n+2)\alpha-1]}{\Gamma(n\alpha-1)}\rho^{n\alpha}+\sum_{n=\frac{1}{\alpha}}^{\infty} C_{n-\frac{1}{\alpha}+1}\frac{\Gamma[(n+1)\alpha]}{\Gamma(n\alpha)}\rho^{n\alpha}+A^2\sum_{n=0}^{\infty} C_n\rho^{n\alpha}=0. \quad (32)$$

The upper and lower limits of the summations are respectively set to ∞ and $\frac{2}{\alpha}$ and they are separated into infinite and finite summations, as represented by Equation (33):

$$\begin{gathered}\sum_{n=\frac{2}{\alpha}}^{\infty} C_{n-\frac{2}{\alpha}+2}\frac{\Gamma[(n+2)\alpha-1]}{\Gamma(n\alpha-1)}\rho^{n\alpha}+\sum_{n=\frac{2}{\alpha}}^{\infty} C_{n-\frac{1}{\alpha}+1}\frac{\Gamma[(n+1)\alpha]}{\Gamma(n\alpha)}\rho^{n\alpha}\\+\sum_{n=\frac{1}{\alpha}}^{\frac{2}{\alpha}-1} C_{n-\frac{1}{\alpha}+1}\frac{\Gamma[(n+1)\alpha]}{\Gamma(n\alpha)}\rho^{n\alpha}+A^2\sum_{n=0}^{\frac{2}{\alpha}-1} C_n\rho^{n\alpha}+A^2\sum_{n=\frac{2}{\alpha}}^{\infty} C_n\rho^{n\alpha}=0.\end{gathered} \quad (33)$$

The possible terms are highlighted, yielding Equation (34):

$$\begin{gathered}\sum_{n=\frac{2}{\alpha}}^{\infty}\left\{C_{n-\frac{2}{\alpha}+2}\frac{\Gamma[(n+2)\alpha-1]}{\Gamma(n\alpha-1)}+C_{n-\frac{1}{\alpha}+1}\frac{\Gamma[(n+1)\alpha]}{\Gamma(n\alpha)}+A^2C_n\right\}p^{n\alpha}+A^2\sum_{n=0}^{\frac{2}{\alpha}-1} C_n\rho^{n\alpha}\\+\sum_{n=\frac{1}{\alpha}}^{\frac{2}{\alpha}-1} C_{n-\frac{1}{\alpha}+1}\frac{\Gamma[(n+1)\alpha]}{\Gamma(n\alpha)}\rho^{n\alpha}=0.\end{gathered} \quad (34)$$

By grouping the terms in the infinite summation with the index n starting at $\frac{2}{\alpha}$, it is possible to set the sum of these terms equal to 0 to obtain the recurrence formula. This formula directly depends on the value of the summation index n and the parameter α, while also involving the Gamma function, which generalizes the factorial.

The recurrence formula related to the ODE described by Equation (19) for obtaining the coefficients for $n \geq \frac{2}{\alpha}$ is represented by Equation (35):

$$C_{n-\frac{2}{\alpha}+2}=-\frac{C_{n-\frac{1}{\alpha}+1}\Gamma[(n+1)\alpha]\Gamma(n\alpha-1)}{\Gamma(n\alpha)\Gamma[(n+2)\alpha-1]}-A^2C_n\frac{\Gamma(n\alpha-1)}{\Gamma[(n+2)\alpha-1]}. \quad (35)$$

It is worth noting that the choice of n values, which is a counting parameter or a counter, is related to the values of α. The choice of α values, which are non-integers, may lead to integer counters. The values chosen for α in this study when $n=\frac{2}{\alpha}$ are $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, and 1. Therefore, n respectively corresponds to 16, 8, 6, 4, and 2. When $n=\frac{1}{\alpha}$ the respective n values are 8, 4, 3, 2, and 1. This approach explores heat conduction for some non-integer orders in the differential equation that governs this physical phenomenon.

