Boolean Real Semigroups

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To Paulo Veloso, dear friend and colleague of the first author.

The first four sections of this paper will appear in [MR], but we thought it adequate that the full version be submitted to the annals of DDGS, including, in section 5, an exposition of the quotients of Boolean Real Semigroups.

Paulo Veloso is not amongst us anymore, having passed away in the ides of November, 2020. The first author has lost a cherished friend and we have lost a distinguished intellectual, logician and computer scientist.

Our purpose here is fourfold: firstly to give, employing the languages of special groups (SG) and real semigroups (RS), new, oftentimes conceptually different and clearer, proofs of the characterization of RSs whose space of characters is Boolean in the natural Harrison (or spectral) topology, originally appearing in section 7.6 and 8.9 of [M], therein named zero-dimensional abstract real spectra and here called Boolean Real Semigroups. Secondly, to give a natural *Horn-geometric* axiomatization of Boolean RSs (in the language of RSs) and establish the closure of this class by certain important constructions: Boolean powers, arbitrary filtered colimits, products, reduced products and RS-sums and by surjective RS-morphisms (in particular, quotients). Thirdly, to characterize morphisms between Boolean RSs and fourthly, to present a rather complete account of the quotients of these structures. Hence, the present paper considerably extends the original work by which it was motivated, namely the references in [M] mentioned above.

Recall that if G is an RS, $G^{\times} = \{u \in G : u^2 = u\}$ is the pre-reduced special group of units in G, and $\text{Id}(G) = \{e \in G : e^2 = e\}$ is the distributive lattice of idempotents in G.

In section 1 we recall, for the benefit of the reader, the basic properties of reduced special groups (RSG) and real semigroups. Section 2 discusses the factorization of a reduced special group into a direct product. Section 3 presents our account of the characterization of Boolean RSs, showing in particular that $G^{\times}$ is a RSG. The third paragraph at the beginning of section 3 explains the differences between our approach and that in [M]. In section 4 we show that if G_1, G_2 are Boolean RSs, there is a natural bijective correspondence, preserving composition and isomorphism, between $Hom_{RS}(G_1, G_2)$ and compatible pairs $\langle f, h\rangle$, where $f : G_1^{\times} \longrightarrow G_2^{\times}$ is a RSG-morphism and $h : \text{Id}(G_1) \longrightarrow \text{Id}(G_2)$ is a bounded lattice morphism (cf. Definition 4.6). In section 5, we characterize quotients of Boolean Real Semigroups.

For the notions of geometric and Horn-geometric formulas and theories in a first-order language with equality we refer to reader to section 2 of Chapter 1 in [DM2].

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If K is **reduced**, then X_K is not empty, being called the **space of orders** of K. It carries a natural Boolean space topology, having as a sub-basis

$$\{[\![x = 1]\!] : x \in K\}, \text{ where } [\![x = 1]\!] \ = \ \{\sigma \in X_K\colon \sigma(x) = 1\}.$$

Note that $[\![-y = 1]\!] = [\![y = -1]\!]$, for all $y \in K$.

c) If $K_1 \xrightarrow{f} K_2$ is SG-morphism and φ, ψ are forms of the same dimension on K_1, then $\varphi \equiv_{K_1} \psi$ implies $f \star \varphi \equiv_{K_2} f \star \psi$, where, for $\theta = \langle a_1, \dots, a_n \rangle \in K_1^n$, $f \star \theta = \langle f(a_1), \dots, f(a_n) \rangle$. ∎

1.7 Pfister's local-global principle. If K is a RSG, $\varphi = \langle a_1, \dots, a_n \rangle$ is a form over K and $\sigma \in X_K$, **the signature of φ at σ**, is defined by $sgn_\sigma(\varphi) = \sum_{k=1}^{n} \sigma(a_k)$ (sum in $\mathbb{Z}$). A very important result is the following (cf. Prop. 3.7, [DM1])

Theorem 1.8 (Pfister's local-global principle) *Let K be a RSG and let φ, ψ be forms of the same dimension over K. Then,*

$$\varphi \equiv_K \psi \quad \textit{iff} \quad \forall\, \sigma \in X_K,\ sgn_\sigma(\varphi) = sgn_\sigma(\psi). \qquad ∎$$

1.9 Duality. We shall not describe the theory of Abstract Order Spaces (AOS), referring the reader to [M]. We recall Thm. 3.19 in [DM1], establishing a **natural duality** between the categories of Reduced Special Groups and Abstract Order Spaces, allowing the transfer of certain results between these categories. ∎

If K is a RSG, then $D_K(1, b)$ is a subgroup of K and the relation $a \in D_K(1, b)$ is a **partial order** ($a \preceq b$). It is proven in Prop. 1.2 of [DMM] that this idea leads to an first-order axiomatization of the theory **reduced** SGs, that will be very useful in what follows:

Proposition 1.10 *A structure $\langle K, \cdot, \preceq, 1, -1 \rangle$ for the first-order language $\{\cdot,\ 1,\ -1,\ \preceq\}$, where $\cdot$ is a binary operation, $\preceq$ is a binary relation, and 1, -1 are constants, is a reduced special group iff it satisfies the following set of axioms:*

[R 0] *$\langle K, \cdot, 1 \rangle$ is a group of exponent two, with $1 \neq -1$.*

[R 1] (Partial order): *$\preceq$ is partial order on K, with first element 1 and last element -1.*

[R 2] (Involution): *For all a, $b \in K$, $a \preceq b \ \Rightarrow\ -b \preceq -a$.*

[R 3] (Subgroup): *For all $b \in K$, $\{x : x \preceq b\}$ is a subgroup of K, that is, x, $y \preceq b$ implies $xy \preceq b$.*

[R 4] (Weak Compatibility): *For all a, b, c, $d \in K$, $a \preceq b$ and $bd \preceq cd$ imply $\exists\ e \in K$ so that $e \preceq d$ and $ae \preceq ce$.* ∎

Remarks 1.11 a) By 1.5.(a), every RSG satisfies axiom [R 4].

b) In the case of a RSG, K, the transitivity of $a \in D_K(1, b)$ is tantamount to this subgroup being saturated (which may fail if K is not reduced).

c) Given a structure, $\langle K, \cdot, \preceq, 1, -1 \rangle$, satisfying the conditions in 1.10, binary isometry may be obtained, as suggested by Lemma 1.4.(a), by $\langle a, b \rangle \equiv_K \langle c, d \rangle$ iff $ab = cd$ and $ac \preceq cd$. ∎

1.12 Saturated Subgroups. This theme is discussed in some detail in Chapter 2 of [DM1]. Here we register, for the convenience of the reader, only the most basic facts. Let K be a RSG.

A subgroup, Δ, of K is **saturated** if $a \in \Delta$ implies $D_K(1, a) \subseteq \Delta$. By 1.6.(b), the kernel of any SG-character is a saturated subgroup of K. If $-1 \in \Delta$, we say Δ is **improper**, and proper otherwise (i.e., $\Delta \neq K$). Clearly, the intersection and union of a upwards directed family of saturated subgroups is saturated.

a) (Lemma 2.4.(b), [DM1]) A subgroup Δ of K is saturated iff for all Pfister forms $\mathcal{P}$ with coefficients in Δ, $D_K(\mathcal{P}) \subseteq \Delta$.

b) If $\mathcal{P}$ is a Pfister form over K, the $D_K(\mathcal{P})$ is a saturated subgroup of K.

Cor. 2.29 and Thm. 2.11 in [DM1] yield the following important

Theorem 1.13 (Separation) *Let K be a RSG, let Δ be a saturated subgroup of K and let a be an element of G such that $a \notin \Delta$. Then, there is $\sigma \in X_K$ so that $\Delta \subseteq \ker \sigma$ and $\sigma(a) = -1$. In particular, for a, $b \in K$, $b \in D(1, a)$ iff for all $\sigma \in X_G$, $\sigma(a) = 1$ entails $\sigma(b) = 1$.* ◼

c) Let $S \subseteq K$ and write $\mathfrak{s}(S)$ for the saturated subgroup of K generated by S, called the **saturation** of S (in [DM1] this is written $\overline{S}$). The Separation Theorem 1.13 and item (a) yield

$$\begin{aligned} \mathfrak{s}(S) &= \bigcap \{\ker \sigma : \sigma \in X_K \text{ and } S \subseteq \ker \sigma\} \\ &= \bigcup \{D_K(\otimes_{a \in F} \langle 1, a \rangle : F \text{ is a finite subset of } S\}. \end{aligned}$$

d) Saturated subgroups of a RSG K classify all RSG-quotients of K (cf. Prop. 2.28, [DM1]). If Δ is a saturated subgroup of K, write $K/\Delta = \{a/\Delta : a \in K\}$ for the quotient RSG and $p_\Delta : K \longrightarrow K/\Delta$, for the canonical projection. Note that $a/\Delta = \{x \in K : ax \in \Delta\}$. Def. 2.27 and Prop. 2.28 in [DM1] show that for a, b, c, $d \in K$, the following are equivalent:

(*i*) $\langle a/\Delta, b/\Delta \rangle \equiv_{K/\Delta} \langle c/\Delta, d/\Delta \rangle$;

(*ii*) There are a', b', c', $d' \in K$ so that aa', bb', cc', $dd' \in \Delta$ and $\langle a', b' \rangle \equiv_K \langle c', d' \rangle$;

(*iii*) There is a Pfister form $\mathcal{P}$ over Δ so that $\langle a, b \rangle \otimes \mathcal{P} \equiv_K \langle c, d \rangle \otimes \mathcal{P}$.

e) Recall that a **subspace** of X_K (in the sense of Abstract Order Spaces (AOS)) is a subset of the form $\bigcap_{i \in I} [\![a_i = 1]\!]$, $a_i \in K$, $i \in I$ (cf. Definition in section 2.4, p. 32 of [M]). Write Sub(X_K) for the family of subspaces of X_K and Sat(K) for the set of saturated subgroups of K. Consider the maps:

$$\Sigma : \text{Sub}(X_K) \longrightarrow \text{Sat}(K) \quad \text{and} \quad S : \text{Sat}(K) \longrightarrow \text{Sub}(X_K),$$

where $\Sigma(X) = \bigcap_{\sigma \in X} \ker \sigma$ and $S(\Delta) = \bigcap_{a \in \Delta} [\![a = 1]\!]$.

The Separation Theorem 1.13 guarantees Σ and S to be inverse, order reversing, bijective correspondences between Sub(X_K) and Sat(K). ◼

B. Real Semigroups

Definition 1.14 (Def. 1.1, [DP1]) *A* **ternary semigroup (TS)** *is a structure $S = \langle S, 1, 0, -1, \cdot \rangle$, such that $\forall\, x \in S$:*

[TS1] *S is a commutative semigroup with unit 1;* [TS2] $x^3 = x$;

[TS3] $-1 \neq 1$ *and* $(-1)(-1) = 1$; [TS4] $x \cdot 0 = 0$;

[TS5] $x = (-1) \cdot x \;\Rightarrow\; x = 0$.

Write $-x$ for $(-1) \cdot x$. A **TS-morphism** *is a map preserving product and the constants 0, 1, −1. Note that for all $x \in S$, $x^4 = x^2$.*

Definition 1.15 *a)* (Def. 2.1, [DP1]) *A* **real semigroup (RS)** *is a ternary semigroup G, together with a ternary relation D on G, to be written $a \in D(b, c)$, called* **binary representation**, *and with* **transversal binary representation**, *D^t, defined by* [t-rep] $a \in D^t(b, c) \Leftrightarrow a \in D(b, c)$ and $-b \in D(-a, c)$ and $-c \in D(b, -a)$,

satisfying, for all $a, b, d, e \in G$:

[RS 0] $c \in D(a,b) \Leftrightarrow c \in D(b,a)$; [RS 1] $a \in D(a,b)$;

[RS 2] $a \in D(b,c) \Rightarrow ad \in D(db, dc)$;

[RS 3] (*Strong associativity*) $a \in D^t(b,c)$ *and* $c \in D^t(d,e) \Rightarrow \exists\, x \in D^t(b,d)$ *s.t.* $a \in D^t(x,e)$;

[RS 4] $e \in D(c^2a, d^2b) \Rightarrow e \in D(a,b)$;

[RS 5] $ad = bd$, $ae = be$ *and* $c \in D(d,e) \Rightarrow ac = bc$;

[RS 6] $c \in D(a,b) \Rightarrow c \in D^t(c^2a, c^2b)$;

[RS 7] (Reduction) $D^t(a,-b) \cap D^t(b,-a) \neq \emptyset \Rightarrow a = b$.

[RS 8] $a \in D(b,c) \Rightarrow a^2 \in D(b^2, c^2)$.

b) Representation and transversal representation (t-representation) are extended, by induction, to forms of dimension $n \geq 3$*: if* $\varphi = \langle a_1, \ldots, a_n \rangle$ *is a form over a real semigroup, G, and* $x \in G$,

$$x \in D_G(\varphi) \Leftrightarrow \exists\, u \in D_G(a_2, \ldots, a_n) \text{ s. t. } x \in D_G(a_1, u).$$

Similarly, for $D^t_G(\varphi)$: $x \in D^t_G(\varphi) \Leftrightarrow \exists\, u \in D^t_G(a_2, \ldots, a_n)$ *s. t.* $x \in D^t_G(a_1, u)$.

c) Write $\mathcal{L}_{RS} = \langle \cdot, 1, 0, -1, D \rangle$ *for the first-order language of real semigroups; a structure for this language satisfying all the axioms in (a), with the possible exception of* [RS3], *is called a* **pre-Real Semigroup (pRS)**.

d) If G is a pRS, write $G^\times = \{u \in G : u^2 = 1\}$ *for the* **group of units** *in G.*

Remark 1.16 If G is a pRS, Prop. I.2.10 in [DP3] establishes the following equivalence:

(1) G is a RS (i.e., satisfies axiom [RS3]);

(2) [RS 3′] : For all $a, b, c, d \in G$,

$$D^t_G(a, b) \cap D^t_G(c, d) \neq \emptyset \Rightarrow D^t_G(a, -c) \cap D^t_G(-b, d) \neq \emptyset.$$

The proof for RSs is entirely analogous to that of Lemma 1.5. ∎

Remark 1.17 The ternary semigroup $\mathbf{3} = \{1, 0, -1\}$ (with product induced by the integers) has a unique structure of RS, with representation and t-representation given by (cf. Cor. 2.4, p. 109, [DP1]):

$$(D) \quad \begin{cases} D_3(0,0) = \{0\}; \quad D_3(0,1) = D_3(1,0) = D_3(1,1) = \{0,1\}; \\ D_3(0,-1) = D_3(-1,0) = D_3(-1,-1) = \{0,-1\}; \\ D_3(1,-1) = D_3(-1,1) = \mathbf{3}. \end{cases}$$

$$(D^t) \quad \begin{cases} D^t_3(0,0) = \{0\}; \quad D^t_3(0,1) = D^t_3(1,0) = D^t_3(1,1) = \{1\}; \\ D^t_3(0,-1) = D^t_3(-1,0) = D^t_3(-1,-1) = \{-1\}; \\ D^t_3(1,-1) = D^t_3(-1,1) = \mathbf{3}. \end{cases}$$ ∎

Definition 1.18 *a) A map* $f : G \longrightarrow H$*, where G, H are pRSs, is a* **RS-morphism** *if it is a TS-morphism preserving representation* (or equivalently, t-representation), *i.e., for all* $a, b, c \in G$, $a \in D(b,c) \Rightarrow f(a) \in D(f(b), f(c))$ (*or the corresponding statement with D replaced by* D^t).

b) If G is a pRS, write X_G *for the set of RS-morphisms from G to* **3***, called the space of* **characters** *of G. For* $g \in G$ *and* $j \in 3 = \{0, 1, -1\}$*, set* $[\![g = j]\!] = \{\sigma \in X_G : \sigma(g) = j\}$.

Some of the basic consequences of the axioms concerning forms over a RS appear in Propositions 2.3 (p. 107) and 2.7 (p. 110) of [DP1]; to ease presentation and reference, we register the Lemma and the Proposition that follow, giving proofs only for those items not established in the above mentioned references. These basic properties of representation and t-representation in RSs, will be used frequently, sometimes without explicit mention.

Lemma 1.19 *Let G be a pRS. For all a, b, c, $d \in G$, we have*

(0) $D^t(a,b) = D^t(b,a) \subseteq D(a,b)$;

(1) $a \in D^t(b,c) \Rightarrow -b \in D^t(-a,c)$.

(2) $0 \in D(a,b)$;

(3) $a \in D^t(b,c) \Rightarrow ad \in D^t(bd, cd)$;

(4) $a \in D(0,1) \cup D(1,1) \Rightarrow a = a^2$; (5) $d \in D(ca, cb) \Rightarrow d = c^2 d$;

(6) $a^2 \in D(1,b)$; in particular, $D(1,1) = \mathrm{Id}(G) = \{a \in G : a = a^2\}$;

(7) $a \in D^t(b,b) \Rightarrow a = b$; (8) $a \in D(0,0) \Rightarrow a = 0$; (9) $1 \in D^t(1,a)$;

(10) $D^t(1,-1) = G$; (11) $ab \in D(1,-a^2)$;

(12) $0 \in D^t(a,b) \Rightarrow a = -b$; (12.a) $a \in D^t(b,0) \Rightarrow a = b$.

