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The 3-dimensional Poincaré conjecture

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Abstract

We show that every homotopy 3-sphere has a simply connected, noncollapsible pseudo-spine. This result conducts us to search for counterexamples to the Poincaré conjecture. By a slight modification of the standard spine of the Poincaré sphere, we obtain a 2-simplicial complex which is a pseudo-spine of a homotopy 3-sphere nonhomeomorphic to S^3. A second example is constructed by taking a closed, orientable surface of genus 3 and attaching 2-cells to this surface.

Introduction

At the end of the second complement to his Analysis Situs, H. Poincaré asserts that a 3-dimensional manifold with trivial homology is simply connected and therefore homeomorphic to the standard 3-sphere, S^3. Precisely, he announces in [P2], p.308,

> *"Tout polyèdre qui a tous ses nombres de Betti égaux a 1 et tous ses tableaux Tq bilatères est simplement connexe, c'est-á-dire homéomorphe á l'hypersphère".*

The sentence "le polyèdre a tous ses tableaux Tq bilatères" means that the torsion coefficients of the polyhedron are all nulls. Thereafter he gave an example of a non simply connected homology 3-sphere, called the sphere of Poincaré,

> *"On pourrait alors se demander si la considération de ses coefficients suffit; si une variété dont tous les nombres de Betti et les coefficients de torsion sont égaux a 1 est pour cela simplement connexe au sens propre du mot, c'est-á-dire homéomorphe á l'hypersphère; ou si, au contraire, il est nécessaire, avant d'affirmer*

qu'une variété est simplement connexe, d'étudier son groupe fondamentale, tel que je l'ai défini dans le Journal de l'École Polytechnique, §12, page 60.

Nous pouvons maintenant repondre a cette question; j'ai formé en effect un example d'une variété dont tous les nombres de Betti et les coefficients de torsion sont égaux a 1, et pourtant n'est pas simplement connexe" ([P3], p.46).

From these two assertions, we can state that Poincaré considered S^3 as the only one closed, connected and simply connected 3-manifold. Perhaps, this could explain why Poincaré has not settled completly this problem. However, since that time, many efforts have been made to solve his celebrated conjecture. We remark that one of the first attempts to prove it was made by J.H.C.Whitehead in [W].

We should like to mention yet two facts:

a) After Bing [B1], a closed, connected 3-manifold M^3 is homeomorphic to S^3, if and only if each simple closed curve in M^3 lies in a 3-cell in M^3. We remark that this caracterization of S^3 is stronger than the simply connectedness and puts doubt in the Poincaré assertion. In fact, we shall exhibit a homotopy 3-sphere and a closed curve in it, which does not lie in a 3-cell, and

b) In [F], Fox suggests that the method of irregular branched coverings of S^3 ought provide counterexamples to the Poincaré conjecture. Coincidently, we use this method to show that any homotopy 3-sphere has a noncollapsible, simply connected pseudo-spine. This result was a guide to the formulation of our fundamental theorem.

Our purpose in this paper is to construct examples of 3-manifolds homotopy equivalent to S^3, called homotopy 3-spheres, which are nonhomeomorphic to S^3.

Our approach is based on the theory of simplicial complexes and our basic idea is to construct a finite 2-dimensional simplicial complex satisfying the following conditions:

i) It is connected and simply connected;

ii) It is piecewise linear embedded in a 3-dimensional manifold, and

iii) It can not be piecewise linear embedded in the 3-dimensional euclidean space E^3.

In theorem 1.2, we shall show that the existence of a simplicial complex satisfying the above-mentioned conditions implies the existence of a homotopy 3-sphere nonhomeomorphic to S^3.

It is remarkable that the first one of our examples is obtained by a simple modification of the standard spine of the Poincaré sphere [P3],[ST].

§1 The fundamental theorem

Next, we shall assume the basic definitions concerning the theory of complexes, which can be found in the references [Hu], [L], [RS], [ST].

However, to make our following text clearer, we remember some results and definitions which are directly related to this work.

After Moise [Mo] and Bing [B2], every topological 3-manifold M^3 can be triangulated. This means that there is a pair $(\mathcal{T}, h)$, where $\mathcal{T}$ is a 3-dimensional simplicial complex and $h:|\mathcal{T}|\rightarrow M^3$ is a homeomorphism. The underlying set $|\mathcal{T}|$ is a polyhedron embedded into a n-dimensional euclidean space E^n.

This result allows us to define in M^3 à piecewise linear structure and piecewise linear maps between sets with piecewise linear structures and topological 3-manifolds. We shall denote piecewise linear simply by p.l. Thus, if $\mathcal{K}$ is a 2-dimensional simplicial complex and M^3 is a 3-manifold, we say that $|\mathcal{K}|$ is p.l. embedded in M^3, if there is a triangulation $(\mathcal{T}, h)$ of M^3 and a map $f:|\mathcal{K}|\rightarrow M^3$ such that $h^{-1} \circ f$ is a p.l. embedding, that is, $h^{-1} \circ f(|\mathcal{K}|)$ is a subpolyhedron of $|\mathcal{T}|$ ([RS], p.7).

