CONTROLLED RETURNING POINTS FOR A LINEAR VOLTERRA-STIELTJES PROCESS

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1 - INTRODUCTION

We consider the property $x(T-) = x(0+)$ for solution of equation

$$(K) \qquad x(t) - x(0) + \int_0^t \cdot d_s K(t,s)x(s) = u(t) - u(0),$$

where $G_0^\sigma SV^u$ has a unique resolvent, $R \in G_I^\sigma SV^u$ — we obtain a necessary condition (see Thm. 1) and a sufficient one (see Thm. 2) — and the question: $x(0) = x(T)$ implies $u(0) = u(T)$?

2 - THE RESULTS

The conditions $x(T-) = x(0+)$ (i.e. $x(0)$ is a returning point) can be viewed as a linear constraint on the solution, $F_\alpha[x] = 0$,

$$\int_0^T \cdot d_s \alpha(s)x(s) = F_\alpha[x] = x(T-) - x(0+).$$

So, let us consider the system $(K) + (F_\alpha)$ where $F_\alpha[x] = c$, $c \in X$. Here $\alpha \in SV_0([0,T], L(X))$.

Let

$$J = \int_0^T \cdot d_\sigma \alpha(\sigma) \circ R(\sigma,0) \in L(X),$$

and let us assume that JX is closed in X (which is always the case in the finite dimensional context).

By Thm. 3.4 (9), [1], we have

$$u(t) = u(t) + R_0(t)(x_0 - u_0) - \int_0^t \cdot d_s R(t,s) u(s)$$

where $x_0 = x(0)$, $u_0 = u(0)$ and $R_0(t) = R(t,0)$.

So

$$\text{(A)} \qquad \int_0^t \cdot d_s R(t,s) = u(t) + R_0(t)(x_0 - u_0) - x(t).$$

We consider the equation $(K_\alpha^*)_0$ (see §2,[2]) which is

$$(K_\alpha^*)_0 \qquad y(s) + \int_s^t \cdot K(t,s)^* d_t y(t) - \int_0^T \cdot K(t,0)^* d_t y(t) = \alpha(s)^* \psi,$$

$\psi \in X'$, $0 \le s \le T$.

By Thm. 3.1 [2], $(K_\alpha^*)_0$ has a unique solution $y \in BV_0([0,T],X')$ given by

$$y(s) = \int_0^T \cdot R(\sigma,0)^* d_\sigma \alpha(\sigma)^* \psi + \int_s^T \cdot R(\sigma,s)^* d_\sigma \alpha(\sigma)^* \psi =$$

$$= -\psi \circ \int_0^T \cdot d_\sigma \alpha(\sigma) \circ R(\sigma,0) + \psi \circ \int_s^T \cdot d_\sigma \alpha(\sigma) \circ R(\sigma,s), \quad 0 \le s \le T.$$

<u>LEMMA</u> 1 - For a solution x of (K) one has:

$$\int_0^T \cdot \langle d_s y(s), u(s)\rangle = \langle \psi, F_\alpha[R_0(x_0-u_0)] - F_\alpha[x]\rangle.$$

Here $R_0(x_0-u_0)$ is the function $t \longrightarrow R(t,0)(x_0-u_0)$ and y is the solution of $(\,\dot{*}\,)_0$ with $\psi\in X'$.

PROOF -

$$\text{(B)}\quad \begin{aligned}\int_0^T \cdot \langle d_s y(s), u(s)\rangle &= \int_0^T \cdot \langle d_s \Big[\psi\circ \int_s^T \cdot d_\sigma \alpha(\sigma)\circ R(\sigma,s)\Big], u(s)\rangle = \\ &= \langle\, \psi, \int_0^T \cdot d_s \Big[\int_s^T \cdot d_\sigma \alpha(\sigma)\circ R(\sigma,s)\Big] u(s)\rangle.\end{aligned}$$

By Thm. 2.6 [1] we get (B) =

$$= \langle \psi, \int_0^T \cdot d_\sigma \alpha(\sigma)\Big[\int_0^\sigma \cdot d_s R(\sigma,s)u(s) - u(\sigma)\Big]\rangle$$

since $R(\sigma,\sigma) = I_X$.

Substituing $\int_0^\sigma \cdot d_s R(\sigma,s)u(s)$ by (A)

we get (B) =

$$= \langle \psi, \int_0^T \cdot d_\sigma \alpha(\sigma)[u(\sigma) + R_0(\sigma)(x_0-u_0) - x(\sigma) - u(\sigma)]\rangle =$$

$$= \langle \psi, F_\alpha[R_0(x_0-u_0)] - F_\alpha[x]\rangle. \qquad \square$$

Now taking into account that (y,ψ) is a solution of $(K_\alpha^*)_0$ iff $\psi\in \operatorname{Ker} J^*$ (see Prop. 7.1, [3]) we get

THEOREM 1 - If x is a solution of $(K) + (F_\alpha)$, then $\forall\psi\in \operatorname{Ker} J^*$ one has

$$\langle\psi, F_\alpha[R_0(u_0-u_0)]\rangle = 0.$$

<u>COROLLARY</u> 1 - If x is a solution of (K), with $x(T-) = x(0+)$ then $\forall\psi\in \mathrm{Ker}\ J^*$, one has:

$$\langle\psi, [R(T\dot{-},0) - R(0\dot{+},0)](x_0-u_0)\rangle = 0.$$

Next suppose that $u(T-) = u(0+)$.