(13) $c \in D(a,b) \Leftrightarrow c \in D^t(c^2a, c^2b)$. In particular,

(13.a) D and D^t are interdefinable; (13.b) $D(a,b) \cap G^\times = D^t(a,b) \cap G^\times$.

(14) $D(a,-a) = D^t(a,-a) = a^2 \cdot G$.

(14.a) $0 \in D^t(a,b) \Leftrightarrow b = -a$; (14.b) $a \in D^t(b,0) \Leftrightarrow a = b$.

(15) $\{a,-a\} \subseteq D^t(b,c) \Rightarrow c = -b$.

Proof. Item (0) follows immediately from the [RS 0] and the definition of D^t (cf. [t-rep] in Definition 1.15). Items (1) – (12) are proven in Proposition 2.3, pp. 107-108, in [DP1], upon noting that the arguments given therein do not employ axiom [RS 3] (strong associativity). For (12.a), note that its hypothesis entails $0 \in D^t(-a,b)$ and so (12) yields $a = b$.

(13) The implication ($\Rightarrow$) follows immediately from [RS 6] in 1.15; for the converse, item (0) and [RS 4] in 1.15 yield $c \in D^t(c^2a, c^2b) \subseteq D(c^2a, c^2b)$, and so $c \in D(a,b)$, as needed. The statement in (13.a) is clear, while if $c \in G^\times$, then (13) guarantees that $c \in D(a,b)$ iff $c \in D^t(a,b)$.

(14) Given $x \in G$, by item (9) we have $1 \in D^t(1,-ax)$, and so (3) entails $a \in D^t(a,-a^2x)$, whence, from (1) we obtain $a^2x \in D^t(a,-a)$, showing that $a^2 \cdot G \subseteq D^t(a,-a)$. Now, from $x \in D(a,-a) = D(a\cdot 1, a\cdot(-1))$, (5) yields $x = a^2x$, whence $D(a,-a) \subseteq a^2 \cdot G$. But then, item (0) and the preceding argument imply $a^2 \cdot G \subseteq D^t(a,-a) \subseteq D(a,-a) \subseteq a^2 \cdot G$, establishing (14). Notice that (14) guarantees that $0 \in D^t(x,-x)$ for all $x \in G$. But then, (14.a), (14.b) and are direct consequences of (1) together with of (14), (12), and (14), (12.a), respectively.

(15) By (3), $-a \in D^t(b,c)$ entails $a \in D^t(-b,-c)$ and so $a \in D^t(b,-(-c)) \cap D^t(-c,-b)$; from axiom [RS 7] (reduction) we obtain $c = -b$. ∎

Proposition 1.20 *Let H be a RS, $\varphi = \langle x_1, \ldots, x_n \rangle$, ψ be forms over H and let a, b, $c \in H$.*

a) $D_H(\varphi)$ and $D_H^t(\varphi)$ do not depend on the order of their entries, i.e, for any permutation σ of those entries, $D_H(\varphi) = D_H(\varphi^\sigma)$ and $D_H^t(\varphi) = D_H^t(\varphi^\sigma)$.

b) (1) $a \in D_H(\varphi) \Rightarrow ac \in D_H(c\varphi)$ and $a \in D_H^t(\varphi) \Rightarrow ac \in D_H^t(c\varphi)$;

(2) $a \in D_H(c\varphi) \Rightarrow a = ac^2$;

(3) $a \in D_H(\varphi) \Rightarrow a \in D^t_H(a^2\varphi)$. *In particular,* $D_H(\varphi) \cap H^\times = D^t_H(\varphi) \cap H^\times$.

c) $a \in D_H(\varphi \oplus \psi) \Leftrightarrow$ *there are* $b \in D_H(\varphi)$, $c \in D_H(\psi)$ *so that* $a \in D_H(b, c)$.

A similar statement holds replacing D_H *by* D^t_H.

d) If a *is an entry in* φ, *then* $a \in D_H(\varphi)$.

e) $a \in D_H(\varphi)$ *and* $b \in D_H(\psi) \Rightarrow ab \in D_H(\varphi \otimes \psi)$. *A similar statement holds for* D^t_H.

f) For $b \in G$ *and* $n \geq 1$, $D^t_G(\underbrace{b, \ldots, b}_{n\times}) = \{b\}$.

g) $D^t_H(\langle 1, a^2 \rangle \otimes \varphi) = D^t_H(\varphi)$.

j) If $H \xrightarrow{f} H'$ *is a RS-morphism* (cf. 1.18.(a)), *then* $a \in D_H(\varphi) \Rightarrow f(a) \in D_{H'}(f \star \varphi)$, *where* $f \star \varphi = \langle f(x_1), \ldots, f(x_n) \rangle$. *Similarly, replacing* D_H *by* D^t_H.

h) (1) $a \in D_H(\varphi) \Rightarrow a \in D_H(\varphi \oplus \psi)$; (2) $a \in D^t_H(\varphi) \Rightarrow a \in D^t_H(a^2\varphi \oplus a^2\psi)$.

Proof. See Prop. 2.7, pp. 110 – 112 in [DP1]. ∎

Remark 1.21 If G is a **real semigroup**, Theorems 4.3 and 4.4. (p. 116) of [DP1] guarantee that its space of **RS-characters**, X_G, separates points in G. Moreover, the representation relations induced by X_G in G coincide with the original ones carried by G, i.e., for all a, b, $c \in G$, we have the following equivalences:

[D_G] $\quad a \in D_G(b, c) \Leftrightarrow \forall\, \sigma \in X_G,\ \sigma(a) \in D_3(\sigma(b), \sigma(c))$,

[D^t_G] $\quad a \in D^t_G(b, c) \Leftrightarrow \forall\, \sigma \in X_G,\ \sigma(a) \in D^t_3(\sigma(b), \sigma(c))$. ∎

1.22 Topologies on X_G. If G is a RS, the collection $\{[\![g = 1]\!] : g \in G\}$ (cf. 1.18.(b)) constitutes sub-basis for a (completely normal or root system, cf. [DST], p. 290) spectral topology on X_G. For the constructible topology on X_G, with which X_G is a Boolean space, we have:

Fact 1.23 *Let* G *be a RS and let* a, b, $c \in G$.

a) (Cor. III.7.3, [DP3] and Prop. 6.1.5, [M]) *There is a unique* $z \in G$, *such that* $z = z^2$ *and* $z \in D^t_G(a^2, b^2)$. *Moreover,* $[\![z = 0]\!] = [\![a = 0]\!] \cap [\![b = 0]\!]$.

b) The sets of the form $[\![b = 0]\!] \cap \bigcap_{k=1}^n [\![a_k = 1]\!]$, *for* $a_1, \ldots, a_n$, $b \in G$ *constitute a basis for the constructible (or patch) topology on* X_G. □

Hence, the spectral topology on X_G is Boolean iff for all $z \in G$, the closed set $[\![z = 0]\!]$ is clopen in the spectral topology of X_G. ∎

1.24 Duality. By Thm. 4.1 in [DP1] (cf. also Thm. I.5.1, [DP2]), the category **RS** is isomorphic to **ARS**$^{\mathrm{op}}$, where **ARS** is the category of abstract real spectra in the sense of M. Marshall. Note that the category **RS** is Horn-geometrically axiomatizable, while **ARS** is not. Clearly, this duality allows the transfer of statements between these categories. ∎

1.25 The group of units of a RS. Let G be a RS and let $G^\times$ be its group of units (1.15.(d)). For any a, $b \in G$, 1.19.(13.b) entails $D^t_G(a, b) \cap G^\times = D_G(a, b) \cap G^\times$.

Define a binary relation on $G^\times$ by $a \preceq b$ iff $a \in D_G(1, b) = D^t_G(1, b)$

Lemma 1.26 (cf. 1.2.11, [DP3]) *Notation as above, the structure* $\langle G^\times, \preceq, 1, -1 \rangle$ *satisfies the axioms* [R 0], [R 1], [R 2] and [R3] *in the statement of Proposition* 1.10. *In particular,* $G^\times$ *is a pSG.*

Proof. Clearly $G^\times$ is a group of exponent 2 ([R 0]). To show $\preceq$ is a partial order ([R 1]), if $a \in G^\times$, then $a \preceq a$ follows from axiom [RS 1] and, as observed above, $D_G \cap G^\times = D^t_G \cap G^\times$. If $a \in D_G(1, b)$ and $b \in D_G(1, a)$, then, 1.19.(1) entails $-1 \in D^t_G(-a, b) \cap D^t_G(-b, a)$ and the reduction axiom [RS 7] of RSs implies $a = b$. For the transitivity of $\preceq$, let $a \in D^t_G(1, b)$ and $b \in D^t_G(1, c)$. By axiom [RS 3], there is $z \in G$ such that $z \in D^t_G(1, 1)$ and $a \in D^t_G(z, c)$. Now, 1.19.(7) yields $z = 1$, whence, $a \in D^t_G(1, c)$.

Axiom [RS 1] yields $1 \in D_G(1, a)$, i.e, $1 \preceq a$, while 1.19.(10) gives $D^t_G(1, -1) = G$ and so $a \preceq -1$, for all $a \in G^\times$. Axiom [R 3] in 1.10 is immediate from 1.19.(1): $a \in D^t_G(b, c) \Rightarrow -b \in D^t_G(-a, c)$.

It remains to establish [R 3]: for $a \in G^\times$, $\{x \in G^\times : x \preceq a\}$ is a subgroup of $G^\times$. If $y, z \in D^t_G(1, a)$, then 1.20.(e) entails, $xy \in D^t_G(\langle 1, a \rangle \otimes \langle 1, a \rangle) = D^t_G(1, a, a, a^2) = D^t_G(1, 1, a, a)$. By 1.20.(c), there are $t \in D^t_G(1, 1)$ and $z \in D^t_G(a, a)$ so that $xy \in D^t_G(t, z)$. As above, we obtain $t = 1$ and and $z = a$. ∎

When treating Boolean RSs we shall use Proposition 1.10 to show that their group of units is a RSG. By 1.26, it suffices to verify the weak associativity axiom ([R 4]) associated to the partial order $\preceq$ defined above. ∎

1.27 Idempotents in Real Semigroups. Let G be a RS and let $\mathrm{Id}(G) = \{a \in G : a^2 = a\} = G^2$ be the set of **idempotents** in G; for $f, g \in \mathrm{Id}(G)$ define $f \leq g$ iff $fg = g$. Then, $\leq$ is a partial order, endowing $\mathrm{Id}(G)$ with a distributive lattice structure, wherein

$$f \wedge g = \text{the unique element in } D^t_G(f, g) \text{ (cf. 1.23.(a)) and } f \vee g = fg,$$

whose bottom element is 1 and top element is 0 (Prop. I.6.8, [DP3]). Note: this is the *opposite* of the usual lattice structure associated to idempotents in a *commutative unitary ring.* ∎

2 Direct Sum Decompositions of Reduced Special Groups

In this section, K is a reduced special group; notation is as in 1.12.

Definition 2.1 *Let* K *be a RSG. A saturated subgroup of* K, Δ, *is a* **direct sum subgroup (DSS)** *of* K *if there is a saturated subgroup,* $\Delta^\perp$, *such that the natural map* $\gamma_\Delta : K \longrightarrow K/\Delta \times K/\Delta^\perp$, *given by* $a \longmapsto (a/\Delta, a/\Delta^\perp)$ *is a RSG-isomorphism, where* $K/\Delta \times K/\Delta^\perp$ *has its canonical product structure.* K *and* $D_K(1, 1) = \{1\}$ *are both also considered as DSS, the* **trivial DSS** (*although* $K/K = \{1\}$ *is not a RSG*).

Remark 2.2 Notation as 2.1, Δ is a DSS iff $\Delta^\perp$ is a DSS; since γ_Δ is injective, we get $\Delta \cap \Delta^\perp = \{1\}$ ($\gamma_\Delta(x) = (1, 1)$ iff $x \in \Delta \cap \Delta^\perp$ iff $x = 1$). ∎

Here is an improved special group version of Lemma 8.9.1 in [M]:

Proposition 2.3 *For a saturated subgroup* Δ *of* K, *the following are equivalent:*

(1) Δ *is a direct sum subgroup of* K;

(2) *There is* $a \in K$ *so that:*

(*i*) $\Delta = D_K(1, a)$ (*and* $\Delta^\perp = D(1, -a)$);

(*ii*) *For all* $b \in K$, *there is* $c \in K$ *so that* $\langle 1, b \rangle \otimes \langle 1, a \rangle \equiv_K \langle 1, 1 \rangle \otimes \langle 1, c \rangle$.

Proof. Notation is as in 2.1 and 2.2. To ease presentation, write $D(\cdot, \cdot)$ for $D_K(\cdot, \cdot)$ and T for $K/\Delta \times K/\Delta^\perp$.

(1) $\Rightarrow$ (2) : Since γ_Δ is a RSG-isomorphism, there is $a \in K$ so that $\gamma_\Delta(a) = (1, -1)$ in T. Hence, $a \in \Delta$ and so $D(1, a) \subseteq \Delta$. If $x \in \Delta$, then $\gamma_\Delta(x) = (1, x/\Delta^\perp)$; since isometry and representation in T are coordinatewise defined, we get $\gamma_\Delta(x) \in D_T((1,1), (1, -1)) = D_T(\gamma_\Delta(1), \gamma_\Delta(a))$, whence $x \in D(1, a)$, establishing 2.(*i*). Note that $\gamma_\Delta(-a) = (-1, 1) \in T$ and reasoning as above obtains $\Delta^\perp = D(1, -a)$. For 2.(*ii*), if $b \in K$, there $c \in K$ such that $\gamma_\Delta(c) = (b/\Delta, -1) \in T$. Thus,

(*) $$cb \in \Delta = D(1, a) \quad \text{and} \quad -c \in D(1, -a).$$

To show $\langle 1, a \rangle \otimes \langle 1, b \rangle = \langle 1, a, b, ab \rangle \equiv_K 2\langle 1, c \rangle = \langle 1, 1, c, c \rangle$, we compare signatures (Pfister's local-global principle for RSGs, 1.8): if $\sigma(a) = 1$, then the first relation in (*) yields $\sigma(cb) = 1$ and so $\sigma(b) = \sigma(c)$, which in turn entails the equality of the signatures of $\langle 1, a \rangle \otimes \langle 1, b \rangle$ and $2\langle 1, c \rangle$ at σ; if $\sigma(-a) = 1$, the second relation in (*) yields $\sigma(c) = -1$ and the signatures of $\langle 1, a \rangle \otimes \langle 1, b \rangle$ and $2\langle 1, c \rangle$ at σ are both 0, establishing (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) : Since $D(1, a) \cap D(1, -a) = \{1\}$, it is clear that $\gamma_\Delta : K \longrightarrow T$ is injective (cf. 2.2.(a)) and a RSG-morphism. To show it is an embedding, assume, for $x, y \in K$, $\gamma_\Delta(x) \in D_T((1,1), \gamma_\Delta(y))$. Hence,

$$x/\Delta \in D_{K/\Delta}(1, y/\Delta) \quad \text{and} \quad x/\Delta^\perp \in D_{K/\Delta^\perp}(1, y/\Delta^\perp).$$

By the equivalence in item 1.12.(d), there are Pfister forms, $\mathcal{P}$ and $\mathcal{P}^\perp$ over Δ and $\Delta^\perp$, respectively such that

(**) $$\langle x, xy \rangle \otimes \mathcal{P} \equiv_K \langle 1, y \rangle \otimes \mathcal{P} \quad \text{and} \quad \langle x, xy \rangle \otimes \mathcal{P}^\perp \equiv_K \langle 1, y \rangle \otimes \mathcal{P}^\perp.$$

Now if $\sigma(a) = 1$, since $\mathcal{P}$ has entries in $D(1, a)$, the first relation in (**) yields $2^n(\sigma(x) + \sigma(xy)) = 2^n(1 + \sigma(y))$ and the signatures of $\langle x, xy \rangle$ and $\langle 1, y \rangle$ are the same for all $\sigma \in [\![a = 1]\!]$; similarly, if $\tau \in [\![a = -1]\!] = [\![-a = 1]\!]$, the second relation in (**) entails the signature of $\langle x, xy \rangle$ and $\langle 1, y \rangle$ to be the same for all $\tau \in [\![a = -1]\!]$. Hence, $\langle x, xy \rangle$ and $\langle 1, y \rangle$ have the same total signature and 1.8 entails $\langle x, xy \rangle \equiv_K \langle 1, y \rangle$ and $x \in D_T(\langle 1, y \rangle)$, establishing that γ_Δ is a RS-embedding.