Let $\mathcal{K}$ be a simplicial complex, $\mathcal{K}_1$ a subcomplex of $\mathcal{K}$, σ a simplex and τ a face of σ, with $\dim \sigma = \dim \tau + 1$.

We say $\mathcal{K}_1$ is obtained from $\mathcal{K}$ by an elementary collapsing if $\mathcal{K}_1 = \mathcal{K} - \{\sigma, \tau\}$. If $\mathcal{L}$ is a subcomplex of $\mathcal{K}$ obtained from $\mathcal{K}$ by a finite sequence of elementary collapsings, we say that $\mathcal{K}$ collapses to $\mathcal{L}$. If $\mathcal{L}$ is the zero-dimensional complex with only one vertex, we say that $\mathcal{K}$ is collapsible.

Let P be a polyhedron p.l. embedded in a p.l. n-manifold M^n. A subset N of M^n is called a regular neighborhood of P in M^n if N is a closed neighborhood of P in M^n, N is an n-manifold (with boundary) and N collapses to P.

The following result about existence and unicity of regular neighborhoods is sufficient to our needs.

Let P be a compact polyhedron p.l. embedded in a 3-manifold M^3. Then there is a regular neighborhood N of P in M^3. Moreover if N_1 and N_2 are two regular neighborhoods of P in M^3, there is a p.l. homeomorphism $f: N_1 \to N_2$ which is the identity on P ([Hu]).

In the literature, the word "spine" means, for a closed 3-manifold M^3, a 2-dimensional subcomplex $\mathcal{K}$, of M^3 (for an allowable triangulation of M^3) such that M^3 - $|\mathcal{K}|$ is a 3-cell (see [Ma]). If M^3 is a compact 3-manifold with nonvoid boundary, $\mathcal{K}$ is a spine of M^3 if M^3 collapses to $|\mathcal{K}|$ and there is no elementary collapse of $\mathcal{K}$ (see [Ca],[Z]). Let M^3 be a closed 3-manifold and $\mathcal{K}$ a subcomplex of M^3. We adopt the terminology of [Po], and we say that $\mathcal{K}$ is a *pseudo-spine* of M^3, if M^3 - Int(N) is a disjoint union of 3-balls, where N is a regular neighborhood of $| \mathcal{K} |$ and Int(N) is the interior of N.

Let P_1 and P_2 be two polygons in E^3. We say that the link (P_1, P_2) is *unlinked* if there is a ball B in E^3 such that P_1 lies interior to B and P_2 lies interior to $E^3 - B$. If such a B does not exist, we say that (P_1, P_2) is *linked* (see [RS]). Some authors use the terms *splittable, unsplittable* in this case (see [R]). Certainly, if P_1 is the boundary of a disc D, p.l. embedded in E^3 and such that P_2 does not intercept D, then (P_1, P_2) is unlinked. An example of a link, which is linked, is (boundary of M, central path of M) where M is a Möebius band, p.l. embedded in E^3. This example will be very useful in this work.

After some attempts to prove (not to disprove) the Poincaré conjecture, we have learned about two important results in 3-dimensional topology. The first one due to Hilden-Montesinos and the second one due to Haken. These two results together allow us to state the following result, which is a kind of noncollapsible pseudo-spine theorem, like the Poénaru theorem in [Po]:

Proposition 1.1

Let Σ^3 be a homotopy 3-sphere. Then Σ^3 has a noncollapsible, simply connected pseudo-spine.

Proof:

Let M^3 be a closed, orientable 3-manifold, then M^3 is a 3-fold irregular covering space of S^3, branched over a knot Γ, so that the inverse image of a point of Γ consists of a point of branch-index 2 and a point of branch-index 1 (see [Hi],[Mon]). The knot Γ is the boundary of a singular disc D in S^3, whose singularities are one branch point P and double arcs originating from P, being pairwise disjoint and terminating in Γ (see [Ha]). Let $p: M^3 \to S^3$ be the 3-fold irregular map of Hilden-Montesinos and let N be a regular neighborhood of D such that D is contained in the interior of N. We denote $f^{-1}(D)$ by $\hat{D}$ and $f^{-1}(N)$ by $\hat{N}$. Then $\hat{N}$ is a regular neighborhood of $\hat{D}$ and $\partial \hat{N}$ is a disjoint union of three 2-dimensional spheres. M^3 is obtained from $\hat{N}$ by capping off each one of these 2-spheres with a 3-cell. Now,if M^3 is the homotopy 3-sphere Σ^3, it follows by Seifert-Van Kampen theorem that $\hat{N}$ is simply connected and therefore $\hat{D}$ is also simply connected. Because of the branch-index 2 points, $\hat{D}$ is noncollapsible. Thus, $\hat{D}$ is a simply-connected, noncollapsible pseudo-spine of Σ^3.

We remark that in this situation $\hat{D}$ is the underlying set of a 2-simplicial complex, which satisfies the i) and ii) conditions given in the introduction. One attempt to prove the Poincaré conjecture is to show that $\hat{D}$ can be piecewise linear embedded in E^3. On the contrary, one is conducted to find counterexamples.

Now, we may present our fundamental theorem.

