Renewal Contact Processes: Phase transition and survival

Luiz Renato Fontes[a], Thomas S. Mountford[b], Daniel Ungaretti[c,a,*],
Maria Eulália Vares[c]

[a] *Instituto de Matemática e Estatística, Universidade de São Paulo, SP, Brazil*
[b] *Département de Mathématiques, 1015 Lausanne, Switzerland*
[c] *Instituto de Matemática, Universidade Federal do Rio de Janeiro, RJ, Brazil*

Abstract

We refine previous results concerning the Renewal Contact Processes. We significantly widen the family of distributions for the interarrival times for which the critical value can be shown to be strictly positive. The result now holds for any dimension $d \geq 1$ and requires only a moment condition slightly stronger than finite first moment. For heavy-tailed interarrival times, we prove a Complete Convergence Theorem and examine when the contact process, conditioned on survival, can be asymptotically predicted knowing the renewal processes. We close with an example of distribution attracted to a stable law of index 1 for which the critical value vanishes.
© 2023 Published by Elsevier B.V.

1. Introduction

In this note we address natural questions arising from the papers [10,11] that deal with an extension of the classical contact process introduced by Harris in [13] as a model for the spread of a contagious infection. The sites of $\mathbb{Z}^d$ are thought as the individuals, the state of the population being represented by a configuration $\xi \in \{0, 1\}^{\mathbb{Z}^d}$, where $\xi(x) = 0$ means that the individual x is healthy and $\xi(x) = 1$ that x is infected. A Markovian evolution was then considered: infected individuals get healthy at rate 1 independently of everything else,

---

* Corresponding author at: Instituto de Matemática, Universidade Federal do Rio de Janeiro, RJ, Brazil.
  E-mail addresses: lrfontes@usp.br (L.R. Fontes), thomas.mountford@epfl.ch (T.S. Mountford),
daniel@im.ufrj.br (D. Ungaretti), eulalia@im.ufrj.br (M.E. Vares).

and healthy individuals get sick at a rate that equals a given parameter λ times the number of infected neighbours. Harris contact process, as it is usually called, is one of the most studied interacting particle systems (see e.g. [18,20]) and has also opened a very wide road to multiple generalizations that have distinct motivations and potential applications, including space or time inhomogeneities, more general graphs and random graphs. A variety of random environments may also be modelled by considering suitable families of random rates. Results regarding survival and extinction for contact processes with random environments can be found in [2,4,17,19,21]. Also, Garet and Marchand [12] prove a shape theorem in this context.

In [14], Harris introduced a percolation structure on which the contact process was built, also known as graphical representation, in terms of a system of independent Poisson point processes. This has shown to be extremely useful not only to prove various basic properties of the process, but also for renormalization arguments (see e.g. [3,6]). It was exactly this angle that motivated the investigation started in [10,11], leading to the consideration of more general percolation structures, where the Poisson times would give place to more general point processes, so that the Markov property is lost, but the percolation questions continue to be meaningful and pose new challenges.

The extension of the contact process that we consider is what we call Renewal Contact Process (RCP). It is a modification of the Harris graphical representation in which transmissions are still given by independent Poisson processes of rate $\lambda > 0$, but cure times are given by i.i.d. renewal processes with interarrival distribution μ, a model we denote by $\mathrm{RCP}(\mu)$. For definiteness, we take the starting times of all renewal processes to be zero, but this choice does not affect our arguments.

In this paper we improve the current understanding of survival and extinction in $\mathrm{RCP}(\mu)$ provided by [10,11]. The critical parameter for $\mathrm{RCP}(\mu)$ is defined as

$$\lambda_c(\mu) := \inf\{\lambda : \ P(\tau^0 = \infty) > 0\},$$

where $\tau^0 := \inf\{t : \ \xi_t^{\{0\}} \equiv 0\}$ and $\xi_t^{\{0\}}$ is the process started from the configuration in which only the origin is infected. (As usual, we make the convention that $\inf \emptyset = \infty$.)

Ref. [11] considered sufficient conditions on μ to ensure that $\lambda_c(\mu) > 0$. The first contribution of the present paper is a new construction, simpler than the one in [11], that results in two meaningful improvements. Firstly, the present construction works for every dimension $d \geq 1$. Secondly, we significantly relax the assumptions on μ, as described by the following result:

Theorem 1.1. *Consider a probability distribution μ satisfying*

$$\int_1^\infty x \exp\left[\theta(\ln x)^{1/2}\right] \mu(\mathrm{d}x) < \infty \quad \text{for some } \theta > 4\sqrt{d \ln 2}. \tag{1}$$

Then, the $\mathrm{RCP}(\mu)$ has $\lambda_c(\mu) > 0$. In particular, $\lambda_c(\mu) > 0$ whenever $\int x^\alpha \mu(\mathrm{d}x) < \infty$ for some $\alpha > 1$.

The construction that leads to Theorem 1.1 is presented in Section 2. Essentially, it shows that if the probability that a renewal process $\mathcal{R}$ with interarrival distribution μ has a large gap is sufficiently small, then the critical parameter for the RCP is strictly positive. The moment condition in (1), together with Lemma 2.3, can be seen as a quantitative control on the probability of having large gaps.

Let us first discuss previous results that hold for the RCP on $\mathbb{Z}^d$ with any spatial dimension $d \geq 1$. Theorem 1 of [11] proves that $\lambda_c(\mu) > 0$ if μ has finite second moment. On the other

hand, in [10] it is proved that if there are ϵ, $C_1 > 0$ and $t_0 > 0$ such that $\mu([t, \infty)) \geq C_1/t^{1-\epsilon}$ for all $t \geq t_0$, then (under some auxiliary regularity hypothesis) $\lambda_c(\mu) = 0$. Notice that for general dimension these previous results leave a large gap between distributions μ for which $\lambda_c(\mu) > 0$ has been proven and those for which we know $\lambda_c(\mu) = 0$.

In the specific case of spatial dimension $d = 1$ this gap was considerably smaller. Theorem 2 of [11] proves that $\lambda_c(\mu) > 0$ if μ satisfies $\int t^\alpha \mu(dt) < \infty$ for some $\alpha > 1$, has a density and a decreasing hazard rate. Therefore, Theorem 1.1 represents a considerable improvement on conditions for $\lambda_c(\mu) > 0$.

In the proof of Theorem 2 of [11], the density and decreasing hazard rate of μ are used to show that RCP(μ) satisfies an FKG inequality, a tool repeatedly used in the proof of that theorem, combined with a crossing property of infection paths which holds only in $d = 1$. The construction used for proving Theorem 1.1 has a similar overall structure, with the crucial difference that it does not require the path crossing property or FKG, and thus allows more general distributions and dimensions.

We stress that the moment condition in (1) shows that there are distributions μ on the domain of attraction of a stable law with index 1 for which $\lambda_c(\mu) > 0$. On the other hand, in Section 5 we give an example (see Theorem 5.1) of a measure μ in the domain of attraction of stable with index 1 for which the critical parameter vanishes. One may be tempted to conjecture that $\lambda_c(\mu) > 0$ is equivalent to μ having a finite first moment. Up until now we have not been able to find a counter-example to this statement.

The discussion so far is concerned with sufficient conditions to ensure that $\lambda_c(\mu)$ is zero or positive, and this is indeed one of the main goals of this paper. Nevertheless, it is also natural to ask whether $\lambda_c(\mu) < \infty$, so that we may speak of a *phase transition*. Clearly, for a degenerate μ (e.g. $\mu(\{1\}) = 1$) the infection always dies out (at time 1), so that $\lambda_c(\mu) = \infty$. This pathologic behaviour should not occur once we avoid the phenomenon of simultaneous extinction. Proving a precise mathematical result demands care, and we still do not have a complete answer. Of course, since $\lambda_c(\mu)$ is clearly non-increasing in d, it suffices to consider the case $d = 1$. A simple sufficient condition can be given if we restrict to the class of measures μ considered in [11]. If μ has a density and a bounded and decreasing hazard rate, then $\lambda_c(\mu)$ is finite. This is further explained in Remark 2.9 in the next section. When $d \geq 2$ the situation is much simpler, and one can avoid the dependencies within each renewal process, simply by using each of them only once to construct an infinite infection path, i.e. through a coupling with supercritical oriented percolation, analogously to what was done in the proof of Theorem 1.3(ii) in [15].

The other results in the paper focus on the long time behaviour of the RCP(μ) for μ such that $\lambda_c(\mu) = 0$. Ref. [10] provides the following conditions on μ to ensure that a RCP(μ) has critical value equal to zero:

(A) There is $1 < M_1 < \infty$, $\epsilon_1 > 0$ and $t_1 > 0$ such that

$$\text{for every } t > t_1, \qquad \epsilon_1 \int_{[0,t]} s\mu(ds) < t\mu(t, M_1 t).$$

(B) There is $1 < M_2 < \infty$, $\epsilon_2 > 0$ and $r_2 > 0$ such that

$$\text{for every } r > r_2, \qquad \epsilon_2 \mu[M_2^r, M_2^{r+1}] \leq \mu[M_2^{r+1}, M_2^{r+2}].$$

(C) There is $M_3 < \infty$, $\epsilon_3 > 0$ such that

$$\text{for } t \geq M_3, \qquad t^{-(1-\epsilon_3)} \leq \mu(t, \infty) \leq t^{-\epsilon_3}.$$

These conditions require that μ has a heavy, mildly regular tail. Under them, it is shown in [10] that for any infection rate $\lambda > 0$, one can find an event of positive probability in which the infection survives — but see Remark 3.6, where we argue that the upper bound in (C) may be dropped as a hypothesis for this result to hold; we will need it for the results of the present article, though (as explained in Remark 3.7). In the event just mentioned, the path along which the infection survives goes to "infinity" as time diverges, so there is no information in that event about strong survival of the process (in whichever way this may be defined, see [22]). In Section 3 we show the following result.

Theorem 1.2. *Let interarrival distribution μ satisfy conditions (A)–(C) of Theorem 1 of [10]. Then, for a RCP starting from any initial condition ξ_0 we have that ξ_t converges in law, as $t \to \infty$, to*

$$P(\tau < \infty)\delta_{\underline{0}} + P(\tau = \infty)\delta_{\underline{1}}, \tag{2}$$

where $\tau = \inf\{t > 0 : \xi_t \equiv 0\}$, and $\delta_{\underline{0}}$ and $\delta_{\underline{1}}$ represent the Dirac measure on the configuration with all sites healthy and all sites infected, respectively.

Given Theorem 1.2, it is natural to see the sites (conditional upon survival of the process) as being a solid growing block of points which lose their infection ever more rarely and are quickly reinfected by their infected neighbours. Section 4 develops this picture further, under stricter regularity conditions for the tail of μ, demanding that it be attracted to an α-stable law with $0 < \alpha < 1$, with some extra regularity for $\alpha < 1/2$. Given a fixed site (e.g. the origin), it is natural to expect that given the information supplied by the renewal process, and in the event of survival of the infection started at the origin, the conditional probability that $\xi_t(0) = 1$ will be close to $1 - e^{-2\lambda d Y_t(0)}$, where $\mathcal{R}_0$ is the renewal process at the origin and $Y_t(0) := t - \sup\{\mathcal{R}_0 \cap [0, t]\}$ is the *age* of $\mathcal{R}_0$ at time t, or, in other words, the time elapsed up to time t since the most recent renewal of $\mathcal{R}_0$ prior to t.

We will effectively confirm this expectation for $\alpha < 1/2$, showing that in this case

$$\lim_{t \to \infty} \left| P(\xi_t(0) = 1 \mid \mathcal{R}, \text{survival}) - (1 - e^{-2\lambda d Y_t(0)}) \right| = 0,$$

see Theorem 4.1. For $\alpha \geq 1/2$, things get more complex, and indeed we show (in the same theorem) that

$$\overline{\lim_{t \to \infty}} \left(1 - e^{-2\lambda d Y_t(0)} - P(\xi_t(0) = 1 \mid \mathcal{R}, \text{survival}) \right) > 0$$

for $\alpha > 1/2$. A more precise result is stated in Theorem 4.4.

We close this introduction with a discussion on related papers. There are affinities between our RCP and the treatment of contact processes in a class of random environment as in [17,21]. The main novel aspect of RCP is the loss of the Markov property. Similarities are also present in the renormalization arguments used in Section 2 and those in [3].

In [8], RCP has been studied in the context of finite graphs. It deals with the RCP(μ) on finite connected graphs, say of size k, with μ attracted to an α-stable law with $0 < \alpha < 1$. Estimates close to optimal are derived for the critical size of the graph at and above which we have $\lambda_c(\mu) = 0$ (and below which $\lambda_c(\mu) = \infty$): except for countably many such α's, the estimates are sharp; for the exceptional α's, there is exactly one value of k for which the value of λ_c is undetermined. Similar ideas appear in connection with quantum versions of the Ising model and highly anisotropic Ising models [1,9,16].

Finally, motivated by different random environments for the contact process, other variations of RCP(μ) have been considered in [15], where the transmissions are also given by renewal processes.

2. Extinction

2.1. Main events

Our construction relates the probability of crossing a box in some direction for a well-chosen sequence of boxes that we define below. One important difference from the previous construction from [11] is a crossing event which we call a *temporal half-crossing*. A general space–time crossing is defined in [11] as follows.

Definition 2.1 (*Crossing*). Given space–time regions $C, D, H \subset \mathbb{Z}^d \times \mathbb{R}$ we say there is a crossing from C to D in H if there is a path $\gamma : [s, t] \to \mathbb{Z}^d$ such that $(\gamma(s), s) \in C$, $(\gamma(t), t) \in D$ and for every $u \in [s, t]$ we have $(\gamma(u), u) \in H$.

Given a space–time box $B := \left(\prod_{i=1}^{d}[a_i, b_i]\right) \times [s, t]$ we usually denote its space projection as $[a, b]$ where $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$. Also, we refer to its faces at direction $1 \leq j \leq d$ by

$$\partial_j^- B := \{(x, u) \in B; \; x_j = a_j\} \quad \text{and} \quad \partial_j B := \{(x, u) \in B; \; x_j = b_j\}.$$

Using this notation, we have three crossing events of box $B = [a, b] \times [s, t]$ that are important in our investigation.

Temporal crossing. Event $T(B)$ in which there is a path from $[a, b] \times \{s\}$ to $[a, b] \times \{t\}$ in B.

Temporal half-crossing. Event $\tilde{T}(B) := T([a, b] \times [s, \frac{t+s}{2}])$. In words, we have a temporal crossing from the bottom of B to the middle of its time interval.

Spatial crossing. For some fixed direction $j \in \{1, \ldots, d\}$ we define event $S_j(B)$ in which there is a crossing from $\partial_j^- B$ to $\partial_j B$ in B, i.e., there is a crossing connecting the opposite faces of direction j.

These events are the basis of our analysis of phase transition in RCP. Consider sequences a_n, b_n and fix a sequence of boxes $B_n = [0, a_n]^d \times [0, b_n]$. We want to relate

1. Crossings of box B_n to crossings of boxes at smaller scales.
2. Event $\{\tau^0 = \infty\}$ to crossings of boxes at some scale n.

From 1. we will obtain recurrence inequalities showing that the probability of crossing a box of scale n is very small for large n and this in turn will imply that in 2. we have $\mathbb{P}(\tau^0 = \infty) = 0$.

Considering a box $B = [-a_n/2, a_n/2]^d \times [0, b_n]$, we can see that if the infection of the origin survives till time b_n then either we have $T(B)$ or the infection must leave box B through some of its faces $\partial_j B$ or $\partial_j^- B$ for $1 \leq j \leq d$. Fix some direction j and notice that $\{(x, u) \in \mathbb{Z}^d \times \mathbb{R}; \; x_j = 0\}$ divides box B into two halves. Denote by $\tilde{B}_j$ the half containing face $\partial_j B$. Since the infection path is càdlàg, if we have a path leaving B through $\partial_j B$ then event $S_j(\tilde{B}_j)$ occurred. Thus, by symmetry and the union bound one can write

$$\mathbb{P}(\tau^0 = \infty) \leq \mathbb{P}(T(B)) + 2d \cdot \mathbb{P}(S_1(\tilde{B}_1)). \tag{3}$$

This quite simple relation already tells us that it suffices to prove that the probability of temporal crossings of B and spatial crossings of half-boxes in the short direction go to zero as $n \to \infty$.

2.2. General moment condition

We consider the sequence of space–time boxes $B_n = [0, a_n]^d \times [0, b_n]$. Also, we denote by $\tilde{B}_j(n)$ the half-box of B_n that contains the face $\partial_j B_n$. We are concerned with the probability of the following events:

$$S_j(B_n), \quad T(B_n), \quad \tilde{T}(B_n) \quad \text{and} \quad S_j(\tilde{B}_j(n)). \tag{4}$$

Notice that the probability of events in which some direction j appear are actually independent of j by symmetry. Another important remark is that whenever we translate a box by $(x, 0) \in \mathbb{Z}^d \times \mathbb{R}$ the probability of any of these crossing events remains the same. However, in order to disregard the specific position of our boxes in space–time and also the possible knowledge of some renewal marks below the box in consideration, it is useful to define the following uniform quantities.