It remains to show γ_Δ is surjective. Let $u, v \in K$; then, there is $c \in K$ so that $\langle 1, -uv \rangle \otimes \langle 1, a \rangle \equiv_K \langle 1, 1 \rangle \otimes \langle 1, c \rangle = \langle 1, c \rangle \oplus \langle 1, c \rangle$. We claim that

(***) $$-(cv)v = -c \in D(1, -a) \quad \text{and} \quad (-cv)u = -cuv \in D(1, a).$$

By Prop. 2.2 in [DM1], $a \in D(\langle 1, -uv \rangle \otimes \langle 1, a \rangle) = D(\langle 1, c \rangle \oplus \langle 1, c \rangle)$, and so Prop. 1.6.(e) in [DM1] and 1.19.(d) yield $a \in D(1, c)$. Thus, $-c \in D(1, -a)$; if $\tau \in X_K$ is such that $\tau(a) = 1$, the isometry $\langle 1, -uv \rangle \otimes \langle 1, a \rangle \equiv_K 2\,\langle 1, c \rangle$ and Theorem 1.8 entail $2\tau(-uv) = 2\tau(c)$ and so $\tau(-cuv) = 1$, for each $\tau \in [\![a = 1]\!]$, whence $-cuv \in D(1, a)$ (by 1.13), establishing (***). Thus, $-cv/\Delta = u/\Delta$, $-cv/\Delta^\perp = v/\Delta^\perp$ and therefore $\gamma_\Delta(-cv) = (u/\Delta, v/\Delta^\perp) \in T$, completing the proof. ∎

Remarks 2.4 (1) In the terminology of section 8.9 in [M], a subset A of X is **complemented or a direct summand** if both A and $X \setminus A$ are subspaces of X and the natural injection

$$K \longrightarrow K{\restriction}A \times K{\restriction}X \setminus A, \text{ given by } A \longmapsto \langle a{\restriction}A, a{\restriction}X \setminus A \rangle,$$

is an isomorphism. Given the duality between the categories of RSGs and AOSs (cf. 1.9), in the language of RSG and with notation as in items (c) and (d) of 1.12 , this can be equivalently rephrased as follows: write Δ for $\Sigma(A)$ and $\Delta^\perp$ for $\Sigma(X \setminus A)$; then, the natural map

$$K \longrightarrow K/\Delta \times K/\Delta^\perp, \text{ given by } a \longmapsto \langle a/\Delta, a/\Delta^\perp \rangle$$

is an isomorphism. In fact, the AOS $\langle K\lceil A, A\rangle$ corresponds, by duality, to the RSG-quotient $K \xrightarrow{p_\Delta} K/\Delta$. Hence, all complemented subspaces, A, of X are of the form $[\![a = 1]\!]$, for some $a \in K$ (and so $\Sigma(A) = D_K(1, a)$) and satisfy condition 2.(*ii*) of 2.3. In fact, Proposition 2.3 and Lemma 8.9.1 in [M] entail the existence of a bijective correspondence between DSSs of K and direct summands of X_K.

(2) Sometimes, it is easier to deal with certain statements in the language of AOSs; an example follows. For $A \subseteq X$ and $\mu \in \{0, 1\}$, set

$$A^\mu = \begin{cases} A & \mu = 0; \\ \neg A = X \setminus A & \text{if } \mu = 1;. \end{cases}$$

Fact 2.5 [Note (3), p. 179, [M], without proof] *If $A_1, \ldots, A_n$ are direct summands of $\langle K, X\rangle$, set $E_\mu = \bigcap_{i=1}^n A_i^{\mu(i)}$, $\mu \in \{0,1\}^n$; let*

$$\mathfrak{q} = \{\mu \in \{0,1\}^n : E_\mu \neq \emptyset\},$$

be the set of indices corresponding to the non-empty atoms of the BA generated by $\{A_1, \ldots, A_n\}$. Let $\Delta_\mu = \Sigma(E_\mu)$, $\mu \in \mathfrak{q}$. Then, all E_μ are direct summands of X and the natural map from K to $\prod_{\mu \in \mathfrak{q}} K/\Delta_\mu$, $a \longmapsto \langle a/\Delta_\mu\rangle_{\mu\in\mathfrak{q}}$, is an isomorphism.

Proof. It is enough to deal with the case $n = 2$ and proceed by induction. Let A, B be direct summands of X; we may assume all four atoms, $A \cap B$, $A \cap \neg B$, $B \cap \neg A$, $\neg A \cap \neg B$ to be non-empty. First, we prove that the natural map from $K\lceil B$ to $[K\lceil(A \cap B) \times K\lceil(B \cap \neg A)]$ is an isomorphism. Given $u, v \in K$, since A is a direct summand of $\langle K, X\rangle$, there is $x \in K$ such that $x\lceil A = u\lceil A$ and $x\lceil\neg A = v\lceil\neg A$. Hence,

$$\begin{cases} x\lceil(A \cap B) = (x\lceil A)\lceil B = (u\lceil A)\lceil B = u\lceil(A \cap B); \\ x\lceil(B \cap \neg A) = = (x\lceil\neg A)\lceil B = (v\lceil\neg A)\lceil B = v\lceil(B \cap \neg A), \end{cases}$$

as desired. Similarly, one shows $K\lceil\neg B$ and $[K\lceil(\neg B \cap A) \times K\lceil(\neg B \cap \neg A)]$ to be isomorphic; thus, all atoms above are direct summands and K is naturally isomorphic to the product of the ensuing four factors. ∎

3 The Structure of Boolean Real Semigroups

Definition 3.1 *A real semigroup, G, is* **Boolean** *if its space of RS-characters, X_G, endowed with the spectral topology, is Boolean, i.e, if the spectral and constructive topologies on X_G coincide.*

Example 3.2 There are two important examples of Boolean RSs:

(1) If K is a reduced special group, let G be K together with a new absorbing element, 0. By Remark 2.2.4 in [DP1] (see also Corollary 2.5, [DP1]), $G = K \cup \{0\}$ can be made into a RS, whose space is RS-characters is the Boolean space X_K, the space of orders of K;

(2) Post algebras of order 3, i.e., the structure of all continuous maps from a Boolean space X to $3 = \{1, 0, -1\}$, the latter endowed with the discrete topology (cf. section 3 in [DP2]). If $P = \mathbb{C}(X, 3)$, then $X_P = X$.

The forthcoming discussion exhibits a significant number of other examples of Boolean RSs. ∎

In this section, among other things, we give new proofs of the following results in [M]:

• Thm. 7.6.4 corresponds to the first statement in item (3) of Theorem 3.6, below; our proof employs Proposition 1.10, an alternative axiomatic for RSGs;

• Thm. 8.9.2 corresponds to Proposition 3.10, below, which is phrased using clopen direct summands of the space of orders of the RSG K (spaces of orders do not have zero sets);

• Theorem 8.9.3 corresponds to Theorem 3.13, below, but our proof uses the exchange principle in RSs (cf. Remark 1.16).

We also describe a Horn-geometric axiomatization of Boolean RSs, prove the preservation of this class of RSs by important constructions (Theorem 3.18) and characterize the rings whose associated RS is Boolean (Proposition 3.16).

3.3 Notation and Remarks. As usual, if G is a RS, $a \in G$ and $\varepsilon \in 3$:

a) By Proposition 1.10 and Lemma 1.26, $G^\times$ is a reduced pre-special group.

b) $[\![a = \varepsilon]\!] = \{\tau \in X_G : \tau(a) = \varepsilon\}$. If $\varepsilon = 0$, we may write $Z(a)$ for $[\![a = 0]\!]$.

c) $\mathbb{Z}_2 = \{1, -1\} = 3^\times$ is the RSG of units in $3 = \{0, 1, -1\}$. ∎

For a proof of the following beautiful result in the language of ARSs, see Thm. 6.8.1, p.129 of [M] (the same proof applies to RSs):

Theorem 3.4 (Hörmander-Lojasiewcz Inequality) *Let G be a RS and let $a, b \in G$. Let Y be a closed subspace of X_G such that $Y \cap Z(a) \subseteq Z(b)$. Then, there is $c \in D^t_G(a, b)$ so that $c{\restriction}Y = a{\restriction}Y$* (i.e., for $\sigma \in Y$, $\sigma(c) = \sigma(a)$). ∎

Lemma 3.5 *Let G be a Boolean RS.*

a) For each $a \in G$, there is $\nabla a \in G$ such that

(1) $[\![\nabla a = 1]\!] = [\![a = 1]\!]$ *and* $[\![\nabla a = -1]\!] = Z(a) \cup [\![a = -1]\!]$. *In particular,* $\nabla a \in G^\times$.

(2) $a = \nabla a \cdot a^2$, *and so* $G = G^\times \cdot \mathrm{Id}(G)$. (3) $a,\ a^2 \in D^t_G(1, \nabla a)$.

b) For all $u \in G^\times$, $\nabla u = u$.

c) For all $a, b, c \in G$, $a \in D^t_G(b, c) \;\Rightarrow\; \nabla a \in D^t_G(\nabla b, \nabla c) \cap G^\times = D_{G^\times}(\nabla b, \nabla c)$.

d) For all $a \in G$ and $\tau \in X_G$,

(1) $Z(a) = [\![\nabla(a^2) = -1]\!]$; (2) $\tau \in [\![\nabla(a^2) = 1]\!]$ *iff* $\tau(a) = \tau(\nabla a)$.

Proof. a) For $a \in G$, since X_G is Boolean, $[\![a = 1]\!]$ is clopen in X_G. Now, observe that $-1 \in G$ satisfies

$$[\![a = 1]\!] \cap Z(a) = \emptyset = Z(-1).$$

By Theorem 3.4, there is $\nabla a \in G$ so that $\nabla a \in D^t_G(a, -1)$ and $a{\restriction}[\![a = 1]\!] = \nabla a{\restriction}[\![a = 1]\!]$. Now notice that:

• For $\tau \in Z(a)$, $\tau(\nabla a) \in D^t_3(\tau(a), -1) = D^t_3(0, -1) = \{-1\}$, and $\tau(\nabla a) = -1$;

• For $\tau \in [\![a = -1]\!]$, the same argument as above shows that $\tau(\nabla a) = -1$.

Consequently, the equalities in (1) are verified and $\nabla a \in G^\times$. Items (2) and (b) follow easily from the equalities in (1) (note: $\forall\, \tau \in X_G$, $\tau(\nabla a \cdot a^2) = \tau(a)$; clearly, $a^2 \in \mathrm{Id}(G)$). For (3), since $\nabla a \in D^t_G(-1, a)$, it follows that $-a \in D^t_G(-1, -\nabla a)$, and so, scaling by -1 obtains $a \in D^t_G(1, \nabla a)$; since $D^t_G(1, \nabla a)$ is a subsemigroup of G, the preceding relation yields $a^2 \in D^t_G(1, \nabla a)$.

c) We employ the equalities in item (a.1); for $\tau \in X_G$, we discuss two cases:

(i) $\tau(\nabla a) = 1$: Then, $\tau(a) = 1$ and so, since $\tau(a) \in D_3^t(\tau(b), \tau(c))$, there are only the following possibilities:

(*) $\qquad \tau(b) = \tau(c) = 1$; $\tau(b) = 1$, $\tau(c) = 0$; $\tau(b) = 0$, $\tau(c) = 1$; or $\tau(ab) = -1$.

In the first case in (*), $\tau(\nabla b) = \tau(\nabla c) = 1$, while in the others $\tau(\nabla b)\tau(\nabla c) = -1$ and so $\tau(\nabla a) \in D_3^t(\tau(\nabla b), \tau(\nabla c))$.

(ii) $\tau(\nabla a) = -1$: Then, either $\tau \in Z(a)$ or $\tau \in [\![a = -1]\!]$; in the former case, either $\tau(b) = \tau(c) = 0$ and so $\tau(\nabla b) = \tau(\nabla c) = -1$, or else, $\tau(b) = -\tau(c) \neq 0$ (i.e., the last alternative in (*) above), yielding $\tau(\nabla b)\tau(\nabla c) = -1$, hence $\tau(\nabla a) = -1 \in D_3^t(\tau(\nabla b), \tau(\nabla c))$. In the latter case, $\tau(a) \in D_3^t(\tau(b), \tau(c))$, leads to a list of possibilities as in (*), with -1 replacing 1 in the first three alternatives, whilst the fourth remains the same; hence we obtain either $\tau(\nabla b) = \tau(\nabla c) = -1$ or $\tau(\nabla b)\tau(\nabla c) = -1$, and thus $\tau(\nabla a) \in D_3^t(\tau(\nabla b), \tau(\nabla c))$, as needed.

d) Item (1) is immediate from (a.1); for (2), (a.1) entails $[\![\nabla a^2 = 1]\!] = [\![a^2 = 1]\!]$. If $\tau \in [\![a^2 = 1]\!]$, then $\tau \notin Z(a)$ and either $\tau \in [\![a = 1]\!] = [\![\nabla a = 1]\!]$ or $\tau \in [\![a = -1]\!] = [\![\nabla a = -1]\!] \setminus Z(a)$, whence $\tau(a) = \tau(\nabla a)$, as needed. ∎

Theorem 3.6 *For a real semigroup G, the following are equivalent:*

(1) *For all $x \in G$, there is $u \in G^\times$ so that $x = ux^2$ and $u \in D_G^t(-1, x)$.*

(2) *The spectral topology on X_G is Boolean;*

(3) *$G^\times$, with the representation relation induced by G, is a RSG, and the restriction map $\tau \in X_G \longmapsto \tau\restriction G^\times \in X_{G^\times}$ is a homeomorphism* [1].

Proof. Item (a.2) in Lemma 3.5 yields (2) $\Rightarrow$ (1). Moreover, since the space of orders of any RSG is Boolean, it is clear that (3) $\Rightarrow$ (2). To complete the proof it suffices to establish (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3).

(1) $\Rightarrow$ (2). Fix $a \in G$ and let $u \in G^\times$ satisfy the conditions in (1) with $x = a^2$. For $\tau \in X_G$:

- If $\tau(a) = 0$, then $u \in D_G^t(-1, a^2)$ entails $\tau(u) \in D_3^t(-1, 0)$ and $\tau(u) = -1$;
- If $\tau(u) = -1$, then $\tau(a^2) = -\tau(a^2)$, hence $\tau(a^2) = 0 = \tau(a)$,

and so $[\![a = 0]\!] = [\![a^2 = 0]\!] = [\![u = -1]\!]$; thus, the closed set $[\![a = 0]\!]$ in the spectral topology of X_G is, in fact, clopen in this topology. Hence, (cf. 1.23.(b)), the spectral and constructible topologies in X_G coincide, establishing (2).

(2) $\Rightarrow$ (3). We start by proving:

$G^\times$ is a RSG. By Lemma 1.26 it suffices to verify $G^\times$ satisfies [R4] in Proposition 1.10. Since a, b, c, $d \in G^\times$, the hypothesis in [R4] is equivalent to

(I) $$a \in D_{G^\times}(1, b) \text{ and } b \in D_{G^\times}(c, d).$$

Since $D_G^t \cap G^\times = D_{G^\times}$, (I) yields

(II) $$a \in D_G^t(1, b) \quad \text{and} \quad b \in D_G^t(c, d).$$

But then, (II) together with items (a) and (c) of Proposition 1.20 yield

(III) $$a \in D_G^t(1, c, d) = D_G^t(1, d, c).$$

Hence, (III) and 1.20.(c) yield $z \in D_G^t(1, d)$, such that $a \in D_G^t(z, c)$. Now, items (b) and (c) of Lemma 3.5 imply $\nabla z \in D_{G^\times}(1, d)$ and $a \in D_{G^\times}(\nabla z, c)$. Just take $y = \nabla z \in G^\times$ to establish the consequent of [R 4], completing the proof that $G^\times$ is a reduced special group.

[1] $X_{G^\times}$ is the space of orders of the reduced special group $G^\times$.

X_G is (naturally) homeomorphic to $X_{G^\times}$. Set $\eta : X_G \longrightarrow X_{G^\times}$ given by $\eta(\tau) = \tau\lceil G^\times$; clearly, $\eta(\tau)$ is a group morphism from $G^\times$ to $\{1, -1\}$, taking -1 to -1 and respecting binary representation (because the same is true of τ). Moreover:

• η is continuous: For $u \in G^\times$, just note that $\eta^{-1}\Big[[\![u = 1]\!]_{X_{G^\times}}\Big] \;=\; [\![u = 1]\!]_{X_G}$;

• η is injective: If $\tau \neq \tau'$ in G, there is $a \in G$ such that $\tau(a) \neq \tau'(a)$.

Case 1. If $\tau(a) = 1$, then $\tau'(a) \in \{0, -1\}$, then 3.5.(a.1) implies $\tau(\nabla a) = 1$ and $\tau'(\nabla a) = -1$;

Case 2. If $\tau(a) = -1$ and $\tau'(a) = 0 \in \{0, 1\}$, then $\tau(-a) = 1$, while $\tau'(-a) \in \{0, -1\}$ and so 3.5.(a.1) yields $\tau(\nabla{-a}) = 1$ and $\tau'(\nabla{-a}) = -1$,

therefore, $\tau \neq \tau'$ implies $\eta(\tau) = \tau\lceil G^\times \neq \tau'\lceil G^\times = \eta(\tau')$, as claimed.

It remains to show:

• η is surjective. Since compact sets in a Hausdorff space are closed, we shall establish the surjectivity of η by showing that its image is dense in $X_{G^\times}$. For this, it is enough to check that Im η meets every non-empty basic clopen in $X_{G^\times}$, that is, for $u_1, \ldots, u_n \in G^\times$,

(IV) $\qquad \bigcap_{j=1}^n [\![u_j = 1]\!]_{X_{G^\times}} \neq \emptyset \;\Rightarrow\; \exists\, \tau \in X_G$, so that $\tau(u_j) = 1, \;\; 1 \leq j \leq n$.