As a result, for different α values, distinct analytical solutions for the problem are highlighted. In this context, by choosing a value of α, for example $\alpha=\frac{1}{2}$, it is emphasized that the solution $R(\rho)$ is obtained based on such choice. By substituting α into Equation (35), we obtain Equation (36):

$$C_{n-2}=-\frac{C_{n-1}\Gamma\left[\frac{(n+1)}{2}\right]\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left[\frac{(n+2)}{2}-1\right]}-A^2C_n\frac{\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left[\frac{(n+2)}{2}-1\right]}. \quad (36)$$

Considering $n \geq \frac{2}{\alpha}$, as previously mentioned for obtaining the recurrence formula, and then $n \geq 4$, the coefficient values can be obtained.

For $n=4$:

$$C_2=-\frac{C_3\Gamma\left(\frac{5}{2}\right)\Gamma(1)}{\Gamma(2)\Gamma(2)}-A^2C_4\frac{\Gamma(1)}{\Gamma(2)}. \quad (37)$$

For $n = 5$:

$$C_3 = -\frac{C_4 \Gamma(3)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)} - A^2 C_5 \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}. \tag{38}$$

For $n = 6$:

$$C_4 = -\frac{C_5 \Gamma\left(\frac{7}{2}\right)\Gamma(2)}{\Gamma(3)\Gamma(3)} - A^2 C_6 \frac{\Gamma(2)}{\Gamma(3)}. \tag{39}$$

For $n = 7$:

$$C_5 = -\frac{C_6 \Gamma(4)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{7}{2}\right)} - A^2 C_7 \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}. \tag{40}$$

For $n = 8$:

$$C_6 = -\frac{C_7 \Gamma\left(\frac{9}{2}\right)\Gamma(3)}{\Gamma(4)\Gamma(4)} - A^2 C_8 \frac{\Gamma(3)}{\Gamma(4)}. \tag{41}$$

The analytical solution $R(\rho)$ for $\alpha = \frac{1}{2}$ is given by Equation (42):

$$R(\rho) = -\sum_{t=2}^{\infty} \frac{\Gamma\left(\frac{t}{2}\right)}{\Gamma\left(\frac{t}{2}+1\right)} \left[G_{t+1} \frac{\Gamma\left(\frac{t+3}{2}\right)}{\Gamma\left(\frac{t}{2}+1\right)} + A^2 G_{t+2} \right] \rho^{\frac{t}{2}}. \tag{42}$$

The second step consists of analytically solving the second ODE, Equation (20), using FC. Once again, using the concept of power series, we have that $\Phi(\varphi)$ is represented as shown in Equation (43), and by applying the derivative, we obtain the equation described in Equation (44):

$$\Phi(\varphi) = \sum_{n=0}^{\infty} C_n \varphi^{n\alpha}, \tag{43}$$

$$D_{\varphi}^{2\alpha} \Phi(\varphi) = C_n \sum_{n=2}^{\infty} \frac{\Gamma(n\alpha+1)}{\Gamma[(n-2)\alpha+1]} \varphi^{(n-2)\alpha}. \tag{44}$$

By applying Equations (43) and (44) to the ODE from Equation (20) one obtains Equation (45):

$$\sum_{n=2}^{\infty} C_n \frac{\Gamma(n\alpha+1)}{\Gamma[(n-2)\alpha+1]} \varphi^{(n-2)\alpha} - A^2 \sum_{n=0}^{\infty} C_n \varphi^{n\alpha} = 0. \tag{45}$$

One proposes the change of variable $n - 2 = m$ so that $n \to 2$ yields $m \to 0$ and $n \to \infty$ yields $m \to \infty$.

$$\sum_{m=0}^{\infty} C_{m+2} \frac{\Gamma[(m+2)\alpha+1]}{\Gamma(m\alpha+1)} \varphi^{m\alpha}, \tag{46}$$

$$\sum_{n=0}^{\infty} C_{n+2} \frac{\Gamma[(n+2)\alpha+1]}{\Gamma(n\alpha+1)} \varphi^{n\alpha} - A^2 \sum_{n=0}^{\infty} C_n \varphi^{n\alpha} = 0, \tag{47}$$

$$\sum_{n=0}^{\infty} \left[C_{n+2} \frac{\Gamma[(n+2)\alpha+1]}{\Gamma(n\alpha+1)} - A^2 C_n \right] \varphi^{n\alpha} = 0, \tag{48}$$

$$C_{n+2} \frac{\Gamma[(n+2)\alpha+1]}{\Gamma(n\alpha+1)} - A^2 C_n = 0, \tag{49}$$

$$C_{n+2} = \frac{C_n A^2 \Gamma(n\alpha+1)}{\Gamma[(n+2)\alpha+1]}. \tag{50}$$

Different values are assigned to n. We have the following values.