Definition 2.2. We define

$$
\begin{aligned}
s_n &:= \sup \hat{\mathbb{P}}(S_j((x, t) + B_n)), & t_n &:= \sup \hat{\mathbb{P}}(T((x, t) + B_n)), \\
h_n &:= \sup \hat{\mathbb{P}}(S_j((x, t) + \tilde{B}_j(n))), & \tilde{t}_n &:= \sup \hat{\mathbb{P}}(\tilde{T}((x, t) + B_n)),
\end{aligned}
\tag{5}
$$

where the suprema above are over all $(x, t) \in \mathbb{Z}^d \times \mathbb{R}_+$ and all product renewal probability measures $\hat{\mathbb{P}}$ with interarrival distribution μ and renewal points starting at (possibly different) time points strictly less than zero.

Notice also that the quantities in which some direction j appear are actually independent of j by symmetry. Using (3) and the uniform quantities defined in (5), we can estimate

$$\mathbb{P}(\tau^0 = \infty) \leq t_n + 2d \cdot h_n \leq \tilde{t}_n + 2d \cdot h_n.$$

We just have to show the right hand side goes to zero, giving upper bounds to the quantities $\tilde{t}_n$ and h_n. This is done recursively, relating quantities from consecutive scales. Heuristically, we prove that whenever we have a crossing on scale n we must have two 'independent' crossings (either spatial crossings or temporal half-crossings) of boxes of the previous scale that are inside the original box.

Notice that if we are moving on a spatial direction, then this independence is immediate. For instance, it is clear that in order to cross B_n on the first coordinate direction we must cross both $\tilde{B}_1(n)$ and $B_n \setminus \tilde{B}_1(n)$. Since these events rely on independent processes, we have that $s_n \leq h_n^2$.

However, when moving on the time direction we might have dependencies; here, the uniform quantities prove their usefulness. The next lemma gives a uniform estimate on the probability of not having renewal marks on an interval, making it useful to adjust our choice of sequence b_n that represents the time length of our sequence of boxes B_n.

Lemma 2.3 (*Moment condition*). *Let μ be any probability distribution on $\mathbb{R}_+$ and $\mathcal{R}$ be a renewal process with interarrival μ started from some $\mathfrak{t} \leq 0$. Let $f : [0, \infty) \to [0, \infty)$ be non-decreasing, differentiable and satisfying $f(0) = 0$ and $f(x) \uparrow \infty$ as $x \to \infty$. If $\int x f(x) \mu(\mathrm{d}x) < \infty$, then uniformly on $\mathfrak{t}$ we have*

$$\sup_{t \geq 0} \mathbb{P}(\mathcal{R} \cap [t, t + u] = \varnothing) \leq \frac{C}{f(u)}, \tag{6}$$

for some positive constant $C = C(\mu, f)$ whenever $f(u) > 0$.

Proof. The proof is a standard application of renewal theorem. We can assume $\mathfrak{t} = 0$ since the case $\tilde{\mathfrak{t}} < 0$ is the same as taking a supremum over intervals $[t, t + u]$ with $t \geq -\tilde{\mathfrak{t}}$ and a renewal started from 0.

Let us first assume μ is non-arithmetic. Denote by F the cumulative distribution function of μ and let $\bar{F} = 1 - F$. Moreover, denote the overshooting at t for renewal $\mathcal{R}$ (i.e. the time till the next renewal mark after t) by Z_t and let $H(t) := \mathbb{E}[f(Z_t)]$. Conditioning with respect to the first renewal T_1, we have

$$
\begin{aligned}
H(t) &= \mathbb{E}[f(T_1 - t)\mathbb{1}\{T_1 > t\}] + \mathbb{E}\big[\mathbb{1}\{T_1 \leq t\}\mathbb{E}[f(Z_t) \mid T_1]\big] \\
&= \mathbb{E}[f(T_1 - t)\mathbb{1}\{T_1 > t\}] + \mathbb{E}\big[\mathbb{1}\{T_1 \leq t\}\mathbb{E}[f(Z_{t-T_1})]\big] \\
&= \int_t^\infty f(x - t)\,\mathrm{d}F(x) + \int_0^t H(t - x)\,\mathrm{d}F(x).
\end{aligned}
$$

Denoting the first integral above by $h(t)$, the equality above is the renewal equation $H = h + H * F$. Some alternative expressions for $h(t)$ are

$$
h(t) = \int_t^\infty f'(x - t)\bar{F}(x)\,\mathrm{d}x = \int_0^\infty f'(s)\bar{F}(s + t)\,\mathrm{d}s. \tag{7}
$$

To justify integration by parts in this step, we write $\int_t^\infty f(x - t)\,\mathrm{d}F(x) = \lim_{L\to\infty} \int_t^L f(x - t)\,\mathrm{d}F(x)$, and then perform the latter integral by parts (assuming $L > t$), obtaining $\int_t^L f'(x - t)\bar{F}(x)\,\mathrm{d}x + f(0)\bar{F}(t) - f(L - t)\bar{F}(L)$. Using the monotonicity of f and Markov's inequality, we find that

$$
f(L - t)\bar{F}(L) \leq f(L)\bar{F}(L) \leq \mathbb{E}[T_1 f(T_1)]/L,
$$

and the justification follows immediately from our other assumptions on f.

Let X be a random variable with distribution μ. From (7) it is easy to see that $h(0) = \mathbb{E}f(X) < \infty$ and that h is decreasing in t. Also, we can evaluate

$$
\begin{aligned}
\int_0^\infty h(t)\,\mathrm{d}t &= \int_0^\infty \int_0^\infty f'(s)\bar{F}(s + t)\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^\infty f'(s) \int_0^\infty \bar{F}(s + t)\,\mathrm{d}t\,\mathrm{d}s \\
&\leq \int_0^\infty f'(s)\mathbb{E}\big[X\mathbb{1}\{X > s\}\big]\,\mathrm{d}s \\
&= \mathbb{E}\Big[X \int_0^X f'(s)\,\mathrm{d}s\Big] \\
&= \mathbb{E}[Xf(X)].
\end{aligned}
$$

Thus, we have that h is directly Riemann integrable when $\mathbb{E}[Xf(X)] < \infty$ and the renewal theorem implies

$$
H(t) = \mathbb{E}[f(Z_t)] \to \frac{\mathbb{E}[Xf(X)]}{\mathbb{E}X}
$$

as $t \to \infty$. Separating the cases in which t is large and t is small, we have a uniform bound on t for $H(t)$. For the latter control, notice that $H(t) = \mathbb{E}[f(Z_t)]$ may be written as $\int_0^t U(\mathrm{d}s) \int_{t-s}^\infty \mu(\mathrm{d}r) f(r - (t - s))$, where U is the renewal measure associated to μ; from our assumptions on f, it follows that the innermost integral above is bounded by $\int_0^\infty \mu(\mathrm{d}r) f(r) = c < \infty$, and thus $H(t) \leq c \cdot U(0, t)$ is finite for every $t > 0$. From the fact that U is nondecreasing, we get that $H(t)$ is bounded on bounded intervals.

Since f is non-negative and non-decreasing, by Markov inequality we can write

$$\mathbb{P}(Z_t \geq u) \leq \frac{\mathbb{E}f(Z_t)}{f(u)} \leq \frac{C}{f(u)}.$$

The conclusion in (6) follows, under the assumption that μ is non-arithmetic. The arithmetic case is even simpler and a minor change in the argument yields the same bound. □

When we know that in a box $[0, a_n]^d \times [s, t]$ every site $x \in [0, a_n]^d$ has a renewal mark, analysing crossing events on $[0, a_n]^d \times [t, \infty)$ gets easier since we are able to forget all information from time interval $[0, s]$. Our next result uses Lemma 2.3 to estimate the probability that such event does not occur.

Corollary 2.4. *Let μ satisfy (1) and $f(x) := e^{\theta(\ln x)^{1/2}} \mathbb{1}_{\{x \geq 1\}}$. Define $J_n(t, s)$ as the event in box $[0, a_n]^d \times [t, t + s]$ in which there is some site $x \in [0, a_n]^d$ with no renewal marks on $[t, t + s]$. Then, for any $s \geq 2$ we have*

$$\sup_{t \geq 0} \mathbb{P}(J_n(t, s)) \leq \frac{Ca_n^d}{f(s)}.$$

Proof. Modify f in $[1, 2]$ to ensure differentiability and use the union bound. □

2.3. Relating successive scales

In this section we prove uniform upper bounds for $\tilde{t}_n$ and h_n in terms of h_{n-1} and $\tilde{t}_{n-1}$. From here on we consider boxes B_n with $a_n = 2^n$.

Temporal half-crossings. Let us upper bound the quantity $\tilde{t}_n$. For this part we work under the assumption that μ satisfies (1). Define

$$G_i := T\big([0, 2^n]^d \times [ib_{n-1}, (i + 1)b_{n-1}]\big)$$

and notice that event G_i is measurable with respect to the σ-algebra that looks all renewal processes and Poisson processes of B_n up to time $(i + 1)b_{n-1}$. Moreover, consider event $J = J_n(b_{n-1}, b_{n-1})$ defined in Corollary 2.4 and notice J is depends on the point processes up to time $2b_{n-1}$.

Assuming that $b_n/2 > 3b_{n-1}$, notice that we have

$$\tilde{T}(B_n) \subset J \cup (G_0 \cap J^c \cap G_2),$$

implying that we can write

$$\hat{\mathbb{P}}(\tilde{T}(B_n)) \leq \hat{\mathbb{P}}(J) + \hat{\mathbb{P}}(G_0) \cdot \hat{\mathbb{P}}(G_2 \mid G_0 \cap J^c).$$

Corollary 2.4 provides an upper bound for $\hat{\mathbb{P}}(J)$. Moreover, we can estimate the conditional probability by integrating over all possible collections $\{t_x; x \in [0, 2^n]^d\}$ of time points in $[b_{n-1}, 2b_{n-1}]$ the probability of event G_2. For any fixed choice of such collection, denote by $\tilde{\mathbb{P}}$ the probability measure with starting renewal marks given by $(x, t_x - 2b_{n-1})$. This leads to the bound

$$\hat{\mathbb{P}}(\tilde{T}(B_n)) \leq \frac{C2^{dn}}{f(b_{n-1})} + \hat{\mathbb{P}}(G_0) \cdot \sup_{\{t_x\}} \tilde{\mathbb{P}}(G_0).$$

The last product on the right hand side may be estimated by

$$\sup_{\{t_x\}} \big(\hat{\mathbb{P}}(T([0, 2^n]^d \times [0, b_{n-1}]))\big)^2,$$

where in the supremum we now consider any possible starting collection of time points $\{t_x; \; x \in \mathbb{Z}^d, t_x \le 0\}$ and we use notation $\hat{\mathbb{P}}$ to emphasize this. We look for an upper bound that is valid for any starting renewal marks. In order to bound $\hat{\mathbb{P}}(T([0, 2^n]^d \times [0, b_{n-1}]))$, we partition $[0, 2^n]^d$ into sub-boxes of side length 2^{n-2}. Considering projections of our crossing into space, we can prove

Lemma 2.5 (*Temporal Half-Crossing*). *Suppose μ satisfies* (1). *For every $n \ge 2$ it holds that*

$$\tilde{t}_n \le \frac{C 2^{dn}}{f(b_{n-1})} + (3^d t_{n-1} + 2d \cdot 3^{d-1} h_{n-1})^2. \tag{8}$$

Proof. For $v \in \{0, 1, 2, 3\}$ let us define

$$I_v := 2^{n-2}v + [0, 2^{n-2}].$$

This collection of 4 intervals of length 2^{n-2} covers $[0, 2^n]$. On $T([0, 2^n]^d \times [0, b_{n-1}])$ we can choose a path $\gamma : [0, b_{n-1}] \to [0, 2^n]^d$ that realizes the temporal crossing and consider its range $\mathcal{I} = \gamma([0, b_{n-1}])$. Project set $\mathcal{I}$ in each coordinate direction j, obtaining a discrete interval $\mathcal{I}_j \subset [0, 2^n]$, and define the *box count* of $\mathcal{I}_j$ by

$$c_j := \min\{|I|; \; I \subset \{0, 1, 2, 3\}, \; \mathcal{I}_j \subset \cup_{v \in I} I_v\}. \tag{9}$$

We decompose our event with respect to what is observed on each $\mathcal{I}_j$.

If for every $1 \le j \le d$ we have $c_j \le 2$ then the whole path γ is contained inside a d-dimensional box with side length 2^{n-1}. In this case, we have some choice of $v \in \{0, 1, 2\}^d$ such that

$$\mathcal{I} \subset 2^{n-2}v + [0, 2^{n-1}]^d,$$

and the number of possible v is given by 3^d.

Now, let us consider the case in which some $c_j \ge 3$ and thus $\mathcal{I}$ is not contained in some of the boxes with side length 2^{n-1} described above. In this case, we refine the argument by considering time. For any time $t \in [0, b_{n-1}]$ we define $\mathcal{I}(t) := \gamma([0, t])$ and for any fixed direction j we consider its projection $\mathcal{I}_j(t)$ and its box count $c_j(t)$. Define

$$t_1 := \inf\{t \in [0, b_{n-1}]; \; \exists 1 \le j \le d \text{ such that } c_j(t) \ge 3\}.$$
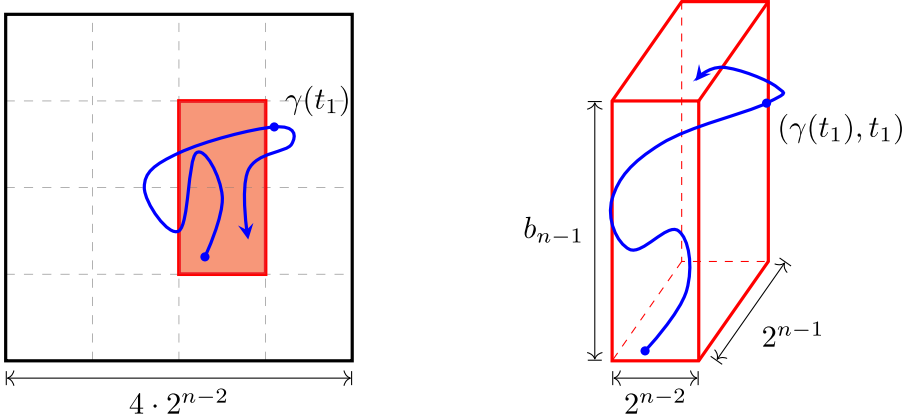
Since γ can only change value when there is transmission to a neighbouring site, at time t_1 we have $c_{j_0}(t_1-) = 2$ and $c_{j_0}(t_1) = 3$ for some special direction j_0 and $c_j(t_1) \le 2$ for every other direction. Thus, there is $v \in \{0, 1, 2\}^d$ such that

$$\mathcal{I}(t_1-) \subset 2^{n-2}v + [0, 2^{n-1}]^d, \quad \text{but} \quad \mathcal{I}_{j_0}(t_1) \not\subset 2^{n-2}v + [0, 2^{n-1}]^d$$
$$\text{and} \quad c_{j_0}(t_1) = 3.$$

Notice that this means path γ must have crossed a half-box of $2^{n-2}v + [0, 2^{n-1}]^d$ on direction j_0 during time interval $[0, t_1] \subset [0, b_{n-1}]$, see Fig. 1. There are $2d \cdot 3^{d-1}$ possible half-boxes to be crossed, which implies

$$\hat{\mathbb{P}}(T([0, 2^n]^d \times [0, b_{n-1}])) \le 3^d t_{n-1} + 2d \cdot 3^{d-1} h_{n-1}.$$

Since the bound above holds for any choice of renewal starting points $\{t_x; \; t_x \le 0, x \in [0, 2^n]^d\}$, taking the supremum over all such collections the result follows. $\quad\square$

Fig. 1. Depiction of the argument in Lemma 2.5 for the case $d = 2$. When the space projected temporal crossing is not contained in one of the 3^d sub-boxes of side length 2^{n-1} we must have a spatial crossing of a half-box of scale $n - 1$.

Spatial crossing. Now we prove a similar bound for quantity h_n. Recall that independence of the Poisson processes implies that for crossing B_n in some fixed spatial direction we need to perform two independent crossings of half B_n in that direction, implying

$$s_n \leq h_n^2.$$

A similar bound for h_n implies the following lemma.