Theorem 1.2

Let $\mathcal{K}$ be a finite 2-dimensional simplicial complex such that:

i) $|\mathcal{K}|$ is connected and simply connected;

ii) $|\mathcal{K}|$ is p.l. embedded in a 3-dimensional manifold, and

iii) $|\mathcal{K}|$ can not be p.l. embedded in E^3.

Then $|\mathcal{K}|$ is the pseudo-spine of a homotopy 3-sphere M^3, which is nonhomeomorphic to S^3.

Proof: Let Q^3 be a 3-manifold, in which $|\mathcal{K}|$ is piecewise linear embedded. This means that there exists a triangulation $\mathcal{L}$ of Q^3 such that $\mathcal{K}$ is a subcomplex of $\mathcal{L}$. Let N be a regular neighborhood of $|\mathcal{K}|$ in Q^3. Then N is a 3-dimensional manifold with boundary. Since N is collapsible to $|\mathcal{K}|$ and $|\mathcal{K}|$ is compact, connected and simply connected, we have also that N is compact, connected and simply connected. For the last assertion, we remind that $|\mathcal{K}|$ is a strong deformation retract of N.

Therefore ∂N is a finite disjoint union of 2-dimensional spheres. We obtain M^3 by capping off each one of the 2-spheres in ∂N with a 3-ball. Then $|\mathcal{K}|$ is a pseudo-spine of M^3 and by an easy application of the Seifert-van Kampen theorem, we obtain that M^3 is a homotopy 3-sphere.

The property (iii) of $\mathcal{K}$ asserts us that M^3 is nonhomeomorphic to S^3.

§2 The Poincaré Sphere. A slight modification of its standard spine

Any 3-dimensional manifold can be obtained as the quotient space of a closed 3-cell, whose boundary is a polyhedron with an even number of faces, by a suitable identification of its faces.

If the closed 3-cell is the regular dodecahedron, in which the 12 faces are arranged in pairs in such a way that the components of each pair are parallel and on diametrically opposite sides of the dodecahedron and one component of each pair is identified with the opposite by rotating the first component one-tenth of a turn counterclockwise about the axis perpendicular to its surface (see this description in [TW]), the quotient space P^3 is, as we shall see, a homology 3-sphere, which is called the Poincaré sphere.

After the identification, the boundary of the regular dodecahedron is transformed in a 2-dimensional cell complex $\mathcal{L}$, with five vertices, ten edges and six 2-dimensional cells. Each vertex is a face of each one of the six 2-cells and each edge is a face of exactly three 2-cells. This 2-dimensional cell complex is a standard complex, [C].

The figure 1 gives an idea of the 2-dimensional complex $\mathcal{L}$. You have only to perform the identifications, which are indicated in this figure.

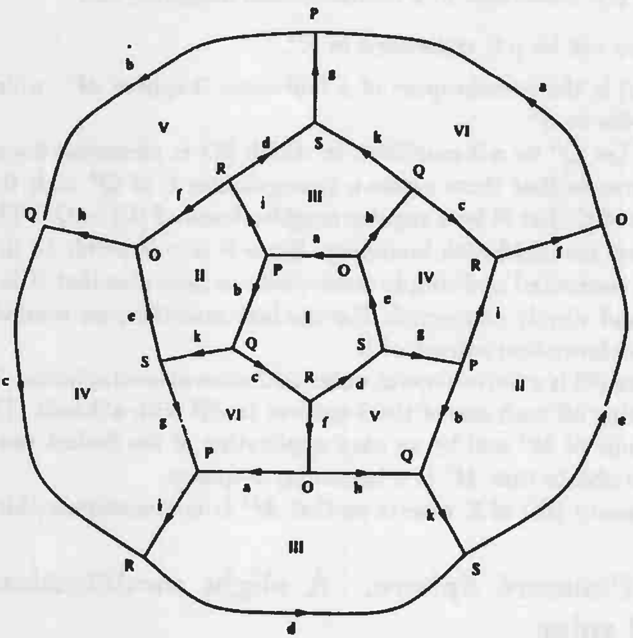

Figure 1: This is the same figure of [ST].

We remark that $P^3 - |\mathcal{L}|$ is homeomorphic to an open 3-cell and $\mathcal{L}$ is a spine of P^3. Then $\mathcal{L}$ carries all the information about the algebraic invariants of P^3, such as the Euler characteristic, the homology and homotopy groups of P^3. Next, to calculate these algebraic invariants, we shall use the same notation and we shall repeat the arguments of Seifert-Threfall book [ST]. The following result will be useful:

" For any point $0 \in |\mathcal{L}|$, $\pi_1(P^3,O)$ is isomorphic to $\pi_1(\mathcal{L},O)$([ST], p.214)."

We denote by O, P, Q, R, S the five vertices of $\mathcal{L}$, by a = OP, b = PQ, c = QR, d = RS, e = SO, f = RO, g = SP, h = OQ, i = PR and k = QS the ten edges of $\mathcal{L}$, and by I, II, III, IV, V, VI the six 2-cells of $\mathcal{L}$, where $\partial I = a.b.c.d.e$, $\partial II = f^{-1}.i^{-1}.b.k.e$, $\partial III = a.i.d.k^{-1}.h^{-1}$, $\partial IV = h.c.i^{-1}.g^{-1}.e$, $\partial V = f^{-1}.d.g.b.h^{-1}$, $\partial VI = ag^{-1}.k^{-1}.c.f$.