For $n = 0$:

$$C_2 = \frac{C_0 A^2 \Gamma(1)}{\Gamma(2\alpha+1)} = \frac{C_0 A^2}{\Gamma(2\alpha+1)}, \tag{51}$$

For $n = 1:$

$$C_3 = \frac{C_1 A^2 \Gamma(\alpha+1)}{\Gamma(3\alpha+1)}, \tag{52}$$

For $n = 2:$

$$\begin{aligned} C_4 &= \frac{C_2 A^2 \Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} \\ &= \frac{C_0 A^2}{\Gamma(2\alpha+1)} \frac{A^2 \Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} = \frac{A^4 C_0}{\Gamma(4\alpha+1)}, \end{aligned} \tag{53}$$

For $n = 3:$

$$\begin{aligned} C_5 &= \frac{C_3 A^2 \Gamma(3\alpha+1)}{\Gamma(5\alpha+1)} \\ &= \frac{C_1 A^2 \Gamma(\alpha+1)}{\Gamma(3\alpha+1)} \frac{A^2 \Gamma(3\alpha+1)}{\Gamma(5\alpha+1)} = \frac{A^4 C_1 \Gamma(\alpha+1)}{\Gamma(5\alpha+1)}, \end{aligned} \tag{54}$$

For $n = 4:$

$$\begin{aligned} C_6 &= \frac{C_4 A^2 \Gamma(4\alpha+1)}{\Gamma(6\alpha+1)} \\ &= \frac{C_0 A^4}{\Gamma(4\alpha+1)} \frac{A^2 \Gamma(4\alpha+1)}{\Gamma(6\alpha+1)} = \frac{A^6 C_0}{\Gamma(6\alpha+1)}, \end{aligned} \tag{55}$$

For $n = 5:$

$$\begin{aligned} C_7 &= \frac{C_5 A^2 \Gamma(5\alpha+1)}{\Gamma(7\alpha+1)} \\ &= \frac{C_1 A^4 \Gamma(\alpha+1)}{\Gamma(5\alpha+1)} \frac{A^2 \Gamma(5\alpha+1)}{\Gamma(7\alpha+1)} = \frac{A^6 C_1 \Gamma(\alpha+1)}{\Gamma(7\alpha+1)}. \end{aligned} \tag{56}$$

It is possible to generalize for terms with even and odd n, namely terms B_k and E_t, respectively, obtaining the final solution as follows:

$$B_k = C_0 \sum_{k=0}^{\infty} \frac{A^{2k+2}}{\Gamma[(2k+2)\alpha+1]}, \tag{57}$$

$$E_t = C_1 \Gamma(\alpha+1) \sum_{t=0}^{\infty} \frac{A^{2t+2}}{\Gamma[(2t+3)\alpha+1]}. \tag{58}$$

We can rewrite $\Phi(\varphi)$ as a power series as follows:

$$\Phi(\varphi) = \sum_{n=0}^{\infty} C_n \varphi^{n\alpha} = C_0 + C_1 \varphi^{\alpha} + C_2 \varphi^{2\alpha} + C_3 \varphi^{3\alpha} + \cdots. \tag{59}$$

Considering the necessary substitutions, the following expression for the second ODE is obtained as follows:

$$\Phi(\varphi) = C_0 + C_1 \varphi^{\alpha} + \sum_{k=0}^{\infty} A^{2k+2} \left\{ \frac{C_0 \varphi^{(2k+2)\alpha}}{\Gamma[(2k+2)\alpha+1]} + \frac{C_1 \Gamma(\alpha+1) \varphi^{(2k+3)\alpha}}{\Gamma[(2k+3)\alpha+1]} \right\}. \tag{60}$$