Let us also assume that

$$\forall s\forall\tau\forall\sigma \quad y(s)K(\tau,\sigma)z = 0, \ \forall z\in\Gamma,$$

where $\Gamma\subset X$ is such that $u(t)\in\Gamma$, for all t.

Recall that

$$y(s) = \int_s^T\!\!\cdot(-K)(t,s)^*d_t y(t) - \int_0^T\!\!\cdot(-K)(t,0)^*d_t y(t) - \alpha(s)^*\psi, \ \ \psi\in X'.$$

We let α be such that $F_\alpha[x] = u(T-) - u(0+)$. Then

$$\int_0^T\!\!\cdot d_s y(s)u(s) = \int_0^T\!\!\cdot d_s\left[\int_s^T\!\!\cdot(-K)(t,s)^*d_t y(t) - \alpha(s)^*\psi\right]u(s).$$

Now

$$\int_0^T\!\!\cdot d_s\alpha(s)^*\psi u(s) = \langle\psi, \int_0^T\!\!\cdot d_s\alpha(s)u(s)\rangle =$$

$$= \psi(u(T-) - u(0+)) = 0$$

Hence

$$\int_0^T \bullet\langle d_s y(s), u(s)\rangle = \int_0^T \bullet d_s\left[\int_s^T \bullet d_t y(t)\circ(-K)(t,s)\right]u(s) =$$

$$= \int_0^T \bullet d_t y(t)\left[\int_0^t \bullet d_s (-K)(t,s)u(s)\right].$$

But our hypothesis imply that

$$\int_0^T \bullet d_t y(t)\left[\int_0^t \bullet d_s (-K)(t,s)u(s)\right] = 0.$$

Since this is for any $\psi \in X'$, we get that: for all (y,ψ) solution of $(K_\alpha^*)_0$ in $BV_0^{0,T}([0,T],X') \times X'$,

$$\int_0^T \bullet\langle d_s y(s), u(s)\rangle = 0.$$

We proved the

THEOREM 2 - If $u(T-) = u(0+)$ and

$\forall s$ $\quad\quad\quad y(s)\Gamma = 0$

or

$$\forall s \forall \tau \forall \sigma \quad y(s)K(\tau,\sigma)\Gamma = 0$$

then $x(T-) = x(0+)$ for any solution of (K).

THEOREM 3 - Let be (K) with:

1) u forcing a solution x with $u(0) = x(0) = x(T)$.

2) $\forall (t,s)$ $K(t,s)X \subset Z$, a closed subspace of X.

3) there exists an $c>0$ such that $\forall\gamma,\gamma'\in\Gamma$, $\exists\eta\in Z^0$, $|\eta|=1$ with $c\|\gamma-\gamma'\| \le \langle\gamma-\gamma',\eta\rangle$.

Then: $\mu(0)=\mu(T)$.

PROOF - By 2) and 3)

$$c\|u(t)-u(0)\| \le \|x(t)-x(0)\|$$

(see Prop. 3, [4]).

In this direction we can state also:

THEOREM 4 - Let be (K) with $x(0)=u(0)$ in the G^- context, and suppose:

1) $R(T\dot{-},T)=R(T\dot{-},0)$ (see Def. 1.6, [1]).

2) $\forall x\in X$, $R(T,s)x\in G^-([0,T],X)$ for $0\le s\le T$.

Then

$$x(0) = x(T) = 0 \text{ implies } u(0) = u(T) = 0.$$

PROOF - Let be $F_\alpha[x]=x(T)=0$, $(\alpha(s)=\chi_{\{T\}}(s))$.

Then $\forall\psi\in X'$, $(y(s),\psi)$ is solution of $(K_\alpha^*)_0$, where:

$$y(s) = \psi\circ[R(T\dot{-},s)-R(T\dot{-},0)].$$

In fact: by (4) of [2]

$$y(s) = \psi\circ\left[-\int_0^T \cdot d_\sigma\alpha(\sigma)R(\sigma,0)+\int_s^T \cdot d_\sigma\alpha(\sigma)R(\sigma,s)\right] =$$

$$= \psi\circ[R(T\dot{-},s)-R(T\dot{-},0)].$$

By Corol. 2.3 of [2] if u gives the solution $u(0) = x(0) = x(T) = 0$ we have

$$\int_0^T \cdot d_s \langle y(s), u(s)\rangle = \langle \psi, \int_0^T d_\sigma [R(T^{\div},\sigma) - R(T^{\div},0)] u(\sigma)\rangle = 0.$$

By hypothesis this implies

$$\langle \psi, \int_0^T \cdot d_\sigma R(T,\sigma) u(\sigma)\rangle = 0, \ \forall \psi \in X'.$$

Hence

$$x(T) = u(T) + \int_0^T \cdot d_s R(T,s) u(s) = 0.$$

Then

$$\mu(T) = 0.$$

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