Lemma 2.6 (*Spatial Crossing*). *For $n \geq 2$ it holds that*

$$h_n \leq 4 \cdot 36^{d-1} \cdot \left\lceil \frac{b_n}{b_{n-1}} \right\rceil^2 \cdot (h_{n-1} + \tilde{t}_{n-1})^2. \tag{10}$$

Proof. Independence of Poisson processes implies that

$$h_n \leq \sup \hat{\mathbb{P}}\big(S_1([0, 2^{n-2}] \times [0, 2^n]^{d-1} \times [0, b_n])\big)^2.$$
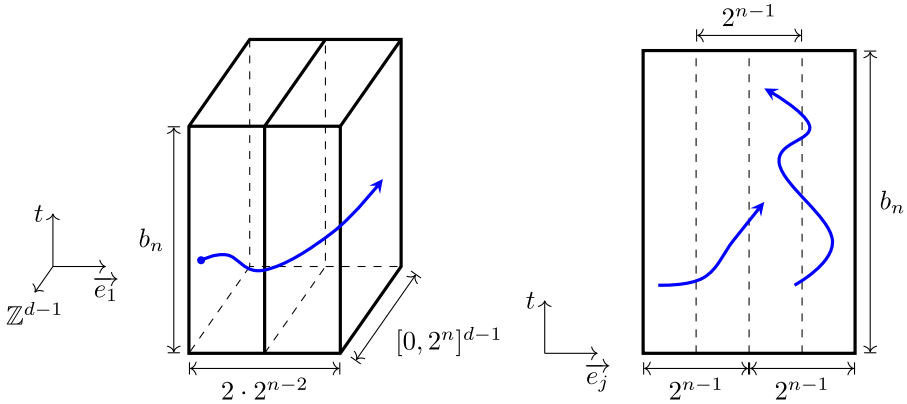
Let us simplify notation here. Since in a first moment we will work with boxes with time length $[0, b_n]$ we omit it from the notation. Also, on space coordinates we only work with intervals of length $2^n, 2^{n-1}$ or 2^{n-2}, so we write simply

$$B(l_1, \ldots, l_d) = \left(\prod_{i=1}^{d} [0, 2^{n-l_i}] \right) \times [0, b_n] \quad \text{for } l_i \in \{0, 1, 2\}.$$

We refer to a crossing of such box on direction j as $S_j(l_1, \ldots, l_d)$. Using this notation we want to show that on $S_1(2, 0, \ldots, 0)$ we can find some crossing of boxes whose side lengths are all at most 2^{n-1}, leading to an estimate of the form

$$\hat{\mathbb{P}}\big(S_1(2, 0, \ldots, 0)\big) \leq C(d) \cdot \hat{\mathbb{P}}\big(S_1(2, 1, \ldots, 1)\big),$$

recalling that $\hat{\mathbb{P}}$ refers to a probability measure starting from some fixed collection $\{\mathfrak{t}_x; \mathfrak{t}_x \leq 0, x \in \mathbb{Z}^d\}$ of starting renewal marks. The main step in this simplification is the following. Consider event $S_1(2, l_2, \ldots, l_d)$ and suppose that in direction j we have $l_j = 0$, meaning that the

Fig. 2. Crossing of a half box at scale n implies two independent spatial crossings. For each crossing, on direction $2 \leq j \leq d$ there are 2 possibilities: either the crossing traverses some interval of length 2^{n-2} or it remains inside an interval of length 2^{n-1}.

interval length in that direction is 2^n. Consider a path $\gamma : [s_1, t_1] \to \mathbb{Z}^d$ with $[s_1, t_1] \subset [0, b_n]$ that realizes event $S_1(2, l_2, \ldots, l_d)$ and let $\mathcal{I}_j$ be the projection of $\gamma([s_1, t_1])$ on direction j and c_j be its box count, i.e.,

$$c_j := \min\{|I|; \ I \subset \{0, 1, 2, 3\}, \ \mathcal{I}_j \subset \cup_{v \in I} I_v\}.$$

When $c_j \leq 2$ we can ensure that $\mathcal{I}_j$ is contained in $[v2^{n-2}, (v+2)2^{n-2}]$ for some $v \in \{0, 1, 2\}$. Thus, instead of the original box $B(2, l_2, \ldots, l_d)$ we can observe the same crossing on the smaller box in which on direction j we replace $[0, 2^n]$ by $[v2^{n-2}, (v+2)2^{n-2}]$, an interval with length 2^{n-1}. Similarly, if $c_j \geq 3$ we know that $\mathcal{I}_j$ must have crossed either I_1 or I_2, implying the crossing on direction j of a smaller box, since now the interval length on direction j is 2^{n-2}.

In both cases, the crossing of our original box implies the occurrence of some crossing of a smaller box inside it, see Fig. 2. Abusing notation, we do not specify the exact position of these smaller boxes, since in the final bound we use the uniform quantities from (5). Thus, we have

$$
\begin{aligned}
\hat{\mathbb{P}}(S_1(2, l_1, &\ldots, 0, \ldots, l_d)) \\
&\leq 3\hat{\mathbb{P}}(S_1(2, l_1, \ldots, 1, \ldots, l_d)) + 2\hat{\mathbb{P}}(S_j(2, l_1, \ldots, 2, \ldots, l_d)) \\
&= 3\hat{\mathbb{P}}(S_1(2, l_1, \ldots, 1, \ldots, l_d)) + 2\hat{\mathbb{P}}(S_1(2, l_1, \ldots, 2, \ldots, l_d))
\end{aligned}
$$

where the equality above follows from symmetry. For $l \in \{1, 2\}^{d-1}$ let us denote $a(l) = \#\{i; \ l_i = 1\}$. Applying the reasoning above to directions $2 \leq j \leq d$ successively, we can write

$$\hat{\mathbb{P}}(S_1(2, 0, \ldots, 0)) \leq \sum_{l \in \{1, 2\}^{d-1}} \hat{\mathbb{P}}(S_1(2, l)) \cdot 3^{a(l)} \cdot 2^{d-1-a(l)}.$$

Finally, notice that any $\hat{\mathbb{P}}(S_1(2, l))$ with $l \in \{1, 2\}^{d-1}$ is upper bounded by $\hat{\mathbb{P}}(S_1(2, 1, \ldots, 1))$ since increasing the box in some direction $2 \leq j \leq d$ can only make it easier to find a crossing.

This leads to the bound

$$\hat{\mathbb{P}}(S_1(2, 0, \ldots, 0)) \leq \hat{\mathbb{P}}(S_1(2, 1, \ldots, 1)) \cdot 2^{d-1} \sum_{l \in \{1, 2\}^{d-1}} (3/2)^{a(l)}$$

$$\leq 6^{d-1} \cdot \hat{\mathbb{P}}(S_1(2, 1, \ldots, 1)).$$

Returning to our previous notation, now we want to bound

$$\hat{\mathbb{P}}(S_1(2, 1, \ldots, 1)) = \hat{\mathbb{P}}(S_1([0, 2^{n-2}] \times [0, 2^{n-1}]^{d-1} \times [0, b_n]))$$

in terms of h_{n-1} and so we need to fix the time scale above. We use a collection of overlapping boxes

$$R_i = [0, 2^{n-2}] \times [0, 2^{n-1}]^{d-1} \times [ib_{n-1}, (i+1)b_{n-1}] \quad \text{for } i \in \tfrac{1}{2}\mathbb{Z}$$

to cover the time interval $[0, b_n]$. Then, either our path γ ensures we have $S_1(R_i)$ for some i or it must make a temporal crossing of some box $[0, 2^{n-2}] \times [0, 2^{n-1}]^{d-1} \times [ib_{n-1}, (i+1/2)b_{n-1}]$, which is event $\tilde{T}(R_i)$. Thus, we can write

$$\hat{\mathbb{P}}(S_1([0, 2^{n-2}] \times [0, 2^{n-1}]^{d-1} \times [0, b_n])) \leq 2 \left\lceil \frac{b_n}{b_{n-1}} \right\rceil \cdot (h_{n-1} + \tilde{t}_{n-1}).$$

Putting the bounds above together and taking the supremum over all possible collections of starting times, we obtain (10). $\square$

Simplifying recurrence. Looking at the expressions obtained in Lemmas 2.5 and 2.6, it seems useful to work with a simpler recurrence based on the quantity

$$u_n := h_n + \tilde{t}_n.$$

Noticing that $t_n \leq \tilde{t}_n$ we can write

$$u_n \leq \left[C(d) \cdot (b_n/b_{n-1})^2 \cdot (h_{n-1} + \tilde{t}_{n-1})^2 \right] + \left[C(d)(\tilde{t}_{n-1} + h_{n-1})^2 + \frac{C2^{dn}}{f(b_{n-1})} \right]$$

$$\leq C(d) \cdot (b_n/b_{n-1})^2 \cdot u_{n-1}^2 + \frac{C2^{dn}}{f(b_{n-1})}. \tag{11}$$

Lemma 2.7. *Let μ be any probability distribution on $\mathbb{R}_+$ and $\mathcal{R}$ be a renewal process with interarrival μ started from some $\mathfrak{t} \leq 0$. Suppose*

$$\int_1^\infty x e^{\theta(\ln x)^{1/2}} \mu(dx) \quad \text{for some } \theta > 4\sqrt{d \ln 2}. \tag{12}$$

There is a choice of sequence b_n and a natural number $n_0(\mu, \theta, d)$ such that if $u_{n_0} \leq 2^{-dn_0}$ then for every $n \geq n_0$ we have $u_n \leq 2^{-dn}$. Consequently, there exists $\lambda_0(\mu, \theta, d) > 0$ such that $\mathbb{P}(\tau^0 = \infty) = 0$ for any $\lambda \in (0, \lambda_0)$.

Proof. Consider the sequence of boxes $B_n = [0, 2^n]^d \times [0, b_n]$. Recall function f is given by

$$f(x) := e^{\theta(\ln x)^{1/2}} \cdot \mathbb{1}\{x \geq 1\}.$$

We want to take $f(b_{n-1}) := e^{\alpha(n-1)}$ for $\alpha > 0$ a parameter to be chosen later so that $2^{nd}/f(b_{n-1})$ tends to zero sufficiently fast. This can be accomplished by taking $b_n := e^{(\alpha/\theta)^2 n^2}$. Recurrence relation (11) then becomes

$$u_n \leq C(d) \left(\frac{b_n}{b_{n-1}} \cdot u_{n-1} \right)^2 + C(\mu, \theta) \exp[(d \ln 2)n - \alpha(n-1)].$$

Because of the error term above, the decay of u_n cannot be faster than $e^{-\alpha(n-1)}$. Based on this, we suppose $u_{n-1} \le e^{-\beta(n-1)}$ for some parameter $\alpha > \beta > 0$. Under this assumption we can estimate

$$\left(\frac{b_n}{b_{n-1}} \cdot u_{n-1}\right)^2 = \left(e^{(\alpha/\theta)^2(2n-1)} \cdot u_{n-1}\right)^2 \le e^{2(\alpha/\theta)^2(2n-1)-2\beta(n-1)},$$

which leads to

$$
\begin{aligned}
u_n &\le C(d)e^{2(\alpha/\theta)^2(2n-1)-2\beta(n-1)} + Ce^{(d\ln 2-\alpha)n+\alpha} \\
&\le C(d,\alpha,\beta,\theta)e^{[4(\alpha/\theta)^2-\beta]n} \cdot e^{-\beta n} + C(\mu,\theta,\alpha)e^{(\beta+d\ln 2-\alpha)n} \cdot e^{-\beta n}.
\end{aligned}
\tag{13}
$$

The induction will follow once we ensure

$$
\begin{cases}
4(\alpha/\theta)^2 - \beta < 0 \\
\beta + d\ln 2 - \alpha < 0
\end{cases}
\quad \text{or, equivalently,} \quad
\begin{cases}
\theta^2 > \frac{4\alpha^2}{\beta} \\
\alpha > \beta + d\ln 2.
\end{cases}
$$

We want to choose parameters α, β in order to make θ as small as possible while still being able to perform the induction. Notice that combining the two inequalities above we have

$$\theta^2 > 4\left(\sqrt{\beta} + \frac{d\ln 2}{\sqrt{\beta}}\right)^2 \ge 16d\ln 2,$$

by AM-GM inequality, with equality when $\beta = d\ln 2$. So, hypothesis (12) is the best we can hope in this setup. Fix $\beta = d\ln 2$. Looking at the possible values of α, we need to choose

$$2d\ln 2 < \alpha < \sqrt{\frac{\theta^2 d\ln 2}{4}}.$$

Since (12) implies $\sqrt{\frac{\theta^2 d\ln 2}{4}} > 2d\ln 2$, we can take for instance $\alpha(d,\theta) := \frac{1}{2}\left(2d\ln 2 + \sqrt{\frac{\theta^2 d\ln 2}{4}}\right)$. Take $n_0 = n_0(\mu, d, \theta)$ sufficiently large so that

$$
\begin{cases}
C(d,\alpha,\beta,\theta)e^{[4(\alpha/\theta)^2-\beta]n} & \le \frac{1}{4} \\
C(\mu,\theta,\alpha)e^{(\beta+d\ln 2-\alpha)n} & \le \frac{1}{4},
\end{cases}
\quad \text{for all } n \ge n_0.
\tag{14}
$$

This is possible since both left hand sides tend to zero as $n \to \infty$. Suppose that $u_{n_0} \le e^{-\beta n_0} = 2^{-dn_0}$, recalling that $\beta = d\ln 2$. Then, we have by (13) that

$$u_n \le \frac{1}{4}e^{-\beta n} + \frac{1}{4}e^{-\beta n} \le e^{-\beta n} \quad \text{for every } n \ge n_0.$$

The induction just described will hold if we can ensure that the base case $n = n_0$ holds. But if $n_0(\mu, \theta, d)$ is fixed we can take λ_0 sufficiently small for it to hold. Indeed, just notice that for any box $(x,t) + B_{n_0}$ if we denote by N the number edges contained in $[0, 2^{n_0}]^d$ we have that

$$\hat{\mathbb{P}}(H) := \hat{\mathbb{P}}(\text{no transmission on } (x,t) + B_{n_0}) = e^{-\lambda b_{n_0} \cdot N} \to 1$$

as $\lambda \to 0$. Clearly, we have $h_{n_0} \le \hat{\mathbb{P}}(H^c)$. Moreover, we can control $\tilde{t}_{n_0}$ similarly, since if there is no transmission the only possibility for a temporal half-crossing of box $(x,t) + B_{n_0}$ is achieving it by a single site, an event which we recall was denoted $J = J_{n_0}(t, b_{n_0}/2)$ in

Corollary 2.4. Hence, we have

$$\hat{\mathbb{P}}(H \cap J) \leq \frac{C(\mu, \theta)2^{dn_0}}{f(b_{n_0}/2)} = C2^{dn_0} \exp\left[-\theta\left(\frac{\alpha^2}{\theta^2}n_0^2 - \ln 2\right)^{\frac{1}{2}}\right]$$

$$= C2^{dn_0} \exp\left[-\alpha n_0\left(1 - \frac{\theta^2 \ln 2}{(\alpha n_0)^2}\right)^{\frac{1}{2}}\right] = Ce^{(d\ln 2 - \alpha)n_0 + O(n_0^{-1})}$$

as $n_0 \to \infty$. Making a small change in (14), we can increase n_0 if needed to ensure $\hat{\mathbb{P}}(H \cap J) \leq \frac{1}{4}e^{-\beta n_0}$ and write

$$\max\{h_{n_0}, \tilde{t}_{n_0}\} \leq \sup\{\hat{\mathbb{P}}(H^c) + \hat{\mathbb{P}}(H \cap J)\} \leq 1 - e^{-\lambda b_{n_0} N} + \frac{1}{4}e^{-\beta n_0} \leq \frac{1}{2}e^{-\beta n_0}$$

for λ sufficiently small. We conclude $u_{n_0} \leq e^{-\beta n_0}$. $\quad\square$

Proof of Theorem 1.1. It follows from the conclusion of Lemma 2.7. $\quad\square$

Remark 2.8. The exponent $1/2$ in the definition of function f is the best possible, meaning that the same reasoning does not work for a function $g = \exp[\theta(\ln x)^\delta]$ with $\delta < 1/2$.

Remark 2.9. We recall (see e.g. Section 2 of [11]) that when the interarrival distribution μ has a density f and $h_\mu(t) = f(t)/\mu(t, \infty)$ is the hazard rate function, the corresponding renewal process starting at some point $t_0 \in \mathbb{R}$ can be easily obtained in terms of a homogeneous Poisson point process on $\mathbb{R} \times \mathbb{R}_+$ with intensity 1. The construction shows that when the hazard rate is decreasing, the corresponding renewal point process, hereby denoted by $\mathcal{R}_\mu$, is an increasing function of points in the Poisson point process. As already mentioned, this property was used in [11] to guarantee the FKG property. Moreover, as easily verified, it also yields the following:

If ν is another probability measure on $(0, \infty)$ with a density g and hazard rate h_ν, and $h_\nu(t) \geq h_\mu(t)$ for all t, then the two renewal processes starting at some $t_0 \in \mathbb{R}$ can be coupled in such a way that $\mathcal{R}_\mu \subset \mathcal{R}_\nu$ with probability one.

Using this observation with ν being an exponential distribution, we conclude that if h_μ is decreasing and bounded, then $\mathcal{R}_\mu$ can be embedded in a Poisson point process. Thus, the classical result on Harris contact process yields $\lambda_c(\mu) < \infty$. It is easy to come up with a wide range of examples of such μ's.