With the respective values for the solutions of the ODEs, $\Phi(\varphi)$ and $R(\rho)$, the general solution $u(\rho,\varphi) = R(\rho)\Phi(\varphi)$ for the example where $\alpha = \frac{1}{2}$ is given by Equation (61):

$$\begin{aligned} u(\rho,\varphi) = &\left\{ -\sum_{t=2}^{\infty} \frac{\Gamma\left(\frac{t}{2}\right)}{\Gamma\left(\frac{t}{2}+1\right)} \left[G_{t+1} \frac{\Gamma\left(\frac{t+3}{2}\right)}{\Gamma\left(\frac{t}{2}+1\right)} + A^2 G_{t+2} \right] \rho^{\frac{t}{2}} \right\} \times \\ &\left\{ C_0 + C_1 \varphi^{\frac{1}{2}} + \sum_{k=0}^{\infty} A^{2k+2} \left[\frac{C_0 \varphi^{k+1}}{\Gamma(k+2)} + \frac{C_1 \Gamma\left(\frac{3}{2}\right) \varphi^{k+\frac{3}{2}}}{\Gamma\left(k+\frac{5}{2}\right)} \right] \right\}. \end{aligned} \tag{61}$$

The analytical solution to the PDE represented by Equation (12) is directly influenced by the choice of the parameter $\boldsymbol{\alpha}$, mainly due to the dependence of the recurrence Equation (35) on the fractional order. Despite the fact that this study employed a specific value of $\boldsymbol{\alpha}$, there are still several possible avenues to be explored, such as the derivation of new general solutions by studying and varying the order of differentiation.

In the same context, future papers also aim to obtain results related to this problem through numerical simulations. This would complement the present study, which focuses on analytical solutions, by enabling clear comparisons of the system's behavior when using different values of α and distinct formulations of fractional derivatives.

4. Final Remarks

In this paper, the use of power series and separation of variables was investigated as a methodology for solving different problems involving fractional-order differential equations. The objective of this study was to present an easily approachable, didactic perspective for obtaining analytical solutions associated with FC, applied to two cases of interest in engineering.

The studied cases were arranged in order of complexity. The first one involved the 1-D Helmholtz equation with an additional derivative term, namely the constant k. The second case addressed the application of the Laplace equation to a steady-state heat conduction problem in a circular plate. In both demonstrations, the analytical solution was obtained, clearly outlining the necessary steps in an instructional manner.

The application of FC to interdisciplinary problems is of great importance, especially when considering the use of well-established methods, namely, the method of separation of variables and the method of power series expansion of the solution, both based on FC, along with a didactic presentation of its applicability from an instructional perspective. It is noteworthy that future perspectives related to this line of study are essential for the mathematical development of these problems, and for enhancing the understanding of those interested in their investigation.

Author Contributions: Conceptualization, S.A.D.; methodology, G.A.F. and S.A.D.; software, G.A.F.; validation, G.A.F., C.A.V. and S.A.D.; formal analysis, G.A.F., C.A.V. and S.A.D.; investigation, G.A.F. and S.A.D.; writing—original draft preparation, G.A.F.; writing—review and editing, G.A.F., C.A.V. and S.A.D.; visualization, G.A.F.; supervision, S.A.D. All authors have read and agreed to the published version of the manuscript.

Funding: This research was, in part, funded by São Paulo Research Foundation (FAPESP), grant number 2024/23322-8.

Data Availability Statement: This is an analytical study where no new numerical data were created.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Proof. Given that, by employing the separation of variables methods, u can be written as $u(x,t) = G(t)F(x)$, where G depends only on t and F only on x, and considering it as a possible solution to Equation (2), one obtains the following equation upon substitution into the PDE of interest:

$$F(x)D_t^{\alpha}G(t) = G(t)\Delta F(x),$$

thus

$$\frac{D_t^{\alpha}G(t)}{G(t)} = \frac{\Delta F(x)}{F(x)} = -A^2, \text{ where } A = \text{const.}$$

Hence, two ODE are established as follows:

$$D_t^{\alpha}G(t) = -A^2G(t), \qquad t \geq 0, \tag{A1}$$

$$\Delta F(x) = -A^2F(x), \qquad x \in \mathcal{D}; F(x)|_{\partial\mathcal{D}} = 0. \tag{A2}$$