To that end we introduce the following notation: if $\varphi = \langle a_1, \ldots, a_n \rangle$ is a form over G, write $\nabla\varphi$ for the n-form over $G^\times$ given by $\langle \nabla a_1, \ldots, \nabla a_n \rangle$. We also need:

Fact 3.7 *Let φ be a form of dimension ≥ 2 over G and let $a \in G$. Then,*

$$a \in D^t_G(\varphi) \quad \Rightarrow \quad \nabla a \in D_{G^\times}(\nabla\varphi).$$

Proof. By Prop. 1.6.(c) in [DM1], if K is special group, then for all forms θ_1, θ_2 over K and $x \in K$,

(I) $\;\; x \in D_K(\theta_1 \oplus \theta_2) \Leftrightarrow \exists\, u \in D_K(\theta_1)$ and $v \in D_K(\theta_2)$ so that $x \in D_K(u, v)$.

For $\dim(\varphi) = 2$, the statement is 3.5.(c); we proceed by induction on the dimension of φ. Assume the result true for $\dim(\psi) = m$ and let $c \in G$. If $a \in D^t_G(\langle c \rangle \oplus \psi)$, then 1.20.(c) yields $x \in D^t_G(\psi)$ such that $a \in D^t_G(c, x)$. Hence, 3.5.(c) and the induction hypothesis entail $\nabla a \in D_{G^\times}(\nabla c, \nabla x)$ and $\nabla x \in D_{G^\times}(\nabla\psi)$. Since $G^\times$ is a RSG, (I) above yields $\nabla a \in D_{G^\times}(\langle \nabla c \rangle \oplus \nabla\psi)$, completing the induction step. □

Consider the Pfister form $\mathcal{P} = \bigotimes_{i=1}^n \langle 1, u_j \rangle$; in $G^\times$, we have $-1 \notin D_{G^\times}(\mathcal{P})$, because, by assumption, there is $\sigma \in X_{G^\times}$ sending all u_j to 1. [2]

Now notice that $-1 \notin D_G(\mathcal{P})$; otherwise, by 1.20.(b.3), we would get $-1 \in D^t_G((-1)^2\mathcal{P}) = D^t_G(\mathcal{P})$, and Fact 3.7 would then entail, since $\mathcal{P} = \nabla\mathcal{P}$ (by 3.5.(b)), $-1 \in D_{G^\times}(\mathcal{P})$, an impossibility.

By Prop. 2.7.(6) in [DP1] and Cor. 4.7 in [DP2], $D_G(\mathcal{P})$ is a saturated subsemigroup of G, containing $\{u_1, \ldots, u_n\}$, but not -1. Now, Thm. 4.2 in [DP1] furnishes a RS-character, τ, whose values in $D_G(\mathcal{P})$ are in $\{0, 1\}$. Since the $u_i \in G^\times$, we must have $\tau(u_i) = 1$, $1 \leq i \leq n$, establishing the density of Im η in $X_{G^\times}$, as needed. ■

Remark 3.8 Theorem 3.6.(1) yields, together with the axioms for RSs in 1.15, a Horn-geometric axiomatization for the class of Boolean RSs in the language real semigroups, $\langle \cdot, 1, -1, D \rangle$ (D^t is defined from D by a conjunction of atomic formulas, cf. [t-rep] in 1.15.(a)). ■

[2] In any RSG, -1 is not represented by $2^n = \bigotimes_{i=1}^n \langle 1, 1 \rangle$.

Before complementing the results in Theorem 3.6, we set down

3.9 Remarks and Notation. a) Let G be a Boolean RS and let $\eta : X_G \longrightarrow X_{G^\times}$ be the homeomorphism given, as above, by $\tau \in X_G \longmapsto \tau \upharpoonright G^\times$. For each $a \in G$, set

$$L(a) = [\![\nabla a^2 = -1]\!]_{X_G^\times} \subseteq X_{G^\times} \text{ and let } L(G^\times) = \{L(a) : a \in G\}.$$

In particular, for e in Id(G), $L(e) = [\![\nabla e = -1]\!]_{X_G^\times}$. Fix $e \in$ Id(G); note that for each $\tau \in X_G$,

$$\tau(e) = 0 \text{ iff } \tau(\nabla e) = -1 \text{ iff } \tau \upharpoonright G^\times(\nabla e) = -1,$$

whence $\eta\,[Z(e)] = [\![\nabla e = -1]\!]_{X_G^\times}$. Since η is a homeomorphism, image by η preserves unions and intersections, and so $L(G^\times)$ is a distributive sub-lattice of the BA of clopens of $X_{G^\times}$ (isomorphic to the BA of clopens of X_G). Hence, the lattices of idempotents in G, that of zero sets in X_G and $L(G^\times)$ are all isomorphic. Note that $L(G^\times)$ is a lattice of **clopens in** $X_{G^\times}$.

b) For $a, b \in G$,

(σ) $$\sigma \in \neg L(a) \cap \neg L(b) = \neg L(ab) \Rightarrow \sigma(\nabla(ab)) = \sigma(\nabla a)\sigma(\nabla b).$$

To see this, let $\mu \in X_G$ be such that $\mu \upharpoonright G^\times = \sigma$. Then, $\mu \in [\![\nabla(a^2) = 1]\!]_{X_G} \cap [\![\nabla(b^2) = 1]\!]_{X_G} = [\![\nabla(a^2b^2) = 1]\!]_{X_G}$ and 3.5.(d) yields $\mu(x) = \mu(\nabla x)$, for $x \in \{a, b, ab\}$. Hence, $\mu(\nabla(ab)) = \mu(ab) = \mu(a)\mu(b) = \mu(\nabla a)\mu(\nabla b)$; since $\mu \upharpoonright G^\times = \sigma$ and $\nabla x \in G^\times$, the preceding equality obtains (σ), as needed. ∎

Proposition 3.10 *If the equivalent conditions in Theorem 3.6 hold, then for all $e \in$ Id(G) and $u \in G^\times$, $\langle 1, u\rangle \otimes \langle 1, \nabla e\rangle \equiv_{G^\times} \langle 1, 1\rangle \otimes \langle 1, \nabla(ue)\rangle$. In particular* (cf. Proposition 2.3)*:*

(1) *$D_{G^\times}(1, \nabla e)$ and $D_{G^\times}(1, -\nabla e)$ are DSSs in the RSG $G^\times$;*

(2) *$L(G^\times) = \{L(a) \subseteq X_{G^\times} : a \in G\}$ is a lattice of direct summands of $X_{G^\times}$* (isomorphic to Id_G).

Proof. Fix $e \in$ Id(G) and $u \in G^\times$. To show $\varphi = \langle 1, u\rangle \otimes \langle 1, \nabla e\rangle \equiv_{G^\times} \langle 1, 1\rangle \otimes \langle \nabla(ue)\rangle = \psi$, it suffices to check their total signatures to be the same; taking into account the homeomorphism η in Theorem 3.6, it suffices to verify the signatures of φ and ψ to be the same at each $\tau \in X_G$ (by Theorem 1.8). For $\tau \in X_G$:

- If $\tau(\nabla e) = -1$, then $\tau \in [\![e = 0]\!] = [\![ue = 0]\!]$ $(u \in G^\times)$ and so $\tau(\nabla(ue)) = -1$, and the signatures of φ and ψ are both 0 at τ;
- If $\tau(\nabla e) = 1$, then $\tau \in [\![e = 1]\!]$, whence $0 \neq \tau(u) = \tau(ue) = \tau(\nabla(ue))$ (recall: by 3.5.(d.2), $\tau \notin Z(ue)$ entails $\tau(ue) = \tau(\nabla(ue))$); thus,

$$\begin{aligned}\tau(\varphi) &= \tau(1) + \tau(u) + \tau(\nabla e) + \tau(u\nabla e) = 2 + 2\,\tau(u) = 2 + 2\tau(\nabla(ue)) \\ &= \tau(\psi),\end{aligned}$$

as needed. Item (1) is an immediate consequence of Proposition 2.3, while (2) follows from Remark 2.4.(1), recalling that for $e \in$ Id(G), $L(e) = [\![-\nabla e = 1]\!]_{X_G^\times}$ is the subspace associated to the DSS $D_{G^\times}(1, -\nabla e)$, ending the proof. ∎

We shall now show how given a RSG, K, and a non-empty bounded sublattice, L, of direct summands of X_K (equivalently, DSSs of K, cf. 2.4.(1)), one can obtain a real semigroup, $\mathcal{G} := \mathcal{G}(K, L)$, such that $X_{\mathcal{G}}$ is naturally homeomorphic to X_K, $\mathcal{G}^\times$ is isomorphic to K and Id$(\mathcal{G})$ is isomorphic to L (with the partial order defined in 1.27).

3.11 Construction To simplify exposition, write X for X_K.

a) Let $P = \mathbb{C}(X, 3)$, the Post algebra of continuous maps from X to 3, where 3 is endowed with the discrete topology. Consider the map

$$(\gamma) \qquad \gamma : K \times L \longrightarrow P, \text{ given by } \langle v, A \rangle \longmapsto \gamma(v, A)(\sigma) = \begin{cases} \sigma(v) & \text{if } \sigma \notin A; \\ 0 & \text{if } \sigma \in A. \end{cases}$$

For $A \in L$ and $a \in K$, set

- $e_A : X \longrightarrow 3$ given by $e_A(\sigma) = \begin{cases} 0 & \text{if } \sigma \in A; \\ 1 & \text{if } \sigma \notin A. \end{cases}$
- $\widehat{a} : X \longrightarrow 3$, given by $\widehat{a}(\sigma) = \sigma(a)$.

b) With notation as in (a), note that:

(1) for $A, B \in L$, $e_A e_B = e_{(A \cup B)}$. Moreover, $e_\emptyset = 1$ and $e_X = 0$.

(2) For each $\langle a, A \rangle \in K \times L$, $\gamma(a, A) = \widehat{a}\, e_A$.

(3) To ease notation, write the elements of $\operatorname{Im} \gamma \subseteq P$ as ae_A, instead of $\widehat{a}\, e_A$ $(a \in K,\ A \in L)$.

c) It is straightforward to check that $ae_A \cdot be_B = abe_Ae_B = abe_{A \cup B}$. Hence,

$$\operatorname{Im} \gamma \ := \ \mathcal{G}(K, L) \ = \ \{ae_A \in P : \langle a, A \rangle \in K \times L\},$$

is a ternary subsemigroup of P, with $1 = e_\emptyset$, $0 = e_X$ and $-1 = -1e_\emptyset$. As usual, write $-v$ for $-1 \cdot v$.

Endow $\mathcal{G} := \mathcal{G}(K, L)$ with the representation and transversal representation induced by the Post algebra P; therefore, for $a, b, c \in K$ and $A, B, C \in L$, $D^t_{\mathcal{G}}$ is given by:

$$ae_A \in D^t_{\mathcal{G}}(be_B, ce_C) \ \text{ iff } \ \text{for all } \sigma \in X, \ \ ae_A(\sigma) \in D^t_3(be_B(\sigma), ce_C(\sigma)).$$

Note: The values in 3 of $\sigma \in X$ at each element of $\mathcal{G}$ appearing in the preceding expression is described by formula (γ) above. ∎

Since all axioms of RSs, with the exception of [RS 3], are universal, and $\mathcal{G} \subseteq P$ (a real semigroup, cf. 3.2.(b)), **$\mathcal{G}$ is pre-real semigroup.** To show it is, in fact, a real semigroup, we will employ an equivalent to [RS 3], namely, the exchange principle (cf. 1.16).

Let $A, B, C, D \in L$ and $a, b, c, d \in K$ and set

$$\mathfrak{a} = ae_A, \quad \mathfrak{b} = be_B, \quad \mathfrak{c} = ce_C \quad \text{and} \quad \mathfrak{d} = de_D.$$

For $W \in L$, write $\neg W$ for $X \setminus W$.

We say that the *exchange principle holds for* $\langle \mathfrak{a}, \mathfrak{b}; \mathfrak{c}, \mathfrak{d} \rangle$ if

$$\text{(exch)} \qquad D^t_{\mathcal{G}}(\mathfrak{a}, \mathfrak{b}) \cap D^t_{\mathcal{G}}(\mathfrak{c}, \mathfrak{d}) \neq \emptyset \ \Rightarrow \ D^t_{\mathcal{G}}(\mathfrak{a}, -\mathfrak{c}) \cap D^t_{\mathcal{G}}(-\mathfrak{b}, \mathfrak{d}) \neq \emptyset.$$

We start with the following observations:

Lemma 3.12 *With notation as above,*

a) $ae_A = be_B$ *iff* $A = B$ *and for all* $\sigma \in \neg A = \neg B$, $\sigma(a) = \sigma(b)$. *In other words,* $\mathfrak{a} = \mathfrak{b}$ *iff* $A = B$ *and* $a/\Delta = b/\Delta$, *where* $\Delta = \Sigma(\neg A)$, *the saturated subgroup associated to* $\neg A$ (cf. 1.12.(e)).

b) $\operatorname{Id}(\mathcal{G}) = \{e_A : A \in L\}$. *Moreover, the map* $A \in L \longmapsto e_A \in \operatorname{Id}(\mathcal{G})$ *is a lattice isomorphism between* L (cf. 1.27) *and* $\operatorname{Id}(\mathcal{G})$

c) We may identify $\mathcal{G}^\times$ *with* K, *that is,* $ae_A \in \mathcal{G}^\times$ *iff* $A = \emptyset$ *(and so* $ae_A = a$*).*

d) For $\mathfrak{a} = ae_A \in \mathcal{G}$, *there is* $u \in K = \mathcal{G}^\times$ *so that* $\mathfrak{a} = u\mathfrak{a}^2$ *and* $u \in D^t_{\mathcal{G}}(-1, \mathfrak{a})$. *Thus, the pre-real semigroup* $\mathcal{G}$ *satisfies condition* (1) *in Theorem* 3.6.

e) $D^t_{\mathcal{G}}(\mathfrak{a}, \mathfrak{b}) \neq \emptyset$.

f) If any one among $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ *is equal to* 0, *then the exchange principle holds for* $\langle \mathfrak{a}, \mathfrak{b}; \mathfrak{c}, \mathfrak{d} \rangle$.

g) If $0 \in D^t_{\mathcal{G}}(\mathfrak{a}, \mathfrak{b}) \cap D^t_{\mathcal{G}}(\mathfrak{c}, \mathfrak{d})$, *then the exchange principle holds for* $\langle \mathfrak{a}, \mathfrak{b}; \mathfrak{c}, \mathfrak{d} \rangle$.

Proof. a) If $\sigma \in A$, then $\sigma(ae_A) = 0$ iff $\sigma \in A$ (recall: a, $b \in K$). It is then clear that $\mathfrak{a} = \mathfrak{b}$ implies $A = B$ and $\sigma(a) = \sigma(b)$ for all $\sigma \notin A = B$. The converse is clear. The second statement in (a) is just a rephrasing of the proven equivalence. Item (b) is straightforward.

c) Clearly, $ae_A \in \mathcal{G}^{\times}$ iff its value at each $\sigma \in X$ is non-zero, i.e., $A = \emptyset$.

d) Let $\mathfrak{a} = ae_A \in \mathcal{G}$; note that $\mathfrak{a}^2 = e_A$ $(a \in K)$. Let $\Delta = \Sigma(A)$ and $\Delta^{\perp} = \Sigma(\neg A)$ be the saturated DSSs associated to A and $\neg A$ (cf. 1.12.(e)). Since $x \in K \longmapsto (x/\Delta,\ x/\Delta^{\perp})$ is an isomorphism, there is $u \in K$ so that $u/\Delta = -1/\Delta$ and $u/\Delta^{\perp} = a/\Delta^{\perp}$. Hence, for all $\sigma \in X$:

$$(1)\ \sigma \in \neg A \ \Rightarrow\ \sigma(u) = \sigma(a); \qquad (2)\ \sigma \in A \ \Rightarrow\ \sigma(u) = -1.$$

Note that (1) and item (a) entail $u\mathfrak{a}^2 = ue_A = ae_A = \mathfrak{a}$. Moreover, (1) and (2) obtain $u \in D^t_{\mathcal{G}}(-1, \mathfrak{a})$. Indeed, for $\sigma \in X_G$, we have:

- If $\sigma \in A$, then $\sigma(u) = -1 \in D^t_3(-1, 0)$;
- If $\sigma \in \neg A$, then $\sigma(u) = \sigma(a)$; since u, $a \in K$, then $-\sigma(u)^2 = \sigma(a)^2 = 1$. Whence, by 1.19.(14), $1 \in D^t_3(-\sigma(u), \sigma(a)) = D^t_3(-\sigma(a), \sigma(a)) = 3$ and 1.19.(1)) entails $\sigma(u) \in D^t_3(-1, \sigma(a))$.

e) Let $E_1 = A \cap B$, $E_2 = A \cap \neg B$, $E_3 = \neg A \cap B$ and $E_4 = \neg A \cap \neg B$ be the four atoms of the BA generated by A and B in $B(X)$. Let $\Delta_k = \Sigma(E_k)$ be the associated saturated subgroup of K, $1 \le k \le 4$ (as in 1.12.(e)). For u, $v \in K$ and $1 \le k \le 4$, the expression "$u = v$ in E_k" stands for $u/\Delta_k = v/\Delta_k$. We now consider the conditions required to construct a witness for the claim in (e) in each of the atoms E_k:

(1) $E_1 = A \cap B$. In this case, we take $y_1 = 1$ in E_1;

(2) $E_2 = A \cap \neg B$. If $\sigma \in E_2$, then a witness t for our claim must satisfy $\sigma(t) \in D^t_3(0, \sigma(b))$, i.e., $\sigma(t) = \sigma(b)$. In this case, we take $y_2 = b$ in E_2;

(3) $E_3 = \neg A \cap B$. With the same reasoning as in (3), we take $y_3 = a$ in E_3;

(4) $E_4 = \neg A \cap \neg B$. If $\sigma \in E_4$, then a witness t for (e) must verify $\sigma(t) \in D^t_3(\sigma(a), \sigma(b))$; in this case, we take $y_4 = a$ in E_4.