3. Complete convergence

In this section we prove Theorem 1.2, which relies on a variant of the argument of [10]. We begin with a sketch of the argument. In the following, λ is a fixed strictly positive infection rate. For our RCP equipped with its natural filtration $(\mathcal{A}_t)_{t \geq 0}$, we say a stopping time T is *extreme* if

$$\max\{\|x\|_\infty : \xi_T(x) = 1\} > \max_{s < T}\{\|x\|_\infty : \xi_s(x) = 1\},$$

where $\|x\|_p$ denotes the usual ℓ^p-norm on $\mathbb{Z}^d$.

An extreme stopping time T is useful as it implies the existence of a site x_T such that $\xi_T(x_T) = 1$ and a Euclidean unit vector $\vec{e}$ in $\mathbb{Z}^d$ so that all renewal processes $(\mathcal{R}_{x_T + m\vec{e}}; m \geq 1)$ are conditionally i.i.d. independent of $\mathcal{A}_T$.

On Lemma 3.2 we show that the probability of the infection surviving only inside a finite cylinder is zero. This means that we can find arbitrarily large extreme stopping times, which gives us a way to build independent events.

Then, on Corollary 3.3 we build a tunnelling event inspired in the construction of [10]. We prove that almost surely we can find a sequence of (random) sites that do not have any cure marks for a very long time. These intervals have a sizeable overlap, allowing the infection to travel from one to another and making them to work as hubs for sustaining the infection indefinitely. Corollary 3.3 provides a quantitative control for the space–time position of such hubs.

Finally, on Lemma 3.5 we leverage the existence of these hubs and show they can be used to infect every site in a suitably sized region around the hub. This construction is based on the following definition.

Definition 3.1. We say (x, u) freely-infects (y, v) in the set $A \subset \mathbb{Z}^d$ if there exists a sequence of points $x = x_0, x_1, \ldots, x_n = y$ and times $u < t_1 < t_2 < \cdots < t_n < v$ so that for each i the sites x_{i-1} and x_i are nearest neighbours and $x_i \in A$, and there is an infection mark from x_{i-1} to x_i at time t_i.

We stress that we are not assuming that $\xi_u(x) = 1$. The event "(x, u) freely-infects (y, v) in A" depends purely on the collection of Poisson processes and does not concern the renewal processes, i.e. it does not take into account the recovery times.

The proof of Theorem 1.2 puts all these pieces together to conclude that, on the event that the infection survives, the probability that any fixed finite set $K \subset \mathbb{Z}^d$ is infected at a large time tends to 1.

Lemma 3.2. *For a RCP with $\tau = \inf\{s > 0 : \xi_s \equiv 0\}$ we have*

$$P\Big(\{\tau = \infty\} \cap \Big\{ \big|\{x : \int_0^\infty \xi_s(x) ds > 0\}\big| < \infty \Big\}\Big) = 0.$$

This lemma implies that for all time t there a.s. exists on the set $\{\tau = \infty\}$ an extreme time $T > t$. Indeed, just take the next time after t when the process encounters a site whose norm is a new maximum among sites infected or previously infected.
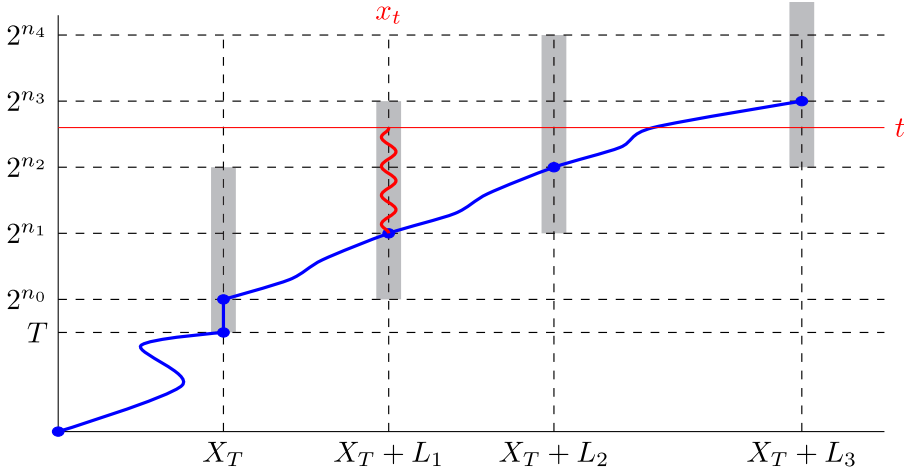
Proof. Without loss of generality we assume that $\sum_x \xi_0(x) < \infty$ as otherwise the result is trivial. The idea is that if the result were not true, then for some $m < \infty$ the infected sites would remain a subset of $[-m, m]^d$ for ever. But as time becomes large there will be arbitrarily large intervals of time on which there are no renewal points for any $x \in [-m, m]^d$ and so no opposition to the process infecting sites outside this finite set. It is enough to prove that for each $m \in \mathbb{N}$, the event

$$\{\tau = \infty\} \cap \big\{\xi_s(x) = 0, \ \forall s \geq 0, \ \forall x \notin [-m, m]^d\big\}$$

has probability zero. But by Proposition 7 in [10], for all n large enough the probability that at least one of the renewal processes on $[-m, m]^d$ intersects the time interval $[2^n, 2^n + 2^{n\epsilon_1}]$ is less than $(2m + 1)^d 2^{-n\epsilon_1}$, provided ϵ_1 is fixed strictly positive but small enough. Furthermore, the probability (conditional upon $\xi_{2^n}(x) = 1$ for some $x \in [-m, m]^d$) that there does not exist a sequence $x = x_0, x_1 \ \cdots \ x_k$ with $k \leq m + 1$ of nearest neighbour sites that satisfy

(i) $\xi_{2^n + 2^{n\epsilon_1}}(x_k) = 1$, with $x_k \notin [-m, m]^d$;
(ii) $\|x_i - x_{i-1}\|_1 = 1$, for all $1 \leq i \leq k$; and
(iii) $\xi_{2^n}(x_0) = 1$ with $x_0 \in [-m, m]^d$.

tends to zero as $n \to \infty$, which implies the lemma. $\quad\square$

Fig. 3. On H_T we have an infinite infected path (in blue) that passes through points $(X_T + L_i, 2^{n_i})$. The grey areas represent absence of cure marks. For $2^{n_{i-1}} < t \leq 2^{n_i}$ we choose $x_t = X_T + L_{i-2}$, which ensures $\xi_s(x_t) = 1$ on the whole interval $[2^{n_{i-2}}, t] \supset [t/2, t]$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Corollary 3.3. *On the event $\{\tau = \infty\}$, for all t large there is a site x_t within distance $\ln^3 t$ of the origin so that $\xi_s(x_t) = 1$ for all $s \in [t/2, t]$.*

Proof. For purely notational reasons we suppose that the dimension, d, is equal to one. We first treat the case $\sum_x \xi_0(x) < \infty$ and then note how the argument given can be extended to the infinite case. Given an extreme stopping time T, we define a suitable tunnelling event H_T (see Fig. 3). What is important is that its conditional probability given $\mathcal{A}_T$ should be bounded away from zero on $\{T > N\}$, where N is a large constant. In this description and calculation of probability bounds, we suppose

$$X_T > \max\{x : \exists s < T \text{ so that } \xi_s(x) = 1\}.$$

If $X_T < \min\{x : \exists s < T \text{ so that } \xi_s(x) = 1\}$ then we simply reflect the definitions and all probability bounds will be the same. Define

$$H_T := \bigcap_{n=0}^{\infty} H_{T,n}, \qquad (15)$$

where the events $H_{T,n}$ are defined recursively via the random integers $\{L_j\}_{j=0}^{\infty}$ and $\{n_j\}_{j=0}^{\infty}$: $H_{T,0}$ is simply the event $\{\mathcal{R}_{X_T} \cap [T, 2^{n_0+2}] = \emptyset\}$, where $2^{n_0} = \inf\{2^n : 2^n > T\}$. By Lemma 2 in [10], there exists $c > 0$ so that

$$P(H_{T,0}|\mathcal{A}_T) \geq c > 0 \quad \text{on } \{T > N\}$$

for all N fixed. We take $L_0 := 0$. Given $n_0, \ldots, n_{i-1}$ and $L_0, L_1, \ldots, L_{i-1}$ we set $n_i = n_{i-1}+1$ and define

$$L_i := \inf\{k > L_{i-1} : \mathcal{R}_{X_T+k} \cap [2^{n_i-1}, 8 \cdot 2^{n_i-1}] = \emptyset\}.$$

Our event $H_{T,i}$ is given by the following conditions:

(i) $L_i - L_{i-1} \leq i n_0$;
(ii) There exists an infection path from $(X_T + L_{i-1}, 2^{n_{i-1}})$ to $(X_T + L_i, 2^{n_i})$ in the space–time rectangle $[X_T + L_{i-1}, X_T + L_i] \times [2^{n_{i-1}}, 2^{n_i}]$.

From the argument in Section 4 of [10], we have that if N is fixed sufficiently large then

$$P(H_T \mid \mathcal{A}_T) = P\Big(\bigcap_{i=0}^{\infty} H_{T,i} \Big| \mathcal{A}_T \Big) \geq c_1 > 0,$$

for some c_1, uniformly on $\{T > N\}$. The argument of [10] uses Markov properties in the environment on $[X_T + L_{i-1})$ given our realization of L_{i-1}. The heavy-tailed distribution for the renewals ensures that for each $x > X_T + L_{i-1}$, the corresponding renewal process has probability (uniformly in x and i) bounded away from zero of satisfying the emptiness condition entering in the definition of L_i, and so condition (i) holds outside exponentially small probability. We then show that the process will infect site $X_T + L_i$ outside exponentially small probability in i. From this, we easily obtain

$$P\big(\{\tau = \infty\} \cap \{\nexists \text{ extreme } T \text{ such that } H_T \text{ occurs}\}\big) = 0. \tag{16}$$

Indeed, whenever event H_T does not happen we have a random finite index U such that $H_{T,U}$ is the first event $H_{T,i}$ that did not happen. Consider the random time $S = 2^{n_U}$. By Lemma 3.2 we can find an extreme stopping time $T_2 > S$ and once again we have $P(H_{T_2} \mid \mathcal{A}_{T_2}) \geq c_1$. Iterating this reasoning, we deduce (16).

By (16) we can conclude that a.s. there is some extreme time T for which H_T happens. Consider the sequences $\{L_j\}_{j=0}^{\infty}$ and $\{n_j\}_{j=0}^{\infty}$ associated with T and let $t > 2^{n_1}$. Let i be the unique index such that $2^{n_{i-1}} < t \leq 2^{n_i}$ and define $x_t := X_T + L_{i-2}$. By construction of event H_T, we have $\xi_s(x_t) = 1$ on the whole interval $[2^{n_{i-2}}, t] \supset [t/2, t]$. We estimate x_t by noticing

$$x_t = X_T + L_{i-2} \leq X_T + \sum_{j=1}^{i-2} j n_0 \leq X_T + \frac{n_0}{2}(i-1)^2 \leq X_T + \frac{n_0}{2}\Big(\frac{\ln t}{\ln 2} - n_0\Big)^2 \ll \ln^3 t,$$

as $t \to \infty$. This implies the corollary for finite initial configurations. If $\sum_x \xi_0(x) = \infty$ then it is easy to see that there exists x so that taking $T = 0$ and $X_T = x$ the event H_T occurs (though of course T is not extreme in this case). $\quad\square$

Remark 3.4. It should be noted that the event H_T in higher dimensions involves a direction along one of the coordinate axes in $\mathbb{Z}^d$ away from the origin.

Before proving Theorem 1.2 we need some definitions and basic lemmas.

Lemma 3.5. *Let $d \geq 1$ and $\mathsf{B}(r) = [-r, r]^d$. Let V_t be the event that for every $x, y \in \mathsf{B}(\ln^3 t)$ we have $(x, t - t^{\epsilon_1})$ freely-infects (y, t) in $\mathsf{B}(\ln^3 t)$, where again ϵ_1 arises from Proposition 7 of [10]. Then $\lim_{t \to \infty} P(V_t) = 1$.*

Proof. Fix x and y and take a shortest path $x = x_0, x_1, \ldots, x_n = y$ from x to y, where x_i is a nearest neighbour of x_{i-1}, and $x_i \in \mathsf{B}(\ln^3 t)$ for every i. Clearly, $n \leq C(d) \ln^3 t$ for some positive constant C. Then we see that

$$P\left(\begin{array}{l} (x, t - t^{\epsilon_1}) \text{ does not freely-} \\ \text{infect } (y, t) \text{ in } \mathsf{B}(\ln^3 t) \end{array} \right) \leq P\big(\mathrm{Poi}(\lambda t^{\epsilon_1}) \leq C(d) \ln^3 t\big) \leq e^{-c t^{\epsilon_1}}, \tag{17}$$

for some positive $c = c(\lambda)$ as $t \to \infty$, where $\mathrm{Poi}(u)$ denotes a Poisson random variable of rate u. Thus, we can write

$$P(V_t) \geq 1 - \sum_{x, y \in \mathsf{B}(\ln^3 t)} P\left(\begin{array}{c} (x, t - t^{\epsilon_1}) \text{ does not freely-} \\ \text{infect } (y, t) \text{ in } \mathsf{B}(\ln^3 t) \end{array} \right) \geq 1 - C(d)(\ln^3 t)^{2d} e^{-c\, t^{\epsilon_1}}. \quad \square$$

Proof of Theorem 1.2. Notice that on $\{\tau = \infty\}$ if we also ensure the occurrence of events V_t,

$$W_t := \left\{ \exists x \in \mathsf{B}(\ln^3 t) : \xi_s(x) = 1 \text{ on } [t/2, t] \right\}, \quad \text{and}$$
$$U_t := \left\{ \mathcal{R}_x \cap (t - t^{\epsilon_1}, t) = \emptyset, \forall x \in \mathsf{B}(\ln^3 t) \right\}$$

then every site of $\mathsf{B}(\ln^3 t)$ is infected at time t. Then, we can write for any fixed $x \in \mathbb{Z}^d$ that

$$P(\tau = \infty, \xi_t(x) = 0) \leq P(V_t^{\mathsf{c}}) + P(\{\tau = \infty\} \cap W_t^{\mathsf{c}}) + P(U_t^{\mathsf{c}})$$

for sufficiently large t. Notice that all three terms on the right hand side tend to zero as $t \to \infty$. Indeed, the first one tends to zero by Lemma 3.5, the second one by Corollary 3.3, and for the third one we have by Proposition 7 of [10] that it has probability less than $C(d)(\ln^3 t)^d t^{-\epsilon_1}$. We conclude that for any finite set $K \subset \mathbb{Z}^d$ we have

$$P(\{\tau = \infty\} \cap \{\xi_t(x) = 1, \forall x \in K\}) \geq P(\tau = \infty) - \sum_{x \in K} P(\tau = \infty, \xi_t(x) = 0)$$

$$\to P(\tau = \infty)$$

as $t \to \infty$ and Theorem 1.2 follows. $\quad \square$

Remark 3.6. As brought up at the introduction, the upper bound on the tail of μ prescribed in Condition C), may indeed be dropped as a hypothesis in Theorem 1 of [10]. We briefly explain an alternative argument for that result, dispensing with the upper bound. For short, we refer to the notation and passages in [10].

In the argument leading up to the proof of Theorem 1 in [10], we use Proposition 7 of [10], for which we require the upper bound; but we may instead use Corollary 4 [10], for which that bound is not required. In that corollary, there figures an interval, denoted by I, and a subinterval $J \subset I$. We apply it with $I = I_i = [t_0 2^i, t_0 2^{i+1}]$ and obtain $J = J_i = (\sigma_i, \sigma_i + (t_0 2^i)^{\gamma'})$, $i = 1, 2, \ldots$, where $\gamma' := \epsilon_3/2$, with ϵ_3 as in Corollary 4 [10], for some $\sigma_i \in I_i$ such that $J_i \subset I_i$. We then modify the argument on page 2911 of [10] as follows:

1. We replace (II) by (II′), where the interval $[t_0 2^i - (t_0 2^i)^{\gamma}, t_0 2^i]$ in (II) is replaced with J_i in (II′);
2. We next replace (III) by (III′), where the interval

$$\left((t_0 2^i - (t_0 2^i)^{\gamma} + \frac{k - L_{i-1}}{i \log t_0}(t_0 2^i)^{\gamma} , \; t_0 2^i - (t_0 2^i)^{\gamma} + \frac{k + 1 - L_{i-1}}{i \log t_0}(t_0 2^i)^{\gamma}) \right)$$

in (III) is replaced by

$$\left(\sigma_i + \frac{k - L_{i-1}}{i \log t_0}(t_0 2^i)^{\gamma'} , \; \sigma_i + \frac{k + 1 - L_{i-1}}{i \log t_0}(t_0 2^i)^{\gamma'} \right)$$

in (III′);
3. We finally replace the last paragraph in the proof of Lemma 9 in [10] by the following (parallel) paragraph: By Corollary 4 in [10], the probability of (II′) occurring and

$L_i \leq L_{i-1} + i \log t_0$ is bounded by $i \log(t_0)(t_0 2^i)^{-\gamma''}$, where $\gamma'' := 2\gamma'/3 = \epsilon_3/3$, (again supposing t_0 is large). Similarly the intersection of (III$'$) and $L_i \leq L_{i-1} + i \log t_0$ has a probability bounded by

$$i \log(t_0) e^{-\lambda(t_0 2^i)^{\gamma''}/i \log(t_0)}.$$

Remark 3.7. As pointed out above, the argument in Remark 3.6 dispenses with Proposition 7 of [10], relying exclusively on Corollary 4 of [10] to establish Theorem 1 of [10]. But to argue Theorem 1.2, as we did above in this section, we require the full force of Proposition 7 of [10]. Indeed, in the proof of Theorem 1.2, in the event U_t we are not able to replace the *fixed* interval $(t - t^{\epsilon_1}, t)$, for which we may claim the absence of cure marks with high probability using Proposition 7, by an interval with similar properties provided by Corollary 4, since (the closure of) the latter interval might not contain t, and that would invalidate our argument.