As a result, the solution to Equation (A1) is as follows:

$$G(t) = E_\alpha(-A_n^2 t^\alpha).$$

Before obtaining $F(x)$, that is, the eigenvalue problem in Equation (A2), one resorts to an infinite series of pairs (u_n, ϕ_n), $n \geq 1$, where ϕ_n corresponds to a sequence of functions that form a complete orthonormal set in the considered domain. Such a function is represented as follows:

$$F(x) = \sum_{n=1}^{\infty} \overline{f}(n)\phi_n(x),$$

where $\overline{f}(n)$ was used to satisfy the initial condition of the addressed problem. □

Consequently, as $u(x,t) = G(t)F(x)$, the analytical solution to the problem is as follows:

$$u(x,t) = \sum_{n=1}^{\infty} \overline{f}(n)E_\alpha(-A_n^2 t^\alpha)\phi_n(x).$$

Additional details can be found in [33,44].

References

1. Boyce, W.E.; DiPrima, R.C.; Meade, D.B. *Elementary Differential Equations and Boundary Value Problems*, 12th ed.; International Adaptation; John Wiley & Sons, Inc: Hoboken, NJ, USA, 2022; ISBN 978-1-119-82051-2.
2. Kreyszig, E.; Kreyszig, H.; Norminton, E.J. *Advanced Engineering Mathematics*, 10th ed.; John Wiley: Hoboken, NJ, USA, 2011; ISBN 978-0-470-45836-5.
3. Kakac, S.; Yener, Y.; Naveira-Cotta, C.P. *Heat Conduction, Fifth Edition*; CRC Press: Boca Raton, FL, USA, 2018; ISBN 978-1-351-71418-1.
4. Guo, X.; Yang, H.; He, Y. An Efficient Numerical Algorithm to Solve Steady State Heat Conduction Problems with Local Uncertainty. *Eng. Anal. Bound. Elem.* **2024**, *163*, 593–605. [CrossRef]
5. Sundqvist, J. Analytical Heat Conduction Modelling for Shaped Laser Beams. *J. Mech. Work. Technol.* **2017**, *247*, 48–54. [CrossRef]
6. Xu, H.J.; Xing, Z.B.; Wang, F.Q.; Cheng, Z.M. Review on Heat Conduction, Heat Convection, Thermal Radiation and Phase Change Heat Transfer of Nanofluids in Porous Media: Fundamentals and Applications. *Chem. Eng. Sci.* **2019**, *195*, 462–483. [CrossRef]
7. David, S.A.; Linares, J.L.; Pallone, E.M.J.A. Fractional Order Calculus: Historical Apologia, Basic Concepts and Some Applications. *Rev. Bras. Ensino Física* **2011**, *33*, 4302. [CrossRef]
8. Luchko, Y.; Kochubei, A. *Handbook of Fractional Calculus with Applications. Volume 1: Basic Theory*, 1st ed.; Tenreiro Machado, J.A., Ed.; De Gruyter: Berlin, Germany; Boston, MA, USA, 2019; ISBN 978-3-11-057162-2.
9. David, S.; Valentim, C.A., Jr. Fractional Euler-Lagrange Equations Applied to Oscillatory Systems. *Mathematics* **2015**, *3*, 258–272. [CrossRef]
10. Valentim, C.A.; Rabi, J.A.; David, S.A. Fractional Mathematical Oncology: On the Potential of Non-Integer Order Calculus Applied to Interdisciplinary Models. *Biosystems* **2021**, *204*, 104377. [CrossRef]
11. David, S.A.; Rabi, J.A. Can Fractional Calculus Be Applied to Relativity? *Axiomathes* **2020**, *30*, 165–176. [CrossRef]
12. Podlubny, I. Numerical Solution of Fractional Differential Equations. In *Mathematics in Science and Engineering*; Technical University of Munich: Munich, Germany, 1999; Volume 198, pp. 223–242, ISBN 978-0-12-558840-9.
13. Singh, H.; Srivastava, H.M.; Pandey, R.K. *Special Functions in Fractional Calculus and Engineering*; CRC Press: Boca Raton, FL, USA, 2023; ISBN 978-1-032-43602-9.
14. Ortigueira, M.D.; Tenreiro Machado, J. Fractional Derivatives: The Perspective of System Theory. *Mathematics* **2019**, *7*, 150. [CrossRef]
15. Valério, D.; Ortigueira, M.D.; Lopes, A.M. How Many Fractional Derivatives Are There? *Mathematics* **2022**, *10*, 737. [CrossRef]
16. Podlubny, I. On Some Geometric Interpretations of Fractional-Order Operators. *arXiv* **2024**, arXiv:2411.12494.
17. Tarasov, V.E. Fractional Derivative As Fractional Power of Derivative. *Int. J. Math.* **2007**, *18*, 281–299. [CrossRef]
18. Muthaiah, S.; Baleanu, D. Existence of Solutions for Nonlinear Fractional Differential Equations and Inclusions Depending on Lower-Order Fractional Derivatives. *Axioms* **2020**, *9*, 44. [CrossRef]
19. Al-Refai, M.; Syam, M.I.; Baleanu, D. Analytical Treatments to Systems of Fractional Differential Equations with Modified Atangana–Baleanu Derivative. *Fractals* **2023**, *31*, 2340156. [CrossRef]