Since K is isomorphic to $\prod_{k=1}^{4} K/\Delta_k$, there is $v \in K$ so that $v/\Delta_k = y_k/\Delta_k$, $1 \le k \le 4$. Then,

(*)
$$t = ve_{A \cap B} \in D^t_{\mathcal{G}}(\mathfrak{a}, \mathfrak{b}).$$

To prove (*), it suffices to show that for each $\sigma \in X$, we have $\sigma(t) \in D^t_{\mathcal{G}}(\sigma(\mathfrak{a}), \sigma(\mathfrak{b}))$. Since $X = \bigcup_{k=1}^{4} E_k$, we prove (*) holds for σ in each of the atoms E_k, taking into account the selections made in (1) – (4) above.

(1*) If $\sigma \in E_1$, then $\sigma(t) = 0 \in D^t_3(\sigma(\mathfrak{a}), \sigma(\mathfrak{b})) = D^t_3(0, 0)$;

(2*) If $\sigma \in E_2$, then $\sigma(t) = \sigma(v) = \sigma(b) \in D^t_3(\sigma(\mathfrak{a}), \sigma(\mathfrak{b})) = D^t_3(0, \sigma(b))$, as needed; a similar reasoning applies if $\sigma \in E_3$;

(4*) If $\sigma \in E_4$, then $\sigma(t) = \sigma(v) = \sigma(a) \in D^t_3(\sigma(\mathfrak{a}), \sigma(\mathfrak{b})) = D^t_3(\sigma(a), \sigma(b))$, which holds because a, $b \in K$ and so $\sigma(a)$, $\sigma(b) \in \{1, -1\}$,

establishing (*) and completing the proof of (e).

f) Without loss of generality, we may assume $\mathfrak{a} = 0$; if $t \in \mathcal{G}$ satisfies $t \in D^t_{\mathcal{G}}(0, \mathfrak{b}) \cap D^t_{\mathcal{G}}(\mathfrak{c}, \mathfrak{d})$, then $t = \mathfrak{b}$ [3]. But $\mathfrak{b} \in D^t_{\mathcal{G}}(\mathfrak{c}, \mathfrak{d})$ entails $-\mathfrak{c} \in D^t_{\mathcal{G}}(-\mathfrak{b}, \mathfrak{d})$, whence $-\mathfrak{c} \in D^t_{\mathcal{G}}(0, -\mathfrak{c}) \cap D^t_{\mathcal{G}}(-\mathfrak{b}, \mathfrak{d})$, as needed.

[3] Recall (1), (14a) and (14b) in 1.19.

g) If $0 \in D^t_{\mathcal{G}}(\mathfrak{a}, \mathfrak{b}) \cap D^t_{\mathcal{G}}(\mathfrak{c}, \mathfrak{d})$, then (cf. 1.19.(12)), $\mathfrak{b} = -\mathfrak{a}$ and $\mathfrak{d} = -\mathfrak{c}$, and the exchange principle leads to $D^t_{\mathcal{G}}(\mathfrak{a}, -\mathfrak{c}) \neq \emptyset$, that is guaranteed by (e). ∎

We now have

Theorem 3.13 *$\mathcal{G} = \mathcal{G}(K, L)$ is a Boolean real semigroup, with $\mathcal{G}^\times = K$, $\mathrm{Id}(\mathcal{G})$ lattice isomorphic to L and $X_{\mathcal{G}}$ naturally isomorphic to X_K, via the restriction map $\tau \longmapsto \tau\restriction K$.*

Proof. In view of Lemma 3.12 and the equivalences in Theorem 3.6, it remains only to establish that $\mathcal{G}$ is a RS. Moreover, again due to 3.12, in this proof we may assume, for $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d} \in \mathcal{G}$:

(!) There is $t \in D^t_{\mathcal{G}}(\mathfrak{a}, \mathfrak{b}) \cap D^t_{\mathcal{G}}(\mathfrak{c}, \mathfrak{d})$, and $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, t are all distinct from 0.

To ease presentation, we introduce the following

3.14 Notation and Remarks. a) If $t \in \mathcal{G} \setminus \{0\}$, to ease the discussion that follows, write t^* for a unit in K (cf. 3.12.(a)) that determines t, i.e., $t = t^* e_W$, for some $W \in L$. Let $\mathfrak{a} = ae_A$, $\mathfrak{b} = be_B$, $\mathfrak{c} = ce_C$ and $\mathfrak{d} = de_D$. Note that

(&) $$(A \cap B) \cup (C \cap D) \subseteq W,$$

since if σ belongs to this union, then $\sigma(t) = 0$.

b) If G is a RS, $x \in G^\times$ and $y \in G$, it follows easily from [RS 1] and [RS 6] in 1.15.(a) that $x \in D^t_G(x, y)$.

c) We shall construct two tables, the first corresponding to case in which $\sigma \in \neg W$ and the second for $\sigma \in W$ (and so $\sigma(t) = 0$).

For $Z \in L$ and $x, y, z \in K$, the tables below uses the following conventions, with $\Delta = \Sigma(Z)$ (cf. 1.12.(e)):

- "$x = y$ in Z" stands for $x/\Delta = y/\Delta$ or equivalently, for all $\sigma \in Z$, $\sigma(x) = \sigma(y)$.
- "$x \in D(y, z)$ in Z" stands for $x/\Delta \in D_{K/\Delta}(y/\Delta, z/\Delta)$;
- A "1" in a column means that we are outside that set, while a "0" means we are in that set. For instance, the sequence "1 0 1 0" in the columns marked A, B, C, D corresponds to $\neg A \cap B \cap \neg C \cap D$, which in first table is the line E_6 and line E_{22} in Table 2.
- The column marked "$\bigcap$" has either a $\surd$, meaning that the atom E_k is not necessarily empty, or $\emptyset$ (whose meaning is obvious).
- We assume there is a witness, $t \in \mathcal{G}$, for the antecedent in [RS 3′]; its possible values and the constraints it imposes on the coefficients in each of the atoms E_k, $1 \leq k \leq 32$, of the BA generated by A, B, C, D, W is registered in the corresponding column in each table.
- The column corresponding to $y \in K$ yields the values and conditions for it to be a witness of the consequent of [RS 3′]. For instance, the first line of the first table indicates that in E_1, $D(a, b) \cap D(c, d) \neq \emptyset$, with $a, b, c, d \in K$. But then 1.5.(b) yields $D(a, -c) \cap D(-b, d) \neq \emptyset$ in E_1, and so it is possible select y_1 in this intersection in E_1.
- In the table for W, the expression "impossible" in the last column indicates that the condition on the column "constraints (in E_k)" cannot hold; thus, the antecedent of our implication is false in E_k.
- **Note:** The intersection of the sets in any line containing a unique 1 (all other entries are 0) *must be empty*: e.g., consider $E_8 = \neg A \cap B \cap C \cap D \cap \neg W$; if $\sigma \in E_8$, then we would have
$$\sigma(t) \in D^t_3(\sigma(\mathfrak{a}), \sigma(\mathfrak{b})) \cap D^t_3(\sigma(\mathfrak{c}), \sigma(\mathfrak{d})) = D^t_3(\sigma(a), 0) \cap D^t_3(0, 0),$$
that is impossible: $D^t_3(0, 0) = \{0\}$, while $D^t_3(\sigma(a), 0) = \{\sigma(a)\} \subseteq \{1, -1\}$. ∎

Table 1. $\sigma \in \neg W$ and for $1 \le k \le 16$, we assume $E_k \subseteq \neg W$; recall (&) in 3.14.(a).

	A	B	C	D	$\bigcap$	t^* and constraints (in E_k)	A	C	B	D	y_k and constraints (in E_k)
E_1	1	1	1	1	$\surd$	$t^* \in D(a, b) \cap D(c, d)$	1	1	1	1	$y_1 \in D(a, -c) \cap D(-b, d)$
E_2	1	1	1	0	$\surd$	$t^* = c \in D(a, b)$	1	1	1	0	$y_2 = -b \in D(a, -c)$
E_3	1	1	0	1	$\surd$	$t^* = d \in D(a, b)$	1	0	1	1	$y_3 = a \in D(-b, d)$
E_4	1	1	0	0	$\emptyset$	———	1	0	1	0	———
E_5	1	0	1	1	$\surd$	$t^* = a \in D(c, d)$	1	1	0	1	$y_5 = d \in D(a, -c)$
E_6	1	0	1	0	$\surd$	$t^* = a = c$	1	1	0	0	$y_6 = 1$; $a = -(-c) = c$
E_7	1	0	0	1	$\surd$	$t^* = a = d$	1	0	0	1	$y_7 = a = d$
E_8	1	0	0	0	$\emptyset$	———	1	0	0	0	———
E_9	0	1	1	1	$\surd$	$t^* = b \in D(c, d)$	0	1	1	1	$y_9 = -c \in D(-b, d)$
E_{10}	0	1	1	0	$\surd$	$t^* = b = c$	0	1	1	0	$y_{10} = -c = -b$
E_{11}	0	1	0	1	$\surd$	$t^* = b = d$	0	0	1	1	$y_{11} = 1$; $b = -(-b) = d$
E_{12}	0	1	0	0	$\emptyset$	———	0	0	1	0	———
E_{13}	0	0	1	1	$\emptyset$	———	0	1	0	1	———
E_{14}	0	0	1	0	$\emptyset$	———	0	1	0	0	———
E_{15}	0	0	0	1	$\emptyset$	———	0	0	0	1	———
E_{16}	0	0	0	0	$\emptyset$	———	0	0	0	0	———

Table 2. $\sigma \in W$ and so $\sigma(t) = 0$; for $16 \le k \le 32$, we assume $E_k \subseteq W$.

	A	B	C	D	$\bigcap$	constraints (in E_k)	A	C	B	D	y_k and constraints (in E_k)
E_{17}	1	1	1	1	$\surd$	$0 \in D^t_G(a, b) \cap D^t_G(c, d)$	1	1	1	1	$b = -a$, $d = -c$, $y_{17} = a$
E_{18}	1	1	1	0	$\emptyset$	$0 = c \in D(a, b)$	1	1	1	0	impossible
E_{19}	1	1	0	1	$\emptyset$	$0 = d \in D(a, b)$	1	0	1	1	impossible
E_{20}	1	1	0	0	$\surd$	$t = 0$; $b = -a$	1	0	1	0	$y_{20} = a = -b$
E_{21}	1	0	1	1	$\emptyset$	$0 = a \in D(c, d)$	1	1	0	1	impossible
E_{22}	1	0	1	0	$\emptyset$	$0 = a = c$	1	1	0	0	impossible
E_{23}	1	0	0	1	$\emptyset$	$0 = a = d$	1	0	0	1	impossible
E_{24}	1	0	0	0	$\emptyset$	———	1	0	0	0	———
E_{25}	0	1	1	1	$\emptyset$	$0 = b \in D(c, d)$	0	1	1	1	impossible
E_{26}	0	1	1	0	$\emptyset$	$0 = b = c$	0	1	1	0	impossible
E_{27}	0	1	0	1	$\emptyset$	$0 = b = d$	0	0	1	1	impossible
E_{28}	0	1	0	0	$\emptyset$	———	0	0	1	0	———
E_{29}	0	0	1	1	$\surd$	$t = 0$; $d = -c$	0	1	0	1	$y_{29} = -c = d$
E_{30}	0	0	1	0	$\emptyset$	———	0	1	0	0	———
E_{31}	0	0	0	1	$\emptyset$	———	0	0	0	1	———
E_{32}	0	0	0	0	$\surd$	$t = 0$	0	0	0	0	$y_{32} = 1$

Since we are assuming that t is a witness for the antecedent of the implication corresponding to the exchange principle, and $W = \bigcup_{k=17}^{32} E_k$, Table 2 shows that except for E_{17}, E_{20}, E_{29} and E_{32}, all other E_k ($17 \le k \le 32$) must be empty. Thus, $W = E_{17} \cup E_{20} \cup E_{29} \cup E_{32}$ (disjoint union).

Notation as above, let $\mathfrak{q} = \{k \in \{1, \ldots, 32\} : E_k \neq \emptyset\}$; set $\Delta_k = \Sigma(E_k)$, for $k \in \mathfrak{q}$. By Fact 2.5, there is a natural isomorphism between K and $\prod_{k \in \mathfrak{q}} K/\Delta_k$. Hence, there is $v \in K$ so that $v/\Delta_k = y_k/\Delta_k$, for each $k \in \mathfrak{q}$.

Let $V = (A \cap C) \cup (B \cap D)$; we shall verify that

(#)
$$z = ve_V \in D^t_G(ae_A, -ce_C) \cap D^t_G(-be_B, de_D) = D^t_G(\mathfrak{a}, -\mathfrak{c}) \cap D^t_G(-\mathfrak{b}, \mathfrak{d}),$$

establishing [RS 3′] and showing that $\mathcal{G}$ is a real semigroup.

We have $V = V \cap (\neg W \cup W) = (V \cap \neg W) \cup (V \cap W)$. We discuss the following cases:

I. In $A \cap C \cap \neg W$. If $\sigma \in A \cap C \cap \neg W$, then $\sigma(z) = 0$; the pertinent line in Table 1 is E_{11} ($E_{16} = E_{12} = E_{15} = \emptyset$).

• If $\sigma \in E_{11}$, then $\sigma(z) = 0 \in D^t_3(0, 0) \cap D^t_3(\sigma(-b), \sigma(d)) = D^t_3(\sigma(-b), \sigma(b))$, as needed. Similarly, one treats the case of each $\sigma \in B \cap D \cap \neg W$.

II. In $A \cap C \cap W$. The pertinent line here is E_{32} in Table 2. But then we have $\sigma(z) = 0 = \sigma(\mathfrak{a}) = \sigma(\mathfrak{b}) = \sigma(\mathfrak{c}) = \sigma(\mathfrak{d})$ and the desired conclusion is immediate. A similar argument applies of the case $\sigma \in B \cap D \cap W$.

If $\sigma \in X \setminus V$, then

$$\sigma \in (\neg A \cap \neg B) \cup (\neg A \cap \neg D) \cup (\neg C \cap \neg B) \cup (\neg C \cap \neg D),$$

which may be written as $(X \setminus V) \cap (\neg W \cup W)$.

III. In $\neg A \cap \neg B \cap \neg W$. The pertinent lines of Table 1 are $E_1 - E_3$.

III.1. If $\sigma \in E_1$, then

$$\begin{aligned}\sigma(z) = \sigma(v) = \sigma(y_1) &\in D^t_3(\sigma(a), \sigma(-c)) \cap D^t_3(\sigma(-b), \sigma(d)) \\ &= D^t_3(\sigma(\mathfrak{a}), \sigma(-\mathfrak{c})) \cap D^t_3(\sigma(-\mathfrak{b}), \sigma(\mathfrak{d}));\end{aligned}$$

III.2. If $\sigma \in E_2$, then

$$\begin{aligned}\sigma(z) = \sigma(y_2) = \sigma(-b) &\in D^t_3(\sigma(a), \sigma(-c)) \cap D^t_3(\sigma(-b), 0) \\ &= D^t_3(\sigma(\mathfrak{a}), \sigma(-\mathfrak{c})) \cap D^t_3(\sigma(-\mathfrak{b}), 0).\end{aligned}$$

Similarly, one treats the cases in which $\sigma \in E_3$ and $\sigma \in \neg C \cap \neg D \cap \neg W$.