Taking O as base point and a, h, f^{-1}, $f^{-1}d$ as auxiliary paths, we obtain that $\pi_1(P^3,0)$ is generated by the homotopy classes of the following closed paths: $A = a.a^{-1}$, $B = a.b.h^{-1}$, $C = h.c.f$, $D = f^{-1}.d.(d^{-1}.f)$, $E = (f^{-1}.d).e$, $F = f^{-1}.f$, $G = (f^{-1}.d).g.a^{-1}$, $H = h.h^{-1}$, $J = a.i.f$ and $K = h.k.(d^{-1}.f)$.

Obviously the homotopy classe of each one of the paths A, D, F, H is the identity. Moreover the 2-cells I, II, III, IV, V, VI give us the following relations: $A.B.C.D.E \simeq 1$, $B.K.E.F^{-1}.J^{-1} \simeq 1$, $A.J.D.K^{-1}.H^{-1} \simeq 1$, $C.J^{-1}.G^{-1}.E.H \simeq 1$, $B.H^{-1}.F^{-1}.D.G \simeq 1$, $A.G^{-1}.K^{-1}.C.F \simeq 1$, where 1 is the constant path at O and the sign $\simeq$ means the paths are homotopic.

By an easy calculation we obtain the following values for the Betti numbers of P^3:

$p^0 = 1$, $p^1 = 0$, $p^2 = 0$, $p^3 = 1$. Also the torsion coefficients are all nulls.

The fundamental group $\pi_1(P^3,0)$ is a group of order 120 and it is generated by the homotopy classes of the paths $B = a.b.h^{-1}$ and $C = h.c.f$.

Now, let $\mathcal{L}'$ be $\mathcal{L}$ minus the 2-cell I.

Let D be a closed disc and $\mathcal{K}$ the cell complex such that $|\mathcal{K}| = |\mathcal{L}'| \cup D/\sim$, where we identify ∂D with the closed path C.

Lemma 2.1.: The cell complex $\mathcal{K}$ is connected and simply connected.

Proof: It is immediate that $\mathcal{K}$ is connected. Regarding the simply connectedness, we remind that, as in the case of $\mathcal{L}$, we have that $\pi_1(\mathcal{K},O)$ is generated by the homotopy classes of the closed paths A, B, C, D, E, F, G, H, J, K and A.B.C.D.E. Obviously $A \simeq D \simeq F \simeq H \simeq 1$ and moreover the 2-cells D, II, III, IV, V, VI give us the following relations:

$$\begin{array}{lcl} C \simeq 1 & & C \simeq 1 \\ B.K.E.F^{-1}.J^{-1} \simeq 1 & & B.K.E.J^{-1} \simeq 1 \\ A.J.D.K^{-1}.H^{-1} \simeq 1 & & J.K^{-1} \simeq 1 \\ C.J^{-1}.G^{-1}.E.H \simeq 1 & \text{or} & J^{-1}.G^{-1}.E \simeq 1 \\ B.H^{-1}.F^{-1}.D.G \simeq 1 & & B.G \simeq 1 \\ A.G^{-1}.K^{-1}.C.F \simeq 1 & & G^{-1}.K^{-1} \simeq 1 \end{array}$$

Thus, using $B.G \simeq 1$ and $G^{-1}.K^{-1} \simeq 1$, we obtain $G^{-1} \simeq K \simeq B$. It follows from $J.K^{-1} \simeq 1$, that $J \simeq K$ and from $J^{-1}.G^{-1}.E \simeq 1$ and $G^{-1} \simeq K \simeq J$ that $E \simeq 1$.

The relation $B.K.E.J^{-1} \simeq 1$ is equivalent to $K.K.E.K^{-1} \simeq 1$ and so $K \simeq 1$.

Thus, $B \simeq E \simeq G \simeq J \simeq K \simeq 1$ and, of course, $A.B.C.D.E \simeq 1$. Therefore $\mathcal{K}$ is simply connected.

For a 2-dimensional simplicial complex $\mathcal{K}$, it is possible to determine whether it can be embedded in some 3-dimensional manifold (see remark 1 of Kranjc in [K], p. 310). Let $\mathcal{K}^{(1)}$ be its intrinsic 1-skeleton (i.e. the subcomplex of $\mathcal{K}$, which corresponds to the nonmanifold points of $|\mathcal{K}|$) and let U be a regular neighborhood of $|\mathcal{K}^{(1)}|$ in $|\mathcal{K}|$. We shall suppose, in particular, that there exists a piecewise linear embedding of U in E^3 and the closure of $|\mathcal{K}| - U$ in $|\mathcal{K}|$ is a disjoint union of n closed discs $D_1,...,D_n$.

Let V be a regular neighborhood of U in E^3 such that $Fr(U) \subset \partial V$.

It is easily seen that V is an orientable handlebody and $|\mathcal{K}^{(1)}|$ is a spine of V.