20. Lotfi, E.M.; Zine, H.; Torres, D.F.M.; Yousfi, N. The Power Fractional Calculus: First Definitions and Properties with Applications to Power Fractional Differential Equations. *Mathematics* **2022**, *10*, 3594. [CrossRef]
21. Shah, P.; Agashe, S. Review of Fractional PID Controller. *Mechatronics* **2016**, *38*, 29–41. [CrossRef]
22. Viera-Martin, E.; Gómez-Aguilar, J.F.; Solís-Pérez, J.E.; Hernández-Pérez, J.A.; Escobar-Jiménez, R.F. Artificial Neural Networks: A Practical Review of Applications Involving Fractional Calculus. *Eur. Phys. J. Spec. Top.* **2022**, *231*, 2059–2095. [CrossRef]
23. Agarwal, R.P.; Baleanu, D.; Nieto, J.J.; Torres, D.F.M.; Zhou, Y. A Survey on Fuzzy Fractional Differential and Optimal Control Nonlocal Evolution Equations. *J. Comput. Appl. Math.* **2018**, *339*, 3–29. [CrossRef]
24. Debbouche, N.; Ouannas, A.; Grassi, G.; Al-Hussein, A.-B.A.; Tahir, F.R.; Saad, K.M.; Jahanshahi, H.; Aly, A.A. Chaos in Cancer Tumor Growth Model with Commensurate and Incommensurate Fractional-Order Derivatives. *Comput. Math. Methods Med.* **2022**, *2022*, 1–13. [CrossRef]
25. Debbouche, N.; Ouannas, A.; Batiha, I.M.; Grassi, G. Chaotic Dynamics in a Novel COVID-19 Pandemic Model Described by Commensurate and Incommensurate Fractional-Order Derivatives. *Nonlinear Dyn.* **2022**, *109*, 33–45. [CrossRef]
26. Guilhon, I.; Admiral, T.D. Solução Numérica Da Equação de Laplace Através Do Método Da Relaxação. *Rev. Bras. Ensino Fís.* **2023**, *45*, e20220256. [CrossRef]
27. Leite, E.J.F.; Montenegro, M. Nontrivial Solutions to Affine p -Laplace Equations via a Perturbation Strategy. *Nonlinear Anal. Real World Appl.* **2024**, *80*, 104154. [CrossRef]
28. Upadhyay, P.; Upadhyay, S.K.; Shukla, K.K. Magnetic Resonance Images Denoising Using a Wavelet Solution to Laplace Equation Associated with a New Variational Model. *Appl. Math. Comput.* **2021**, *400*, 126083. [CrossRef]
29. Ramšak, M. On Numerical Solution Stability of the Laplace Equation in Absence of the Dirichlet Boundary Condition: Benchmark Problem Proposal. *Eng. Anal. Bound. Elem.* **2022**, *144*, 475–481. [CrossRef]
30. Kar, M.; Lin, Y.-H.; Zimmermann, P. Determining Coefficients for a Fractional P-Laplace Equation from Exterior Measurements. *J. Differ. Equ.* **2024**, *406*, 338–365. [CrossRef]
31. Duan, Y.; Wei, Y. Properties of Fractional P-Laplace Equations with Sign-Changing Potential. *Nonlinear Anal.* **2024**, *249*, 113628. [CrossRef]
32. Diethelm, K.; Ford, N.J.; Freed, A.D. A Predictor-Corrector Approach for the Numerical. *Nonlinear Dyn.* **2002**, *29*, 3–22. [CrossRef]