IV. In $\neg A \cap \neg D \cap \neg W$. The pertinent lines in Table 1 are E_1, E_3, E_5, E_7.

IV.1. If $\sigma \in E_1$, the argument is just as case III.1. above;

IV.2. If $\sigma \in E_3$, then $\sigma(z) = \sigma(y_3) = \sigma(a) \in D^t_3(\sigma(a), 0) \cap D^t_3(\sigma(-b), \sigma(d))$
$$= D^t_3(\sigma(\mathfrak{a}), 0) \cap D^t_3(\sigma(-\mathfrak{b}), \sigma(\mathfrak{d}));$$

IV.3 If $\sigma \in E_5$, then $\sigma(z) = \sigma(y_5) = \sigma(d) \in D^t_3(\sigma(a), \sigma(-c)) \cap D^t_3(0, \sigma(d))$
$$= D^t_3(\sigma(\mathfrak{a}), \sigma(-\mathfrak{c}) \cap D^t_3(0, \sigma(\mathfrak{d}));$$

IV.4. If $\sigma \in E_7$, $\sigma(z) = \sigma(y_7) = \sigma(a) = \sigma(d) \in D^t_3(\sigma(a), 0) \cap D^t_3(0, \sigma(d))$
$$= D^t_3(\sigma(\mathfrak{a}), 0) \cap D^t_3(0, \sigma(\mathfrak{d})),$$

as needed. The case in which $\sigma \in \neg B \cap \neg C \cap \neg W$ is handled similarly.

V. In $\neg A \cap \neg B \cap W$. The relevant lines in Table 2 are E_{17} and E_{20}.

• If $\sigma \in E_{17}$, then $\sigma(z) = \sigma(v) = \sigma(a) \in D^t_3(\sigma(a), \sigma(-c))$, because $\sigma(a)$ is a unit in 3.

• If $\sigma \in E_{20}$, then $\sigma(z) = \sigma(v) = \sigma(y_{20}) = \sigma(a) = \sigma(-b) \in D^t_3(\sigma(\mathfrak{a}), 0) \cap D^t_3(\sigma(-\mathfrak{b}), 0)$, as needed. The case $\neg C \cap \neg D \cap W$ can be treated similarly.

VI. In $\neg A \cap \neg D \cap W$ and $\neg B \cap \neg C \cap W$. The relevant line is E_{17} in Table 2 and same argument used for $\sigma \in E_{17}$ in (V) above also applies here.

This completes the proof of (#) and that $\mathcal{G}$ is a real semigroup. ∎

We can now state, with notation as in 3.9 and recalling that $L(G^{\times})$ is lattice-isomorphic to Id_G:

Theorem 3.15 (Structure Theorem for Boolean RS) *If G is a Boolean RS, there is a* *natural RS-isomorphism between G and* $\mathcal{G}(G^\times, L(G^\times))$, *given by the map* $a \in G \stackrel{f}{\longmapsto} \nabla a e_{L(a)} \in \mathcal{G}(G^\times, L(G^\times))$.

Proof. To make matters clearer, we maintain a distinction between the homeomorphic spaces X_G and $X_{G^\times}$. Let $\eta : X_G \longrightarrow X_{G^\times}$, $\eta(\tau) = \tau \restriction G^\times$, be the homeomorphism in Theorem 3.6.(3). By 3.9, for each $a \in G$, $L(a) = \eta[Z(a)]$ and so, $\eta^{-1}[\neg L(a)] = [\![a = 1]\!]_{X_G} \cup [\![a = -1]\!]_{X_G} = [\![a^2 = 1]\!]_{X_G}$. Hence, recalling 3.5.(d.2), for all $a \in G$ and $\tau \in X_G$,

$$\text{(I)} \quad \begin{cases} (1)\ \tau \in Z(a) \ \text{ iff } \ \tau \restriction G^\times \in L(a); \\ (2)\ \tau \in [\![a^2 = 1]\!]_{X_G} = [\![\nabla(a^2) = 1]\!]_{X_G} \text{ iff } \tau(a) = \tau(\nabla a) \text{ iff } \tau \restriction G^\times \in \neg L(a). \end{cases}$$

We first note that $f(0) = \nabla 0 e_X = 0$, $f(1) = \nabla 1 e_\emptyset = 1$ and $f(-1) = \nabla(-1)e_\emptyset = -1$. Next, we show that f preserves products. For $a, b \in G$, we have

$$\begin{cases} f(ab) = \nabla(ab)e_{L(ab)} = \nabla(ab)e_{L(a)\cup L(b)} & \text{and} \\ f(a)f(b) = \nabla a \nabla b e_{L(a)} e_{L(b)} = \nabla a \nabla b e_{L(a)\cup L(b)}. \end{cases}$$

For $\sigma \in X_{G^\times}$:

- If $\sigma \in L(a) \cup L(b)$, then both $f(ab)$ and $f(a)f(b)$ are zero;
- If $\sigma \notin L(a) \cup L(b)$, let τ_s be the unique element of X_G so that $\sigma = \tau_s \restriction G^\times$. Then, by (I).(2) we get $\tau_s \in [\![a^2 = 1]\!]_{X_G} \cap [\![b^2 = 1]\!]_{X_G} = [\![(ab)^2 = 1]\!]_{X_G}$ and so

$$(*) \qquad \tau_s(a) = \tau_s(\nabla a),\ \tau_s(b) = \tau_s(\nabla b) \text{ and } \tau_s(ab) = \tau_s(\nabla(ab)) = \tau_s(\nabla a)\tau_s(\nabla b).$$

Since $\nabla x \in G^\times$ for $x \in G$ and $\tau_s \restriction G^\times = \sigma$, (*) entails $\sigma(\nabla(ab)) = \sigma(\nabla a)\sigma(\nabla b)$ and so, for all $\sigma \in X$, $\sigma(f(ab)) = \sigma(f(a)f(b))$, yielding $f(ab) = f(a)f(b)$ in $\mathcal{G}$, as needed.

<u>f is injective.</u> If, for $a, b \in G$, we have $\nabla a e_{L(a)} = \nabla b e_{L(b)}$, then 3.12.(a) entails $L(a) = L(b)$ and for all $\sigma \notin L(a) = L(b)$, the equalities $\sigma(\nabla a) = \sigma(\nabla b)$. Note that $L(a) = L(b)$ entails $Z(a) = Z(b)$ in X_G. We have $a = \nabla a \cdot a^2$ and $b = \nabla b \cdot b^2$ (3.5.(a.2)); for $\tau \in X_G$:

- If $\tau \in Z(a) = Z(b)$, then $\tau(a) = \tau(b) = 0$;
- If $\tau \in [\![a^2 = 1]\!]_{X_G} = [\![b^2 = 1]\!]_{X_G}$, then $\tau \restriction G^\times \in \neg L(a) = \neg L(b)$ and so $\tau(\nabla a) = \tau(\nabla b)$, entailing $\tau(a) = \tau(b)$.

Since τ is arbitrary in X_G, we obtain $a = b$, establishing the injectivity of f.

<u>f is surjective.</u> Let $t = u e_{L(a)} \in \mathcal{G}$, for some $u \in G^\times$ and $a \in G$. Let $b \in G$ be given by $b = ua^2 = \nabla b \cdot b^2$; then, $b^2 = a^2$ and so $L(a) = L(b)$. Thus, $t = u e_{L(b)}$; we show that $f(b) = \nabla b e_{L(b)} = u e_{L(b)} = t$. If $\sigma \in L(b)$, then $\sigma(f(b)) = 0 = \sigma(t)$. If $\sigma \in \neg L(b)$, let $\tau_s \in X_G$ satisfy $\tau_s \restriction G^\times = \sigma$; then, $\tau_s \in [\![b^2 = 1]\!]_{X_G}$ and so, by 3.5.(d.2), $\tau_s(b) = \tau_s(\nabla b) = \tau_s(u)$. Since $u, \nabla b \in G^\times$, we obtain $\sigma(u) = \sigma(\nabla b)$. Now, Lemma 3.12.(a) entails $f(b) = \nabla b e_{L(b)} = u e_{L(b)}$, as desired.

To finish the proof, observe that the arguments presented above show that for each $a \in G$ and all $\tau \in X_G$,

$$(**) \qquad \tau(a) \ = \ \tau \restriction G^\times(\nabla a e_{L(a)}) \ = \ \tau \restriction G^\times(f(a)).$$

Since $\tau \longmapsto \tau \restriction G^\times$ and f are bijections, (**) implies that f must be an isomorphism, ending the proof. ∎

We end this section with two themes: the first is a characterization of semi-real rings (commutative and unitary) whose associated RS is Boolean; the second is to establish that the class of Boolean RSs is closed under a number of important constructions.

Proposition 3.16 *The real spectrum of a reduced semireal unitary commutative ring is Boolean iff its real closure is von Neumann regular.*

Proof. The equivalence is forthcoming from the following two well-known facts:

(1) The real spectrum of a reduced, semireal unitary commutative ring is homeomorphic to the Zariski spectrum of its real closure (cf. 13.6.3, p. 534, [DST]);

(2) The Zariski spectrum of a unitary commutative ring is Boolean iff it is von Neumann regular (cf. second paragraph, p. 71, [DST]). ∎

Remark 3.17 The characterization in 3.16 can, perhaps, be sharpened. To give an example, just consider the ring of integers $\mathbb{Z}$: its real spectra is Boolean, but it is very far from being von Neumann regular. It is an interesting – and seemingly hard – question to obtain a characterization in terms of the original ring. ∎

Theorem 3.18 *a) The class of Boolean real semigroups is closed under arbitrary*

(1) *Boolean extensions;* (2) *filtered colimits;* (3) *Products;*

(4) *RS-sums;* (5) *Reduced products; in particular, ultraproducts.*

b) Let G, H be RSs and let $f : G \longrightarrow H$ be a surjective *RS-morphism. If G is a Boolean RS, then the same is true of H. In particular, the class of Boolean RSs is closed under quotients by RS-congruences.*

Proof. a) (1) Let X be a Boolean space, G be a Boolean RS and let $T = \mathbb{C}(X, G)$ be the Boolean power of G by X, that is, the set of all locally constant G-valued maps on X. By Thm. 2.5 in [DMP], T is a real semigroup, whose space of RS-characters is $X \times X_G$ (with the product topology) and so T is also a Boolean RS.

(2) Since Boolean RSs are Horn-geometric axiomatizable, it follows from Prop. 3.2 in [DMP] that the class of Boolean RSs is closed under arbitrary directed colimits (or inductive limits).

(3) Again, the Horn-geometric axiomatizability of Boolean RSs and a classical result by Kiesler, Galvin and Shelah (Thm. 6.2.5′, p. 366, [CK]), guarantees that the class of Boolean RSs is closed under arbitrary products.

(4) Let $\mathcal{R} = \{G_i \colon i \in I\}$ be a non-empty family of Boolean RSs. For the definition of the RS-sum of $\mathcal{R}$, $\bigoplus_{i\in I} G_i$, we refer the reader to Def. 4.2 in [DMP]. If I is finite, then, $\bigoplus_{i\in I} G_i$ is the product of the G_i (Prop. 4.3.(a), [DMP]), a case already covered by (3). Henceforth, we assume I is *infinite*. Let Fin(I) be the set of all non-empty finite subsets of I; for each $F \in$ Fin(I), define

$$G_F^\flat \;=\; \left(\textstyle\prod_{i\in F} G_i\right) \times \mathbf{3},$$

with its natural product structure; note that $G_F^?$ is a Boolean RS (by (3)). Further, By Lemma 4.1.(b) of [DMP], if $K \subseteq J \in$ Fin(I), there are RS-morphisms (in fact, pure embeddings) $\iota_{KJ} : G_K^? \longrightarrow G_J^?$; moreover, it is shown in the proof of item (b) of Prop. 4.3 in [DMP], that $\bigoplus_{i\in I} G_i$ is the inductive limit of the $G_F^\flat$, with $F \in$ Fin(I), partially ordered under inclusion. The desired conclusion follows from item (2).

(5) It is well-known that reduced powers by a filter are inductive limits of products, and that this construction preserves Horn-geometric theories.

b) It is well-known (and straightforward to check) that positive $\forall\exists$ sentences are preserved by surjective L-morphisms, where L is any first-order language with equality. Hence, by the equivalence in item (a) of Theorem 3.6, the desired conclusion is immediately forthcoming. ∎

4 Morphisms of Boolean Real Semigroups

Since in this section we shall be dealing with several real semigroups, to ease presentation introduce the following

4.1 Notation If G_i is an RS, $K_i := G_i^{\times}$ is its RSG of units, $a \in G_i$, $u \in K_i$, $\varepsilon \in 3$ and $\mu \in \{1, -1\}$, write

$$[\![a = \varepsilon]\!]_i = \{\tau \in X_{G_i} : \tau(a) = \varepsilon\} \quad \text{and} \quad [\![u = \mu]\!]_i^{\times} = \{\sigma \in K_i : \sigma(u) = \mu\}.$$

If $\eta_i : X_{G_i} \longrightarrow X_{K_i}$, $\tau \longmapsto \tau \restriction K_i$, is the homeomorphism in Theorem 3.6.(3) and $a \in G_i$, then, with the notation above and in 3.9, $L(a) = [\![\nabla a^2 = -1]\!]_i^{\times} = \eta_i[Z(a^2)]$, i.e., $\eta^{-1}[L(a)] = Z(a)$. The reader should keep in mind that $L(a^2) = L(a)$ (cf. 3.9.(a)) and $\neg L(a^2) = \neg L(a)$. ∎

Let $F : G_1 \longrightarrow G_2$ be an RS-morphism and let $f = F \restriction K_1 : K_1 \longrightarrow K_2$ (a RSG-morphism). The map f yields, by composition, a continuous map, f_*: $X_{K_2} \longrightarrow X_{K_1}$, $f_*(\sigma) = \sigma \circ f$. We start with the following

Lemma 4.2 *Let e_i be idempotents in G_i, $i = 1, 2$. The following are equivalent:*

(1) *(a) $f_*(L(e_2)) \subseteq L(e_1)$ and (b) $f_*(\neg L(e_2)) \subseteq \neg L(e_1)$;*

(2) $\nabla e_2 = f(\nabla e_1)$.

Proof. (1) ⇒ (2): For $\sigma \in X_{K_2}$, suppose $\sigma(\nabla e_2) = -1$, i.e., $\sigma \in [\![\nabla e_2 = -1]\!]$; then, (1.(a)) entails $f_*(\sigma) = \sigma \circ f \in L(e_1) = [\![\nabla e_1 = -1]\!]$ and so $\sigma(f(\nabla e_1)) = -1$. A similar argument, employing (1.(b)) shows that $\sigma(\nabla e_2) = 1$ entails $\sigma(f(\nabla e_1)) = 1$, and the equality in (2) follows immediately.

(2) ⇒ (1): For $\sigma \in X_{K_2}$, we have two possibilities:

- If $\sigma \in L(e_2)$, then $\sigma(\nabla e_2) = -1 = \sigma(f(\nabla e_1))$ and so $f_*(\sigma) \in L(e_1)$, verifying (1.(a));
- If $\sigma \in \neg L(e_2)$, then $\sigma(\nabla e_2) = 1 = \sigma(f(\nabla e_1))$, and so $f_*(\sigma) \in [\![\nabla e_1 = 1]\!] = \neg L(e_1)$, proving (1.(b)), as needed. ∎

Remarks 4.3 a) An RS-morphism, $F : G_1 \longrightarrow G_2$, gives rise to two maps:

- A RSG morphism $f_F := F \restriction K_1 : K_1 \longrightarrow K_2$;
- A lattice morphism, $h_F : Id(G_1) \longrightarrow Id(G_2)$, given by $h_F(e) = F(e)$. To see h_F is indeed a lattice morphism note that for idempotents x, y in G_1, we have, recalling 1.27,

– $h_F(x \vee y) = h_F(xy) = F(xy) = F(x)F(y) = h_F(x) \vee h_F(y)$;

– $x \wedge y \in D^t_{G_1}(x, y)$, and so $h_F(x \wedge y) = F(x \wedge y) \in D^t_{G_2}(F(x), F(y))$, yielding $h_F(x \wedge y) = h_F(x) \wedge h_F(y)$.

b) F may be obtained back from the pair $\langle f_F, h_F \rangle$: for $a \in G_1$, 3.5.(a.2) yields $a = \nabla a \cdot a^2$ and so $F(a) = f_F(\nabla a)F(a^2) = f_F(\nabla a)h_F(a^2)$. ∎

Lemma 4.4 *Let $F : G_1 \longrightarrow G_2$ be an RS-morphism. To simplify exposition, write f for f_F.*

a) For each $e \in \mathrm{Id}(G_1)$, we have $\nabla F(e) = f(\nabla e)$.

b) The pair $\langle f, h_F \rangle$ satisfies the conditions (1.(a)) and (1.(b)) in 4.2, i.e., for all $e \in Id(G_1)$,

(*) $$(i)\ f_*[L(h_F(e))] \subseteq L(e) \quad \text{and} \quad (ii)\ f_*[\neg L(h_F(e))] \subseteq \neg L(e).$$

Proof. a) By 3.5.(a.3), we have $e \in D^t_{\mathcal{G}_1}(1, \nabla e)$, whence $F(e) \in D^t_{\mathcal{G}_2}(1, f(\nabla e))$, which yields, by 3.5.(c), [4]

(I) $$\nabla F(e) \in D_{K_2}(1, f(\nabla e)).$$

Let $\tau \in X_{G_2}$; then:

- If $\tau(f(\nabla e)) = 1$, relation (I) implies $\tau(\nabla F(e)) = 1$;
- If $\tau \in [\![\nabla F(e) = 1]\!]$, then, by 3.5.(a.1), $\tau \in [\![F(e) = 1]\!]$, i.e., $\tau \circ F \in [\![e = 1]\!] = [\![\nabla e = 1]\!]$, whence $\tau(F(\nabla e)) = \tau(f(\nabla e)) = 1$.