4. Closeness to determinism

In this section we consider a strengthening of Theorem 1.2. This requires greater regularity on our renewal distribution. We require not merely that condition C) of [10] holds but that F has a regular tail power:

$$\bar{F}(t) \equiv 1 - F(t) \in RV(-\alpha)$$

for some $0 < \alpha < 1$, where $RV(\beta)$ denotes the set of functions that for large t are of the form $t^\beta L(t)$ for L slowly varying. If $\alpha \leq 1/2$ we require, in addition, the second condition of Theorem 1.4 of [5] that function

$$I_1^+(\delta; t) := \int_{1 \leq z \leq \delta t} \frac{F(t - \mathrm{d}z)}{z \bar{F}(z)^2} \quad \text{satisfies} \quad \lim_{\delta \to 0} \overline{\lim_{t \to \infty}} \, t \bar{F}(t) \cdot I_1^+(\delta; t) = 0, \tag{18}$$

which in the notation of [5] is saying that $I_1^+(\delta; t)$ is *asymptotically negligible*.

Theorem 1.2 tells us that on the event $\{\tau = \infty\}$ the configuration ξ_t converges to $\delta_{\underline{1}}$ in distribution as $t \to \infty$, which is equivalent to

$$\text{for every } x \in \mathbb{Z}^d, \qquad \xi_t(x) \xrightarrow{P} 1 \text{ as } t \to \infty.$$

This is because the renewal (or healing) points become so sparse as t becomes large that the infection process infects all sites in a bounded region "deterministically" if there are no healing points nearby.

One way of expressing this is to introduce the σ-field $\mathcal{G}$ generated by the renewal processes $(\mathcal{R}_u)_{u \in \mathbb{Z}^d}$ and the extinction random variable τ. We should have that, when the infection survives, the conditional probability $P(\xi_t(x) = 1 \mid \mathcal{G})$ should be close to 1 for large t if there are no points of $\mathcal{R}_x \cap [0, t]$ close to t. Refining further, one might hope that on $\{\tau = \infty\}$ it holds

$$\lim_{t \to \infty} \left| P(\xi_t(x) = 0 \mid \mathcal{G}) - e^{-2\lambda d Y_t(x)} \right| = 0 \quad \text{for every } x \in \mathbb{Z}^d,$$

where $Y_t(x) := t - \sup\{\mathcal{R}_x \cap [0, t]\}$ is the age process. In fact this depends on the power decay of $\bar{F}$.

Theorem 4.1. *If $\bar{F} \in RV(-\alpha)$ for $0 < \alpha < 1$ then for all $x \in \mathbb{Z}^d$:*

(i) If $\alpha < 1/2$ and also (18), it holds on $\{\tau = \infty\}$ that

$$\lim_{t\to\infty} \left| P(\xi_t(x) = 0 \mid \mathcal{G}) - e^{-2\lambda d Y_t(x)} \right| = 0.$$

(ii) If $\alpha > 1/2$ and also $F(t) > 0$, for every $t > 0$, it holds on $\{\tau = \infty\}$ that

$$\overline{\lim_{t\to\infty}} \left(P(\xi_t(x) = 0 \mid \mathcal{G}) - e^{-2\lambda d Y_t(x)} \right) > 0.$$

Remark 4.2. The case $\alpha = 1/2$ is not explicitly treated (though it is treatable) as it depends on how $\bar{F}(t)/t^{1/2}$ behaves as $t \to \infty$.

Remark 4.3. In the case $\alpha > \frac{1}{2}$ we do not need the full force of [5], the preceding strong renewal theorem of [7] suffices.

The same proof can be adapted to reach a more precise conclusion when $\alpha \in (1/2, 1)$.

Theorem 4.4. *Assume that $\bar{F} \in RV(-\alpha)$ for $\alpha \in (1/2, 1)$ and that $F(t) > 0$ for every $t > 0$. For all $x \in \mathbb{Z}^d$:*

(i) If $1 \le k < 2d$ and $1 - \alpha \in \left(\frac{1}{k+2}, \frac{1}{k+1}\right)$, then on $\{\tau = \infty\}$ we have

$$\overline{\lim_{t\to\infty}} \left(P(\xi_t(x) = 0 \mid \mathcal{G}) - e^{-(2d-k)\lambda Y_t(x)} \right) = 0.$$

(ii) If $1 - \alpha < \frac{1}{2d+1}$ then for every $0 \le s < \infty$ we have on $\{\tau = \infty\}$ that

$$\overline{\lim_{t\to\infty}} \mathbb{1}_{\{Y_t(x)=s\}} P(\xi_t(x) = 0 \mid \mathcal{G}) = 1.$$

We provide a detailed proof for Theorem 4.1. The same steps are used (in generalized form) for Theorem 4.4 but the extra details involved do not add any insight to the result. Considering this, we opted to only sketch the proof of Theorem 4.4, pointing out the differences to its simpler version. We require preliminary lemmas first.

Given an integer M and $t \ge 0$ we define the event $H(M, t)$ to be that for some stopping time $T < t$ with $X_T \notin [-M, M]^d$ the event H_T occurs (see Remark 3.4 following Corollary 3.3). The next lemma is immediate from the argument in Corollary 3.3.

Lemma 4.5. *For any $M \in \mathbb{N}$ and finite ξ_0 we have that as $t \to \infty$*

$$P(H(M, t) \mid \mathcal{G}) \xrightarrow{a.s.} \mathbb{1}_{\{\tau=\infty\}}.$$

Proof. From the proof of Corollary 3.3 we know $P\left(\{\tau = \infty\} \cap \left(\cup_{t\ge1} H(M, t)\right)^c\right) = 0$. Since $H(M, t)$ is increasing in t, we have that the limit of $P(H(M, t) \mid \mathcal{G})$ as $t \to \infty$ is almost surely

$$P(\cup_{t\ge1} H(M, t) \mid \mathcal{G}) = P(\tau = \infty \mid \mathcal{G}) - P\left(\{\tau = \infty\} \cap \left(\cup_{t\ge1} H(M, t)\right)^c \mid \mathcal{G}\right)$$
$$= \mathbb{1}_{\{\tau=\infty\}}. \quad \square$$

here $\quad \square$

For the next lemma, let us define θ_t as the time-shift by t of the infection Poisson processes $\{N^{x,y}\}$. It holds

Lemma 4.6. *Given $M \in \mathbb{N}$ and $t_0 > 0$, let A be some event generated by Poisson processes $N^{x,y} \cap [0, t_0]$ for $x, y \in [-M, M]^d$. Then,*

$$\lim_{t \to \infty} P\big(\theta_t(A) \mid \mathcal{G}\big) = P(A) \quad a.s. \text{ on } \{\tau = \infty\}.$$

Proof. The conditional probability $P\big(\theta_t(A) \mid \mathcal{G}\big)$ on the event $\{\tau = \infty\}$ can be written as

$$P(\theta_t(A) \mid \mathcal{G}) = P(\theta_t(A) \cap H(M, t) \mid \mathcal{G}) + P(\theta_t(A) \cap H(M, t)^{\mathsf{c}} \mid \mathcal{G}).$$

We now claim that $P\big(\theta_t(A) \cap H(M, t) \mid \mathcal{G}\big) = P(A) \cdot P(H(M, t) \mid \mathcal{G})$. Indeed, consider the families

$$\mathcal{C} := \big\{C \in \mathcal{G}; \ P(\theta_t(A) \cap H(M, t) \cap C) = P(A) \cdot P(H(M, t) \cap C)\big\},$$
$$\mathcal{P} := \big\{V \cap W; \ V \in \sigma(\mathcal{R}_z; \ z \in \mathbb{Z}^d), \ W \in \sigma(\tau)\big\}.$$

It is straightforward to check that $\mathcal{C}$ is a λ-system and $\mathcal{P}$ is a π-system that generates $\mathcal{G}$. Notice that $H(M, t) \subset \{\tau = \infty\}$. If $W \supset \{\tau = \infty\}$ we have

$$P(\theta_t(A) \cap H(M, t) \cap (V \cap W)) = P(\theta_t(A) \cap (H(M, t) \cap V)) = P(A) \cdot P(H(M, t) \cap V),$$

since A does not depend on renewals, only on a region of infection that is disjoint of the one event $H(M, t) \cap V$ depends. If $W \not\supset \{\tau = \infty\}$, then both sides are zero. Thus, we conclude that $\mathcal{P} \subset \mathcal{C}$ and by Dynkin's $\pi - \lambda$ Theorem the claim follows. The result follows from Lemma 4.5. $\quad\square$

Remark 4.7. The limit also holds a.s. on $\{\tau < \infty\}$, with a simpler proof, but this is not needed in our argument.

Corollary 4.8. *For $x \in \mathbb{Z}^d$ and $t > 0$, let*

$$A_t := \{\forall z \sim x, \ N^{z,x} \cap [t - Y_t(x), t] = \emptyset\},$$

where $\sim$ signifies the relation of being neighbour. Then,

$$\lim_{t \to \infty} \big|P\big(A_t \mid \mathcal{G}\big) - e^{-2d\lambda(Y_t(x))}\big| = 0 \quad a.s. \text{ on } \{\tau = \infty\}.$$

Proof. Fix $0 < M < \infty$ and consider events

$$A_t^{i,M} = \Big\{\forall z \sim x, \ N^{z,x} \cap \Big[t - \frac{i}{M}, t\Big] = \emptyset\Big\}.$$

By Lemma 4.6, for each $0 \leq i \leq M^2$, $P(A_t^{i,M} \mid \mathcal{G}) \to e^{-2d\lambda i/M}$ as t becomes large on $\{\tau = \infty\}$. So by monotonicity for each i and for t large, on $\{\tau = \infty\}$

$$-\frac{1}{M} \leq \mathbb{1}_{\{Y_t(x) \in [\frac{i}{M}, \frac{i+1}{M}]\}}\big(e^{-2d\lambda i/M} - P(A_t \mid \mathcal{G})\big) \leq \frac{3d\lambda}{M}$$

and similarly for t large on $\{\tau = \infty\}$

$$0 \leq \mathbb{1}_{\{Y_t(x) \geq M\}} P(A_t \mid \mathcal{G}) \leq e^{-d\lambda M}.$$

The result now follows by the arbitrariness of M. $\quad\square$

Remark 4.9. The argument generalizes with A_t replaced by

$$\{\forall(z_i, y_i), i = 1, 2, \ldots r, \ N^{z_i, y_i} \cap [t - Y_t, t] = \emptyset\}.$$

We now bring in two probability estimates. The first is a generalization of Lemma 3.5 and follows quickly from the bounds arrived at in its proof.

Corollary 4.10. *Fix $\epsilon_2 \in (0, 1)$. Let $C_n = C_n(\epsilon_2)$ be the event that there exists a (time) interval $I = [T, T + 2^{n\epsilon_2}] \subset [2^{n-1}, 2^{n+1}]$ and sites $x, y \in \mathsf{B}(n^3)$ such that (x, T) does not freely-infect $(y, T + 2^{n\epsilon_2})$ in $\mathsf{B}(n^3)$. There exist constants $c(\lambda), K(d), n_0(\lambda, d, \epsilon_2) > 0$ such that for all $n \geq n_0$ we have*

$$P(C_n) \leq K 2^{n(1-\epsilon_2)} n^{6d} \cdot e^{-c2^{n\epsilon_2}}.$$

Proof. Let $t_j := 2^{n-1} + j \cdot 2^{n\epsilon_2}/2$ and notice that intervals $I_j = [t_j, t_{j+1}]$ for $0 \leq j \leq \lfloor 3 \cdot 2^{n(1-\epsilon_2)} \rfloor$ cover $[2^{n-1}, 2^{n+1}]$. Moreover, if C_n happens then the interval $[T, T + 2^{n\epsilon_2}]$ obtained must contain some I_j. The argument from Lemma 3.5 shows that for any I_j the probability that there are $x, y \in \mathsf{B}(n^3)$ such that (x, t_j) does not freely-infect (y, t_{j+1}) in $\mathsf{B}(n^3)$ is bounded by

$$\sum_{x,y \in \mathsf{B}(n^3)} P\left(\mathrm{Poi}\left(\lambda \cdot (2^{n\epsilon_2}/2)\right) \leq C(d)n^3\right) \leq K(d)n^{6d} \cdot e^{-c(\lambda)2^{n\epsilon_2}}$$

for positive constants $K(d)$ and $c(\lambda)$. The result follows from union bound. $\square$

Lemma 4.11. *Let $\alpha < 1/2$, $\bar{F} \in RV(-\alpha)$ satisfying (18), and fix $\epsilon \in (0, \frac{1}{2} - \alpha)$. The event $B_n = B_n(\epsilon)$ defined by*

$$B_n := \left\{ \begin{array}{c} \exists \text{ distinct } z, z' \in \mathsf{B}(n^3), \ s \in [2^{n-1}, 2^{n+1}] \text{ such that} \\ \mathcal{R}_z \cap [s, s+1] \neq \emptyset, \ \mathcal{R}_{z'} \cap [s, s + 2 \cdot 2^{n\epsilon}] \neq \emptyset \end{array} \right\}$$

satisfies

$$P(B_n) < K \cdot n^{6d} \cdot 2^{-n(1-2\alpha-2\epsilon)}$$

for a positive constant $K = K(\alpha, d)$ and sufficiently large n.

Proof. We simply write event B_n as the union of $B_n(z, z')$ for $z, z' \in \mathsf{B}(n^3)$, where

$$B_n(z, z') := \{\exists s \in [2^{n-1}, 2^{n+1}] : \mathcal{R}_z \cap [s, s+1] \neq \emptyset, \ \mathcal{R}_{z'} \cap [s, s + 2 \cdot 2^{n\epsilon}] \neq \emptyset\}.$$

We then note that $B_n(z, z')$ is in turn a subset of the union

$$\bigcup_{j=2^{n-1}}^{2^{n+1}} \{\mathcal{R}_z \cap [j, j+2] \neq \emptyset, \ \mathcal{R}_{z'} \cap [j, j + 2 \cdot 2^{n\epsilon} + 1] \neq \emptyset\},$$

whose events for fixed z, z' and j will be denoted $B_n(z, z', j)$. By independence, since $z \neq z'$ we have

$$P\left(B_n(z, z', j)\right) = P\left(\mathcal{R}_z \cap [j, j+2] \neq \emptyset\right) \cdot P\left(\mathcal{R}_{z'} \cap [j, j + 2 \cdot 2^{n\epsilon} + 1] \neq \emptyset\right).$$

Let us assume that $\mathcal{R}$ is a non-arithmetic renewal process. The Strong Renewal Theorem (Theorem 1.4 of [5]) provides an estimate

$$P\left(\mathcal{R} \cap [j, j+2] \neq \emptyset\right) \leq U([j, j+2]) \sim C(\alpha)\frac{L(j)}{j^{1-\alpha}} \quad \text{as } j \to \infty,$$

where U denotes the renewal measure associated to F, L is a slowly varying function, and $C(\alpha)$ is a positive constant. Also, the definition of slowly varying function implies the bounds

$$P\big(\mathcal{R} \cap [j, j+2] \neq \emptyset\big) \ll 2^{-n(1-\alpha-\epsilon/2)}, \quad \text{and}$$

$$P\big(\mathcal{R} \cap [j, j+2 \cdot 2^{n\epsilon} + 1] \neq \emptyset\big) \leq \sum_{k=j}^{j+2 \cdot 2^{n\epsilon} - 1} P\big(\mathcal{R} \cap [k, k+2] \neq \emptyset\big)$$

$$\ll (2 \cdot 2^{n\epsilon}) \cdot 2^{-n(1-\alpha-\epsilon/2)}.$$

The result now follows from the usual union bound, at least for the non-arithmetic case. For the arithmetic case, we just have to consider intervals $[j, j+h]$ for h being the span of $\mathcal{R}$ and the same reasoning applies. $\quad \square$

Finally, the following estimate, a result which is similar to Lemma 3 of [10], shows that even in the case in which there are renewal marks on some interval $[2^{n-1}, 2^{n+1}]$, the probability that these marks are too dense on this interval decays rapidly with n.