$|\mathcal{K}|$ is obtained from U by attaching the n discs $D_1,...D_n$ to Fr(U) along $\partial D_1 \cup ... \cup \partial D_n$ and Fr(U) is a disjoint union of n simple closed paths $\beta_1, ..., \beta_n$ in ∂V. We choose n sheets $F_1,...,F_n$, with $F_i \subset \partial V$, $F_i \cap F_j = \emptyset$, if $i \neq j$ and also each F_i a neighborhood of β_i, homeomorphic to $\beta_i \times [-1,1]$ for i = 1, 2, ..., n.

We define $W = V \cup (\cup_{i=1}^{n} B_i)/\sim$, where $B_i = D_i \times [-1,1]$ and $\sim$ is the identification of F_i with $\partial D_i \times [-1,1]$, such that β_i is identified with $\partial D_i \times 0$.

Therefore, with these hipothesis, we obtain an embedding of $|\mathcal{K}|$ in W, which is a 3-manifold with boundary.

We come back now to our simply connected 2-dimensional cell complex $\mathcal{K}$.

Taking a normal subdivision of $\mathcal{K}$, we can suppose $\mathcal{K}$ is a 2-dimensional simplicial complex.

A regular neigborhood U of $|\mathcal{K}^{(1)}|$ can be easily embedded in E^3. In practice we construct such a regular neigborhood U embedded in E^3 with sheets of paper. Firstly we construct the neighborhoods of each one of the five vertices O, P, Q, R, S as it is shown in the following figures:

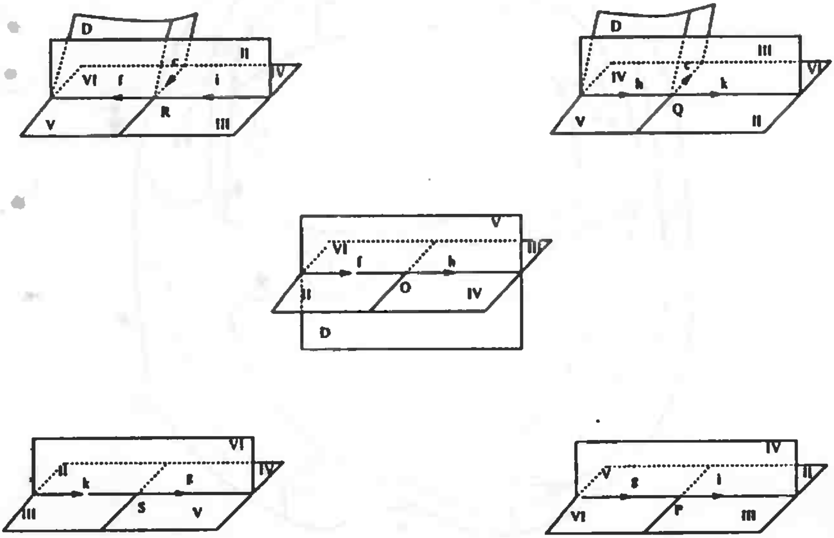

Figure 2: Pieces of a regular neighborhood of $|\mathcal{K}^{(1)}|$ in $|\mathcal{K}|$.

Then, we glue these neighborhoods, as it is indicated in figure 2, to obtain the regular neighborhood U piecewise linear embedded in E^3. At the end, the regular neighborhood U we have constructed is such that Fr(U) is a disjoint union of two closed paths and at least one of these paths, precisely the one which is not contained in D, is linked with $|\mathcal{K}^{(1)}|$. We denote this path by β_1 and the other by β_2. To show that our last statement is true, we remark that the embedding of U into E^3, which is showed in the figure 3 , is certainly the simpler of all the embeddings of U in E^3 and in this embedding the path β_1 is linked with $|\mathcal{K}^{(1)}|$. So, it is not possible to attach discs to this neigborhood U to obtain a p.l. embedding of $|\mathcal{K}|$ in E^3. For a definitive argument to show that $|\mathcal{K}|$ can not be p.l. embedded in E^3, see the remark 1) at the end of this paragraph. Now, let V be a regular neigborhood of U p.l. embedded in E^3 and such that Fr(U) $\subset \partial$V. Then V is an orientable handlebody of genus 2. $|\mathcal{K}|$ is obtained from U by attaching discs D_1 and D_2 to Fr(U). We proceed as in general case, described before, to obtain $|\mathcal{K}|$ p.l. embedded in a 3-manifold.

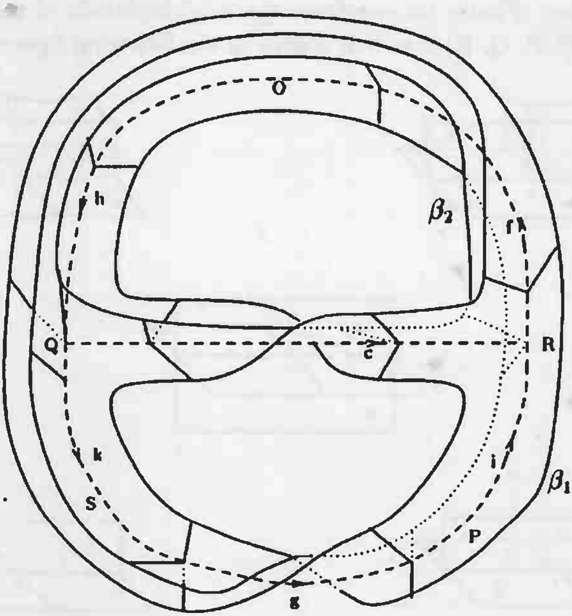

Figure 3: A regular neighborhood of $|\mathcal{K}^{(1)}|$ in $|\mathcal{K}|$.