Thus, for all $\tau \in X_{G_2}$, $\tau(\nabla F(e)) = 1$ iff $\tau(f(\nabla e)) = 1$; since both $\nabla F(e)$ and $f(\nabla e) \in K_2$, we conclude $\nabla F(e) = f(\nabla e)$, as desired.

Item (b) is immediate from (a) and 4.2: just take $e_2 = F(e) = h_F(e)$. ∎

Remark 4.5 Let G_1, G_2 be RSs, let $f : G_1^\times \longrightarrow G_2^\times$ be an RSG-morphism and h: $\mathrm{Id}(G_1) \longrightarrow \mathrm{Id}(G_2)$ be a lattice morphism, such that a pair $\langle f, h \rangle$ satisfies the conditions in (*) of 4.4.(b). Let $x, y \in \mathrm{Id}(G_1)$ and suppose $\sigma \in L(h(x)) \cap L(h(y)) \subseteq X_{K_2}$. Then, (*i*) in (*) of 4.4.(b), yields

$$\sigma \circ f \in L(x) \text{ and } \sigma \circ f \in L(y) \text{ and so } \sigma \circ f \in L(x) \cap L(y).$$

This applies to any Boolean combination of $L(h(x))$, $L(h(y))$, employing (*i*) and (*ii*) in (*) of 4.4.(b), and used below without comment. ∎

Definition 4.6 *Let G_i, $i = 1, 2$ be RSs. A pair $\langle f, h \rangle$, where $f : K_1 \longrightarrow K_2$ is an RSG-morphism and $h : \mathrm{Id}(G_1) \longrightarrow \mathrm{Id}(G_2)$ is a* (bounded) *lattice morphism, is* **compatible** *if they satisfy the conditions in (*) of 4.4.(b). Write $\mathcal{C}(G_1, G_2)$ for the set of all compatible pairs between G_1 and G_2.*

By 4.4, every RS-morphism, $F : G_1 \longrightarrow G_2$ yields a compatible pair, while 4.3.(b) shows that F may be obtained back from its compatible pair. To establish a bijective correspondence between $Hom_{RS}(G_1, G_2)$ and $\mathcal{C}(G_1, G_2)$ it suffices to establish:

Theorem 4.7 *If $\langle f, h \rangle \in \mathcal{C}(G_1, G_2)$, the map $F(f, h) : G_1 \longrightarrow G_2$, given by $a \in G_1 \mapsto f(\nabla a)h(a^2)$ is an RS-morphism. Moreover:*

a) $F(f, h) \restriction K_1 = f$ and $F(f, h) \restriction Id(G_1) = h$;

b) $G = G_1 = G_2$, then $F(f, h) = Id_G$ iff $f = Id_{K_{G^\times}}$ and $h = Id_{Id(G)}$.

Proof. We first verify (a) and (b). Once again, to simplify presentation, write F for $F(f, h)$. For $u \in K_1$, $F(u) = f(\nabla u)u^2 = f(u)$ and so $F \restriction K_1 = f$; next, if $a \in G_1$, then $F(a^2)^2 = F(a^2) = (f(\nabla a^2))^2 h(a^2)^2 = h(a^2)$ (recall: $f(\nabla a^2) \in K_2$), completing the verification of (a). Item (b) is clear.

We now turn to the proof that F is an RS-morphism. Note that $F(1) = 1$, $F(-1) = -1$ and $F(0) = 0$. We must show that F preserves products and representation.

I. <u>F preserves products.</u> For $a = \nabla a \cdot a^2$ and $b = \nabla b \cdot b^2 \in G_1$, we have [5] :

(*i*) $F(ab) = f(\nabla(ab))h(a^2b^2) = f(\nabla(ab))h(a^2 \vee b^2) = f(\nabla(ab))[h(a^2) \vee h(b^2)]$
$\qquad = f(\nabla(ab))h(a^2)h(b^2)$;

(*ii*) $F(a)F(b) = f(\nabla a \nabla b)h(a^2)h(b^2)$

[4] Recall: if u is a unit in a Boolean RS, $\nabla u = u$ and 1, $f(\nabla e) \in K_2$.
[5] Recall that for idempotents x, y, we have $x \vee y = xy$.

For $\tau \in X_{G_2}$:

(1) If $\tau \in [\![h(a^2) = 0]\!]_2$ or $\tau \in [\![h(b^2) = 0]\!]_2$ then the both terms in (i) and (ii) are zero, as needed;

(2) If $\tau \in [\![h(a^2) = 1]\!]_2 \cap [\![h(b^2) = 1]\!]_2 = [\![h(a^2b^2) = 1]\!]_2$, then $\tau \restriction K_2 \in \neg L(h(a^2)) \cap \neg L(h(b^2))$; now compatibility entails $\tau \restriction K_2 \circ f \in \neg L(a) \cap \neg L(b) = \neg L(ab)$. The equality in (σ) of 3.9.(b) obtains

$$\tau(F(ab)) = \tau(f(\nabla(ab))) = \tau(f(\nabla a)f(\nabla b)) = \tau(F(a)F(b)),$$

completing the proof of that F preserves products.

<u>II. F preserves representation.</u> For a, b, $c \in G_1$, suppose that $a \in D^t_{G_1}(b, c)$. Fix $\tau \in X_{G_2}$; set $\sigma := \tau \restriction G_2^\times$ and let $\mu \in X_{G_1}$ be such that $\mu \restriction G_1^\times = f \circ \sigma$.

We discuss two cases:

<u>II.1. $\tau \in [\![h(a^2) = 0]\!]_2 \cup [\![h(b^2) = 0]\!]_2 \cup [\![h(c^2) = 0]\!]_2$.</u> Since $a \in D^t_{G_1}(b, c)$ iff $-b \in D^t_{G_1}(-a, c)$ iff $-c \in D^t_{G_1}(b, -a)$, it suffices to discuss the case $\tau \in [\![h(a^2) = 0]\!]_2$, for the others can be similarly treated.

So assume $\tau(h(a^2)) = 0$ (and so $\sigma \in L(h(a^2))$, hence, by compatibility, $\sigma \circ f \in L(a)$. We claim that in this case

(#) $$\tau(h(b^2)) = \tau(h(c^2)).$$

Indeed, suppose $\tau(h(b^2)) = 0$ and $\tau(h(c^2)) = 1$ (i.e., $\sigma \in L(h(b^2)) \cap \neg(L(h(c^2)))$. Then, compatibility yields

$$\sigma \circ f \in L(a) \cap L(b) \cap \neg L(c).$$

Since $\mu \restriction G_1^\times = \sigma \circ f$, we get $\mu(a) \in Z(a)$, $\mu(b) \in Z(b)$, while $\mu \in [\![\nabla c^2 = 1]\!]_1 = [\![c^2 = 1]\!]_1$, whence $\mu(c) \neq 0$. But then, $a \in D^t_{G_1}(b, c)$ implies $\mu(a) = 0 \in D^t_3(0, \mu(c))$, which is impossible since $\mu(c) \neq 0$. A similar argument shows that $\tau(h(b^2)) = 1$ and $\tau(h(c^2)) = 0$ to be untenable, establishing (#).

If $\tau(h(a^2)) = \tau(h(c^2)) = 0$, then $\tau(F(a)) = \tau(F(b)) = \tau(F(c)) = 0$, and we are done.

Henceforth, assume $\tau \in [\![h(b^2) = 1]\!]_2 \cap [\![h(c^2) = 1]\!]_2$, whence, $\sigma \in L(h(a^2))$ and $\sigma \in \neg L(h(b^2)) \cap \neg L(h(c^2))$. Compatibility yields $\sigma \circ f \in L(a)$ and $\sigma \circ f \in \neg L(b) \cap \neg L(c)$, hence $\mu \in Z(a)$ and $\mu \in [\![\nabla b^2 = 1]\!]_1 = [\![b^2 = 1]\!]_1$ and $\mu \in [\![\nabla c^2 = 1]\!]_1 = [\![c^2 = 1]\!]_1$. In particular, $\mu(b)$, $\mu(c) \neq 0$. From $a \in D^t_{G_1}(b, c)$, we obtain $\mu(a) = 0 \in D^t_3(\mu(b), \mu(c))$, and so

(*) $$\mu(c) = -\mu(b) \neq 0.$$

If $\mu \in [\![b = 1]\!]_1 \cap [\![c = -1]\!]_1$, 3.5.(a.1) yields $\mu(\nabla b) = 1$ and $\mu(\nabla c) = -1$ (recall: $\mu(c) \neq 0$); a similar argument applies in case $\mu(b) = -1$ and $\mu(c) = 1$, and so (*) entails $\mu(\nabla b \cdot \nabla c) = -1$, that is, $\mu \restriction G_1^\times(\nabla c) = \sigma \circ f(\nabla c) = -\mu \restriction G_1^\times(\nabla b) = -(\sigma \circ f(\nabla b))$. Unraveling notation yields $\tau(f(\nabla c)) = -\tau(f(\nabla b))$, and so (recall: $\tau(h(b^2)) = \tau(h(c^2)) = 1$), $\tau(F(a)) = 0 \in D^t_3(\tau(f(\nabla b)), -\tau(f(\nabla b)))$, ending the discussion of case II.1.

<u>II.2. $\tau \in [\![h(a^2) = 1]\!]_2 \cap [\![h(b^2) = 1]\!]_2 = [\![h(a^2b^2) = 1]\!]_2$</u> and $\tau \in [\![h(c^2) = 1]\!]_2$. For $x \in \{a, b, c\}$, $\tau(F(x)) = \tau(f(\nabla x))$. By 3.5.(c), we have $\nabla a \in D^t_{G_1}(\nabla b, \nabla c) \cap K_1 = D_{K_1}(\nabla b, \nabla c)$ and so $f(\nabla a) \in D^t_{G_2}(f(\nabla b), f(\nabla c))$, because f is a RSG-morphism. Hence, $\tau(F(a)) \in D^t_3(\tau(F(b)), \tau(F(c)))$. Since τ is arbitrary in X_{G_2}, the proof is complete. ∎

We now have

Proposition 4.8 *Let G_i be Boolean RSs and let $K_i = G_i^\times$, $i = 1, 2, 3$.*

a) If $\langle f, h \rangle \in \mathcal{C}(G_1, G_2)$ and $\langle g, k \rangle \in \mathcal{C}(G_2, G_3)$, then

(1) $\langle g \circ f, k \circ h \rangle \in \mathcal{C}(G_1, G_3)$.

(2) $F(g \circ f, k \circ h) = F(g, k) \circ F(f, h)$.

b) For an RS-morphism $F : G_1 \longrightarrow G_2$, the following are equivalent,

(1) *F is an RS-isomorphism;*

(2) *f_F is a RSG-isomorphism and $h_F : \mathrm{Id}(G_1) \longrightarrow \mathrm{Id}(G_2)$ is a lattice isomorphism* (and $\langle f_F, h_F \rangle \in \mathcal{C}(G_1, G_2)$).

Proof. a) (1) Clearly, it suffices to show that $\langle g \circ f, h \circ k \rangle$ satisfy conditions (i) and (ii) in (*) of Lemma 4.4.(b). Note that $(g \circ f)_* = f_* \circ g_*$. If $e \in \mathrm{Id}(G_1)$, then $h(e) \in \mathrm{Id}(G_2)$ and so

$$f_*[g_*[L(k(h(e)))]] \subseteq f_*[L(h(e)] \subseteq L(e),$$

proving (i) in (*) of 4.4.(b). The same argument will yield (ii) in (*) of 4.4.(b), establishing (a.1). Item (a.2) is clear.

b) (1) $\Rightarrow$ (2). Clearly, if F is an RS-isomorphism, then $f_F : K_1 \longrightarrow K_2$ is an RSG-isomorphism and $h_F : \mathrm{Id}(G_1) \longrightarrow \mathrm{Id}(G_2)$ is a lattice isomorphism.

(2) $\Rightarrow$ (1). To simplify exposition, write f for f_F and h for h_F. Let $g : K_2 \longrightarrow K_1$ and $k : \mathrm{Id}(G_2) \longrightarrow$ be the inverse isomorphisms of f and h, respectively. Since f is a RS-isomorphism, its dual $f_* : X_{K_2} \longrightarrow X_{K_1}$ is a homeomorphism, whose inverse is g_*. Note that $\{L(e), \neg L(e)\}$ and $\{L(h(e)), \neg L(h(e)\}$ are clopen partitions of X_{K_1} and X_{K_2} respectively; but then, because $\langle f, h \rangle$ satisfies conditions (i) and (ii) in (*) of Lemma 4.4.(b), we conclude that for all $e \in \mathrm{Id}(G_1)$,

(I) $$f_*[L(h(e)] = L(e) \text{ and } f_*[\neg L(h(e)] = \neg L(e).$$

We now show that $\langle g, k \rangle \in \mathcal{C}(G_2, G_1)$. Indeed, if $e' \in \mathrm{Id}(G_2)$, then $k(e') \in \mathrm{Id}(G_1)$ and so the first equality in (I) obtains $f_*[L(h(k(e')))] = L(k(e'))$, whence (recall: g_* is the inverse of f_*) $g_*[L(k(e'))] = L(h(k(e'))) = L(e')$ as needed. A similar argument shows that $\langle g, k \rangle$ satisfies (ii) in (*) of 4.4.(b) and so $\langle f, k \rangle \in \mathcal{C}(G_2, G_1)$. Since both f, g and h, k are inverses to one another, items (a) and yield (recall: $F = F(f, h)$), $F(g, k) \circ F = Id_{G_1}$, $F \circ F(g, h) = Id_{G_2}$ and F is an RS-isomorphism, ending the proof. ∎

5 Quotients of Boolean Real Semigroups

Here we characterize RS-quotients in the class of Boolean RSs, showing that if G is a Boolean RS, any RS-congruence on G is determined by a subsemigroup of G [6], generated by a saturated subgroup of $G^\times$ (Theorem 5.5) [7].

5.1 Notation. a) In this section we fix:

- A real semigroup, G, and let $K := G^\times$ be the RSG of units in G (cf. 3.6.(3));
- An RS-congruence on G, $\equiv$, writing $G_\equiv$ for the quotient RS and $\pi : G \longrightarrow G_\equiv$ for the natural quotient RS-morphism. Let $K_\equiv := G_\equiv^\times$ be the RSG of units of the Boolean RS $G_\equiv$ (by 3.18.(b)) and let $X_\equiv$ be the space of RS-characters of $G_\equiv$.

b) Let $X_G \xrightarrow{\eta} X_K$ be the homeomorphism $\tau \in X_G \longmapsto \tau \restriction K \in X_K$ (3.6.(3)).

c) Let $\Sigma = \ker(\pi \restriction K)$, a (proper) saturated subgroup of K (recall: $\pi \restriction K$ is a RSG-morphism).

d) Let $\mathcal{H} = \{\sigma \circ \pi \in X_G : \sigma \in X_\equiv\}$. Thus, if $\pi_* : X_\equiv \longrightarrow X_G$ is the dual of π, induced by composition, we have $\mathcal{H} = \pi_*[X_\equiv]$ [8].

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[6] Called subspaces in [M].

[7] Conversely, by Theorem II.3.8, [DP3], any saturated subgroup of an RS H, rise to an RS-congruence on H.

[8] In [DP3], (cf. II.2.7.(i)), π_* is written π^*, while $\mathcal{H}$ is sometimes written $\mathcal{H}_\equiv$

e) If $\tau \in X_G$, set $P(\tau) = \{x \in G : \tau(x) \in \{0, 1\}\} = [\![x = 1]\!]_{X_G} \cup [\![x = 0]\!]_{X_G} := [\![x \geq 0]\!]_{X_G}$. ■

Remarks 5.2 a) By definition (cf. II.2.1 and I.1.25, [DP3]), an RS-congruence on G is also a *proper* congruence of the ternary semigroup underlying G, i.e., $\equiv$ is a proper subset of $G \times G$ and for all $x \in G$, $x \equiv -x$ implies $x = 0$. In particular,

(*i*) $\pi(0)$, $\pi(1)$ and $\pi(-1)$ are pairwise distinct;

(*ii*) For $a, b, c, d \in G$, $\quad a \equiv b$ and $c \equiv d \cdot \Rightarrow \quad ac \equiv bd$.

Item (*ii*) above will be frequently used below without explicit mention.

b) By items (*ii*) and (*v*) of Proposition II.2.8, [DP3], we have:

(1) $\mathcal{H}$ is a proconstructible subset of X_G, whence closed in X_G (X_G is Boolean).

(2) For all $a, b \in G$,

(#) $\quad\quad a \equiv b$ iff For all $p \in \mathcal{H}$, $p(a) = p(b)$.

c) If $e, f \in \mathrm{Id}(G)$, it follows straightforwardly from Lemma 3.5.(a.1) (or its item (d)) that $\nabla e = \nabla f$ iff $e = f$.

d) Recall (cf. Notation I.1.4 and Definition I.4.1, [DP3] or Definition 3.1, p. 112, [DP1]) that a subset A of G is:

- A **subsemigroup** of G if it is closed under products and contains 1;
- **Saturated** if for all $a, b \in G$, $\quad a, b \in A \quad$ implies $\quad D_G(a, b) \subseteq A$. Similarly, one defines when A is **transversally saturated (or t-saturated)**, replacing D_G by D^t_G.