Lemma 4.12. *Fix $\alpha, \epsilon \in (0, 1)$. There is $g(\alpha) \in (0, 1)$ such that the event $D_n = D_n(\epsilon)$ defined by*

$$D_n := \{\exists z \in \mathsf{B}(n^3), \ I \subset [2^{n-1}, 2^{n+1}] : |I| = 2^{n\epsilon}, \ |\mathcal{R}_z \cap I| \geq n^2 2^{n\epsilon g(\alpha)}\}$$

satisfies

$$P(D_n) \leq K(d) n^{3d} \cdot 2^n \cdot 2^{-c\epsilon^2 n^2}$$

for constants $c > 0$ and $K(d) > 0$ and sufficiently large n.

Proof. Consider the collection of intervals $I_j = [2^{n-1} + j, 2^{n-1} + j + 2^{n\epsilon} + 1]$ for integer j satisfying $0 \leq j \leq 3 \cdot 2^{n-1}$. Then $[2^{n-1}, 2^{n+1}] \subset \cup_j I_j$ and whenever event $D_n(\epsilon)$ happens the interval I obtained must be contained in some I_j and implies there are many renewal marks inside I_j. Denoting $|I_j| = 2^{n\epsilon} + 1$ by l, the proof of Lemma 3 of [10] gives the following estimate

$$P(|\mathcal{R} \cap I_j| \geq l^{1-\epsilon_3} \ln^2 l) \leq 2 \cdot e^{-\ln^2 l} \leq 2^{-c\epsilon^2 n^2} \qquad \text{for large } n, \tag{19}$$

where constant $\epsilon_3 > 0$ satisfies $t^{-(1-\epsilon_3)} \leq \bar{F}(t)$ for large t (the proof of Lemma 3 of [10] only uses the lower bound of condition C). Since $\bar{F}(t) \in RV(-\alpha)$ and $\alpha \in (0, 1)$, we can take $\epsilon_3 := (1 - \alpha)/2$. Let us define $g(\alpha) := 1 - \epsilon_3/2$, so that $g(\alpha) > 1 - \epsilon_3$. It is straightforward to check that

$$n^2 2^{n\epsilon g(\alpha)} \gg l^{1-\epsilon_3} \ln^2 l \qquad \text{as } n \to \infty.$$

Using (19) we conclude that

$$P(D_n) \leq \sum_{z \in \mathsf{B}(n^3)} \sum_{j=0}^{3 \cdot 2^{n-1}} P\big(|\mathcal{R} \cap I_j| \geq n^2 2^{n\epsilon g(\alpha)}\big) \leq K(d) n^{3d} \cdot 2^n \cdot 2^{-c\epsilon^2 n^2}. \quad \square$$

Proof (*Proof of Theorem 4.1, Part (i)*). We assume without loss of generality that x is the origin and denote $Y_t(0)$ simply by Y_t and recall that our estimates hold a.s. on the event $\{\tau = \infty\}$. For $t > 0$ define $n = n(t) := \lfloor \log_2 t \rfloor$, so that $t \in [2^n, 2^{n+1})$. Fix $\epsilon \in (0, 1/2 - \alpha)$ and consider events $B_k(\epsilon)$ and $D_k(\epsilon)$, which are both $\mathcal{G}$-measurable. These events can be used to ensure that

as $t \to \infty$ the renewal marks near $\{0\} \times \{t\}$ are relatively sparse, almost surely. Indeed, by Lemmas 4.11 and 4.12, we have

$$\sum_{k \geq 1} P(B_k \cup D_k) < \infty, \quad \text{implying that} \quad \mathbb{1}_{B_k^c \cap D_k^c} \xrightarrow{\text{a.s.}} 1 \quad \text{as } k \to \infty.$$

On the event $G_n := B_n^c \cap D_n^c$ there is at most one site $z \in \mathsf{B}(n^3)$ with $\mathcal{R}_z \cap [t - 2^{n\epsilon}, t] \neq \emptyset$, since otherwise event B_n happens. Moreover, on D_n^c we must have some interval $I \subset [t - 2^{n\epsilon}, t]$ that has no cure marks of $\mathcal{R}_z$ with length

$$|I| \geq \frac{2^{n\epsilon}}{n^2 2^{n\epsilon g(\alpha)}} = \frac{1}{n^2} \cdot 2^{n\epsilon(1 - g(\alpha))} \gg 2^{n\epsilon'} \qquad \text{as } t \to \infty, \tag{20}$$

for $\epsilon' := \epsilon(1 - g(\alpha))/2$. This implies $\mathsf{B}(n^3) \times I$ is free of renewals.

Event $G_n \in \mathcal{G}$ provides some control on the renewal structure of $\mathsf{B}(n^3) \times [2^{n-1}, 2^{n+1}]$. Now, we discuss two other events that are not $\mathcal{G}$ measurable but will help us handle the infections in $\mathsf{B}(n^3) \times [2^{n-1}, 2^{n+1}]$. The first event is $C_n = C_n(\epsilon')$. By the estimates for $P(C_n)$ from Corollary 4.10 and the Borel–Cantelli Lemma we have that

$$P(\varlimsup_n C_n) = 0 \quad \text{implies} \quad P(\varlimsup_n C_n \mid \mathcal{G}) = 0 \quad \text{a.s.} ,$$

$$\text{and hence} \quad \lim_n P(C_n \mid \mathcal{G}) = 0 \quad \text{a.s.}$$

by Fatou's Lemma. The second event is denoted H_n and defined as

$$H_n := \{ \forall \, m \geq n, \ \exists \, y \in \mathsf{B}(m^3) \setminus \mathsf{B}(2) \text{ with } \xi_s(y) = 1, \ \forall \, s \in [2^m, 2^{m+1}] \}.$$

By Corollary 3.3 and monotone convergence for conditional expectations, we have $P(H_n \mid \mathcal{G}) \to 1$ on event $\{\tau = \infty\}$.

We consider $P(\xi_t(0) = 1 \mid \mathcal{G})$ on survival and on event G_n. There are two cases. We first suppose that $Y_t \leq 2^{n\epsilon}$. The fact that the renewal environment belongs to G_n implies that there are no renewals on $\mathsf{B}(n^3) \times (t - Y_t, t]$. Obviously, the event $\{\xi_t(0) = 0\}$ contains the event

$$A_t = \{ \forall z \sim 0, \ N^{z,0} \cap [t - Y_t(0), t] = \emptyset \} \quad \text{(recall Corollary 4.8)}.$$

Also, on G_n we have

$$H_{n-1} \cap C_n^c \cap A_t^c \subset \{\xi_t(0) = 1\}.$$

Indeed, if $2^{n\epsilon'} \leq Y_t \leq 2^{n\epsilon}$ then the infected site given by H_{n-1} will infect the origin on event C_n^c. If $Y_t < 2^{n\epsilon'}$ then by our discussion next to (20) we can find an interval $I \subset (t - 2^{n\epsilon}, t - Y_t)$ such that the infection provided by H_{n-1} will spread throughout I and guarantee that all neighbours of the origin, which we denote by Γ_0, will be infected at time $\sup I \leq t - Y_t$. Moreover, on G_n the only cure mark in $[t - 2^{n\epsilon}, t]$ is the one at the origin at time $t - Y_t$, implying that Γ_0 is infected at time $t - Y_t$. Hence, the transmission given by A_t^c will infect the origin and we can write $\{\xi_t(0) = 0\} \setminus A_t \subset H_{n-1}^c \cup C_n$, implying

$$P(A_t \mid \mathcal{G}) \leq P(\xi_t(0) = 0 \mid \mathcal{G}) \leq P(A_t \mid \mathcal{G}) + P(H_{n-1}^c \mid \mathcal{G}) + P(C_n \mid \mathcal{G}).$$

By Corollary 4.8, on survival and G_n we have

$$\mathbb{1}_{\{Y_t \leq 2^{n\epsilon}\}} \left| P(\xi_t(0) = 0 \mid \mathcal{G}) - e^{-2d\lambda \cdot Y_t} \right| \to 0 \qquad \text{as } t \to \infty.$$

If $Y_t \geq 2^{n\epsilon}$, then the interval $I \subset [t - Y_t, t]$ provided by (20) is long enough for event C_n^{c} to infect the origin. Thus, on $Y_t \geq 2^{n\epsilon}$ we have

$$0 \leq P(\xi_t(0) = 0 \mid \mathcal{G}) \leq P(H_{n-1}^{\mathsf{c}} \mid \mathcal{G}) + P(C_n \mid \mathcal{G}),$$
$$0 \leq e^{-2d\lambda \cdot Y_t} \leq e^{-2d\lambda 2^{n\epsilon}}$$

implying that $|P(\xi_t(0) = 0 \mid \mathcal{G}) - e^{-2d\lambda \cdot Y_t}| \to 0$ as $t \to \infty$. $\quad\square$

Now, we turn to the proof of Theorem 4.1(ii) and fix $\alpha > 1/2$. We rely on two preliminary results. Given $\epsilon > 0$ and $z \in \mathbb{Z}^d$ we say time interval I is an ϵ-block (for $\mathcal{R}_z$) if $I \backslash \mathcal{R}_z$ contains only intervals of length less than ϵ.

In order to motivate our next proposition, we prove:

Lemma 4.13.   *Given $M, \delta > 0$, there is $\epsilon = \epsilon(d, M, \delta, \lambda) > 0$ so that for s sufficiently large and z a fixed site*

$$P(z \text{ infects a neighbour in } [s, s + M] \mid \mathcal{G}) < \delta$$

on the event where $[s, s + M]$ is an ϵ-block (for $\mathcal{R}_z$).

Proof.   Write $I_0, I_1, \ldots, I_K$ for the (ordered) intervals of $[s, s + M] \backslash \mathcal{R}_z$. Denote by N^z (resp. N_i^z) the union of Poisson processes $N^{y,z}$ (resp. $N^{z,y}$) with $y \sim z$. We note that event $\{z \text{ infects a neighbour in } [s, s + M]\}$ is contained on

$$\{N^z \cap I_0 \neq \emptyset\} \cup \cup_{j=1}^K \{I_j \text{ contains points in } N_i^z \text{ and } N^z\}.$$

Consider event

$$D := \left\{ \begin{array}{c} \forall r \in [0, M] \cap N^z, \ \forall u \in [0, M] \cap N_i^z \\ \text{we have } |r - u| > \epsilon, \text{ and } r > \epsilon \text{ and } u > \epsilon \end{array} \right\}.$$

On event $[s, s + M]$ is an ϵ-block we have that

$$P(z \text{ infects a neighbour in } [s, s + M] \mid \mathcal{G}) \leq P(\theta_s(D^{\mathsf{c}}) \mid \mathcal{G}) \to P(D^{\mathsf{c}}) \text{ as } s \to \infty$$

by Lemma 4.6. Now, we simply notice that fixed $\lambda, d, M > 0$ the probability on the right hand side is continuous function of ϵ and converges to 0 as $\epsilon \downarrow 0$. $\quad\square$

Let us fix z a neighbour of 0. We define event $A_{M,\epsilon}^n$ to be the event that in $[2^n, 2^{n+1})$ there exists t such that $[t, t+1] \cap \mathcal{R}_0 \neq \emptyset$, $[t+1, t+M+1] \cap \mathcal{R}_0 = \emptyset$ and $[t, t+M+1]$ is an ϵ-block (for $\mathcal{R}_z$). Our following result will use the following notation for comparing sequences: we say that $f \asymp g$ if there is $K \geq 1$ such that $(1/K)|g(n)| \leq |f(n)| \leq K|g(n)|$ for every $n \geq 1$.

Proposition 4.14.   *Let F satisfy the conditions of Theorem 4.1 with $\alpha > 1/2$. For $\epsilon > 0$, $M < \infty$,*

$$P(\overline{\lim_n} A_{M,\epsilon}^n) = 1.$$

Proof.   It is based on a second moment argument. We assume $\epsilon < 1$. Consider events $A_j = A_j(M, \epsilon)$ defined by

$$A_j := \left\{ \begin{array}{c} [j, j+1] \cap \mathcal{R}_0 \neq \emptyset, \ [j+1, j+M+1] \cap \mathcal{R}_0 = \emptyset, \\ \text{and } [j, j+M+1] \text{ is an } \epsilon\text{-block for } \mathcal{R}_z \end{array} \right\}$$

and for $n \geq 1$ define the random variables

$$X_n := \sum_{j=2^n}^{2^{n+1}-1} \mathbb{1}_{A_j}$$

that count the number of occurrences of events A_j for $2^n \leq j < 2^{n+1}$. Clearly, the event $A_{M,\epsilon}^n$ contains the event $\{X_n > 0\}$.

The largest part of the proof consists of showing the existence of a $\delta > 0$ independent of n so that $P(X_n > 0) > \delta$ for all $n \geq 1$. Once we have this, we simply note that the desired conclusion follows from Hewitt–Savage's 0–1 law, considering that $\overline{\lim}_n A_{M,\epsilon}^n$ is invariant with respect to finite permutations of the family of i.i.d. random variables $\{(T_i^0, T_i^z); i \geq 0\}$.

By Paley–Zygmund inequality, it suffices to find $K = K(M, \epsilon) < \infty$ so that for n large

$$E X_n^2 \leq K (E X_n)^2.$$

The Strong Renewal Theorem of [5] will play a key role in the bounding of both moments. This states (in our context, recalling that $F(t) > 0$ for $t > 0$ implies that our renewal process is non-arithmetic) that as x becomes large

$$U(x, x+h) x^{1-\alpha} L(x) \to c_\alpha h, \tag{21}$$

where $U(I) := E(|\mathcal{R} \cap I|)$ and c_α is a positive constant. Notice that for all intervals $I = [x, x+h]$ with $0 < h \leq 1$ we have that $U(I)$ is comparable to $P(\mathcal{R} \cap I \neq \emptyset)$. Indeed, by Markov inequality we have

$$P(\mathcal{R} \cap I \neq \emptyset) = P(|\mathcal{R} \cap I| \geq 1) \leq U(I).$$

On the other hand, we have

$$U(x, x+h) = \sum_{j \geq 1} P(|\mathcal{R} \cap I| \geq j) \leq \sum_{j \geq 1} P(|\mathcal{R} \cap I| \geq 1) P(T \leq h)^{j-1}$$
$$= \frac{P(|\mathcal{R} \cap I| \geq 1)}{\bar{F}(h)},$$

where we recall $T \overset{d}{=} \mu$ and $\bar{F}(t) > 0$ for any $t > 0$. This leads to the estimate

$$P(\mathcal{R} \cap [x, x+h] \neq \emptyset) \leq U(x, x+h) \leq \bar{F}(1)^{-1} \cdot P(\mathcal{R} \cap [x, x+h] \neq \emptyset).$$

We now show that $P(A_j)$ is comparable to $U(j, j+1)^2$ by decomposing $P(A_j)$ with respect to what happens at the origin and at z.