Then we have established the two following lemmas:

Lemma 2.2.: $|\mathcal{K}|$ can be p.l. embedded in a 3-manifold.

Lemma 2.3.: $|\mathcal{K}|$ can not be p.l. embedded in E^3.

Thus, $\mathcal{K}$ is a 2-simplicial complex which verify the theorem 1.2 conditions.

Since the Euler characteristic of $\mathcal{K}$ is $\chi(\mathcal{K}) = V - E + F = 5 - 10 + 6 = 1$, we have also $\chi(N) = 1$, where N is the regular neighborhood of $|\mathcal{K}|$ as given in theorem 1.2. But $\chi(\partial N) = 2\chi(N)$. Therefore, ∂N is homeomorphic to S^2 and the 3-manifold M^3 is obtained by capping off this 2-dimensional sphere with a 3-cell.

So, we have our principal result.

Theorem 2.1: There is a homotopy 3-sphere nonhomeomorphic to S^3

Remarks: 1) One crucial point in this contruction is the lemma 2.3. So, we should like to emphasize this point with the following observation: $\mathcal{L}'$ is a subcomplex of $\mathcal{K}$ and the polyhedron $|\mathcal{L}'|$ is obtained from a Möebius band M by a suitable identification of its boundary, as it is indicated in figure 4. The polygon which forms the boundary of M is linked with the central curve of M. We identify, initially, the three points of M, marked with P and we obtain a wedge of three closed curves, which must be identified to obtain $|\mathcal{L}'|$. Two of these closed curves are unlinked with the central curve of M and the third one is linked with this central curve. Then it is not possible to perform these identifications to obtain an embedding of $|\mathcal{L}'|$ in E^3. It is not a difficult task to see that the path i.f.h.k.g, which is part of $|\mathcal{K}^{(1)}|$, does not lie in a 3-cell in M^3. Then we use the Bing theorem [B1], to conclude that M^3 is nonhomeomorphic to S^3.

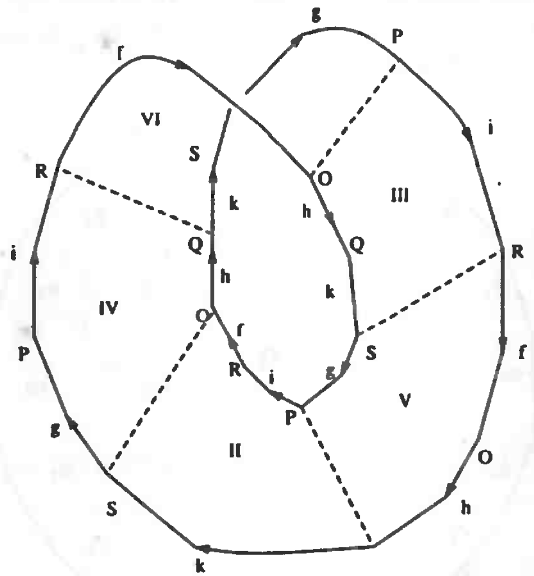

Figure 4: A Möebius band.

2) $\mathcal{K}$ is minimal in the following sense; if $\mathcal{K}_1$ is a subcomplex of a subdivision of $\mathcal{K}$ such that $|\mathcal{K}_1|$ is connected, simply connected and cannot be embedded in E^3,then $|\mathcal{K}_1| = |\mathcal{K}|$. Thus M^3 is a prime 3-manifold and we might construct an infinite number of nonhomeomorphic homotopy 3-spheres by induction, using M^3 and connected sums. Precisely,we define the sequence of nonhomeomorphic homotopy 3-spheres, by $M_1 = M$ and $M_{n+1} = M_n \# M^3$,

for $n = 1, 2, \ldots$ (for the definition of a prime 3-manifold and connected sums, see [He]).

§3 A different construction

Our purpose, in what follows, is to show that we can use our method to construct new counterexamples to the 3-dimensional Poincaré conjecture. It consists simply in taking a compact, connected and orientable surface of genus g, choosing suitable closed paths in this surface and attaching cells to this surface over the paths we have choosed to obtain a 2-simplicial complex satisfying the theorem 1.2 conditions.

For our second example, we consider a plane convex region Ω bounded by a 18-sided polygon, from which we shall obtain, by a suitable identification of its sides, a torus of genus 3.

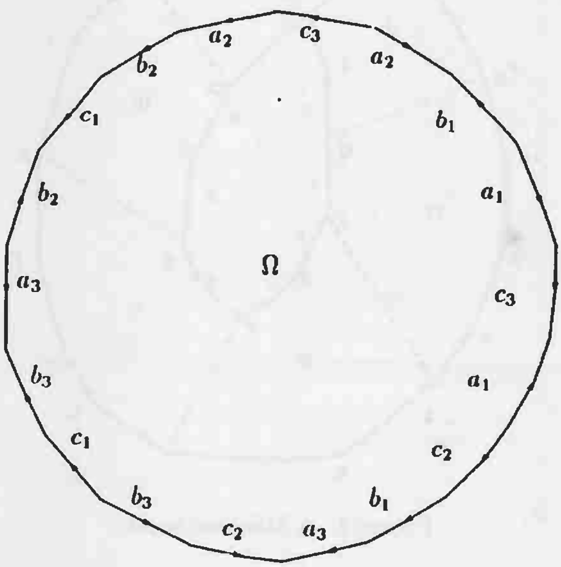

Figure 5: A 18-sided polygon.