We also mention item (2.b) of Proposition I.4.6, [DP3], namely:

Fact 5.3 *If A is a subset of G, closed under products and containing* 1, *then the saturated subsemigroup generated by A, $[A]$, is given by*

$$[A] \;=\; \bigcup \{D_G(\varphi) : \varphi \text{ is a } n\text{-form with coefficients in } A,\ n \geq 1\}. \qquad \square$$

e) If $n \geq 1$ is an integer, write $n\langle 1 \rangle$ for the n-form whose entries are all equal to 1. It follows easily from (D) in 1.17 and 1.20.(c) that $D_3(n\langle 1 \rangle) = \{0, 1\}$. ■

Proposition 5.4 *With notation as above, and for $a, b \in G$*

a) $a \equiv b \;\Leftrightarrow\;$ *(i)* $\nabla a \equiv \nabla b$ *and (ii)* $a^2 \equiv b^2$.

b) If $u \in K$ and $u \equiv a$, then $u \equiv \nabla a$ and $a^2 \equiv 1$.

c) If $w \in K_{\equiv}$, then there is $v \in K$ so that $\pi(v) = w$. In particular, $\pi \restriction K : K \longrightarrow K_{\equiv}$ is surjective. Moreover, for $u, v \in K$, $\quad u \equiv v \quad$ iff $\quad uv \in \Sigma$ (cf. 5.1.(c)).

d) If $e \in \mathrm{Id}(G_{\equiv})$, there is $x \in \mathrm{Id}(G)$ so that $\pi(x) = e$.

Proof. a) Since for all $x \in G$, $x = \nabla x \cdot x^2$ (3.5.(a.2)), only the implication $\Rightarrow$ needs proof. Assume $a \equiv b$; then, termwise multiplication of this relation by itself yields $a^2 \equiv b^2$, i.e., (*ii*). To establish (*i*), fix $p \in \mathcal{H}$, then $a \equiv b$, (#) in 5.2.(b.2) and (*ii*) entail

(I) $$p(a) = p(\nabla a)p(a^2) = p(b) = p(\nabla b)p(b^2) = p(\nabla b)p(a^2).$$

If $p(a^2) = 0 = p(b^2)$, $p \in Z(a) \cap Z(b)$ and so 3.5.(a.1) yields $p(\nabla a) = p(\nabla b) = -1$. If, $p(a^2) = p(b^2) = 1$, then (I) entails $p(a) = p(\nabla a) = p(b) = p(\nabla b)$. Since p is arbitrary in $\mathcal{H}$, (#) in 5.2.(b.2) obtains $\nabla a \equiv \nabla b$, as needed.

Item (b) is immediate from (a), recalling that for all $u \in K$, $\nabla u = u$.

c) Let $w \in K_{\equiv}$; then, there is $a \in G$ so that $\pi(a) = w$. Thus, $\pi(a)^2 = \pi(a^2) = w^2 = 1$, and so $a^2 \equiv 1$. Thus, $a = \nabla a \cdot a^2 \equiv \nabla a \in K$, entailing $\pi(\nabla a) = w$. For the second statement in (c), since $u, v \in K$, we have $\pi(u) = \pi(v) \in K_{\equiv}$. Thus,

$$u \equiv v \quad \text{iff} \quad \pi(u) = \pi(v) \quad \text{iff} \quad \pi(uv) = 1 \quad \text{iff} \quad uv \in \ker(\pi \upharpoonright K) = \Sigma.$$

d) If $e \in \text{Id}(G_{\equiv})$, there is $x \in G$ so that $\pi(x) = e$. But then, $\pi(x)^2 = \pi(x^2) = e^2$, as needed. ∎

The main result in this section is the following

Theorem 5.5 *With notation as in items (c) and (d) of 5.1, let $\Delta(\Sigma)$ be the saturated subsemigroup generated Σ in G. Now define,*

$$\begin{cases} \mathcal{H}_\Sigma & = \{\tau \in X_G : \Sigma \subseteq \ker(\tau \upharpoonright K)\}; \\ \mathcal{H}_{\Delta(\Sigma)} & = \{\tau \in X_G : \Delta(\Sigma) \subseteq P(\tau)\}. \end{cases}$$

Then,

a) For all $a, b \in G$, $a \equiv b \Rightarrow$ For each $\tau \in \mathcal{H}_\Sigma$, $\tau(a) = \tau(b)$.

b) $\mathcal{H} = \mathcal{H}_\Sigma = \mathcal{H}_{\Delta(\Sigma)}$. In particular, all of the these three sets of RS-characters induce the congruence $\equiv$ on G.

c) The map $\widehat{\pi} : K/\Sigma \longrightarrow K_{\equiv}$, given by $\widehat{\pi}(u/\Sigma) = \pi(u)$ is a RSG-isomorphism.

d) (1) For all $e, e' \in (\text{Id}(G)$,

$$(*) \qquad e \equiv e' \quad \textit{iff} \quad \pi(e) = \pi(e') \quad \textit{iff} \quad \nabla e \cdot \nabla e' \in \Sigma.$$

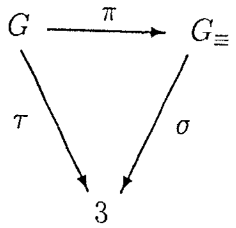
(2) The restriction of $\equiv$ to $\text{Id}(G)$ is a lattice congruence and $\pi \upharpoonright \text{Id}(G) : \text{Id}(G) \longrightarrow \text{Id}(G_{\equiv})$ is the natural quotient map.

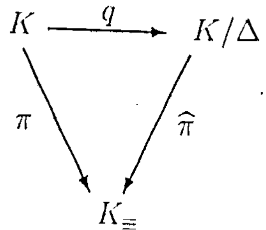
Proof. a) For $a, b \in G$, suppose $a \equiv b$; then 5.4.(a) entails (i) $\nabla a \equiv \nabla b$ and (ii) $a^2 \equiv b^2$. From (i) and 5.4.(c) we obtain $\nabla a \cdot \nabla b \in \Sigma$ and so $\tau(\nabla a) = \tau(\nabla b)$ (recall: $\Sigma \subseteq \ker(\tau \upharpoonright K)$. Now (ii) and 5.4 yield $\nabla a^2 \equiv \nabla b^2$, that just as above yields $\tau(\nabla(a^2)) = \tau(\nabla(b^2))$. Since τ is a RS-morphism, 4.4.(a) entails $\nabla \tau(a^2) = \tau(\nabla(a^2)) = \tau(\nabla(b^2)) = \nabla \tau(b^2)$, and so $\tau(a^2) = \tau(b^2)$ (by 5.2.(c)). But then,

$$\tau(a) = \tau(\nabla a \cdot a^2) = \tau(\nabla a)\tau(a^2) = \tau(\nabla b)\tau(b^2) = \tau(b),$$

as needed.

b) We first verify that $\mathcal{H} = \mathcal{H}_\Sigma$. If $p \in \mathcal{H}$, then since $K = \ker(\pi \upharpoonright K)$, it is clear that $\Sigma \subseteq p \upharpoonright K$, and $\mathcal{H} \subseteq \mathcal{H}_\Sigma$. For the reverse inclusion, suppose $\tau \in \mathcal{H}_\Sigma$; since $\equiv$ is an RS-congruence, item (a) implies that there is a unique RS-morphism, $\sigma : G_{\equiv} \longrightarrow 3$ (i.e., $\sigma \in X_{\equiv}$) making the diagram below left commutative:





and so $\tau = \sigma \circ \pi \in \mathcal{H}$, establishing that $\mathcal{H} = \mathcal{H}_\Sigma$. If $\gamma \in \mathcal{H}_{\Delta(\Sigma)}$, since $\Sigma \subseteq \Delta_\Sigma$, then $\Sigma \subseteq P(\gamma)$; but then, because Σ consists of units, we obtain $\Sigma \subseteq \ker(\gamma \upharpoonright K)$, showing $\mathcal{H}_{\Delta(\Sigma)} \subseteq \mathcal{H}_\Sigma$. Now let $p \in \mathcal{H}$; we already know that $\Sigma \subseteq \ker(p \upharpoonright K)$. If $a \in \Delta_\Sigma$ (the subsemigroup generated by Σ in G), by 5.3, there is a n-form φ over Σ $(n \geq 1)$ so that $a \in D_G(\varphi)$. Hence, $p(a) \in D_3(n\langle 1 \rangle)$ and so by 5.2.(d), $p(a) \in \{0, 1\}$, whence $p \in \mathcal{H}_{\Delta(\Sigma)}$. Therefore,

$$\mathcal{H} \subseteq \mathcal{H}_{\Delta(\Sigma)} \subseteq \mathcal{H}_\Sigma,$$

and the first part of the proof establishes the desired equality.

c) Let $q : K \longrightarrow K/\Sigma$ be the natural RSG quotient morphism and let $\pi \upharpoonright K : K \longrightarrow K_{\equiv}$ be the induced RSG morphism, which is surjective by 5.4.(c). Since the maps q and $\pi \upharpoonright K$ have the same kernel (namely Σ), and q is an RSG-quotient morphism, there is, by Proposition 2.21.(b), p. 43, [DM1], a unique RSG-morphism, $\widehat{\pi} : K/\Delta \longrightarrow K_{\equiv}$, making the diagram above right commutative. To show $\widehat{\pi}$ is a RSG-isomorphism, we prove that $\pi \upharpoonright K$ is *regular* (Definition 2.22, p. 43, [DM1]). Since $\equiv$ is the equivalence relation corresponding to a subspace (by item (b)), we employ the characterization of D^t in the quotient $G_{\equiv}$, appearing in Theorem II.3.8.(c) of [DP3]. Let $u_i \in K$, $i = 1, 2, 3$ and assume $\pi(u_1) \in D_{K_{\equiv}}(\pi(u_2), \pi(u_3)) = D^t_{G_{\equiv}}(\pi(u_2), \pi(u_3)) \cap K_{\equiv}$. By item (c) in Theorem II.3.8, [DP3],

$$\text{(I)} \qquad \begin{cases} \exists\, x_i \in G,\ i = 1, 2, 3, \text{ so that } & (1)\ x_i \equiv u_i^2 = 1; \text{ and} \\ & (2)\ u_1x_1 \in D^t_G(u_2x_2,\, u_3x_3). \end{cases}$$

Then (I.1) yields $u_ix_i \equiv u_i$, whence $\nabla(u_ix_i) \equiv \nabla u_i = u_i$, while (I.2) and 3.5.(c) imply $\nabla(u_1x_1) \in D^t_G(\nabla(u_2x_2), \nabla(u_3x_3))$. Hence, setting $u_i' = \nabla(u_ix_i) \in K$, $i = 1, 2, 3$, obtains

$$u_1' \in D^t_G(u_2', u_3') \cap K = D_K(u_2', u_3'), \text{ with } u_i' \equiv u_i \ (i = 1, 2, 3),$$

establishing the regularity of $\pi \upharpoonright K$. By Proposition 2.22, p. 43, in [DM1], $\widehat{\pi}$ is an RSG-isomorphism, as needed.

d) (1) The first equivalence in is clear; for the second, $e \equiv e'$ entails, by 5.4.(a), $\nabla e \equiv \nabla e'$, and so since these terms are units, 5.4.(c) obtains $\nabla e \cdot \nabla e' \in \Sigma$. For the converse, if this latter condition is satisfied then, recalling 4.4.(a), we get $\nabla\pi(e) = \pi(\nabla e) = \pi(\nabla e') = \nabla\pi(e')$, and so 5.2.(c) implies $\pi(e) = \pi(e')$, as needed.

By 4.3.(a), the RS-morphism π yields a lattice morphism, $\mathrm{Id}(G) \xrightarrow{h_\pi} \mathrm{Id}(G_{\equiv})$, given by $e \in \mathrm{Id}(G) \longmapsto h_\pi(e) = \pi(e) \in \mathrm{Id}(G_{\equiv})$, that by item (d) is surjective. By (*) in item (d.1),

$$e \equiv e' \quad \text{iff} \quad h_\pi(e) = \pi(e) = \pi(e') = h_\pi(e'),$$

and so the restriction of $\equiv$ to $\mathrm{Id}(G)$ is equal to the kernel of the lattice morphism h_π [9], a lattice congruence, whose natural quotient projection is $\pi \upharpoonright Id(G) : \mathrm{Id}(G) \longrightarrow \mathrm{Id}(G_{\equiv})$ [10]. ∎

Remarks 5.6 a) With notation as in 5.1, by Theorem 5.5, any RS-congruence on a Boolean RS is induced by a subspace of X_G of a special kind: note that $\mathcal{H}_\Sigma = \bigcap\{[\![u = 1]\!]_{X_G} : u \in \Sigma\}$. In fact, items (c) and (d.1) of 5.5 show that such an RS-congruence is essentially determined by the saturated subgroup Σ of K.

b) If B is a Boolean algebra, any non-empty closed set, C, of the Stone space of B, $S(B)$, is of the form described in item (a). Indeed, if $U = S(B) \setminus C$, then there is a family of finite subsets of B, $\mathcal{C} = \{J \subseteq B\colon J \text{ is finite}\}$, so that $U = \bigcup_{J \in \mathcal{C}} \bigcap_{u \in J} [\![u = 1]\!]$. For $J \in \mathcal{C}$, set $u_J = \bigwedge_{u \in J} u$; then, $[\![u_J = 1]\!] = \bigcap_{u \in J} [\![u = 1]\!]$, whence $U = \bigcup_{J \in \mathcal{C}} [\![u_J = 1]\!]$ and so $C = \bigcap_{J \in \mathcal{C}} [\![-u_j = 1]\!]$.

In particular, if $P = \mathbb{C}(X, 3)$ is a Post algebra of order 3 and B is the Boolean algebra of units in P, then X_P is naturally homeomorphic to $S(B)$ and so any closed subset of X_P is a subspace of the kind described in (a).

c) It is not clear that *every* closed subset of the space of orders of a Boolean RS is a subspace, in particular of the form described in item (a). In fact, this is not clear even for reduced special groups. ∎

[9] $\ker(h_\pi) = \{\langle x, y\rangle \in Id(G) \times Id(G) : h_\pi(x) = h_\pi(y)\}$.

[10] It is not true, in general, that the congruence of distributive lattices are given by either filters or ideals.

References

[ABR] C. Andradas, L. Bröcker, J. Ruiz, **Constructible Sets is Real Geometry**, Modern Surveys in Math **33**, Springer, 1996.

[AM] M. F. Atiyah, I. G. Macdonald, **Introduction to Commutative Algebra**, Addison-Wesley Publ. Co., London, 1969.

[CK] C. C. Chang, H. J. Keisler, **Model Theory**, North-Holland Publ. Co., Amsterdam, 1990.

[BCR] J. Bochnak, M. Coste, M-F. Roy, **Real Algebraic Geometry**, Ergeb. Math. 36, Springer-Verlag, Berlin, 1998.

[DM1] M. Dickmann, F. Miraglia, **Special Groups : Boolean-Theoretic Methods in the Theory of Quadratic Forms**, Memoirs Amer. Math. Soc. **689**, Providence, R.I., 2000.

[DM2] M. Dickmann, F. Miraglia, **Faithfully Quadratic Rings**, Memoirs of the AMS 1128 (November 2015).

[DMM] M. Dickmann, M. Marshall, F. Miraglia, *Lattice-ordered reduced special groups*, Annals of Pure and Appl. Logic **132** (2005), 27-49.

[DMP] M. Dickmann, F. Miraglia, A. Petrovich, *Constructions in the category of real semi-groups*, Contemporary Math. **697**, American Math. Soc., (2017), 109-136.

[DP1] M. Dickmann, A. Petrovich, *Real Semigroups and Abstract Real Spectra, I*, Contemporary Math. **344** (2004), AMS, 99-119.

[DP2] M. Dickmann, A, Petrovich, *The Three-Valued Logic of Quadratic Form Theory over Real Rings*, in **"Andrezj Mostowski and Foundational Studies"** (A. Ehrenfeucht, V. W. Marek, M. Srebrny, eds.), IOS Press, Amsterdam, 2008, 49-67.

[DP3] M. Dickmann, A. Petrovich, **Real Semigroups, Real Spectra and Quadratic Forms over Rings**, (in preparation); February 10th, 2016 version available at http://www.ime.usp.br/~miraglia/textos/RS-fev-16.pdf.

[DP4] M. Dickmann, A. Petrovich, *Spectral Real Semigroups*, Annales de la Faculté des Sciences de Toulouse, Mathématiques, Serie 6, Vol. XXI (2012), 359-412.

[DST] M. Dickmann, N. Schwartz, M. Tressl, **Spectral Spaces**, New Mathematical Monographs **35**, Cambridge University Press, 2019.

[M] M. Marshall, **Spaces of Orderings and Abstract Real Spectra**, Lecture Notes in Mathematics **1636**, Springer-Verlag, Berlin, 1996.

[MR] F. Miraglia, Hugo, R. O. Ribeiro, *Boolean Real Semigroups*, to appear in the Tribute Series, College Publications, London, 2020.