It is immediate that

$$P(A_j) \leq P(\mathcal{R}_0 \cap [j, j+1] \neq \emptyset) P(\mathcal{R}_z \cap [j, j+\epsilon] \neq \emptyset) \leq U(j, j+1)^2,$$

For a lower bound, we have that

$$P(\mathcal{R}_0 \cap [j+1, j+M+1] = \emptyset, \ \mathcal{R}_0 \cap [j, j+1] \neq \emptyset) \geq K \cdot U(j, j+1) \cdot \bar{F}(M+2)$$

We claim that $P([j, j+M+1]$ is an ϵ-block) satisfies a similar lower bound, for some constant $K = K(M, \epsilon)$. Indeed, notice that we can find $\eta = \eta(\epsilon) > 0$ such that $F(\epsilon) > F(\eta) > 0$. If we have $\mathcal{R}_z \cap [j, j+\epsilon] \neq \emptyset$ and the next $\lceil (M+1)/\eta \rceil$ random variables T_i of the renewal process satisfy $T_i \in [\eta, \epsilon]$ we will have an ϵ-block, which leads to the bound

$$P([j, j+M+1] \text{ is an } \epsilon\text{-block}) \geq U(j, j+\epsilon) \cdot (F(\epsilon) - F(\eta))^{\lceil (M+1)/\eta \rceil}.$$

These estimates imply that $P(A_j) \asymp U(j, j+1)^2$ for some constant $K(M, \epsilon)$. Using the estimate given by the Strong Renewal Theorem (21), defining $n = n(j)$ as the only integer satisfying $2^n \leq j < 2^{n+1}$ we have

$$P(A_j) \asymp \left(\frac{c_\alpha}{L(j)j^{1-\alpha}} \right)^2 = \left(\frac{c_\alpha}{L(2^n)2^{n(1-\alpha)}} \cdot \frac{L(2^n)2^{n(1-\alpha)}}{L(j)j^{1-\alpha}} \right)^2 \asymp L(2^n)^{-2}2^{2n(\alpha-1)}.$$

Thus, EX_n satisfies

$$EX_n = \sum_{j=2^n}^{2^{n+1}-1} P(A_j) \asymp L(2^n)^{-2}2^{(2\alpha-1)n},$$

which by our assumption on α tends to infinity as n becomes large. To finish the proof we must show that EX_n^2 has an upper bound of the same order of magnitude as $(EX_n)^2$. While proving first moment estimates, we concluded that $P(A_j)$ is comparable to the probability of the event

$$A_j' := \{[j, j+1] \cap \mathcal{R}_0 \neq \emptyset\} \cap \{[j, j+1] \cap \mathcal{R}_z \neq \emptyset\}.$$

The same argument shows that $P(A_j \cap A_k) \leq P(A_j' \cap A_k')$, so it suffices to give an appropriate upper bound to

$$E\left[\left(\sum_{j=2^n}^{2^{n+1}-1} \mathbb{1}_{A_j'} \right)^2 \right] = 2 \sum_{2^n \leq j < k < 2^{n+1}} P(A_j' \cap A_k') + \sum_{j=2^n}^{2^{n+1}-1} P(A_j').$$

Our analysis rests on bounding $P(A_k' \mid A_j')$. We note that for $j < k$ an application of the Markov property on the first renewal inside $[j, j+1]$ implies

$$\inf_{x \in [k-j-1, k-j]} U(x, x+1)^2/K^2 \leq P(A_k'|A_j') \leq \sup_{x \in [k-j-1, k-j]} U(x, x+1)^2$$

for some positive $K(\epsilon, M)$. In particular, an upper bound on $P(A_k' \mid A_j')$ will follow from bounding

$$C_r := \sup_{r-1 \leq x \leq r} U(x, x+1)^2 \quad \text{for } r = k - j.$$

Let $\nu = \nu(\alpha) > 1$ be a fixed constant whose precise value we will determine later. We fix $M' \geq M+1$ so that whenever $x \geq M'$ we have in addition that

$$\left\{ \frac{L(y)}{L(x)} : x \leq y \leq 4x \right\} \cup \left\{ U(x, x+1)x^{1-\alpha}\frac{L(x)}{c_\alpha} \right\} \cup \left\{ \frac{U(x,x+1)}{U(x',x'+1)} : |x - x'| \leq 1 \right\} \subset (\frac{1}{\nu}, \nu).$$

Then for $r \geq M'$: $U(r, r+1)^2\nu^{-2} \leq C_r \leq U(r, r+1)^2\nu^2,$

and so: $c_\alpha^2 r^{-2(1-\alpha)}L(r)^{-2}\nu^{-4} \leq C_r \leq c_\alpha^2 r^{-2(1-\alpha)}L(r)^{-2}\nu^4.$

Once again, our choice of M' yields that for $r \geq M'$

$$\frac{C_{2r-1} + C_{2r}}{C_r} \geq \frac{C_{2r}}{C_r} \geq \nu^{-8}\left(\frac{L(r)}{L(2r)} \right)^2 2^{2(\alpha-1)} \geq \nu^{-10}2^{2(\alpha-1)}. \tag{22}$$

Notice that for any fixed $j \in [2^n, 2^{n+1})$, we have

$$\sum_{k=j}^{2^{n+1}-1} P(A_k' \mid A_j') \leq M' + 1 + \sum_{M'+1}^{2^{n+1}-1} C_r \leq M' + 1 + \sum_{l=1}^{R}\sum_{r \in J_l} C_r \tag{23}$$

where $J_l := (M'2^{l-1}, M'2^l]$ and $R := \inf\{l : M'2^l \geq 2^{n+1}\}$. The bound on (22) implies

$$\sum_{r \in J_l} C_r \leq \nu^{10} 2^{-2(\alpha-1)} \sum_{r \in J_{l+1}} C_r, \quad \text{for any } 1 \leq l < R.$$

Choosing $\nu > 1$ so that $q := \nu^{10} 2^{-2(\alpha-1)} < 1$, we have from (23) that

$$\sum_{k=j}^{2^{n+1}-1} P(A'_k \mid A'_j) \leq M' + 1 + (1 + q + \cdots + q^{R-1}) \sum_{r \in J_R} C_r \leq M' + 1 + (1-q)^{-1} \sum_{r \in J_R} C_r.$$

Since J_R has at most $4 \cdot 2^n$ integer points and our conditions for M' ensure that each C_r, for $r \in J_R$, is comparable to one another, we conclude that

$$\sum_{k=j}^{2^{n+1}-1} P(A'_k \mid A'_j) \leq K 2^n C_{2^{n+1}} \leq K L (2^n)^{-2} 2^{(2\alpha-1)n}$$

for some positive $K(\alpha)$ and the proof is completed. $\quad\square$

Proof (*Proof of Theorem 4.1, Part (ii)*). We fix $M = 1$ and postpone the definition of $\delta = \delta(d, \lambda) > 0$ and $\epsilon = \epsilon(\delta) > 0$ that provide a suitable choice of event $A := \overline{\lim}_n A^n_{1,\epsilon}$. By Proposition 4.14, the event A occurs a.s. for any choice of $\epsilon > 0$. On A we can find arbitrarily large times t such that

$$\mathcal{R}_0 \cap [t, t+1] \neq \emptyset, \quad \mathcal{R}_0 \cap [t+1, t+2] = \emptyset, \quad \text{and} \quad [t, t+2] \text{ is } \epsilon\text{-block for } \mathcal{R}_z.$$

The above property ensures that $Y_{t+2} \in [1, 2]$. Recall that Γ_0 denotes the neighbours of the origin. Using Lemma 4.13 we have that on $A \cap \{\tau = \infty\}$

$$P(\xi_{t+2}(0) = 1 \mid \mathcal{G}) \leq \delta + P\big(\cup_{y \in \Gamma_0 \setminus \{z\}} \{N^{y,0} \cap [t+2-Y_{t+2}, t+2] \neq \emptyset\} \mid \mathcal{G}\big)$$
$$\leq \delta + \big(1 - e^{-(2d-1)\lambda Y_{t+2}} + \delta\big)$$

for a suitable time t, where the last inequality follows from Corollary 4.8 and Remark 4.9 when t is sufficiently large. Hence, defining

$$\eta(d, \lambda) := \inf_{x \in [1,2]} \big(e^{-(2d-1)\lambda x} - e^{-2d\lambda x}\big)$$

we can estimate

$$\big(1 - e^{-2d\lambda Y_{t+2}}\big) - P(\xi_{t+2}(0) = 1 \mid \mathcal{G}) \geq \big(e^{-(2d-1)\lambda Y_{t+2}} - e^{-2d\lambda Y_{t+2}}\big) - 2\delta \geq \eta - 2\delta,$$

which is positive once we define $\delta := \eta/4$. The choice of ϵ is made accordingly, using Lemma 4.13. $\quad\square$

The proof of Theorem 4.4 follows the same lines of the proof of Theorem 4.1. Instead of writing down every detail for this similar proof, we give a sketch of the argument, singling out the main differences.

Proof (*Sketch of Proof of Theorem 4.4*). The proof is equivalent to showing that

(a) If $1 - \alpha > \frac{1}{k+2}$ then $\overline{\lim}_t \big(P(\xi_t(0) = 0 \mid \mathcal{G}) - e^{-(2d-k)\lambda Y_t}\big) \leq 0$.

(b) If $1 - \alpha < \frac{1}{k+1}$ then for each $s > 0$

$$\overline{\lim_t} \, \mathbb{1}_{\{Y_t = s\}}\big(P(\xi_t(0) = 0 \mid \mathcal{G}) - e^{-(2d-k)\lambda Y_t}\big) \geq 0.$$

We consider first b), that is $1 - \alpha < \frac{1}{k+1}$. For $0 < \epsilon$, $M < \infty$ and $z_1, \ldots, z_k \in \Gamma_0$ distinct, we define $A^n_{M,\epsilon}(k)$ to be the event that

$$A^n_{M,\epsilon}(k) := \left\{ \begin{array}{l} \exists t \in [2^n, 2^{n+1}); \; [t, t+1] \cap \mathcal{R}_0 \neq \emptyset, \; [t+1, t+M+1] \cap \mathcal{R}_0 = \emptyset, \\ \text{and } [t, t+M+1] \text{ is an } \epsilon\text{-block for } \mathcal{R}_{z_j}, \text{ for every } 1 \leq j \leq k \end{array} \right\}.$$

The next claim follows Proposition 4.14 closely:

Claim 4.15. *For any $M < \infty$ and $\epsilon > 0$ we have $P(\overline{\lim}_n A^n_{M,\epsilon}(k)) = 1$.*

Proof (*Sketch of Proof*). Proposition 4.14 is the claim with $k = 1$. For $k \geq 2$, introduce random a variable $X_n = \sum_{i=2^n}^{2^{n+1}-1} \mathbb{1}_{G(i)}$ where

$$G(i) = \left\{ \begin{array}{l} [i, i+1] \cap \mathcal{R}_0 \neq \emptyset, \; [i+1, i+M+1] \cap \mathcal{R}_0 = \emptyset \text{ and} \\ [i, i+M+1] \text{ is an } \epsilon\text{-block for } \mathcal{R}_{z_j}, \text{ for every } 1 \leq j \leq k \end{array} \right\}.$$

So, $X_n > 0$ implies that event $A^n_{M,\epsilon}(k)$ occurs. As in Proposition 4.14, for all n large and $2^n \leq i < 2^{n+1}$ we have that

$$P(G(i)) \asymp L(2^n)^{-(k+1)} 2^{-n(k+1)(1-\alpha)},$$

so that

$$E(X_n) \asymp L(2^n)^{-(k+1)} 2^{n(1-(k+1)(1-\alpha))}.$$

Also, for $2^n \leq j - i < 2^{n+1} - 1$ we have that

$$P(G(j) \cap G(i)) \leq P(G(i)) \cdot \sup_{x \in [j-i-1, j-i]} U(x, x+1)^{k+1}$$

and similar computations to Proposition 4.14 lead to

$$\sum_{j=i}^{2^{n+1}-1} P(G(j) \mid G(i)) \leq K L(2^n)^{-(k+1)} 2^{n(1-(k+1)(1-\alpha))}$$

for universal K (depending on ϵ and M). This suffices to bound $E(X_n^2)$ by a universal multiple of $(E X_n)^2$ and leads to the proof of the claim. $\square$

Hence, fixing $M, \delta > 0$ we can follow the proof of Theorem 4.1(ii) (with a slight generalization of Lemma 4.13) to show that for any $s \in [0, M] > 0$ there are infinitely many t_n tending to infinity so that $Y_{t_n} = s$ and

$$P(\xi_{t_n}(0) = 0 \mid \mathcal{G}) - e^{-(2d-k)\lambda s} \geq -k\delta.$$

So that by the arbitrariness of M and δ we have

$$\forall s > 0, \quad \overline{\lim_{t \to \infty}} \mathbb{1}_{\{Y_t = s\}} \left(P(\xi_t(0) = 0 \mid \mathcal{G}) - e^{-(2d-k)\lambda Y_t} \right) \geq 0.$$

In particular if $1 - \alpha < \frac{1}{2d+1}$, then for every $s > 0$ we have

$$\overline{\lim_{t \to \infty}} \mathbb{1}_{\{Y_t = s\}} P(\xi_t(0) = 0 \mid \mathcal{G}) = 1.$$

For the proof of a), we essentially follow the same structure of the proof of Theorem 4.1(i). Notice that when $\alpha \in (1/2, 1)$ the estimates for the probability of events $C_n(\epsilon')$ and $D_n(\epsilon)$ are still available (see Corollary 4.10 and Lemma 4.12, respectively) for ϵ and ϵ' to be chosen below. One important difference is that now we have to consider a variation of event B_n defined in Lemma 4.11. The higher value of α will imply that we expect to have a structure of renewals

that is not as extremely sparse as in the case $\alpha \in (0, 1/2)$, but is still sparse nonetheless. We define $B_n^k = B_n^k(\epsilon)$ by

$$B_n^k := \left\{ \begin{array}{c} \exists \text{ distinct } \{z_j\}_{j=0}^{k+1} \in \mathsf{B}(n^3), \ s \in [2^{n-1}, 2^{n+1}]; \\ \mathcal{R}_{z_0} \cap [s, s+1] \neq \emptyset, \ \mathcal{R}_{z_j} \cap [s, s+2 \cdot 2^{n\epsilon}] \neq \emptyset \text{ for } 1 \leq j \leq k+1 \end{array} \right\}.$$

Adapting the argument of Lemma 4.11 shows that for $\epsilon > 0$ fixed so that $1 - \alpha > \frac{1}{k+2} + \epsilon$ we have for some universal $K = K(k)$

$$P(B_n^k) \leq K n^{3d(k+2)} 2^{-n(k+1-(k+2)\alpha-(k+2)\epsilon)}$$

and $P(B_n^k)$ is summable on n. We choose $\epsilon' < \epsilon$ so that on event D_n^{c} for every $z_0, z_1, z_2, \ldots, z_k \in \mathsf{B}(n^3)$ and every interval $I \subset [2^{n-1}, 2^{n+1}]$ of length $2^{n\epsilon}/2$ there is an interval of length $2^{n\epsilon'}$ with no cure points in $\cup_{i=0}^k \mathcal{R}_{z_i}$.

Thus, taking $\epsilon(k, \alpha)$ small we can ensure that

$$\mathbb{1}_{(B_j^k)^{\mathsf{c}} \cap D_j^{\mathsf{c}}} \to 1 \qquad \text{a.s. as } j \to \infty.$$

Arguing as in the proof of Theorem 4.1(i), taking $n(t) = \lfloor \log_2 t \rfloor$ we show that on event $G_n := (B_n^k)^{\mathsf{c}} \cap D_n^{\mathsf{c}}$ there will be at most $k+1$ different sites $z \in \mathsf{B}(n^3)$ such that $\mathcal{R}_z \cap [t - 2^{n\epsilon}, t] \neq \emptyset$, and also some interval $I \subset [t - 2^{n\epsilon}, t - 2^{n\epsilon}/2]$ with $\mathsf{B}(n^3) \times I$ without cure marks and satisfying $|I| = 2^{n\epsilon'}$. We define event $C_n = C_n(\epsilon')$ for this ϵ'. We have, as in the proof of Theorem 4.1(i) that $P(C_n^{\mathsf{c}} \cap H_{n-1} \mid \mathcal{G}) \to 1$ $a.s.$ Given this, we have on event $Y_t \geq 2^{n\epsilon}$ that during interval $I \subset [t - Y_t, t]$ the origin is infected and then

$$\lim_t \mathbb{1}_{\{Y_t > 2^{n\epsilon}\} \cap G_n} \big(P(\xi_t(0) = 1 \mid \mathcal{G}) - 1 \big) = 0.$$

Now let us consider $\{Y_t \leq 2^{n\epsilon}\} \cap G_n$. Here, the origin is one of the $k+1$ sites with renewals on $\mathsf{B}(n^3) \times [t - 2^{n\epsilon}, t]$. There are at most k neighbours of the origin with cure marks on the interval $[t - 2^{n\epsilon}, t]$, so we have that if $C_n \cup H_{n-1}^{\mathsf{c}}$ does not occur, then at least $2d - k$ neighbours of the origin have no cure marks and must be infected during this whole interval. The probability that none of these neighbours has a transmission to the origin during $[t - Y_t, t]$ can be controlled as $t \to \infty$, see Remark 4.9 succeeding Corollary 4.8. If any of these infected neighbours transmits the infection to the origin then the origin must end up infected at time t. In other words, we have on $\{\tau = \infty\}$ that

$$\underline{\lim}_t \mathbb{1}_{\{Y_t \leq 2^{n\epsilon}\} \cap G_n} \left(P(\xi_t(0) = 1 \mid \mathcal{G}) - \left(1 - e^{-(2d-k)\lambda Y_t}\right) \right) \geq 0,$$

or equivalently, $\overline{\lim}_t \mathbb{1}_{\{Y_t \leq 2^{n\epsilon}\} \cap G_n} \left(P(\xi_t(0) = 0 \mid \mathcal{G}) - e^{-(2d-k)\lambda Y_t} \right) \leq 0.$

This finishes the sketch of the proof of Theorem 4.4.   □

5. An example

As anticipated in the Introduction, we now give an example of distribution μ on $(0, \infty)$ that belongs to the domain of attraction of a stable law of index one, but for which the associated contact renewal process has $\lambda_c = 0$. Of course, it suffices to consider $d = 1$. The question of whether infinite first moment could be enough for $\lambda_c = 0$ remains open for the moment.

Theorem 5.1. *Let $t_0 > e$ be fixed, and consider the probability measure μ on $(0, \infty)$, given by*

$$\mu(t, \infty) = \bar{F}(t) := K L(t)/t, \quad t > t_0, \tag{24}$$

where $L(t) = \exp(\ln t / \ln \ln t)$, *and K is the normalizing constant. If we consider the renewal contact process on $\mathbb{Z}$ with interarrival distribution μ as above, then $\lambda_c = 0$.*

The proof follows the same line of argument as in [10], identifying suitable scales for the tunnelling event to happen with positive probability. Before setting the convenient scales, we recall information about the renewal process under consideration.