Using the same terminology of " normal forms", which is useful in the construction of closed 2-dimensional manifolds, we shall note T_3 by the symbol:

$a_1c_3a_1^{-1}c_2b_1a_3c_2^{-1}b_3^{-1}\ c_1b_3a_3^{-1}b_2c_1^{-1}b_2^{-1}a_2^{-1}\ c_3^{-1}a_2b_1^{-1}$

After the identification the torus of genus 3, T_3, will be the underlying set of a 2-dimensional cell complex with only one 2-cell, nine edges and four vertices. The 1-skeleton of this complex carries all the information about the fundamental group of T_3. We remark that we are considering,in this case, cells complexes with open cells. The identification, in pairs, of the sides of this 18-sided polygon, defines in T_3, the closed paths:

$\gamma_i = \bar{c}_i$ for i = 1, 2, 3 ; $\alpha = \bar{a}_1.\bar{a}_2.\bar{a}_3$; $\beta = \bar{b}_1.\bar{b}_2.\bar{b}_3$

$\alpha_1 = \bar{a}_1.\bar{a}_2.(\bar{b}_1)^{-1}$; $\alpha_2 = \bar{b}_1.\bar{a}_3$ and $\alpha_3 = \bar{a}_3^{-1}.\bar{b}_2.\bar{b}_3$ where, for a side a, $\bar{a}$ is the set of points in T_3, image of the two edges a and a^{-1}, after the identification.

The vertices of this cell-decomposition of T_3 are $V_1 = \gamma_1 \cap \beta$, $V_2 = \gamma_3 \cap \alpha$ and $\{V_3, V_4\} = \alpha \cup \beta$. See the figure 6.

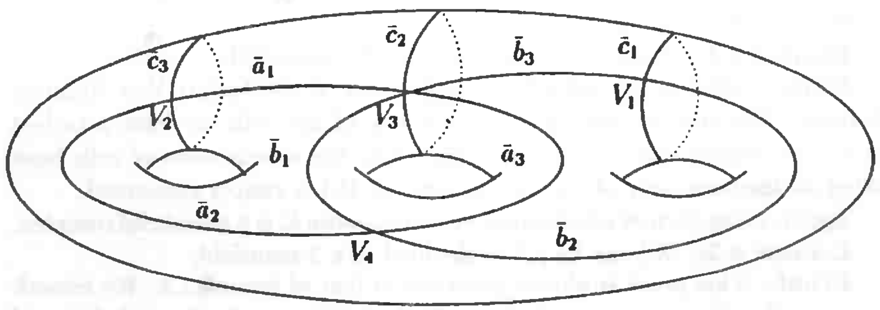

Figure 6: A genus 3 torus.

We need yet another definition: Let D be a closed 2-dimensional disc and let p and q be two distinct points in ∂D. We note by $\tilde{D}$ the quotient space obtained by the identification of p with q, which we will call a folded disc.

Take a simple path in ∂D, which starts at p, goes to q and continues just to end at p. The image of this path in $\tilde{D}$ gives us a well defined path into $\partial\tilde{D}$. We shall denote simply these paths by ∂D and $\partial\tilde{D}$.

We consider now, besides the torus T_3, three 2-dimensional closed discs D_1, D_2, D_3, one folded disc $\tilde{D}$ and two cylinders C_1 and C_2.

On each cylinder C_i we take points $p_i \in (\partial C_i)_1$ and $q_i \in (\partial C_i)_2$ and simple paths δ_i into C_i with ends p_i and q_i, where $(\partial C_i)_1$ and $(\partial C_i)_2$ are the two connected components of ∂C_i, for i = 1, 2.

Let $|\mathcal{K}|$ be the adjunction space, defined by:
$|\mathcal{K}| = T_3 \cup D_1 \cup D_2 \cup D_3 \cup \tilde{D} \cup C_1 \cup C_2/\sim$
where we make the identifications:
$\partial D_1 \sim \alpha$, $\partial D_2 \sim \beta$, $\partial D_3 \sim \alpha_1$, $\partial \tilde{D} \sim \gamma_1 \cdot \alpha_3$
At this point, it is necessary to say some words about the later identification:

We remind that $\partial \tilde{D}$ is a wedge of two simple closed paths joined at $\tilde{p}$; $\gamma_1 \cdot \alpha_3$ is also a wedge of two simple closed paths joined at V_1. We identify then $\tilde{p}$ with V_1; the first one of the simple paths forming $\partial \tilde{D}$ is identified with α_3 and the other with γ_1.