33. David, S.A.; Valentim, C.A.; Debbouche, A. Fractional Modeling Applied to the Dynamics of the Action Potential in Cardiac Tissue. *Fractal Fract.* **2022**, *6*, 149. [CrossRef]
34. Chi, X.; Yu, B.; Jiang, X. Parameter Estimation for the Time Fractional Heat Conduction Model Based on Experimental Heat Flux Data. *Appl. Math. Lett.* **2020**, *102*, 106094. [CrossRef]
35. Gomes, A.V. Transformadas Integrais, Modelagem Fracionária e o Sistema de Lotka-Volterra. Master's Thesis, Universidade do Estado de São Paulo, Botucatu, Brazil, 2014.
36. Wyld, H.W. *Mathematical Methods for Physics*, 2nd ed.; Powell, G., Ed.; CRC Press: Boca Raton, FL, USA, 2020; ISBN 978-1-003-03746-0.
37. Srivastava, S.; Kumar, V.; Gupta, V.K.; Kumar, R.; Rawat, V.K.; Nishad, A.K. Comparison of Fins in IC Engine Using CFD Analysis. In Proceedings of the 2nd International Conference on Smart Sustainable Materials and Technologies (ICSSMT 2023), Trichy, India, 30–31 August 2023; Sumesh, M., Tavares, J.M.R.S., Vettivel, S.C., Oliveira, M.O., Eds.; Advances in Science, Technology & Innovation. Springer Nature: Cham, Switzerland, 2024; pp. 19–30, ISBN 978-3-031-50023-7.
38. Liu, Y.; Li, L.; Zhang, Y. Numerical Simulation of Non-Fourier Heat Conduction in Fins by Lattice Boltzmann Method. *Appl. Therm. Eng.* **2020**, *166*, 114670. [CrossRef]
39. Pradhan, S.; Kumari, K.; Barua, A.; Panicker, R.V.; Singh, H.; Jeet, S. Numerical Simulation of Heat Transfer and Design Optimization of IC Engine Fins Geometry Using Finite Element Analysis. *Int. J. Interact. Des. Manuf.* **2024**, *18*, 479–491. [CrossRef]
40. Meresse, D.; Harmand, S.; Siroux, M.; Watremez, M.; Dubar, L. Experimental Disc Heat Flux Identification on a Reduced Scale Braking System Using the Inverse Heat Conduction Method. *Appl. Therm. Eng.* **2012**, *48*, 202–210. [CrossRef]
41. Carmona, S.; Rouizi, Y.; Quéméner, O.; Joly, F.; Neveu, A. Estimation of Heat Flux by Using Reduced Model and the Adjoint Method. Application to a Brake Disc Rotating. *Int. J. Therm. Sci.* **2018**, *131*, 94–104. [CrossRef]
42. Yevtushenko, A.A.; Kuciej, M.; Och, E. Temperature in Thermally Nonlinear Pad–Disk Brake System. *Int. Commun. Heat Mass Transf.* **2014**, *57*, 274–281. [CrossRef]
43. Ottoni, J.E. Introdução Ao Cálculo Fracionário Com Aplicações. *RMAT* **2018**, *5*, 50–77.
44. Nane, E. Fractional Cauchy Problems on Bounded Domains: Survey of Recent Results. In *Fractional Dynamics and Control*; Baleanu, D., Machado, J.A.T., Luo, A.C.J., Eds.; Springer: New York, NY, USA, 2012; pp. 185–198, ISBN 978-1-4614-0456-9.