Notation. For a renewal process (starting at time zero, say) identified by renewal times $S_k = T_1 + \cdots + T_k$, $k \geq 1$, where the random variables $\{T_i\}_i$ are i.i.d. with distribution μ, we write Z_t and Y_t for the corresponding overshooting and age processes:

$$Z_t = S_{N_t+1} - t; \quad Y_t = t - S_{N_t}, \quad \text{where } N_t \text{ is defined by } S_{N_t} \leq t < S_{N_t+1}. \tag{25}$$

Let also $m(t) = \int_0^t \bar{F}(s)ds$, for $t > 0$. Moreover, when referring to the renewal process attached to site $j \in \mathbb{Z}$ we shall add a superscript j to the corresponding variables.

Remark 5.2. Theorem 6 in [7] implies that if $0 < \theta < 1$, then

$$P\big(Z_t > m^{-1}(\theta m(t))\big) \sim 1 - \theta \quad \text{as } t \to \infty. \tag{26}$$

Lemma 5.3. *For the distribution μ under consideration and $\alpha > 0$ one has*

$$\lim_{t \to \infty} \frac{m(t/(\ln t)^\alpha)}{m(t)} = e^{-\alpha} \tag{27}$$

Proof. We first note that we may apply L'Hôpital's rule to the quotient on the left hand side of (27), thus reducing to looking at the quotient of derivatives of its respective terms, which results in

$$\exp\left[\left\{\frac{\ln t}{\ln(\ln t - \alpha \ln \ln t)} - \frac{\ln t}{\ln \ln t}\right\} - \alpha\left\{\frac{\ln \ln t}{\ln(\ln t - \alpha \ln \ln t)}\right\}\right] \cdot \left(1 - \frac{\alpha}{\ln t}\right) \tag{28}$$

and the claim of the lemma follows readily by checking that the expressions in braces in (28) converge, respectively, to 0 and 1 as $t \to \infty$. □

For the tunnelling event, let us consider the following (time and space) scales: we set $\alpha > 0$, $R_0 \geq e$ and $L_0 = 0$; then, for $k \geq 0$, let

$$R_{k+1} = R_k + \frac{R_k}{(\ln R_k)^\alpha},$$

$$L_{k+1} = \min\{j \geq L_k + 1 : Z_{R_k}^j > R_{k+1} - R_k\}.$$

For convenience, let us write $r_k = R_k - R_{k-1}$ for $k \geq 1$, $r_0 = R_0$, $M_k = \ln r_k$ and $\ell_k = \ln R_k$; let also $\mathcal{I}_0 = [1, L_1]$, and $\mathcal{I}_k = [L_k, L_{k+1}]$, $k \geq 1$. The following statement estimates the growth rate of sequence ℓ_k.

Lemma 5.4. *Let $\beta = (1+\alpha)^{-1}$. Then, provided $R_0 = R_0(\alpha)$ is large enough, we have that $\ell_k \geq (\ell_0 + k)^\beta$ for every $k \geq 0$.*

Proof. We have $R_{k+1} = R_k(1 + \ln^{-\alpha} R_k)$, implying that

$$\ell_k = \ell_{k-1} + \ln\big(1 + \ell_{k-1}^{-\alpha}\big) \geq \ell_{k-1} + \beta \ell_{k-1}^{-\alpha} =: g(\ell_{k-1}), \tag{29}$$

as soon as R_0 is large enough.

We argue the claim of the lemma by induction. It clearly holds true for $k = 0$, since $\ell_0 \geq 1$. Assuming it so does for $k = n \geq 0$, and using the fact that g is increasing in $[1, \infty)$, it follows from (29) that

$$\ell_{n+1} \geq g(\ell_n) \geq g((\ell_0 + n)^\beta) = (\ell_0 + n)^\beta + \beta(\ell_0 + n)^{-\alpha\beta} \geq (\ell_0 + n + 1)^\beta,$$

and the argument for the induction step is complete as soon as the latter inequality is justified. This may be done by writing $(\ell_0 + n + 1)^\beta - (\ell_0 + n)^\beta$, using the Mean Value Theorem, as

$$\frac{\beta}{(\ell_0 + n + \zeta)^{1-\beta}} \leq \frac{\beta}{(\ell_0 + n)^{1-\beta}} = \frac{\beta}{(\ell_0 + n)^{\alpha\beta}},$$

for some $\zeta \in [0, 1]$; the latter identity follows from the choice of β.   □

We also need a lower bound on the growth of $m(R_k)$.

Lemma 5.5.   *Let $\mathcal{L}_k = m(R_{k+1}) - m(R_k)$. Then, if R_0 is large enough, we have that for every $k \geq 0$.*

$$\mathcal{L}_k \geq e^{\sqrt{\ell_k}} \tag{30}$$

Proof.
For R_0 large, we may write $m(R_{k+1}) - m(R_k)$ as

$$K \int_{R_k}^{R_{k+1}} e^{\frac{\ln s}{\ln \ln s}} \frac{ds}{s} = K \int_{\ell_k}^{\ell_{k+1}} e^{\frac{s}{\ln s}} ds \geq K(\ell_{k+1} - \ell_k) e^{\frac{\ell_k}{\ln \ell_k}}. \tag{31}$$

Now,

$$\ell_{k+1} - \ell_k = \ln\left(R_k + \frac{R_k}{\ell_k^\alpha}\right) - \ln(R_k) = \ln\left(1 + \frac{1}{\ell_k^\alpha}\right) \geq \frac{1}{2\ell_k^\alpha}, \tag{32}$$

and the claim of the lemma follows readily by using the above inequality in (31).   □

Proof of Theorem 5.1.   Since the probability of no renewals on $\{0\} \times [0, R_0]$ is positive for any R_0, for the tunnelling it suffices to show that for any value $\lambda > 0$ of the infection rate, we may take R_0 so large that $\sum_{k \geq 0} P(B_k) < 1$, where B_k are events to be shortly defined, such that in the complement of $\cup_{k \geq 0} B_k$ there exists an infection path starting at $\{0\} \times [0, R_0]$ and continuing forever, see Fig. 4. We stress that in order for this strategy to work, R_0 has to be taken sufficiently large along all of the steps of the argument. Similarly to Section 4 of [10], the events B_k are defined as the union of the following events:
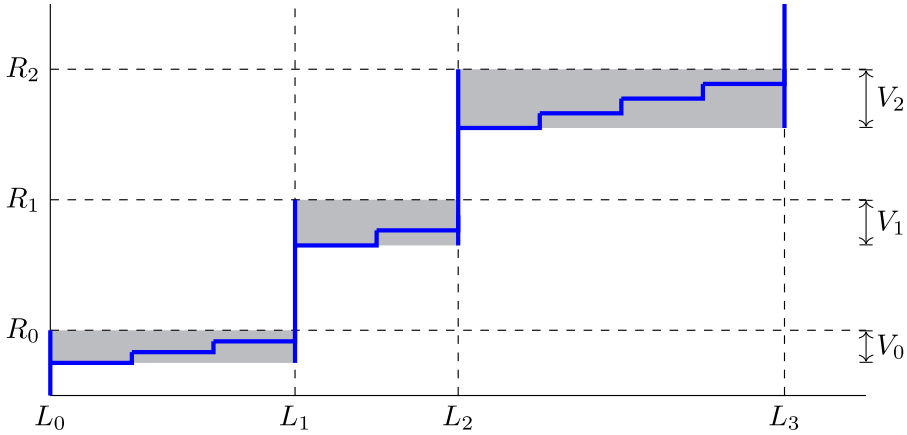
(I) $\{L_{k+1} > L_k + M_k\}$;
(II) For a suitable V_k (as defined below), the rectangle $A_k := \mathcal{I}_k \times [R_k - V_k, R_k]$ is not free of renewal (cure) marks;
(III) $(L_k, R_k - V_k)$ does not freely-infect (L_{k+1}, R_k) in A_k (see Definition 3.1).

In order for this proof strategy to work we need:

(a) To control $P(L_{k+1} > L_k + M_k)$.
(b) To show that for suitable random variables V_k the sum of the probabilities of the events B_k as defined above is indeed less than 1.

Using (26) and Lemma 5.3 we see that for each k, the random variable $L_{k+1} - L_k$ is stochastically dominated by a geometric distribution with parameter $1 - \theta$, where $e^{-\alpha} < \theta < 1$.

Fig. 4. Construction on Theorem 5.1. On the complement of $\cup_{k\geq0}B_k$, grey regions A_k and intervals $\{L_k\} \times [R_{k-1} - V_{k-1}, R_k]$ are free of cure marks, providing sufficient space for the infection from the origin to survive in a straightforward way.

Thus,

$$P(L_{k+1} - L_k > M_k) < \theta^{M_k}. \tag{33}$$

As natural candidate for V_k we have $V_k = \min\{r_k, Y_{R_k}^{L_k+1}, \ldots, Y_{R_k}^{L_k+1}\}$ which we shall explore when $L_{k+1} \leq L_k + M_k$.

Note that if $[a, a+M] \times [s, s+V]$ is a space–time interval free of cure marks and such that site a is infected at time s, then the probability that the infection does not reach the space–time point $(a + M, s + V)$ is bounded by that of $G(M, \lambda) > V$, where $G(M, \lambda)$ has distribution Gamma with parameters M and λ. Indeed, the rightmost infection path will simply move as a Poisson process with rate λ. The result follows if we can prove that, for suitable R_0, the sum over $k \geq 0$ of the probabilities in (33) and those in (34) are less than one,

$$P\left(\min\{r_k, Y_{R_k}^{L_k+1}, \ldots, Y_{R_k}^{L_k+M_k}\} < G(M_k, \lambda)\right) \tag{34}$$

with $G(M_k, \lambda)$ as above, independent of the renewal processes. The probability in (34) is easily seen to be bounded from above by

$$P(G(M_k, \lambda) > r_k) + M_k P(G(M_k, \lambda) > Y_{R_k})$$
$$\leq M_k e^{-\frac{\lambda}{M_k} r_k} + M_k^2 E(e^{-\frac{\lambda}{M_k} Y_{R_k}}). \tag{35}$$

For the second summand on the r.h.s. of (35), we write it in terms of the renewal measure U for μ:

$$E(e^{-\frac{\lambda}{M_k} Y_{R_k}}) = \int_0^{R_k} U(ds)\bar{F}(R_k - s)e^{-\frac{\lambda}{M_k}(R_k - s)}$$
$$\leq e^{-\lambda M_k} + U(R_k) - U(R_k - M_k^2)$$
$$= e^{-\lambda M_k} + \sum_{i=1}^{M_k^2}[U(R_k - M_k^2 + i) - U(R_k - M_k^2 + i - 1)].$$

Using now Lemma 10 (b) in [7] the last sum is bounded from above by $\frac{2M_k{}^2}{m(R_k)}$ so that the second term in the last line of (35) is bounded from above by

$$M_k{}^2 e^{-\lambda M_k} + \frac{2M_k{}^4}{m(R_k)} \leq M_k{}^2 e^{-\lambda M_k} + \frac{2M_k{}^4}{\mathcal{L}_{k-1}}, \tag{36}$$

since $m(R_k) = \mathcal{L}_{k-1} + m(R_{k-1}) \geq \mathcal{L}_{k-1}$. Recall that $M_k = \ln \frac{R_{k-1}}{(\ln R_{k-1})^\alpha}$ and $\ell_k = \ln R_k$. If we choose a sufficiently large $R_0(\alpha)$ then

$$\frac{1}{2}\ell_{k-1} \leq M_k \leq \ell_{k-1}, \qquad \text{for every } k \geq 1.$$

Hence, putting the inequalities above together and using Lemma 5.5, we get

$$\sum_{k \geq 0} P(B_k) \leq \sum_{k \geq 0} \left(\theta^{M_k} + M_k e^{-\frac{\lambda}{M_k} e^{M_k}} + M_k{}^2 e^{-\lambda M_k} + 2M_k{}^4 e^{-M_k^{1/2}} \right)$$

Finally, we notice that the estimate $\ell_k \geq (\ell_0 + k)^\beta$ of Lemma 5.4 ensures that M_k grows fast enough so that the series above converges. Moreover, given $\epsilon > 0$ we may take $\bar{R}(\epsilon)$ so that $\sum_{k \geq 0} P(B_k) < \epsilon$ if $R_0 > \bar{R}(\epsilon)$. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

Funding

References

[1] M. Aizenman, A. Klein, C. Newman, Percolation methods for disordered quantum Ising models, in: R. Kotecký (Ed.), Phase Transitions: Mathematics, Physics, Biology,..., World Scientific, 1994, pp. 1–26, http://dx.doi.org/10.1142/9789814535366.

[2] E.D. Andjel, Survival of multidimensional contact process in random environments, Bol. Soc. Brasil. Mat. (N.S.) 23 (1–2) (1992) 109–119, http://dx.doi.org/10.1007/BF02584814.

[3] C. Bezuidenhout, G. Grimmett, The critical contact process dies out, Ann. Probab. 18 (4) (1990) 1462–1482, http://dx.doi.org/10.1214/aop/1176990627.

[4] M. Bramson, R. Durrett, R.H. Schonmann, The contact processes in a random environment, Ann. Probab. 19 (3) (1991) 960–983, http://dx.doi.org/10.1214/aop/1176990331.

[5] F. Caravenna, R. Doney, Local large deviations and the strong renewal theorem, Electron. J. Probab. 24 (2019) 1–48, http://dx.doi.org/10.1214/19-ejp319.

[6] R. Durrett, D. Griffeath, Supercritical contact processes on Z, Ann. Probab. 11 (1) (1983) 1–15, http://dx.doi.org/10.1214/aop/1176993655.

[7] K.B. Erickson, Strong renewal theorems with infinite mean, Trans. Amer. Math. Soc. 151 (1970) 263–291, http://dx.doi.org/10.2307/1995628.

[8] L.R. Fontes, P.A. Gomes, R. Sanchis, Contact process under heavy-tailed renewals on finite graphs, Bernoulli 27 (3) (2021) 1745–1763, http://dx.doi.org/10.3150/20-bej1290.

[9] L.R. Fontes, D.H.U. Marchetti, I. Merola, E. Presutti, M.E. Vares, Layered systems at the mean field critical temperature, J. Stat. Phys. 161 (1) (2015) 91–122, http://dx.doi.org/10.1007/s10955-015-1307-9.

[10] L.R. Fontes, D.H. Marchetti, T.S. Mountford, M.E. Vares, Contact process under renewals I, Stochastic Process. Appl. 129 (8) (2019) 2903–2911, http://dx.doi.org/10.1016/j.spa.2018.08.007.

[11] L.R. Fontes, T.S. Mountford, M.E. Vares, Contact process under renewals II, Stochastic Process. Appl. 130 (2) (2020) 1103–1118, http://dx.doi.org/10.1016/j.spa.2019.04.008.

[12] O. Garet, R. Marchand, Asymptotic shape for the contact process in random environment, Ann. Appl. Probab. 22 (4) (2012) 1362–1410, http://dx.doi.org/10.1214/11-aap796.

[13] T.E. Harris, Contact interactions on a lattice, Ann. Probab. 2 (6) (1974) 969–988, http://dx.doi.org/10.1214/aop/1176996493.

[14] T.E. Harris, Additive set-valued Markov processes and graphical methods, Ann. Probab. 6 (3) (1978) 355–378, http://dx.doi.org/10.1214/aop/1176995523.

[15] M. Hilário, D. Ungaretti, D. Valesin, M.E. Vares, Results on the contact process with dynamic edges or under renewals, Electron. J. Probab. 27 (2022) 1–31, http://dx.doi.org/10.1214/22-EJP811.

[16] D. Ioffe, A. Levit, Ground states for mean field models with a transverse component, J. Stat. Phys. 151 (6) (2013) 1140–1161, http://dx.doi.org/10.1007/s10955-013-0745-5.

[17] A. Klein, Extinction of contact and percolation processes in a random environment, Ann. Probab. 22 (3) (1994) 1227–1251, http://dx.doi.org/10.1214/aop/1176988601.

[18] T.M. Liggett, Interacting Particle Systems, in: Grundlehren der Mathematischen Wissenschaften, vol. 276, Springer-Verlag, New York, 1985, p. xv+488, http://dx.doi.org/10.1007/978-1-4613-8542-4.

[19] T.M. Liggett, The survival of one-dimensional contact processes in random environments, Ann. Probab. 20 (2) (1992) 696–723, http://dx.doi.org/10.1214/aop/1176989801.

[20] T.M. Liggett, Stochastic Interacting Systems: Contact, Voter and Exclusion Processes, in: Grundlehren der Mathematischen Wissenschaften, vol. 324, Springer-Verlag, Berlin, 1999, p. xii+332, http://dx.doi.org/10.1007/978-3-662-03990-8.

[21] C.M. Newman, S.B. Volchan, Persistent survival of one-dimensional contact processes in random environments, Ann. Probab. 24 (1) (1996) 411–421, http://dx.doi.org/10.1214/aop/1042644723.

[22] R. Pemantle, The contact process on trees, Ann. Probab. 20 (4) (1992) 2089–2116, http://dx.doi.org/10.1214/aop/1176989541.