Finally, we make the following identifications: $(\partial C_1)_1 \sim \gamma_1$, $(\partial C_1)_2 \sim \gamma_2$, $(\partial C_2)_1 \sim \gamma_2$, $(\partial C_2)_2 \sim \gamma_3$ $\delta_1 \sim \bar{b}_3$ and $\delta_2 \sim \bar{a}_1$.

$|\mathcal{K}|$ is the underlying set of a cell complex $\mathcal{K}$ once, $\mathcal{K}$ is obtained from T_3 by attaching a few cells.

Lemma 3.1.: $|\mathcal{K}|$ is connected and simply connected.

Proof.: This is an immediate application of the Seifert-Van Kampen theorem. We observe only that to each one of the cells we have attached to T_3 the identification set is connected. Also the attachments of cells have killed all the homotopy of T_3 and the new set $|\mathcal{K}|$ is simply connected.

Again, using normal subdivision we can suppose $\mathcal{K}$ is a simplicial complex.

Lemma 3.2.: $|\mathcal{K}|$ can be p.l. embedded in a 3-manifold.

Proof.: This proof is almost identical to that of lemma 2.2. We remark however, that the intrinsic 1-skelleton $\mathcal{K}^{(1)}$ of $\mathcal{K}$ is exactly the 1-skeleton of the initial cell decomposition of $\mathcal{K}$ and if U is a regular neighborhood of $|\mathcal{K}^{(1)}|$ piecewise linear embedded in E^3 and V is a regular neighborhood of U in E^3 such that $Fr(U) \subset \partial V$, then V is an orientable handlebody of genus 6, Fr(U) is the disjoint union of seven closed paths in ∂V and each one is the boundary of a disc in $|\mathcal{K}|$. We proceed as in lemma 2.2 to conclude the proof.

Lemma 3.3.: $|\mathcal{K}|$ can not be p.l. embedded in E^3.

Proof.: We suppose on the contrary that there is a p.l. embedding of $|\mathcal{K}|$ into E^3 and we note $A = Int(T_3)$ and $B = Ext(T_3)$. Without loss of generality we can assume $D_2 \subset B$. With this hipothesis, it is an easy consequence of our construction that $D_1 \subset A$, $C_2 \subset A$, $\tilde{D} \subset A$, $C_1 \subset B$ and $D_3 \subset B$.

We will show now that either α_2 is linked with γ_1 or α_2 is linked with γ_3.

We remind that the two cylinders are glued at α_2 and is defined a new cylinder $C_1 \cup C_2/\sim = C_3$ and $\partial C_3 = \gamma_1 \cup \gamma_3$. The closed path α_2 intersects C_3 in only one point, precisely at the vertex V_3. Since $\bar{a}_3 \subset \partial D_1$, $D_1 \subset A$, $\bar{b}_1 \subset \partial D_2$ and $D_2 \subset B$ and further $C_2 \subset A$ and $C_1 \subset B$, we obtain that α_2

pierces the cylinder C_3. Then, either α_2 is linked with γ_1 or α_2 is linked with γ_3.

We shall show that each one of the above-mentioned possibilities give us a contradiction.

We suppose first α_2 is linked with γ_1. The disc D_1 defines a homotopy between α_2 and α_1 and we have $D_1 \cap \gamma_1 = \emptyset$. Then α_1 is also linked with γ_1. Since $\alpha_1 = \partial D_3$ and $D_3 \cap \gamma_1 = \emptyset$, we have a contradiction.

Suppose now that α_2 is linked with γ_3. The disc D_2 defines a homotopy between α_2 and α_3 and $D_2 \cap \gamma_1 = \emptyset$. Therefore α_3 is linked with γ_1. The folded disc $\tilde{D}$ defines a homotopy between γ_1 and α_3 and $\tilde{D} \cap \gamma_3 = \emptyset$, therefore γ_1 is linked with γ_3, but this is a contradiction, once $\gamma_1 \cup \gamma_3 = \partial C_3$.

Again, we have a 2-simplicial complex $\mathcal{K}$, which verify the theorem 1.2 conditions.

In this case, $\chi(\mathcal{K}) = 2$ and $\chi(N) = 2$, where N is the regular neighborhood of $|\mathcal{K}|$ as in theorem 1.2. Thus, ∂N is a disjoint union of two 2-dimensional spheres. We obtain a homotopy 3-sphere M^3 by capping off each one of these 2-dimensional spheres with a 3-cell.

§4 Some conjectures related to the Poincaré conjecture

We call attention to the following two conjectures which are directly related to the Poincaré conjecture.

The first one is the Zeeman conjecture [Z].

" If K is a contractible 2-complex, then K×I is collapsible."

Zeeman shows in his theorem (2) that the above conjecture is stronger than the Poincaré conjecture. Then, we have also counterexamples to the Zeeman conjecture.

The second one is the Bing-Borsuk conjecture [BB].

" Every n-dimensional homogeneous compact ANR-space is an n-dimensional manifold."

In dimension 1 and 2 this conjecture was proved by Bing and Borsuk. In dimension 3, Jacobsche [J] proves that this conjecture is also stronger than the Poincaré conjecture. In fact, he proves the following theorem:

" If there exists a fake 3-cell F, then there exists a 3-dimensional homogeneous compact ANR-space which is not a manifold."

Thus, the Bing-Borsuk conjecture fails to be true in dimension 3.

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